Lecture 12 - Limit Theorems II

Standardization of a random variable

- A random variable X for which $EX = 0$ and $D^2(X) = 1$ is called standardized
- For any random variable X a variable:

$$U = \frac{X - EX}{D(X)}$$

- Is standardized

Bernoulli scheme

- We have:

$$S_n = \sum_{i=1}^{n} X_i, \quad \overline{X}_n = \frac{S_n}{n}$$

- If we standardize S_n (or $\overline{X}_n$):

$$U_n = \frac{S_n - np}{\sqrt{np(1-p)}} = \frac{\overline{X}_n - p}{\sqrt{p(1-p)}}\sqrt{n}$$

- We can see that the standardized version of S_n (or $\overline{X}_n$) cannot converge to a number, because all its elements have unit standard deviation.

De Moivre-Laplace theorem

- Take a sequence of independent random variables $X_1, X_2, X_3, \ldots$ where $X_n \sim B(p)$ for all n

$$S_n = \sum_{i=1}^{n} X_n \sim B(n,p), \quad U_n = \frac{S_n - np}{\sqrt{np(1-p)}}$$

- The c.d.f. of U_n converges to a c.d.f. of a standard normal distribution:

$$\forall x \quad \lim_{n\to\infty} F_{U_n}(x) = \phi(x)$$

- Conclusion: for large n, the distribution of U_n can be approximated by $N(0,1)$:

$$U_n = \frac{S_n - np}{\sqrt{np(1-p)}} \simeq Z, \quad \text{where } Z \sim N(0,1)$$

- Equivalently, the distribution of $S_n \sim B(n,p)$ can be approximated by:

$$S_n \simeq \left(\sqrt{np(1-p)}Z + np\right) \sim N(np, np(1-p))$$

- Conclusion: for large n, binomial distribution $B(n,p)$ can be approximated by a normal distribution $N(np, np(1-p))$

Central Limit Theorem

Lindeberg-Levy theorem

- $X_1, X_2, X_3, \ldots$ — sequence of independent and identically distributed random variables, $EX_i = \mu$, $D^2(X_i) = \sigma^2$.
- According to the law of large numbers:

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \overset{\text{w. pr. 1}}{\to} \mu$$

- We can, however, standardize the sequence of arithmetic means:

$$U_n = \frac{\overline{X}_n - E\overline{X}_n}{D(\overline{X}_n)} = \frac{\overline{X}_n - \mu}{\sigma}\sqrt{n}$$

- Then:

$$\forall x \quad \lim_{n \to \infty} F_{U_n}(x) = \phi(x)$$

- Where ϕ is the c.d.f. of $N(0,1)$
- It can be written as (with notation introduced in the next section):

$$U_n \xrightarrow{D} Z, \ \text{ where } Z \sim N(0,1)$$

- Conclusion: the convergence of the standardized arithmetic mean of independent random variables to the normal distribution is universal.

Convergence in distribution

- We say that a sequence of random variables $X_1, X_2, ...$ converges to a random variable X:
  - With probability one (denoted $X_n \overset{\text{w. pr. 1}}{\rightarrow} X$) if:

$$P(\lim_{n \to \infty} X_n = X) = 1$$

  - In probability (denoted $X_n \xrightarrow{P} X$) if:

$$\forall \epsilon > 0 \quad \lim_{n \to \infty} P(|X_n - X| > \epsilon) = 0$$

  - In distribution (denoted $X_n \xrightarrow{D} X$) if:

$$\lim_{n \to \infty} F_{X_n}(x) = F_X(x) \text{ for each } x \text{ at which } F_x \text{ is continuous}$$

- It holds that:

$$X_n \overset{\text{w. pr. 1}}{\rightarrow} X \implies X_n \xrightarrow{P} X \implies X_n \xrightarrow{D} X$$

- So convergence in distribution is the weakest type of convergence.