Elastic One-Loop Amplitudes

M.E. Irizarry-Gelpí

October 10, 2015

Contents

Chapter 1

Introduction

...

1.1 Euler Gamma Function

The Euler Gamma function is

$$\Gamma(z) = \int_0^\infty \mathrm{d}x \left(\frac{1}{x}\right)^{1-z} \exp\left(-x\right). \tag{1.1}$$

Setting $x = \kappa^2 w$ with $\kappa^2 > 0$ leads to

$$\Gamma(z) = \left(\kappa^2\right)^z \int_0^\infty \mathrm{d}w \left(\frac{1}{w}\right)^{1-z} \exp\left(-\kappa^2 w\right), \tag{1.2}$$

which allows you to write

$$\left(\frac{1}{\kappa^2}\right)^z = \frac{1}{\Gamma(z)} \int_0^\infty \mathrm{d}w \left(\frac{1}{w}\right)^{1-z} \exp\left(-\kappa^2 w\right). \tag{1.3}$$

Here w is a Schwinger modulus.

1.2 Propagators

In the momentum basis, the propagator for a free quantum with mass m is given by

$$\widehat{G}_m(p, q) = \left(\frac{2}{|p|^2 + m^2}\right) \delta(p - q). \tag{1.4}$$

Using a Schwinger modulus, this can be re-written as

$$\widehat{G}_m(p, q) = \delta(p - q) \int_0^\infty \mathrm{d}T \exp\left[-\left(\frac{|p|^2 + m^2}{2}\right) T\right]. \tag{1.5}$$

From the momentum basis, you can go to the position basis via a Fourier transform:

$$G_m(x, y) = \int\int \mathrm{d}p\mathrm{d}q\, \widehat{G}_m(p, q) \exp\left(ip \cdot x - iq \cdot y\right). \tag{1.6}$$

Integration over p and q gives

$$G_m(x, y) = \int_0^\infty \mathrm{d}T \left(\frac{1}{T}\right)^{D/2} \exp\left[-\frac{1}{2T}\left|x - y\right|^2 - \frac{1}{2}m^2 T\right]. \tag{1.7}$$

As a special case, you can take the $m \to 0$ limit to obtain the propagator for a free massless quantum:

$$G_0(x, y) = \int_0^\infty \mathrm{d}T \left(\frac{1}{T}\right)^{D/2} \exp\left[-\frac{1}{2T}\left|x - y\right|^2\right] = \left(\frac{2}{|x - y|^2}\right)^{(D-2)/2} \Gamma\left(\frac{D - 2}{2}\right). \tag{1.8}$$

This is valid as long as $D \neq 2$.

1.3 Kinematics

There are four external quanta; two incoming (labeled 1 and 2) and two outgoing (labeled 3 and 4). In the position basis, each external quantum is associated to a spacetime position. These four spacetime position vectors are independent. Similarly, in the momentum basis, each external quantum is associated to an energy-momentum vector that satisfies an on-shell constraint. Since this process is elastic, you have

$$m_3^2 = -\left|p_1\right|^2 = -\left|p_3\right|^2, \qquad m_4^2 = -\left|p_2\right|^2 = -\left|p_4\right|^2. \tag{1.9}$$

A priori, these four energy-momentum vectors are independent. But as you will see, due to translation invariance, the four energy-momentum vectors satisfy a linear constraint:

$$p_1 + p_2 = p_3 + p_4. \tag{1.10}$$

There are three Mandelstam invariants:

$$s = -\left|p_1 + p_2\right|^2, \qquad t = -\left|p_1 - p_3\right|^2, \qquad u = -\left|p_1 - p_4\right|^2. \tag{1.11}$$

Due to the conservation constraint, it follows that

$$s + t + u = 2m_3^2 + 2m_4^2. \tag{1.12}$$

An important function is

$$\Lambda_{34}(s) = \left[s - (m_3 - m_4)^2\right]\left[s - (m_3 + m_4)^2\right]. \tag{1.13}$$

This is known as the Källén function. Note that $\Lambda_{34}(s)$ can also be written as

$$\Lambda_{34}(s) = \left(s - m_3^2 - m_4^2\right)^2 - 4m_3^2 m_4^2 = \left(m_3^2 - m_4^2\right)^2 - s\,(t+u). \tag{1.14}$$

1.4 Change of Variables

In this section I will discuss different changes of position variables, and the corresponding conjugate momenta.

