Observed data
$x = (x_1, x_2, ..., x_N)$

Hidden variables
$Z = (Z_1, Z_2, ..., Z_N)$

Two hidden states
$1, 2$

Two emission probability
$P(X_i = x| Z_i = 1) = f_{\varphi_1}(x)$
$P(X_i = x| Z_i = 2) = f_{\varphi_3}(x)$

Transition probability
$P(Z_i = j| Z_{i-1} = k) = a_{jk}, \ j, k \in \{1, 2\}$

initial probability
$P(Z_1 = i) = \pi_i, \ i \in \{1, 2\}$

Variables to be inferenced
$\theta = \{\varphi, A, \pi\}$

E step :

$$Q(\theta, \theta_s) = E_{Z|X;\theta_s}[log(P(Z, X; \theta))]$$

$$\begin{aligned} P(Z, X; \theta) &= P(Z_1)P(X_1|Z_1)P(X_2|Z_2)P(Z_2|Z_1) \ldots P(X_N|Z_N)P(Z_N|Z_{N-1}) \\ &= P(Z_1)P(X_1|Z_1)\prod_{i=2}^{N} P(X_i|Z_i)P(Z_i|Z_{i-1}) \end{aligned}$$

$$P(Z_1) = \pi_1^{\mathbb{1}(Z_1 = 1)} \pi_2^{\mathbb{1}(Z_1 = 2)}$$

$$P(X_i|Z_i) = f_{\varphi_1}(X_i)^{\mathbb{1}(Z_i = 1)} f_{\varphi_2}(X_i)^{\mathbb{1}(Z_i = 2)}$$

$$P(Z_i|Z_{i-1}) = a_{11}^{\mathbb{1}(Z_i = 1)\mathbb{1}(Z_{i-1} = 1)} a_{12}^{\mathbb{1}(Z_i = 2)\mathbb{1}(Z_{i-1} = 1)} a_{21}^{\mathbb{1}(Z_i = 1)\mathbb{1}(Z_{i-1} = 2)} a_{22}^{\mathbb{1}(Z_i = 2)\mathbb{1}(Z_{i-1} = 2)}$$

$$\begin{aligned} P(Z, X; \theta) = \ & \pi_1^{\mathbb{1}(Z_1 = 1)} \pi_2^{\mathbb{1}(Z_1 = 2)} \cdot \\ & \prod_{i=1}^{N} f_{\varphi_1}(X_i)^{\mathbb{1}(Z_i = 1)} f_{\varphi_2}(X_i)^{\mathbb{1}(Z_i = 2)} \cdot \\ & \prod_{i=2}^{N} a_{11}^{\mathbb{1}(Z_i = 1)\mathbb{1}(Z_{i-1} = 1)} a_{12}^{\mathbb{1}(Z_i = 2)\mathbb{1}(Z_{i-1} = 1)} a_{21}^{\mathbb{1}(Z_i = 1)\mathbb{1}(Z_{i-1} = 2)} a_{22}^{\mathbb{1}(Z_i = 2)\mathbb{1}(Z_{i-1} = 2)} \\ = \ & \prod_{i=1}^{2} \pi_1^{\mathbb{1}(Z_1 = i)} \prod_{i=1}^{N}\prod_{j=1}^{2} f_{\varphi_j}(X_i)^{\mathbb{1}(Z_i = j)} \prod_{i=2}^{N}\prod_{j=1}^{2}\prod_{k=1}^{2} a_{jk}^{\mathbb{1}(Z_i = k)\mathbb{1}(Z_{i-1} = j)} \end{aligned}$$

$$\begin{aligned} log(P(Z, X; \theta)) = \ & \sum_{i=1}^{2} \mathbb{1}(Z_1 = i) log(\pi_1) + \sum_{i=1}^{N}\sum_{j=1}^{2} \mathbb{1}(Z_i = j) log(f_{\varphi_j}(X_i)) + \\ & \sum_{i=2}^{N}\sum_{j=1}^{2}\sum_{k=1}^{2} \mathbb{1}(Z_i = k)\mathbb{1}(Z_{i-1} = j) log(a_{jk}) \end{aligned}$$

