Lecture 7 - Graph Theory

Introduction to graphs

- A graph is a set of nodes (vertices) and edges (connection between nodes).
- If v_1 and v_2 are connected by an edge e then
    - v_1 and v_2 are endnodes of e.
    - e is incident to v_1 and v_2
    - v_1 and v_2 are adjacent
- A directed graph is a graph where edges have directions (arrows) so when v_1 and v_2 are connected by an edge it doesn't mean that v_2 and v_1 are connected.
- For directed graphs we usually call connections between nodes arcs and for undirected graphs we call them edges.
- Weighted graphs are graphs with numbers assigned to edges (indicating length, cost, probability, etc.)

Terminology

- Parallel edges: edges between same node pair
- Loop: edge starting and ending in same node
- Plain graph: no loops, no parallel edges
- Multigraph: a graph which is not plain
- Subgraph: A graph where the set of nodes is a subset of nodes of a bigger graph and set of edges is a subset of edges of the bigger graph where each edge must connect only the nodes that are in the subgraph's node set.
- Incidence matrix: A matrix of edges x nodes where 1 represents that an edge is incident to a node.

- Adjacency matrix: A matrix of vertices x vertices where a 1 represents a connection between vertices. In a multigraph the adjacency matrix can have numbers bigger than one representing the number of connections.
- Degree of node: number of incident edges.

- Theorem: d_i — degree of v_i

$$|E| = \frac{\sum_i d_i}{2}$$

Where E is set of edges.

- In a directed graph each vertex has two degrees, an in-degree and an out-degree.
- In a directed graph a node with in-degree 0 is called a source and a node with out-degree 0 is called a sink.

$$\sum d_v^i = \sum d_v^o = |A|$$

Where d_v^i means in-degree of a node and d_v^o out-degree of a node, the sums are over all vertices and A is set of all arcs.

- Theorem: In an undirected graph, there is an even number of nodes which have an odd degree.

Isomorphism

- Definition: $G = (V, E)$ and $G^* = (V^*, E^*)$ are isomorphic if:

$$\exists f : V \to V^*[(u, v) \in E \implies (f(u), f(v)) \in E^*] \wedge f \text{ is bijective}$$

- Or in other words two graphs are isomorphic is and only if there exist a bijection between their sets of vertices that preserves adjacency of vertices, i.e. if two nodes are adjacent to one another if G they must be adjacent in G^*.

Homeomorphism

- Definition: $G = (V, E)$ and $G^* = (V^*, E^*)$ are homeomorphic:
    - G and G^* are isomorphic, except that
    - some edges in E^* are divided with additional nodes

Regular graphs

- A regular graphs have all nodes with the same degree.
- An n-regular graph is a graph with all nodes of n degree.

Completely Connected Graph

- A completely connected graph is a graph with every pair of nodes adjacent.
- K_n is a completely connected graph with n nodes.

Bipartite Graph

- A Bipartite graph is a graph where all nodes can be split into two groups and no two nodes are adjacent in the same group, only nodes in different groups can be adjacent.
- A completely bipartite graph is a graph where all nodes in one group are adjacent to all nodes in the second group, and vice versa.
- A $K_{m,n}$ graph is a completely bipartite graph with the first group of nodes of size m and the second of size n.

Walk, Trail, Circuit, Path and Cycle

- Definition: A **walk** is a sequence of nodes and edges from a starting node v_0 to an ending node v_n
- The length of a walk if the number of edges
- If $v_0 = v_n$ the walk is closed
- A **trail** is a walk with no edges repeating
- A **circuit** is a closed trail

- A **spanning trail** covers all edges
- A **path** is a walk with no nodes repeating
- A **cycle** is a closed path
- **Spanning path** visits all nodes

Connected and Disconnected graph

- A connected graph is a graph with a path between every pair of nodes
- A disconnected graph can be divided into connected components
- Not to be confused with completely connected graphs

Distance and Diameter

- A distance between v_i and v_j is length of shortest path between v_i and v_j.
- A diameter of a graph is the largest distance in graph

Cut-Points

- $G - v$: delete v and all its incident edges from G
- v is a cut-point for G iff:
    - G is connected but $G - v$ is not

Directed Walks

- A directed walk is a sequence of nodes that are connected by an arc in directed graph such that all arcs are directed in the same order as the nodes in this sequence
- A semi-walk in directed graph G is a walk in the underlying undirected graph G', but not a directed walk in G. Or in other words it is a walk if we ignore the directions of arcs.
- A semi-trail in directed graph G is a trail in the underlying undirected graph G', but not a directed trail in G. Or in other words it is a trail if we ignore the directions of arcs.