1.4.1 Incoming Basis

Consider the following expression:

$$\mathbb{F} \equiv x_1 \cdot p_1 + x_2 \cdot p_2 - x_3 \cdot p_3 - x_4 \cdot p_4. \tag{1.15}$$

Now make the change of variables

$$X \equiv \frac{x_1 + x_2 + x_3 + x_4}{4}, \qquad x_{34} \equiv x_3 - x_4, \qquad x_{31} \equiv x_3 - x_1, \qquad x_{42} \equiv x_4 - x_2. \tag{1.16}$$

The inverse relation is

$$x_1 = \frac{4X + 2x_{34} - 3x_{31} + x_{42}}{4}, \tag{1.17}$$

$$x_2 = \frac{4X + 2x_{34} + x_{31} - 3x_{42}}{4}, \tag{1.18}$$

$$x_3 = \frac{4X + 2x_{34} + x_{31} + x_{42}}{4}, \tag{1.19}$$

$$x_4 = \frac{4X - 2x_{34} + x_{31} + x_{42}}{4}. \tag{1.20}$$

Then $\mathbb{F}$ can be written as

$$\mathbb{F} \equiv X \cdot P + x_{34} \cdot p_{34} - x_{31} \cdot p_{31} - x_{42} \cdot p_{42}, \tag{1.21}$$

where

$$P = p_1 + p_2 - p_3 - p_4, \tag{1.22}$$

$$p_{34} = \frac{p_1 - p_2 - p_3 + p_4}{2}, \tag{1.23}$$

$$p_{31} = \frac{3p_1 - p_2 + p_3 + p_4}{4}, \tag{1.24}$$

$$p_{42} = \frac{-p_1 + 3p_2 + p_3 + p_4}{4}. \tag{1.25}$$

1.4.2 Outgoing Basis

Consider (1.15) and make the change of variables

$$X \equiv \frac{x_1+x_2+x_3+x_4}{4}, \quad x_{12} \equiv x_1 - x_2, \quad x_{31} \equiv x_3 - x_1, \quad x_{42} \equiv x_4 - x_2. \tag{1.26}$$

The inverse relation is

$$x_1 = \frac{4X + 2x_{12} - x_{31} - x_{42}}{4}, \tag{1.27}$$

$$x_2 = \frac{4X - 2x_{12} - x_{31} - x_{42}}{4}, \tag{1.28}$$

$$x_3 = \frac{4X + 2x_{12} + 3x_{31} - x_{42}}{4}, \tag{1.29}$$

$$x_4 = \frac{4X - 2x_{12} - x_{31} + 3x_{42}}{4}. \tag{1.30}$$

Then $\mathbb{F}$ can be written as

$$\mathbb{F} \equiv X \cdot P + x_{12} \cdot p_{12} - x_{31} \cdot p_{31} - x_{42} \cdot p_{42}, \tag{1.31}$$

where

$$P = p_1 + p_2 - p_3 - p_4, \tag{1.32}$$

$$p_{12} = \frac{p_1 - p_2 - p_3 + p_4}{2}, \tag{1.33}$$

$$p_{31} = \frac{p_1 + p_2 + 3p_3 - p_4}{4}, \tag{1.34}$$

$$p_{42} = \frac{p_1 + p_2 - p_3 + 3p_4}{4}. \tag{1.35}$$

1.5 Massless Propagators Between Two Points

Consider the following correlator consisting of $L+1$ massless propagators in $D-2$ spacetime dimensions connecting two points in spacetime:

$$\mathcal{T}_L(x) = \delta(x_3 - x_1)\delta(x_4 - x_2) \prod_{i=1}^{L+1} G_0(x_3|x_4). \tag{1.36}$$

This correlator describes an L-loop level process. In $D-2$ spacetime dimensions, the massless propagator is

$$G_0(x|y) = \left(\frac{2}{|x-y|^2}\right)^{(D-4)/2} \Gamma\left(\frac{D-4}{2}\right). \tag{1.37}$$

Let $D = 4 + 2\epsilon$. Then $\mathcal{T}_L$ becomes

$$\mathcal{T}_L(x) = \delta(x_3 - x_1)\delta(x_4 - x_2) \left(\frac{2}{|x_{34}|^2}\right)^{(L+1)\epsilon} [\Gamma(\epsilon)]^{L+1}. \tag{1.38}$$

Taking the Fourier transform, leads to

$$\widehat{\mathcal{T}}_L(p) = [\Gamma(\epsilon)]^{L+1} \delta(P) \int \mathrm{d}x_{34} \left(\frac{2}{|x_{34}|^2}\right)^{(L+1)\epsilon} \exp(i x_{34} \cdot p_{34}). \tag{1.39}$$