$$E_{Z|X;\theta_s}\left[\sum_{i=1}^{2} \mathbb{1}(Z_1 = i) log(\pi_1)\right] = \sum_{i=1}^{2} E_{Z|X;\theta_s}[\mathbb{1}(Z_1 = i) log(\pi_1)] = \sum_{i=1}^{2} P(Z_1 = i|X; \theta_s) log(\pi_1)$$

$$\begin{aligned} E_{Z|X;\theta_s}\left[\sum_{i=1}^{N}\sum_{j=1}^{2} \mathbb{1}(Z_i = j) log(f_{\varphi_j}(X_i))\right] &= \sum_{i=1}^{N}\sum_{j=1}^{2} E_{Z|X;\theta_s}[\mathbb{1}(Z_i = j) log(f_{\varphi_j}(X_i))] \\ &= \sum_{i=1}^{N}\sum_{j=1}^{2} P(Z_i = j|X; \theta_s) log(f_{\varphi_j}(X_i)) \end{aligned}$$

$$\begin{aligned} E_{Z|X;\theta_s}\left[\sum_{i=1}^{N}\sum_{j=1}^{2}\sum_{k=1}^{2} \mathbb{1}(Z_i = k)\mathbb{1}(Z_{i-1} = j) log(a_{jk})\right] &= \sum_{i=1}^{N}\sum_{j=1}^{2}\sum_{k=1}^{2} E_{Z|X;\theta_s}[\mathbb{1}(Z_i = k)\mathbb{1}(Z_{i-1} = j) log(a_{jk})] \\ &= \sum_{i=2}^{N}\sum_{j=1}^{2}\sum_{k=1}^{2} P(Z_i = k, Z_{i-1} = j|X; \theta_s) log(a_{jk}) \end{aligned}$$

Let $\gamma_{ij} = P(Z_i = j|X; \theta_s)$; $\beta_{ijk} = P(Z_i = k, Z_{i-1} = j|X; \theta_s)$

$$Q(\theta, \theta_s) = \sum_{i=1}^{2} \gamma_{1i} log(\pi_i) + \sum_{i=1}^{N}\sum_{j=1}^{2} \gamma_{ij} log(f_{\varphi_j}(X_i)) + \sum_{i=2}^{N}\sum_{j=1}^{2}\sum_{k=1}^{2} \beta_{ijk} log(a_{jk})$$

M step :

For initial probability π

$$\frac{dQ(\theta,\ \theta_s)}{d\pi_1} = \frac{\gamma_{11}}{\pi_1} - \frac{\gamma_{12}}{1-\pi_1} = 0$$

$$\gamma_{11}\,(1-\pi_1) = \gamma_{12}\cdot\pi_1$$

$$\pi_1 = \frac{\gamma_{11}}{\gamma_{11} + \gamma_{12}},\ \pi_1 = \frac{\gamma_{12}}{\gamma_{11} + \gamma_{12}}$$

For emission probability

$$\frac{dQ(\theta,\ \theta_s)}{d\varphi_j} = \sum_{i=1}^{N}\gamma_{ij}\frac{dlog(f_{\varphi_j}(X_i))}{d\varphi_j} = 0$$

For transition probability

$$\frac{dQ(\theta,\ \theta_s)}{d\varphi_{i11}} = \sum_{i=2}^{N}\left(\frac{\beta_{i11}}{a_{11}} - \frac{\beta_{i12}}{1-a_{11}}\right) = 0$$

$$\sum_{i=2}^{N}\beta_{i12}\cdot a_{11} = (1-a_{11})\sum_{i=2}^{N}\beta_{i11}$$

$$a_{11} = \frac{\sum_{i=2}^{N}\beta_{i11}}{\sum_{i=2}^{N}(\beta_{i11} + \beta_{i12})} = \frac{\sum_{i=2}^{N}\beta_{i11}}{\sum_{i=2}^{N}(\beta_{i11} + \beta_{i12})} = \frac{\sum_{i=2}^{N}\beta_{i11}}{\sum_{i=1}^{N-1}\gamma_{i1}}$$

$$a12 = \frac{\sum_{i=2}^{N}\beta_{i12}}{\sum_{i=1}^{N-1}\gamma_{i1}}$$