Lecture 4 - Advanced Counting

Permutations

- A permutation of a set of distinct objects is an ordered arrangement of there objects.
- An ordered arrangement of r elements of a set of n elements is called an r-permutation
- Theorem: The number of r-permutations of a set of n distinct elements is:

$$P(n, r) = \frac{n!}{(n-r)!}$$

- In particular $P(n, n) = n!$

Note here that the order is important

The number of r-permutations (variations) of a set of n objects with repetition allowed is n^r.

- Theorem: Suppose a collection consists of n objects of which
  - n_1 are of type 1, indistinguishable from each other
  - n_2 are of type 1, indistinguishable from each other, ...
  - n_k are of type k, indistinguishable from each other
- And $n_! + n_2 + ... + n_k = n$
- Then the number of distinct permutations of the n objects is:

$$\frac{n!}{n_1! n_2! \dots n_k!}$$

Combinations

- Whereas permutations consider order, combinations are used when order does not matter.

- Definition: A k-combination of elements of a set is an unordered selection of k elements from the set.
- Theorem: The number of k-combinations of a set of cardinality n is:

$$C(n,k) = \binom{n}{k} = \frac{n!}{(n-k)!k!}$$

- This reads “n choose k”

- Theorem: The number of ways to fill r slots from n categories with repetition allowed is:

$$\binom{n+r-1}{r} = C(r+n-1, n-1)$$

Alternatively this can be denoted as:

$$\left(\binom{n}{r}\right) = \binom{n+r-1}{r}$$

Integer solutions to equations

- One type of problem that can be solved by the generalized combination formula is of the form:

> 💡 How many non-negative integer solutions are there to the equation a+b+c+d=100?

- As we can see here we have a 100 slots to fill from 4 categories so the solution to this problem is $\left(\binom{100}{4}\right)$.
- How many integer solutions are there to:

$$a+b+c+d=15$$

When $a \geq 3$, $b \geq 0$, $c \geq 2$ and $d \geq 1$?

To solve this problem we need to deduce 6 (where six is the sum of constrains) from the number of slots (15) and we get $\left(\binom{9}{4}\right)$.

- How many integer solutions are there to:

$$a + b + c + d = 15$$

When $a \geq -3$, $b \geq 0$, $c \geq -2$, and $d \geq -1$?

In this case, we alter the restrictions and equation so that negative restrictions “go away”. To do this, we need to set all negative restrictions to zero and add their absolute values to the right hand side of the equation:

$$a + b + c + d = 15 + 3 + 2 + 1 = 21$$

And $a \geq 0, b \geq 0, c \geq 0, d \geq 0$

So the final solution is $\left(\binom{21}{4}\right)$.

Principle of Inclusion-Exclusion

In the general case of Inclusion-exclusion we can count the number of cases when at least one of n events occurs (disjunction). Then using the set theorical notation the number of cases is:

- **Theorem**: Let $A_1, A_2, \ldots, A_n$ be finite sets, then

$$\begin{aligned} |A_1 \cup A_2 \cup \ldots \cup A_n| = &\textstyle\sum_i |A_i| \\ &- \textstyle\sum_{i<j} |A_i \cap A_j| \\ &+ \textstyle\sum_{i<j<k} |A_i \cap A_j \cap A_k| \\ &- \ldots \\ &+ (-1)^{n+1} |A_1 \cap A_2 \cap \ldots \cap A_n| \end{aligned}$$

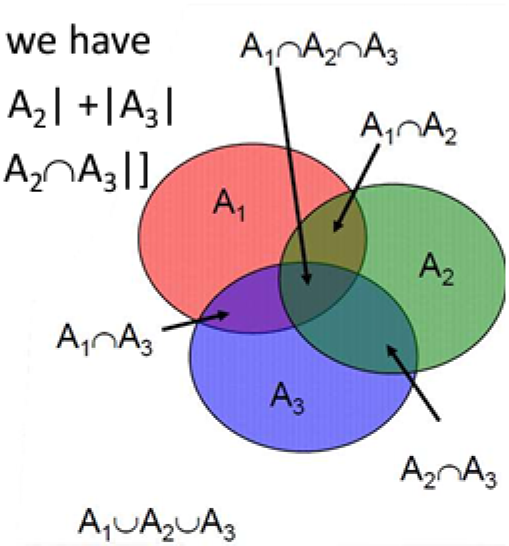
Each summation is over

- all i,
- pairs i,j with i<j,
- triples with i<j<k, etc.

There each set represents the number of different outcomes of an event.

- **Example:** when n=3, we have

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - [|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|] + |A_1 \cap A_2 \cap A_3|$$



- **Example:** To illustrate, when n=4, we have

$$\begin{aligned} |A_1 \cup A_2 \cup A_3 \cup A_4| = {} & |A_1| + |A_2| + |A_3| + |A_4| \\ & - [|A_1 \cap A_2| + |A_1 \cap A_3| + |A_1 \cap A_4| \\ & + |A_2 \cap A_3| + |A_2 \cap A_4| + |A_3 \cap A_4|] \\ & + [|A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| \\ & \quad + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4|] \\ & - |A_1 \cap A_2 \cap A_3 \cap A_4| \end{aligned}$$

Generalized Pigeonhole principle

- Theorem: If N objects are placed into k boxes then there is at least one box containing at least $\lceil \frac{N}{k} \rceil$ objects
- A probabilistic generalization states that:
  - If n objects are randomly put intro m boxes with uniform probability (i.e., each object is placed in a given box with probability $\frac{1}{m}$)
  - Then at least one box will hold more than one object with probability

$$1 - \frac{m!}{(m-n)!m^n}$$

- For n = 0 and n = 1 that probability is zero; in other words, if there is just one pigeon, there cannot be a conflict
- For $n > m$ it is one, in which case it coincides with the ordinary pigeonhole principle.
- For $n \leq m$ there is still a substantial chance that clashes will occur. For example if 2 pigeons are randomly assigned to 4 pigeonholes, there is a 25% chance that at least one pigeonhole will hold more than one pigeon.

Derangements

- Derangements are permutations of objects such that no element is in its original position
- Example: 21453 is a derangement of 12345 but 21543 is not
- The number of derangements of a set with n elements is:

$$D_n = n!\left(1 - \frac{1}{1!} + \frac{2}{2!} - \frac{3}{3!} + \ldots (-1)^n \frac{1}{n!}\right)$$

- Note that $\lim_{n\to\infty} \frac{D_n}{n!} = e^{-1}$

At least and Exactly

> 💡 How many arrangements of CUCURBITAPEPO with exactly one pair of the same consecutive letters exist?

We need to use the general formula:

$$E_m = \binom{m}{0} S_m - \binom{m+1}{1} S_{m+1} + \binom{m+2}{2} S_{m+2} - \ldots \pm \binom{n}{n-m} S_n$$

Where $S_i = \binom{n}{i}\frac{(k-i)!}{a_1!a_2!a_3!\ldots}$ is the number of arrangements of k letter word with at least i pairs of the same consecutive letters and $a_1, a_2, a_3, \ldots$ are the numbers of repeating letters.

💡 How many arrangements of CUCURBITAPEPO with at least one pair of the same consecutive letters exist?

To solve this we need this formula:

$$\alpha_m = \binom{m-1}{0} S_m - \binom{m}{1} S_{m+1} + \binom{m+1}{2} S_{m+2} - \ldots \pm \binom{n-1}{n-m} S_n$$