Using a Schwinger modulus leads to

$$\widehat{\mathcal{T}}_L = \frac{[\Gamma(\epsilon)]^{L+1}}{\Gamma[(L+1)\epsilon]} \delta(P) \int_0^\infty \mathrm{d}T \left(\frac{1}{T}\right)^{1-(L+1)\epsilon} \exp\left[-\frac{1}{2}|x_{34}|^2 T + i x_{34} \cdot p_{34}\right]. \tag{1.40}$$

Integration over x_{34} gives

$$\widehat{\mathcal{T}}_L = \frac{[\Gamma(\epsilon)]^{L+1}}{\Gamma[(L+1)\epsilon]} \delta(P) \int_0^\infty \mathrm{d}T \left(\frac{1}{T}\right)^{2-L\epsilon} \exp\left[-\frac{1}{2T}|p_{34}|^2\right]. \tag{1.41}$$

Finally, integrating over T gives

$$\widehat{\mathcal{T}}_L = \frac{\left[\Gamma\left(\epsilon\right)\right]^{L+1}\Gamma(1-L\epsilon)}{\Gamma\left[(L+1)\epsilon\right]}\delta(P)\left(-\frac{2}{t}\right)\left(-\frac{t}{2}\right)^{L\epsilon}, \tag{1.42}$$

where I have used $\left|p_{34}\right|^2 = -t$.

1.6 Massive Propagators Between Two Points

...

1.7 Outgoing Null Sudakov Momenta

Let k_3 and k_4 be null momenta related to the outgoing momenta via the transformation:

$$p_3 = k_3 + b_3 k_4, \qquad p_4 = k_4 + a_4 k_3. \tag{1.43}$$

Then,

$$m_3^2 = -\left|p_3\right|^2 \implies m_3^2 = -2b_3\left(k_3\cdot k_4\right), \tag{1.44}$$
$$m_4^2 = -\left|p_4\right|^2 \implies m_4^2 = -2a_4\left(k_3\cdot k_4\right), \tag{1.45}$$
$$s = -\left|p_3+p_4\right|^2 \implies s = -2\left(1+b_3\right)\left(1+a_4\right)\left(k_3\cdot k_4\right). \tag{1.46}$$

Solving these equations yield

$$b_3 = \frac{2m_3^2}{s-m_3^2-m_4^2+\sqrt{\Lambda_{34}(s)}}, \qquad a_4 = \frac{2m_4^2}{s-m_3^2-m_4^2+\sqrt{\Lambda_{34}(s)}}; \tag{1.47}$$

and

$$k_3\cdot k_4 = -\frac{1}{4}\left(s-m_3^2-m_4^2+\sqrt{\Lambda_{34}(s)}\right). \tag{1.48}$$

You have

$$b_3 a_4 = \frac{s-m_3^2-m_4^2-\sqrt{\Lambda_{34}(s)}}{s-m_3^2-m_4^2+\sqrt{\Lambda_{34}(s)}}, \tag{1.49}$$

$$1+b_3a_4 = \frac{2\left(s-m_3^2-m_4^2\right)}{s-m_3^2-m_4^2+\sqrt{\Lambda_{34}(s)}}, \qquad 1-b_3a_4 = \frac{2\sqrt{\Lambda_{34}(s)}}{s-m_3^2-m_4^2+\sqrt{\Lambda_{34}(s)}}; \tag{1.50}$$

and

$$b_3+a_4 = \frac{2\left(m_3^2+m_4^2\right)}{s-m_3^2-m_4^2+\sqrt{\Lambda_{34}(s)}}, \qquad b_3-a_4 = \frac{2\left(m_3^2-m_4^2\right)}{s-m_3^2-m_4^2+\sqrt{\Lambda_{34}(s)}}. \tag{1.51}$$

Also

$$1+b_3 = \frac{s+m_3^2-m_4^2+\sqrt{\Lambda_{34}(s)}}{s-m_3^2-m_4^2+\sqrt{\Lambda_{34}(s)}} = \frac{s-m_3^2+m_4^2-\sqrt{\Lambda_{34}(s)}}{2m_4^2}, \tag{1.52}$$
$$1+a_4 = \frac{s-m_3^2+m_4^2+\sqrt{\Lambda_{34}(s)}}{s-m_3^2-m_4^2+\sqrt{\Lambda_{34}(s)}} = \frac{s+m_3^2-m_4^2-\sqrt{\Lambda_{34}(s)}}{2m_3^2}; \tag{1.53}$$

and

$$1-b_3 = \frac{s-3m_3^2-m_4^2+\sqrt{\Lambda_{34}(s)}}{s-m_3^2-m_4^2+\sqrt{\Lambda_{34}(s)}}, \qquad 1-a_4 = \frac{s-m_3^2-3m_4^2+\sqrt{\Lambda_{34}(s)}}{s-m_3^2-m_4^2+\sqrt{\Lambda_{34}(s)}}. \tag{1.54}$$