- A semi-path in directed graph G is a path in the underlying undirected graph G', but not a directed path in G. Or in other words it is a path if we ignore the directions of arcs.
- If between every pair of nodes there is:
  - a semi-path: the graph is weakly connected
  - a path from a to b, but not from b to a: the graph is unilaterally connected
  - a path between a to b and b to a: strongly connected

Traversable Graphs

- Definition: G is traversable if and only if G contains a spanning trail
- A node with an odd degree must be either the starting node or the ending node of the trail
- All nodes except the starting node and the ending node must have even degrees

Euler Graphs

- Definition: An Euler graph is graph that contains a closed spanning trail
- G is an Euler graph if and only if all nodes in G have even degrees
- **Euler circuit** is another name for a closed spanning trail.

Hamilton Graph

- Definition: A Hamiltonian graph contains a closed spanning path
- Definition: A Hamiltonian path is a path that visits each vertex of the graph exactly once. So in other words a spanning path.
- Definition: A Hamiltonian cycle is a cycle that visits each vertex exactly once. So in other words a closed spanning path

Planar Graphs

- Definition: G is planar if and only if G can be drawn on a plane without intersecting its edges
- A map of G: a planar drawing of G
- A map divides plane into regions
- Degree of region: length of closed walk that surrounds region
- Theorem:

$$|E| = \frac{\sum_i d_{r_i}}{2}$$

Where d_{r_i} is the degree or region r_i

Euler's Formula

$$|V| - |E| + |R| = 2$$

Where $G = (V, E)$ is a planar, connected graph and R is the set of regions in a map of G.

- Theorem: if $G = (V, E)$ is a connected planar graph where $|V| \geq 3$, then:

$$|E| \leq 3|V| - 6$$

- Theorem: if $G = (V, E)$ connected planar graph where $|V| \geq 3$

$$\exists v \in V[d_v \leq 5]$$

Kuratowski's Theorem

- Theorem: G contains a subgraph homeomorphic to K_5 or $K_{3,3}$ in and only if G is not planar.

Platonic Solids

- Regular polyhedron: a 3-dimensional solid where faces are identical polygons

- Projection of a regular polyhedron onto a plane is a planar graph where the corners are the nodes, the sides are the edges and the faces are the regions.
- There are only 5 platonic solids

Graph Colouring

- Graph colouring problem is a way of colouring the vertices of a graph such that no two adjacent vertices are of the same colour
- A colouring using at most k colours is called a proper k-colouring. The smallest number of colours needed to colour a graph G is called its chromatic number, and is often denoted $\chi(G)$.
- Theorem (Four Colour Theorem): The regions in a map can be coloured using four colours.
- A chromatic polynomial $P(G, \lambda)$ of a graph G gives the number of ways to colour the graph G with λ colours.
    - The chromatic polynomial of a completely connected graph K_n is $\frac{\lambda!}{(\lambda-n)!}$
    - The chromatic polynomial of a completely disconnected graph with n nodes is λ^n
    - The chromatic polynomial of a graph that looks like a snake (like a path) is equal to $\lambda(\lambda-1)^{n-1}$ where n is the number of nodes.
    - The chromatic polynomial of a disconnected graph is equal to the product of chromatic polynomials of its connected components.

Subgraphs

- A **spanning subgraph** contains all of the vertices from the parent graph and need not contain all of the edges
- An **induced subgraph** contains a subset of the vertices of the parent graph along with all of the edges that connect the vertices that exist in both the parent graph and the subgraph.

- A **subgraph** may or may not contain some vertices and edges of the original graph. So in general subgraph can have less edges between the same vertices than the original one.

Complement of s Graph

- The complement of a graph G is a graph G' on the same set of vertices as of G such that there will be an edge between two vertices (v, e) in G', if and only if there is no edge in between (v, e) in G.

Trees and Forests

- A **forest** is a collection of trees. Or in other words it is a disconnected graph with all of its components being trees.
- The **depth** of a node is the length of the closest path between this node and the root node.
- The **level** of a node is its depth + 1
- The **heigh** of a tree is the length of the longest path of a leaf node to the root node.
- A **balanced tree** is a tree in which the height of the left and right subtree of any node differ by not more than 1.
- The successors of a node in a tree are all nodes below it.