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Written retake exam in Linear Algebra

Solutions

1. Given matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} .$$

a) Compute eigenvalues of $\mathbf{A}$ and define diagonal matrix $\mathbf{\Lambda}$ with them.

In order to compute eigenvalues of matrix $\mathbf{A}$, we need to solve the characteristic equation

$$\det(\lambda\mathbf{I} - \mathbf{A}) = 0 .$$

Taking into account that

$$(\lambda - 0)(\lambda - 3) - 1(-2) = \lambda^2 - 3\lambda + 0 + 2 = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2)$$

we find

$$\lambda_1 = 1 \quad \text{and} \quad \lambda_2 = 2 .$$

Thus we get the following matrix

$$\mathbf{\Lambda} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} .$$

b) Compute eigenvectors of $\mathbf{A}$ and define matrix $\mathbf{W}$ composed of them.

It is easy to see that for the eigenvalue $\lambda_1 = 1$ the eigenvector $\mathbf{w}_1$ can be chosen as

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{or after normalization to a length of 1 as} \quad \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} .$$

It is also easy to see that for the eigenvalue $\lambda_2 = 2$ the eigenvector $\mathbf{w}_2$ can be chosen as

$$\mathbf{w}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{or after normalization to a length of 1 as} \quad \begin{bmatrix} \frac{\sqrt{5}}{5} \\ 2\frac{\sqrt{5}}{5} \end{bmatrix} .$$

Thus we get the following matrix

$$\mathbf{W} = \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} .$$

c) Check the result of the expression $\mathbf{W}\mathbf{\Lambda}\mathbf{W}^{-1}$.

We should first compute

$$\mathbf{W}^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} .$$

Now, we can easily check that

$$\mathbf{W}\mathbf{\Lambda}\mathbf{W}^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} = \mathbf{A} .$$

d) Compute eigenvalues of $\mathbf{A}^{-1}$.

Eigenvalues of matrix $\mathbf{A}^{-1}$ are

$$\lambda_1^{-1} = 1 \quad \text{and} \quad \lambda_2^{-1} = 0.5 .$$

e) Compute eigenvalues of $\mathbf{A}^3$.

Eigenvalues of matrix $\mathbf{A}^3$ are

$$\lambda_1^3 = 1 \quad \text{and} \quad \lambda_2^3 = 8 .$$

2. Given the following matrix and vector:

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 2 \\ 4 & -1 & 5 \\ -6 & 2 & -4 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 5 \\ 10 \\ -12 \end{bmatrix} .$$

a) Compute the LU decomposition of $\mathbf{A}$.

In order to realize the Gaussian elimination of non-zero elements under the main diagonal of the matrix $\mathbf{A}$, we have to start with position $(2, 1)$. This procedure consists in the left-sided multiplication of $\mathbf{A}$ by the elimination matrix

$$\mathbf{E}_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

The result is

$$\mathbf{E}_{21}\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \\ 4 & -1 & 5 \\ -6 & 2 & -4 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 2 \\ 0 & 1 & 1 \\ -6 & 2 & -4 \end{bmatrix} .$$

The next step is elimination in position $(3, 1)$, which consists in the left-sided multiplication by the elimination matrix

$$\mathbf{E}_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -(-6)/2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} .$$

The result is

$$\mathbf{E}_{31}(\mathbf{E}_{21}\mathbf{A}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \\ 0 & 1 & 1 \\ -6 & 2 & -4 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 2 \\ 0 & 1 & 1 \\ 0 & -1 & 2 \end{bmatrix} .$$

Finally, we have to eliminate the non-zero element in position $(3,2)$, which again consists in the left-sided multiplication by the last elimination matrix

$$\mathbf{E}_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -(-1)/1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} .$$

As a result of these eliminations, the following upper triangular matrix is obtained

$$\mathbf{U} = \mathbf{E}_{32}(\mathbf{E}_{31}\mathbf{E}_{21}\mathbf{A}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \\ 0 & 1 & 1 \\ -6 & 2 & -4 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix} .$$

The lower triangular matrix $\mathbf{L}$ can now be computed as

$$\mathbf{L} = \mathbf{A}\mathbf{U}^{-1} = \mathbf{A}\mathbf{A}^{-1}\mathbf{E}_{21}^{-1}\mathbf{E}_{31}^{-1}\mathbf{E}_{32}^{-1} = \mathbf{E}_{21}^{-1}\mathbf{E}_{31}^{-1}\mathbf{E}_{32}^{-1}$$

but the inverses of the elimination matrices have simply opposite elements off the main diagonal, i.e.

$$\mathbf{E}_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} , \qquad \mathbf{E}_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} , \qquad \mathbf{E}_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

and each of them appears "in its place" in the matrix $\mathbf{L}$, i.e.

$$\mathbf{L} = \mathbf{E}_{21}^{-1}\mathbf{E}_{31}^{-1}\mathbf{E}_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix} .$$

Taking into account that all elements on the main diagonal of the matrix $\mathbf{L}$ are known (they are all equal to one), it is worth noting that we do not need to store them and the remaining relevant elements of both matrices $\mathbf{L}$ and $\mathbf{U}$ can be stored in a single data array of the size of these matrices

$$\begin{matrix} 2 & -1 & 2 \\ 2 & 1 & 1 \\ -3 & -1 & 3 \end{matrix} .$$

b) Solve $\mathbf{A}\mathbf{x} = \mathbf{b}$ using the decomposition from the previous subtask.