Thus

$$(1+b_3)(1+a_4) = \frac{2s}{s-m_3^2-m_4^2+\sqrt{\Lambda_{34}(s)}}, \tag{1.55}$$

$$(1-b_3)(1+a_4) = \frac{2\sqrt{\Lambda_{34}(s)}-2\left(m_3^2-m_4^2\right)}{s-m_3^2-m_4^2+\sqrt{\Lambda_{34}(s)}}, \tag{1.56}$$

$$(1+b_3)(1-a_4) = \frac{2\sqrt{\Lambda_{34}(s)}+2\left(m_3^2-m_4^2\right)}{s-m_3^2-m_4^2+\sqrt{\Lambda_{34}(s)}}, \tag{1.57}$$

$$(1-b_3)(1-a_4) = \frac{2\left(s-2m_3^2-2m_4^2\right)}{s-m_3^2-m_4^2+\sqrt{\Lambda_{34}(s)}}. \tag{1.58}$$

These expressions are relevant to the outgoing speeds in the center-of-momentum frame:

$$\frac{1-b_3a_4}{1+2b_3+b_3a_4} = \frac{\sqrt{\Lambda_{34}(s)}}{s+m_3^2-m_4^2}, \qquad \frac{1-b_3a_4}{1+2a_4+b_3a_4} = \frac{\sqrt{\Lambda_{34}(s)}}{s-m_3^2+m_4^2}. \tag{1.59}$$

Other relevant results are

$$\frac{1}{b_3}\left(\frac{1+b_3}{1+a_4}\right) = \frac{s+m_3^2-m_4^2+\sqrt{\Lambda_{34}(s)}}{s+m_3^2-m_4^2-\sqrt{\Lambda_{34}(s)}}, \tag{1.60}$$

$$\frac{1}{a_4}\left(\frac{1+a_4}{1+b_3}\right) = \frac{s-m_3^2+m_4^2+\sqrt{\Lambda_{34}(s)}}{s-m_3^2+m_4^2-\sqrt{\Lambda_{34}(s)}}; \tag{1.61}$$

and

$$\frac{1}{b_3a_4} = \frac{s-m_3^2-m_4^2+\sqrt{\Lambda_{34}(s)}}{s-m_3^2-m_4^2-\sqrt{\Lambda_{34}(s)}}. \tag{1.62}$$

The logarithm of these expressions are related to rapidities in the center-of-momentum frame.

The inverse of (1.43) is

$$k_3 = \frac{p_3-b_3p_4}{1-b_3a_4}, \qquad k_4 = \frac{p_4-a_4p_3}{1-b_3a_4}. \tag{1.63}$$

Thus,

$$p_1\cdot k_3 = \frac{p_1\cdot p_3-b_3\left(p_1\cdot p_4\right)}{1-b_3a_4} = \frac{t-2m_3^2-b_3\left(u-m_3^2-m_4^2\right)}{2\left(1-b_3a_4\right)}, \tag{1.64}$$

$$p_2\cdot k_3 = \frac{p_2\cdot p_3-b_3\left(p_2\cdot p_4\right)}{1-b_3a_4} = \frac{u-m_3^2-m_4^2-b_3\left(t-2m_4^2\right)}{2\left(1-b_3a_4\right)}, \tag{1.65}$$

$$p_3\cdot k_3 = \frac{|p_3|^2-b_3\left(p_3\cdot p_4\right)}{1-b_3a_4} = \frac{b_3\left(s-m_3^2-m_4^2\right)-2m_3^2}{2\left(1-b_3a_4\right)}, \tag{1.66}$$

$$p_4\cdot k_3 = \frac{p_3\cdot p_4-b_3|p_4|^2}{1-b_3a_4} = \frac{m_3^2+m_4^2-s+2b_3m_4^2}{2\left(1-b_3a_4\right)}; \tag{1.67}$$

and

$$p_1\cdot k_4 = \frac{p_1\cdot p_4-a_4\left(p_1\cdot p_3\right)}{1-b_3a_4} = \frac{u-m_3^2-m_4^2-a_4\left(t-2m_3^2\right)}{2\left(1-b_3a_4\right)}, \tag{1.68}$$

