$$x_i | c_i,\ \mu \sim N\left(c_i^T \mu,\ 1\right)$$

$$c_i \sim Categorical(1 / K,\ \ldots,\ 1 / K)$$

$$\mu_k \sim N\left(0,\ \sigma^2\right)$$

$$q(c_i;\ \varphi_i) \sim Categorical(\varphi_{i1},\ \ldots,\ \varphi_{iK})$$

$$q(\mu_k;\ m_k,\ s_k) \sim N\left(m_k,\ s_k^2\right)$$

$$p(\mu_k) = \frac{1}{\sqrt{2\pi}\sigma} exp\left(-\frac{\mu_k^2}{2\sigma^2}\right),\ p(c_i) = \frac{1}{K},\ p(x_i | c_i,\ \mu) = \frac{1}{\sqrt{2\pi}} \prod_k^K exp\left(-\frac{(x_i - \mu_k)^2}{2}\right)^{c_{ik}}$$

$$p(x,\ c,\ \mu) = p(x,\ c \mid \mu) p(\mu) = \left(\prod_i^N p(x_i | c_i,\ \mu)\, p(c_i)\right)\left(\prod_k^K p(\mu_k)\right) =$$

$$= C\left(\prod_i^N \prod_k^K exp\left(-\frac{(x_i - \mu_k)^2}{2}\right)^{c_{ik}}\right)\left(\prod_k^K exp\left(-\frac{\mu_k^2}{2\sigma^2}\right)\right),\ \ where\ C\ is\ a\ constant.$$

Update for μ_k

$$log(p(x,\ c,\ \mu_k, \mu_{-k}))\ =\ C\ +\ \sum_i^N \sum_k^K \left(-\,c_{ik}\frac{(x_i\ -\ \mu_k)^2}{2}\right)\ +\ \sum_k^K -\frac{\mu_k^2}{2\sigma^2}$$

$$=\ C\ -\ \frac{\mu_k^2}{2\sigma^2}\ -\ \sum_i^N \left(c_{ik}\frac{(x_i\ -\ \mu_k)^2}{2}\right)$$

$$E_{c,\,\mu_{-k}}(log(p(x,\ c,\ \mu_k, \mu_{-k})))\ =\ C\ -\ \frac{\mu_k^2}{2\sigma^2}\ -\ \sum_i^N \left(E_{c_i}(c_{ik})\frac{(x_i\ -\ \mu_k)^2}{2}\right)$$

$$=\ C\ -\ \frac{\mu_k^2}{2\sigma^2}\ -\sum_i^N \left(E_{c_i}(c_{ik})\frac{x_i^2\ +\ \mu_k^2\ -\ 2x_i\mu_k}{2}\right)$$

$$=\ C\ -\ \frac{\mu_k^2}{2\sigma^2}\ -\sum_i^N \left(E_{c_i}(c_{ik})\frac{\mu_k^2\ -\ 2x_i\mu_k}{2}\right)$$

$$=\ C\ -\ \frac{\mu_k^2}{2\sigma^2}\ -\sum_i^N \frac{E_{c_i}(c_{ik})\mu_k^2}{2}\ +\ \sum_i^N E_{c_i}(c_{ik})x_i\mu_k$$

$$E_{c_i}(c_{ik})\ =\ \varphi_{ik}$$

$$q^*(\mu_k)\ \propto\ exp(E_{c,\,\mu_{-k}}(log(p(x,\ c,\ \mu_k, \mu_{-k}))))\ \propto\ exp\left(-\ \frac{\mu_k^2}{2\sigma^2}\ -\sum_i^N \frac{\varphi_{ik}\mu_k^2}{2}\ +\ \sum_i^N \varphi_{ik}x_i\mu_k\right)$$

$$=\ exp\left(-\ \left(\frac{1}{2\sigma^2}\ +\sum_i^N \frac{\varphi_{ik}}{2}\right)\mu_k^2\ +\ \sum_i^N \varphi_{ik}x_i\mu_k\right)$$

