Lecture 6 - Recursion

Recurrence Relations

- A recurrence relation for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, a_2, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$ where n_0 is a nonnegative integer.
- A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation.
- In other words, a recurrence relation is like a recursively defined sequence, but without specifying any initial values. Therefore the same recurrence relation can have (and usually has) multiple solutions.

- In general we would prefer to have an explicit formula to compute the value of a_n.
- Those recurrence relations that express the terms of a sequence as linear combination of previous terms can obtain such formulas in a systematic way.

Linear homogeneous recurrence relations

- Definition: A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

Where $c_1, c_2, \ldots, c_k$ are real numbers and $c_k \neq 0$.

- This kind of sequence is uniquely determined by the recurrence relation and the k initial conditions:

$$a_0 = C_0, a_1 = C_1, a_2 = C_2, \ldots, a_{k-1} = C_{k-1}$$

- To determine a linear homogeneous recurrence relation we need to find its characteristic equation:

- Definition: A characteristic equation of a recurrent relation is defined as:

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_{k-1} r - c_k = 0$$

- The solutions of this equation are called the characteristic roots of the recurrence relation.
- Then our recurrence relation has an explicit formula in the form:

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \ldots$$

Where $r_1, r_2, \ldots$ are the characteristic roots and $\alpha_1, \alpha_2, \ldots$ are arbitrary coefficients.

- Theorem: Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0. A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$ for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.