$$p_2\cdot k_4 = \frac{p_2\cdot p_4-a_4\left(p_2\cdot p_3\right)}{1-b_3a_4} = \frac{t-2m_4^2-a_4\left(u-m_3^2-m_4^2\right)}{2\left(1-b_3a_4\right)}, \tag{1.69}$$

$$p_3\cdot k_4 = \frac{p_3\cdot p_4-a_4|p_3|^2}{1-b_3a_4} = \frac{m_3^2+m_4^2-s+2a_4m_3^2}{2\left(1-b_3a_4\right)}, \tag{1.70}$$

$$p_4\cdot k_4 = \frac{|p_4|^2-a_4\left(p_3\cdot p_4\right)}{1-b_3a_4} = \frac{a_4\left(s-m_3^2-m_4^2\right)-2m_4^2}{2\left(1-b_3a_4\right)}. \tag{1.71}$$

1.7.1 Sudakov Null Decomposition

Given a spacetime vector v, the Sudakov null decomposition is given by

$$v = V + a_v k_3 + b_v k_4, \qquad V \cdot k_3 = V \cdot k_4 = 0. \tag{1.72}$$

Here V lives in the orthogonal complement to the Sudakov subspace, and a_v and b_v are the Sudakov moduli. Since the Sudakov momenta are null, you have

$$a_v = \frac{v \cdot k_4}{k_3 \cdot k_4}, \qquad b_v = \frac{v \cdot k_3}{k_3 \cdot k_4}. \tag{1.73}$$

Thus,

$$|V|^2 = |v|^2 - 2a_v b_v \left(k_3 \cdot k_4\right). \tag{1.74}$$

The external momenta split into the incoming set (p_1 and p_2) and the outgoing set (p_3 and p_4). You have already null-decomposed the outgoing set by introducing the Sudakov momenta in (1.43). Hence, by definition, the outgoing set live in the Sudakov subspace. The Sudakov null decomposition of the incoming set is

$$p_1 = P_1 + a_1 k_3 + b_1 k_4, \qquad p_2 = P_2 + a_2 k_3 + b_2 k_4. \tag{1.75}$$

Using the conservation constraint, you find

$$P_1 + P_2 + (a_1 + a_2)\, k_3 + (b_1 + b_2)\, k_4 = (1 + a_4)\, k_3 + (1 + b_3)\, k_4. \tag{1.76}$$

This leads to three relations:

$$P_1 + P_2 = 0, \qquad a_1 + a_2 = 1 + a_4, \qquad b_1 + b_2 = 1 + b_3. \tag{1.77}$$

You have

$$a_1 = \frac{a_4\left(t - 2m_3^2\right) - \left(u - m_3^2 - m_4^2\right)}{\sqrt{\Lambda_{34}(s)}}, \qquad b_1 = \frac{b_3\left(u - m_3^2 - m_4^2\right) - \left(t - 2m_3^2\right)}{\sqrt{\Lambda_{34}(s)}}; \tag{1.78}$$

and

$$a_2 = \frac{a_4\left(u - m_3^2 - m_4^2\right) - \left(t - 2m_4^2\right)}{\sqrt{\Lambda_{34}(s)}}, \qquad b_2 = \frac{b_3\left(t - 2m_4^2\right) - \left(u - m_3^2 - m_4^2\right)}{\sqrt{\Lambda_{34}(s)}}. \tag{1.79}$$

With the null decomposition of the external momenta, you can null-decompose any other combination of external momenta. Let

$$p \equiv p_1 + p_2 = p_3 + p_4, \qquad q \equiv p_1 - p_3 = p_4 - p_2, \qquad r \equiv p_1 - p_4 = p_3 - p_2. \tag{1.80}$$

These are the spacetime vectors that yield the Mandelstam invariants:

$$s = -\left|p\right|^2, \qquad t = -\left|q\right|^2, \qquad u = -\left|r\right|^2. \tag{1.81}$$

The Sudakov null decomposition of these momenta is

$$p = P + a_p k_3 + b_p k_4, \qquad q = Q + a_q k_3 + b_q k_4, \qquad r = R + a_r k_3 + b_r k_4. \tag{1.82}$$

Using the null decomposition of the external momenta yields:

$$P = P_1 + P_2 = 0, \qquad a_p = a_1 + a_2 = 1 + a_4, \qquad b_p = b_1 + b_2 = 1 + b_3; \tag{1.83}$$

$$Q = P_1 = -P_2, \qquad a_q = a_1 - 1 = a_4 - a_2, \qquad b_q = b_1 - b_3 = 1 - b_2; \tag{1.84}$$

$$R = P_1 = -P_2, \qquad a_r = a_1 - a_4 = 1 - a_2, \qquad b_r = b_1 - 1 = b_3 - b_2. \tag{1.85}$$