$$=\ exp\left(-\frac{\mu_k^2\ -\ \frac{\sum_i^N \varphi_{ik}x_i\mu_k}{\frac{1}{2\sigma^2}+\sum_i^N\frac{\varphi_{ik}}{2}}}{\frac{1}{\frac{1}{2\sigma^2}+\sum_i^N\frac{\varphi_{ik}}{2}}}\right)\ =\ exp\left(-\frac{\mu_k^2\ -\ 2\ \cdot\ \frac{\sum_i^N \varphi_{ik}x_i}{\frac{1}{\sigma^2}+\sum_i^N \varphi_{ik}}\mu_k}{2\cdot\ \frac{1}{\frac{1}{\sigma^2}+\sum_i^N \varphi_{ik}}}\right)$$

For an appropraite posterior distribution, $\int q^*(\mu_k)\,d\mu_k\ =\ 1$

$$\therefore\ \ \mu_k \sim N\left(\frac{\sum_i^N \varphi_{ik}x_i}{\frac{1}{\sigma^2}\ +\sum_i^N \varphi_{ik}},\ \frac{1}{\frac{1}{\sigma^2}\ +\sum_i^N \varphi_{ik}}\right)$$

that is: $m_k\ =\ \dfrac{\sum_i^N \varphi_{ik}x_i}{\frac{1}{\sigma^2}\ +\sum_i^N \varphi_{ik}},\ s_k^2\ =\ \dfrac{1}{\frac{1}{\sigma^2}\ +\sum_i^N \varphi_{ik}}$

Update for c_{ik}

$$log(p(x,\ \mu,\ c_i,\ c_{-i}))\ =\ C\ +\ \sum_i^N \sum_k^K \left(-\,c_{ik}\frac{(x_i\ -\ \mu_k)^2}{2}\right)\ +\ \sum_k^K -\frac{\mu_k^2}{2\sigma^2}$$

$$=\ C\ -\sum_k^K\ c_{ik}\frac{(x_i\ -\ \mu_k)^2}{2}$$

$$E_{\mu,\,c_{-i}}(log(p(x,\ \mu,\ c_{ik},\ c_{-ik})))\ =\ C\ -\sum_k^K\left(\frac{c_{ik}x_i^2}{2}\ -\ \frac{c_{ik}E_{\mu_k}\left(\mu_k^2\right)}{2}+\ c_{ik}x_iE_{\mu_k}(\mu_k)\right)$$

$$E_{\mu_k}(\mu_k)\ =\ m_k$$

$$E_{\mu_k}\left(\mu_k^2\right)\ =\ Var(\mu_k)\ +\ E_{\mu_k}^2(\mu_k)\ =\ s_k^2\ +\ m_k^2$$

$$q^*(c_i)\ \propto\ exp(E_{\mu,\,c_{-i}}(log(p(x,\ \mu,\ c_i,\ c_{-i}))))\ \propto\ exp\left(\sum_k^K\left(-\frac{c_{ik}x_i^2}{2}\ -\ \frac{c_{ik}\ \left(s_k^2\ +\ m_k^2\right)}{2}+\ c_{ik}x_im_k\right)\right)$$

$$=\ exp\left(-\frac{x_i^2}{2}\ +\sum_k^K\left(-\ \frac{c_{ik}\ \left(s_k^2\ +\ m_k^2\right)}{2}+\ c_{ik}x_im_k\right)\right)\ \propto exp\left(\sum_k^K\left(-\ \frac{c_{ik}\ \left(s_k^2\ +\ m_k^2\right)}{2}+\ c_{ik}x_im_k\right)\right)$$

$$\varphi_{ik}=\ C\ \cdot\ exp\left(-\ \frac{s_k^2\ +\ m_k^2}{2}+\ x_im_k\right)$$

For an appropraite posterior distribution, $\sum_k^K \varphi_{ik}\ =\ 1$

$$\varphi_{ik}\ =\ \frac{exp\left(-\ \frac{s_k^2\,+\,m_k^2}{2}+\ x_im_k\right)}{\sum_k^K exp\left(-\ \frac{s_k^2\,+\,m_k^2}{2}+\ x_im_k\right)}$$