We have to solve the equation

$$\mathbf{A}\mathbf{x} = \mathbf{L}\mathbf{U}\mathbf{x} = \mathbf{b} .$$

Substituting

$$\mathbf{y} = \mathbf{U}\mathbf{x}$$

we obtain

$$\mathbf{Ly} = \mathbf{b}\ .$$

The solution $\mathbf{x}$ can be obtained in two steps. First, we calculate the unknown vector $\mathbf{y}$ by forward substitution and then we calculate the proper unknown vector $\mathbf{x}$ by backward substitution. This is illustrated by the following calculations

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ -12 \end{bmatrix}\ ,$$

stąd $y_1 = 5$, $y_2 = 0$, $y_3 = 3$.

$$\begin{bmatrix} 2 & -1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ 3 \end{bmatrix}\ ,$$

stąd ostatecznie $x_3 = 1$, $x_2 = -1$, $x_1 = 1$, czyli

$$\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\ .$$

c) Compute the determinant of $\mathbf{A}$.

Notice first that since

$$\mathbf{A} = \mathbf{LU}$$

then

$$\det \mathbf{A} = \det \mathbf{L} \det \mathbf{U}\ .$$

The matrices $\mathbf{L}$ and $\mathbf{U}$ are triangular, so their determinants are simply the products of the elements on their main diagonals. Therefore

$$\det \mathbf{A} = 1 \cdot 6 = 6\ .$$

d) Compute the inverse of $\mathbf{A}$.

Denote columns of matrix $\mathbf{A}^{-1}$ as $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ and notice that

$$\mathbf{AA}^{-1} = \mathbf{A} \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\ .$$

Using the LU decomposition we can compute matrix $\mathbf{A}^(-1)$ column wise in the following three steps

$$\mathbf{LUx}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\ , \quad \mathbf{LUx}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\ , \quad \mathbf{LUx}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\ .$$

The result is

$$\mathbf{A}^{-1} = \begin{bmatrix} -1 & 0 & -0.5 \\ -7/3 & 2/3 & -1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}\ .$$

e) Compute the decomposition $\mathbf{A} = \mathbf{L}\mathbf{D}\mathbf{U}_1$ such that $\mathbf{D}$ is a diagonal matrix while $\mathbf{U}_1$ is an upper triangular matrix with ones on its main diagonal.

The matrix $\mathbf{L}$ always has elements equal to 1 on the main diagonal. However, the matrix $\mathbf{U}$ does not. Introducing designations $d_1 = u_{11}, d_2 = u_{22}, \ldots, d_n = u_{nn}$, we can write

$$\mathbf{U} = \begin{bmatrix} d_1 & & & & \\ & d_2 & & & \\ & & d_3 & & \\ & & & \ddots & \\ & & & & d_n \end{bmatrix} \begin{bmatrix} 1 & u_{12}/d_1 & u_{13}/d_1 & \cdots & u_{1n}/d_1 \\ & 1 & u_{23}/d_2 & \cdots & u_{2n}/d_2 \\ & & 1 & \cdots & u_{3n}/d_3 \\ & & & \ddots & \vdots \\ & & & & 1 \end{bmatrix} = \mathbf{D}\mathbf{U}_1 \ .$$

The matrix $\mathbf{U}_1$ already has all elements on the main diagonal equal to 1. Applying this reasoning to the matrix

$$\mathbf{U} = \begin{bmatrix} 2 & -1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

we get

$$\mathbf{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

and

$$\mathbf{U}_1 = \begin{bmatrix} 1 & -0.5 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \ .$$

3. In order to change basis (from the old one with vectors $\mathbf{e}_i$ to the new one with vectors $\mathbf{a}_i$) a transformation matrix $\mathbf{T}$ is used

$$\mathbf{T} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \ ,$$

with the meaning that if $\mathbf{x}$ is a vector described with the old basis then its description with the new basis is $\mathbf{T}\mathbf{x}$

a) Determine the old basis vectors $\mathbf{e}_i$ represented with the new basis $\mathbf{a}_i$.

Assume that

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and denote columns of the matrix $\mathbf{T}$ by $\mathbf{t}_1$ and $\mathbf{t}_2$. Taking into account that the vector $\mathbf{x}$ representation with the new basis is

$$\mathbf{T}\mathbf{x} = x_1\mathbf{t}_1 + x_2\mathbf{t}_2$$

we conclude that

$$\mathbf{t}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{t}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

are the old basis vectors represented with the new basis.

b) Determine the new basis vectors $\mathbf{a}_i$ represented with the old basis $\mathbf{e}_i$.