Explicitly,

$$a_1 + a_2 = \frac{\left(s - m_3^2 + m_4^2\right) - a_4\left(s + m_3^2 - m_4^2\right)}{\sqrt{\Lambda_{34}(s)}}, \tag{1.86}$$

$$b_1 + b_2 = \frac{\left(s + m_3^2 - m_4^2\right) - b_3\left(s - m_3^2 + m_4^2\right)}{\sqrt{\Lambda_{34}(s)}}. \tag{1.87}$$

Using

$$\frac{s+m_3^2-m_4^2}{\sqrt{\Lambda_{34}(s)}}=\frac{(1+b_3)+b_3\,(1+a_4)}{1-b_3a_4},\qquad \frac{s-m_3^2+m_4^2}{\sqrt{\Lambda_{34}(s)}}=\frac{(1+a_4)+a_4\,(1+b_3)}{1-b_3a_4};\tag{1.88}$$

you find that $a_p = 1 + a_4$ and $b_p = 1 + b_3$. This checks the relations from (1.77). You also have

$$a_q=\frac{(1+a_4)\,t}{\sqrt{\Lambda_{34}(s)}},\qquad b_q=-\frac{(1+b_3)\,t}{\sqrt{\Lambda_{34}(s)}};\tag{1.89}$$

and

$$a_r=\frac{(1+a_4)\,t}{\sqrt{\Lambda_{34}(s)}}+1-a_4,\qquad b_r=-\frac{(1+b_3)\,t}{\sqrt{\Lambda_{34}(s)}}-1+b_3.\tag{1.90}$$

From $t=-\left|q\right|^2$ it follows that

$$\left|Q\right|^2=-t\left[1+\frac{st}{\Lambda_{34}(s)}\right].\tag{1.91}$$

Thus,

$$\left|R\right|^2=\left|P_1\right|^2=\left|P_2\right|^2=\left|Q\right|^2=-t\left[1+\frac{st}{\Lambda_{34}(s)}\right].\tag{1.92}$$

Chapter 2

Massless Medium

In this chapter I will consider one-loop contributions that involve a massless medium.

2.1 Box

The box correlator in a massless medium is:

$$\mathcal{B}_0(x) = G_0(x_1|x_2)G_\Phi(x_3|x_1)G_0(x_3|x_4)G_\Psi(x_4|x_2). \tag{2.1}$$

In terms of four Schwinger moduli you have

$$\mathcal{B}_0 = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \mathrm{d}T_{12}\mathrm{d}T_{31}\mathrm{d}T_{34}\mathrm{d}T_{42} \left(\frac{1}{T_{12}T_{31}T_{34}T_{42}}\right)^{D/2} \exp\left[-\frac{1}{2}B_0(x,T)\right], \tag{2.2}$$

where

$$B_0 = \frac{1}{T_{12}}\left|x_{12}\right|^2 + \frac{1}{T_{31}}\left|x_{31}\right|^2 + m_1^2 T_{31} + \frac{1}{T_{34}}\left|x_{34}\right|^2 + \frac{1}{T_{42}}\left|x_{42}\right|^2 + m_2^2 T_{42}. \tag{2.3}$$

The box amplitude follows from the Fourier transform:

$$\widehat{\mathcal{B}}_0(p) = \int\int\int\int \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4\, \mathcal{B}_0(x) \exp\left[i\mathbb{F}(x,p)\right], \tag{2.4}$$

with $\mathbb{F}$ given by (1.15). Note that

$$x_{12} + x_{31} - x_{34} - x_{42} = 0. \tag{2.5}$$

That is,

$$|x_{34}|^2 = |x_{12} + x_{31} - x_{42}|^2. \tag{2.6}$$

We make a change of variables:

$$\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4 \sim \mathrm{d}X\mathrm{d}x_{12}\mathrm{d}x_{31}\mathrm{d}x_{42} = \int \mathrm{d}X\mathrm{d}x_{12}\mathrm{d}x_{31}\mathrm{d}x_{34}\mathrm{d}x_{42}\delta(x_{12} + x_{31} - x_{34} - x_{42}), \tag{2.7}$$

and use

$$\delta(x_{12} + x_{31} - x_{34} - x_{42}) = \int \mathrm{d}q\, \exp\left[-iq\cdot(x_{12} + x_{31} - x_{34} - x_{42})\right], \tag{2.8}$$

