Lecture 4 - Independence

Independence - definition

- Two events A and B are called independent, if:

$$P(A \cap B) = P(A)P(B)$$

- Notation: $A \perp B$
- Remark: If $P \perp B$ then $P(A|B) = P(A)$ and vice versa
- The independence relation is symmetric
- Two disjoint events can be independent if and only if $P(A) = 0$ or $P(B) = 0$
- An event can be independent from itself if and only if $P(A) = 0$ or $P(A) = 1$
- Any event is independent with events Ω and $\emptyset$.
- If $A \perp B$ then $A' \perp B$, $A \perp B'$ and $A' \perp B'$

More than two independent events

- Consider three events A_1, A_2, A_3 for which:

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$$

- Does it also hold $A_1 \perp A_2$, $A_1 \perp A_3$, $A_2 \perp A_3$?
- Answer: No
- If it holds that $A_1 \perp A_2$, $A_1 \perp A_3$, $A_2 \perp A_3$ then $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$?
- Answer: also no
- Events $A_1, A_2, \ldots, A_n$ are mutually independent, if for every subset of indices $S \subseteq \{1, 2, \ldots, n\}$

$$P(\bigcap_{i\in S} A_i) = \prod_{i\in S} P(A_i)$$

- Example: Events A_1, A_2, A_3 are independent if

$$\begin{gathered} P(A_1 \cap A_2) = P(A_1)P(A_2) \\ P(A_1 \cap A_3) = P(A_1)P(A_3) \\ P(A_2 \cap A_3) = P(A_2)P(A_3) \\ P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3) \end{gathered}$$

Conditional independence

- Events A and B are called conditionally independent given event C if:

$$P(A \cap B|C) = P(A|C)P(B|C)$$

- Notation $A \perp B|C$
- Meaning: Is event C happened then the occurrence of B(A) does not influence the probability of the occurrence of A(B).
- The conditional independence of two events $A \perp B|C$ does not imply the independence $A \not\perp B$, and vice versa, independence $A \perp B$ does not imply the conditional independence $A \not\perp B|C$

Product spaces

- Consider n probabilistic spaces $(\Omega_i, \mathcal{F}_i, P_i)$, $i = 1, \ldots, n$, concerning n **independent** trials of a random experiment
- A product space is a probabilistic space $(\Omega, \mathcal{F}, P)$, where:
  - $\Omega = \Omega_1 \times \Omega_2 \times \ldots \times \Omega_n$
  - $\mathcal{F}$ is a σ-algebra of events generated from events of the form $A_1 \times \ldots \times A_n$, $A_i \in \mathcal{F}_i$
  - $P(A_1 \times \ldots \times A_n) = P_1(A_1) \cdot P_2(A_2) \cdot \ldots \cdot P_n(A_n)$

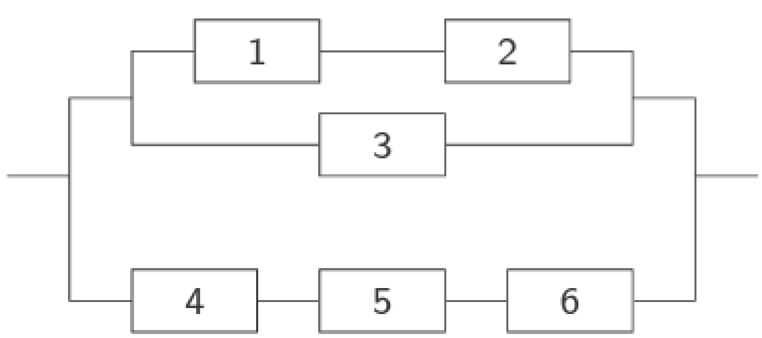
- From the definition, events concerning different trials in the experiment, e.g., $A_1 \times \Omega_2 \times \ldots \times \Omega_n$
- Example: We roll two dice
- $\Omega_1 = \Omega_2 = \{1, 2, 3, 4, 5, 6\}$
- $P_1 = P_2$, $P_1(\{j\}) = \frac{1}{6}$, $j = 1, \ldots, 6$
- $\Omega = \Omega_1 \times \Omega_2$, $\Omega = \{(i, j) : i, j = 1, \ldots, 6\}$
- $P(\{(i, j)\}) = P_1(\{i\})P_2(\{j\}) = \frac{1}{6} \cdot \frac{1}{6}$
- An event concerning the first die $A = A_1 \times \Omega_2$
- E.g. A — we rolled 6 on the 1st die
- $A = \{6\} \times \Omega_2 = \{(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$
- An event concerning the second die $B = \Omega_1 \times A_2$

