Lecture 2 - Functions and Relations

Functions

A function f from a set A to a set B is an assignment of exactly one element of B to each element of A.

We write:

$$f(a) = b$$

if b is the unique element of B assigned by the function f.

Domain and codomain

If $f : A \to B$ we say that A is the domain of f and B is the codomain of f.

Image and preimage

If $f(a) = b$ we say that b is the image of a and that a is the preimage of b.

Range

The range of f is the set of all images of elements of A (the domain).

If we have a subset S of the domain A, we say that the image of S is the set of all images of elements in S. We denote it like this:

$$f(S) = \{f(s) | s \in S\}$$

Properties of a function

Injective (into, one-to-one)

A function f is one-to-one if and only if:

$$\forall x, y \in A \text{ such that if } x = y \text{ then } f(x) = f(y)$$

Surjective (onto)

A function f is onto if and only if for every element $b \in B$ there is an element $a \in A$ such that$f(a) = b$.

Bijective

A function if bijective if and only if it is injective and surjective.

Graphs

A graph of a function f is the set of ordered pairs:

$$\{(a, b) | a \in A, b \in B\}$$

Relation

A relation R is a subset of:

$$R \subset A \times B$$

Where $A \times B$ is a cartesian product of two sets A and B.

When (a, b) belongs to R, a is said to be related to b.

A graph of a function f is also a subset of $A \times B$, and therefore it is a relation. Graphs of functions are relations where $a \in A$ is the first element of the ordered pair only once.

A relation on a set A is a relation from A to A, or in other words it is a subset of $A \times A$. When $n = |A|$ the set $A \times A$ has exactly n^2 elements, and therefore there are 2^{n^2} distinct relations on A

Reflexive Relations

A relation is said to be reflexive if $(a, a) \in R$ for every element in A.

If $n = |A|$ then the number of different reflexive relations that can be defined on A is equal to $2^{n(n-1)}$

Symmetric Relations

A relation is said to be symmetric if $(b, a) \in R$ whenever $(a, b) \in R$.

Antisymmetric Relations

A relation is said to be antisymmetric if $a = b$ whenever $(a, b) \in R$ and $(b, a) \in R$.

Asymmetric Relations

A relation is said to be asymmetric if $(a, b) \in R$ implies that $(b, a) \notin R$.

Transitive Relations

A relation is said to be transitive if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$.

Composite of relations

Let R be a relation from a set A to a set B and S a relation from B to a set C. The composite of R and S is the relation consisting of ordered pairs (a, c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in R$. We denote the composite of R and S by $S \circ R$.

Matrices and relations

If R is a relation from $A = \{a_1, a_2, a_3, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$, then R can be represented by the zero-one matrix $M_R = [m_{ij}]$ with:

$$m_{ij} = 1 \text{ if } (a_i, b_j) \in R$$

$$m_{ij} = 0 \text{ if } (a_i, b_j) \notin R$$

Example

$$R = \{(2,1), (3,1), (3,2)\}$$

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

All matrices representing a relation on some set A are square.

All matrices representing a reflexive relation have 1′s along the main diagonal.

$$M_{ref} = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & 1 \end{bmatrix}$$

All matrices representing a symmetric relation are symmetric along the main diagonal.

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

Directed graphs and relations

Relations can also be represented using a directed graph (graph as in graph theory) or digraphs. An ordered pair $(a, b) \in R$ represents an edge from a vertex

a to a vertex b, where a is called the initial vertex and b is called the terminal vertex.

Equivalence relation

A relation is called an equivalence relation if it is reflexive, symmetric and transitive. Two elements related by an equivalence relation are called equivalent.