to perform the integration over the spacetime positions:

$$\widehat{\mathcal{B}}_0(p) = \delta(P)\int \mathrm{d}q \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \mathrm{d}T_{12}\mathrm{d}T_{31}\mathrm{d}T_{34}\mathrm{d}T_{42}\, \exp\left[-\frac{1}{2}\widehat{B}_0(p,q,T)\right], \tag{2.9}$$

where

$$\widehat{B}_0 = \left|q - p_{12}\right|^2 T_{12} + \left(\left|q + p_{31}\right|^2 + m_1^2\right) T_{31} + \left|q\right|^2 T_{34} + \left(\left|q - p_{42}\right|^2 + m_2^2\right) T_{42}. \tag{2.10}$$

This result is kinematic-exact.

2.1.1 Sudakov Moduli

One can integrate over the Schwinger moduli in (2.9) to obtain:

$$\widehat{\mathcal{B}}_A(p) = \delta(P) \int \mathrm{d}q \left(\frac{2}{|q-p_{12}|^2} \right) \left(\frac{2}{|q+p_{31}|^2 + m_1^2} \right) \left(\frac{2}{|q|^2} \right) \left(\frac{2}{|q-p_{42}|^2 + m_2^2} \right). \tag{2.11}$$

In this expression q plays the role of a (virtual) loop momentum variable.

The Dirac delta enforces the $P = 0$ constraint. Enforcing this constraint leads to

$$p_{12} = p_1 - p_3 = p_4 - p_2, \qquad p_{31} = p_3, \qquad p_{42} = p_4. \tag{2.12}$$

Thus,

$$|p_{12}|^2 = -t, \qquad |p_{31}|^2 = -m_1^2, \qquad |p_{42}|^2 = -m_2^2. \tag{2.13}$$

Let k_3 and k_4 be null spacetime vectors with units of mass. The null Sudakov decomposition of p_3 and p_4 is as follows:

$$p_3 = k_3 + c_{34} k_4, \qquad p_4 = k_4 + c_{43} k_3. \tag{2.14}$$

In terms of p_3 and p_4, you have

$$k_3 = \frac{p_3 - c_{34} p_4}{1 - c_{34} c_{43}}, \qquad k_4 = \frac{p_4 - c_{43} p_3}{1 - c_{34} c_{43}}. \tag{2.15}$$

From $|p_3|^2 = -m_1^2$ and $|p_4|^2 = -m_2^2$ it follows that

$$c_{34} = -\frac{m_1^2}{2(k_3 \cdot k_4)}, \qquad c_{43} = -\frac{m_2^2}{2(k_3 \cdot k_4)} \qquad \Longrightarrow \qquad \frac{c_{34}}{c_{43}} = \frac{m_1^2}{m_2^2}. \tag{2.16}$$

Using $|k_3|^2 = 0$ and $|k_4|^2 = 0$ you find quadratic equations for c_{34} and c_{43}:

$$m_2^2 c_{34}^2 + \left(m_1^2 + m_2^2 - s\right) c_{34} + m_1^2 = 0, \qquad m_1^2 c_{43}^2 + \left(m_1^2 + m_2^2 - s\right) c_{43} + m_2^2 = 0. \tag{2.17}$$

Solving each quadratic equation yields

$$c_{43} = \left(\frac{m_2^2}{m_1^2} \right) c_{34}, \qquad c_{34} = \frac{s - m_1^2 - m_2^2 \pm \sqrt{\Lambda_{12}(s)}}{2m_2^2}. \tag{2.18}$$

Note that c_{34} and c_{43} are (dimensionless) functions that can be written in terms of two (dimensionless) ratios

$$\frac{s}{m_1 m_2}, \qquad \frac{m_1}{m_2}. \tag{2.19}$$

From $s = -|p_3 + p_4|^2$ it follows that

$$2(k_3 \cdot k_4) = -\frac{s}{(1 + c_{34})(1 + c_{43})} = -m_1 m_2 \left[\frac{2m_1 m_2}{s - m_1^2 - m_2^2 \pm \sqrt{\Lambda_{12}(s)}} \right]. \tag{2.20}$$

This can also be written as

$$2(k_3 \cdot k_4) = -m_1 m_2 \left[\frac{s - m_1^2 - m_2^2 \mp \sqrt{\Lambda_{12}(s)}}{2m_1 m_2} \right]. \tag{2.21}$$

Next you decompose the virtual momentum q as

$$q = Q + a_q k_3 + b_q k_4. \tag{2.22}$$

Here a_q and b_q are Sudakov moduli. The integration measure over q becomes

$$\mathrm{d}q = \sqrt{-(k_3 \cdot k_4)^2} \mathrm{d}a_q \mathrm{d}b_q \mathrm{d}Q. \tag{2.23}$$

Note that the volume measure for Q is over $D - 2$ spacetime dimensions.

