Lecture 9 - Transversals & Tournaments

Transversal

- Let $\mathbb{Y}$ be a finite set, and let $A = (A_1, A_2, A_3, \ldots, A_n)$ be a family of n subsets of $\mathbb{Y}$. A family $(e_1, e_2, \ldots, e_n)$ of elements of $\mathbb{Y}$ is called a system of representatives (SR) of A, provided that

$$e_1 \in A_1, e_2 \in A_2, \ldots, e_n \in A_n$$

- In a system of representatives, the element e_1 belongs to A_1 and thus "represents" the set A_1. If, in a system of representatives, the elements $e_1, e_2, \ldots, e_n$ are all different then $(e_1, e_2, \ldots, e_n)$ is called a system of distinct representatives (SDR or transversal).
- In order for a family $A = (A_1, A_2, \ldots, A_n)$ of sets to have a SDR, it is necessary that the following condition holds:
    - For each $k = 1, 2, \ldots, n$ and each choice of k distinct indices $i_1, i_2, \ldots, i_k$ from $\{1, 2, \ldots, n\}$

$$|A_{i_1} \cup A_{i_2} \cup \ldots \cup A_{i_k}| \geq k$$

- In short, every k sets of the family collectively contain at least k elements.
- This is often called the marriage condition.

The Marriage Problem

- There are n men and m women, and all the men are eager to marry. If there were no restrictions on who matties whom, then, in order to marry off all the men, we need only require that the number m of women be at least as large as the number n of men.

- But we would expect that each man and each woman would insist on some compatibility with a spouse, thereby eliminating some of the women as potential spouses for each man. Thus, each man would arrive at a certain set of compatible women from the set of available women.
- Formally, let $(A_1, A_2, \ldots, A_n)$ be the family of subsets of the women, where A_i denotes the set of compatible women for ith man $(i = 1, \ldots, n)$. Then marrying off all the men corresponds to an SDR $(w_1, w_2, \ldots, w_n)$ of $(A_1, A_2, \ldots, A_n)$. The correspondence is that the ith man marries.

Hall's Theorem

- The marriage problem can also be phrased using the language of graph theory.
- If we have a bipartite graph where the first group of vertices denotes the set of men and the second group denotes the set of women, then the edges represent that a man likes a women.
- Theorem (Hall's Theorem):
    - Definition: A matching in a graph G is a set of edges $M \subseteq E(G)$ such that no two edges in M are incident to the same vertex.
    - The vertices incident to an edge in M are said to be covered by M.
    - A matching is said to be perfect when it covers $V(G)$.
    - The set $N(S)$ of neighbours of some set $S \subseteq V(G)$ of vertices is the image of S under the edge-relation, that is

$$N(S) = \{r | < s - r > \in E(G) \text{ for some } s \in S\}$$

    - S is called a bottleneck if $|S| > |N(S)|$
    - The Hall's theorem says that the matching exists if and only of there are no bottlenecks in G.

Tournaments

- Definition: An orientation of a complete graph K_n with n vertices is called a tournament. It is a digraph such that each distinct pair of vertices is joined by exactly one arc. This arc may have either of the two possible directions.
- In other words a tournament is a directed graph obtained by assigning a direction for each edge in an undirected complete graph.
- Definition: Transitive tournaments are those in which it is possible to list all the players in a final ranking order so that each “player” (represented by a vertex) beats all those further down on the list
- Theorem: Every tournament has a Hamilton path.