$$
\begin{gathered}
P(A) = P(A_1 \times \Omega_2) = P_1(A_1) \cdot P_2(\Omega_2) = P_1(A_1) \\
P(B) = P(\Omega_1 \times A_2) = P_1(\omega_1) \cdot P_2(A_2) = P_2(A_2) \\
P(A \cap B) = P(A_1 \times A_2) = P_1(A_1) \cdot P_2(A_2)
\end{gathered}
$$

- So $P(A \cap B) = P(A)P(B) \implies A \perp B$

System reliability

- If each of n independent components of a system can fail with probability p_i, $i = 1, \ldots, n$, what is the probability of failure of the entire system?

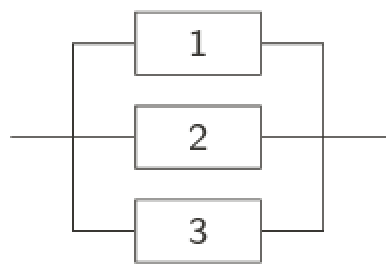


- System fails if there is no path between the source (left) and the sink (right), which is free of any failed components.
- A series system fails if at least one of the components fails

- A_i — ith component failed
- A — system failed, $A = A_1 \cup \ldots \cup A_n$

$$P(A) = 1 - (1 - p_1) \cdot \ldots \cdot (1 - p_n)$$

- (Because those events are independent)
- A parallel system fails if all components fail



- A_i — ith component failed
- A — system failed, $A = A_1 \cap \ldots \cap A_n$

$$P(A) = p_1 \cdot \ldots \cdot p_n$$

Bernoulli scheme

- We perform a series of independent trials (experiments) each of which ends with success with probability p and failure with probability 1-p. Calculate the probability of k successes in n trials.
- We code a success by 1 and failure as 0
- $\Omega = \{(b_1, \ldots, b_n) : b_i \in \{0, 1\}\}$

- If $k = b_1 + b_2 + \ldots + b_n$ is the number of successes then:

$$P(\{(b_1, \ldots, b_n)\}) = \underbrace{p \cdot \ldots \cdot p}_{k} \cdot \underbrace{(1-p) \cdot \ldots \cdot (1-p)}_{n-k} = p^k (1-p)^{n-k}$$

- If A_k — k successes in n trials, then:

$$P(A_k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Random walks

- It is sometimes reasonable to have no predefined number of trials n
- Random walk: We move left or right with probability $\frac{1}{2}$. What is the chance that we ever move more than 10 steps away from the starting point?
- Example: There are n steps from a bar to home. A student is k steps from home. With probability $\frac{1}{2}$ he moves a step towards home or a bar. If he goes back to bar, he will never leave 💀. What is the chance of getting home?
- S_k — student will come home when starting k steps from it
- H_+ — first step towards home
- H_- — first step towards bar
- Since $H_+ \cup H_- = \Omega$ and $H_+ \cap H_- = \emptyset$

$$P(S_k) = P(S_k | H_+) P(H_+) + P(S_k | H_-) P(H_-)$$

- And also $P(S_k | H_+) = P(S_{k-1})$ and $P(S_k | H_-) = P(S_{k+1})$
- Let $s_k = P(S_k)$ and multiplying by 2, we get a recurrence:
- $2s_k = s_{k-1} + s_{k+1}$ with boundary conditions $s_0 = 1$ and $s_n = 0$
- It reduces to $s_k - s_{k-1} = s_{k+1} - s_k$
- Consecutive differences $r = s_{k+1} - s_k$ are identical for all k!

- So, $s_1 = s_0 + r, s_2 = s_0 + 2r, \ldots, s_n = s_0 + nr$
- And since $s_n = 0$ and $s_0 = 1$ then $r = -\frac{1}{n}$
- Therefore $P(S_k) = 1 - \frac{k}{n}$
- But, when $p \neq \frac{1}{2}$, then $P(S_k) = \frac{r^k - r^n}{1 - r^n}$, where $r = \frac{p}{1-p}$