Lecture 1 - Introduction to Probability

Sample Space

- Every possible result of a random experiment it called **outcome** and is denoted with ω.
- The set of all possible outcomes is called the **sample space** Ω.
- An **event** is a subset of sample space. We say that event A occurred if the actual outcome $\omega \in A$.

Classical Probability

- The probability of event A:

$$P(A) = \frac{|A|}{|\Omega|}$$

- Where $|S|$ is the cardinality of the set S.
- Principle of indifference: "every outcome is equally likely".
- It holds that:
  - $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
  - $P(A - B) = P(A) - P(A \cap B)$
  - $P(A') = 1 - P(A)$
- if $A_1, \ldots, A_n$ are mutually exclusive (disjoint) then:

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$$

Birthday Paradox

- What is the probability that in a set of 23 people some pair of them will have the same birthday? (For simplicity, assume 365 days in a year)
- Number of possible outcomes: $|\Omega| = 365^{23}$
- Event A - "at least one pair with the same birthday"
- Event A' - "every person has a different birthday"

$$|A'| = 365 \cdot 364 \cdot \ldots \cdot (365 - 22)$$

- Therefore:

$$P(A') = \frac{365 \cdot 364 \cdot \ldots \cdot 343}{365^{23}} \simeq 0.493$$

$$P(A) = 1 - P(A') \simeq 0.507$$

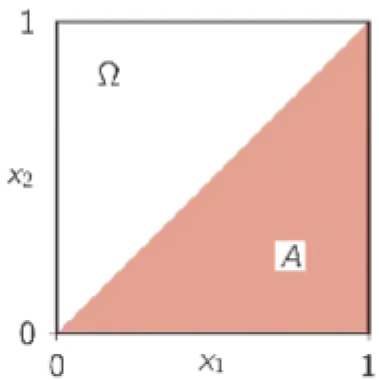
- The probability is surprisingly large (above 50%)
- Direct application to calculating the probability of collision for hash functions.

Geometric Probability

- It is analogous to classical probability, but:
    - The outcomes of an experiment are points in $\mathbb{R}^n$
    - The events are sets in $\mathbb{R}^n$
    - The "size" of a set is its n-dimensional measure: length (n=1), area (n=2), volume (n=3), etc.
- For each subset $A \subset \mathbb{R}^n$, let $|A|$ denote its n-dimensional measure
- Sample space $\Omega \subset \mathbb{R}^n$, where $|\Omega| < \infty$
- Events $A \subset \Omega$ are subsets of Ω
- The probability of A:

$$P(A) = \frac{|A|}{|\Omega|}$$

- Principle of indifference: “Every point in Ω is equally likely”
- Example: We draw two points x_1 and x_2 from a unit interval $[0, 1]$. Calculate the probability that $x_1 > x_2$?



Sample space: set of pairs (x_1, x_2)

$$\Omega = [0,1] \times [0,1] = [0,1]^2$$

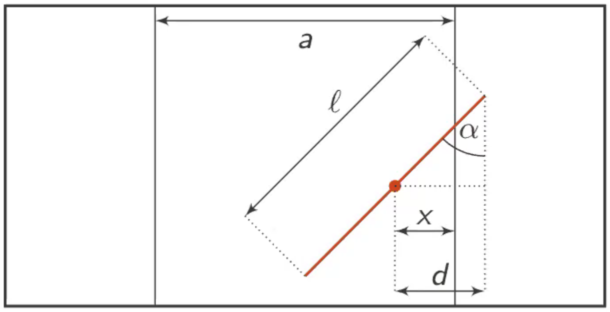
The event we are interested in:

$$A = \{(x_1, x_2) \in \Omega : x_1 > x_2\}$$

$$P(A) = \frac{|A|}{|\Omega|} = \frac{1/2}{1} = \frac{1}{2}$$

Buffon’s Needle

- We drop a needle of length l onto the floor made of parallel strips of wood of the same width $a \geq l$. What is the probability that the needle will lie across a line between two strips?



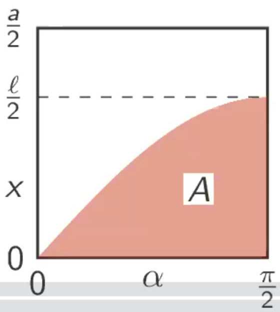
Outcomes are pairs (x, α):

x – distance from the center of the needle to the closest line, $x \in [0, a/2]$

α – acute angle between the needle and one of the lines, $\alpha \in [0, \frac{\pi}{2}]$

$\Omega = [0, a/2] \times [0, \frac{\pi}{2}]$, $|\Omega| = \frac{a\pi}{4}$

The needle will lie across a line if $x \leqslant d = \frac{\ell}{2} \sin\alpha$ (event A)



$$P(A) = \frac{|A|}{|\Omega|} = \frac{4}{a\pi}\frac{\ell}{2}\underbrace{\int_0^{\pi/2} \sin\alpha \, \mathrm{d}\alpha}_{=1} = \frac{2\ell}{a\pi}$$

If $\ell = \frac{a}{2}$, $P(A) = \frac{1}{\pi}$. Can be used to experimentally estimate the number π!

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