We have to compute the inverse to the matrix $\mathbf{T}$, which is

$$\mathbf{T}^{-1} = \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix} = \begin{bmatrix} \mathbf{t}_{1\text{i}} & \mathbf{t}_{2\text{i}} \end{bmatrix} .$$

Vectors

$$\mathbf{t}_{1\text{i}} = \begin{bmatrix} 2/3 \\ -1/3 \end{bmatrix} \quad \text{and} \quad \mathbf{t}_{2\text{i}} = \begin{bmatrix} -1/3 \\ 2/3 \end{bmatrix}$$

are the new basis vectors represented with the old basis.

c) Assuming that vector

$$\mathbf{y} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

is represented with the new basis find its representation $\mathbf{y}_\text{o}$ with the old basis.

$$\mathbf{y}_\text{o} = \mathbf{T}^{-1}\mathbf{y} = \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} .$$

d) Consider vectors described with the old basis, are the new basis vectors orthogonal?

The new basis vectors represented with the old basis are $\mathbf{t}_{1\text{i}}$ and $\mathbf{t}_{2\text{i}}$. Their scalar product equals

$$\mathbf{t}_{2\text{i}}^\text{T}\mathbf{t}_{1\text{i}} = (-1/3) \cdot (2/3) + (2/3) \cdot (-1/3) = -4/9 \neq 0 .$$

Thus these vectors represented with the old basis are not orthogonal.

e) Consider vectors described with the old basis, are the old basis vectors orthogonal?

The basis vectors (not only old basis but any two-vector basis of two elements) represented by themselves are equal to

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} .$$

They are then obviously not only orthogonal but also orthonormal.

4. Assume that vector $\mathbf{a}_1$ from subtask 3.b) constitutes basis of the projection subspace (the subspace onto which we project)

a) Compute orthoprojector (the perpendicularly projecting matrix) $\mathbf{P}_1$ onto this subspace.

We need vector $\mathbf{a}_1$ represented in the old basis. Let us omit the subscript 1 and write

$$\mathbf{a} = \mathbf{a}_1 = \mathbf{t}_{1\text{i}} = \begin{bmatrix} 2/3 \\ -1/3 \end{bmatrix} .$$

The orthoprojector $\mathbf{P}_1$ — we are looking for — is equal to

$$\mathbf{P}_1 = \mathbf{a}(\mathbf{a}^\text{T}\mathbf{a})^{-1}\mathbf{a}^\text{T} = \begin{bmatrix} 0.8 & -0.4 \\ -0.4 & 0.2 \end{bmatrix} .$$

b) Compute the complementary orthoprojector $\mathbf{P}_2$ that projects perpendicularly in the direction of vector $\mathbf{a}_1$.

The answer is straightforward:

$$\mathbf{P}_2 = \mathbf{I} - \mathbf{P}_1 = \begin{bmatrix} 0.2 & 0.4 \\ 0.4 & 0.8 \end{bmatrix} .$$

c) Compute projector $\mathbf{P}_3$ projecting onto the same subspace as $\mathbf{P}_1$ but in the direction of vector $\mathbf{a}_2$ from subtask 3.b).

The vector $\mathbf{a} = \mathbf{a}_1 = \mathbf{t}_{1\mathrm{i}}$ constitutes basis of the column space of the projector $\mathbf{P}_3$ while the vector $\mathbf{a}_2 = \mathbf{t}_{2\mathrm{i}}$ constitutes basis of its null space. However, the null space is not represented by its basis. On the contrary, it is represented by a homogeneous equation (we need one equation only because the whole space is two dimensional and the null space basis is composed of one vector, i.e., thus the number of homogeneous equations defining the null space is $2 - 1 = 1$). This vector, say $\mathbf{b}$, is perpendicular to $\mathbf{a}_2$,

$$\mathbf{b} \perp \mathbf{a}_2 = \begin{bmatrix} -1/3 \\ 2/3 \end{bmatrix}$$

(and simultaneously constitutes basis of the projector $\mathbf{P}_3$ row space). It is easy to notice that it can be chosen to be equal to

$$\mathbf{b} = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} \perp \begin{bmatrix} -1/3 \\ 2/3 \end{bmatrix} .$$

Therefore

$$\mathbf{P}_3 = \mathbf{a}(\mathbf{b}^\mathrm{T}\mathbf{a})^{-1}\mathbf{b}^\mathrm{T} = \begin{bmatrix} 4/3 & 2/3 \\ -2/3 & -1/3 \end{bmatrix} .$$

d) Compute projector $\mathbf{P}_4$ complementary to projector $\mathbf{P}_3$.

The answer is straightforward:

$$\mathbf{P}_4 = \mathbf{I} - \mathbf{P}_3 = \begin{bmatrix} -1/3 & -2/3 \\ 2/3 & 4/3 \end{bmatrix} .$$

e) Returning to the old and new bases in task 3, answer whether the vectors $\mathbf{P}_i\mathbf{x}$, $i = 1, 2, 3, 4$ ($\mathbf{x}$ is any vector) are expressed with the old or with the new basis? — justify your answer.

New basis (expressed with the old basis) is needed only to define projections. Projections are performed on vectors defined with the old basis and the results are obviously also represented with the old basis.