Now you write each of the factors in the denominator in (2.11) in terms of the Sudakov moduli and the transversal momentum Q. First write

$$p_{12} = P_{12} + a_{12}k_3 + b_{12}k_4. \tag{2.24}$$

Since p_{12} is known, a_{12}, b_{12}, and P_{12} are also known. From $k_3 \cdot p_{12}$ and $k_4 \cdot p_{12}$ it follows that

$$b_{12} = \frac{k_3 \cdot p_{12}}{k_3 \cdot k_4}, \qquad a_{12} = \frac{k_4 \cdot p_{12}}{k_3 \cdot k_4}. \tag{2.25}$$

Using (2.15) leads to:

$$k_3 \cdot p_{12} = \frac{(p_3 - c_{34}p_4)\cdot(p_1 - p_3)}{1 - c_{34}c_{43}}, \qquad k_4 \cdot p_{12} = \frac{(p_4 - c_{43}p_3)\cdot(p_1 - p_3)}{1 - c_{34}c_{43}}. \tag{2.26}$$

Recall that

$$s = -\left|p_3 + p_4\right|^2 \quad \Rightarrow \quad p_3 \cdot p_4 = \frac{m_1^2 + m_2^2 - s}{2}, \tag{2.27}$$

$$t = -\left|p_1 - p_3\right|^2 \quad \Rightarrow \quad p_1 \cdot p_3 = \frac{t - 2m_1^2}{2}, \tag{2.28}$$

$$u = -\left|p_1 - p_4\right|^2 \quad \Rightarrow \quad p_1 \cdot p_4 = \frac{u - m_1^2 - m_2^2}{2}. \tag{2.29}$$

$$\tag{2.30}$$

Thus,

$$k_3 \cdot p_{12} = \frac{t}{2}\left(\frac{1 + c_{34}}{1 - c_{34}c_{43}}\right), \qquad k_4 \cdot p_{12} = -\frac{t}{2}\left(\frac{1 + c_{43}}{1 - c_{34}c_{43}}\right). \tag{2.31}$$

Hence,

$$a_{12} = \frac{t}{s}\left[\frac{(1 + c_{34})(1 + c_{43})^2}{1 - c_{34}c_{43}}\right], \qquad b_{12} = -\frac{t}{s}\left[\frac{(1 + c_{34})^2(1 + c_{43})}{1 - c_{34}c_{43}}\right]. \tag{2.32}$$

Using $\left|p_{12}\right|^2 = -t$ it follows that

$$\left|P_{12}\right|^2 = -t - 2a_{12}b_{12}\left(k_3 \cdot k_4\right) = -t - 2\left[\frac{\left(k_3 \cdot p_{12}\right)\left(k_4 \cdot p_{12}\right)}{\left(k_3 \cdot k_4\right)}\right], \tag{2.33}$$

which can be written as

$$\left|P_{12}\right|^2 = -t\left(1 + \frac{t}{s}\frac{(1 + c_{34})^2(1 + c_{43})^2}{(1 - c_{34}c_{43})^2}\right). \tag{2.34}$$

The terms in the denominator in (2.11) become:

$$\left|q - p_{12}\right|^2 = \left|Q - P_{12}\right|^2 + 2\left(a_q - a_{12}\right)\left(b_q - b_{12}\right)\left(k_3 \cdot k_4\right), \tag{2.35}$$

$$\left|q + p_{31}\right|^2 + m_1^2 = \left|Q\right|^2 + m_1^2 + 2\left(a_q + 1\right)\left(b_q + c_{34}\right)\left(k_3 \cdot k_4\right), \tag{2.36}$$

$$\left|q\right|^2 = \left|Q\right|^2 + 2a_q b_q\left(k_3 \cdot k_4\right), \tag{2.37}$$

$$\left|q - p_{42}\right|^2 + m_2^2 = \left|Q\right|^2 + m_2^2 + 2\left(a_q - c_{43}\right)\left(b_q - 1\right)\left(k_3 \cdot k_4\right). \tag{2.38}$$

2.1.2 Feynman Moduli

After integrating over q in (2.9), you find:

$$\widehat{\mathcal{B}}_A(p) = \delta(P)\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{\mathrm{d}T_{13}\mathrm{d}T_{21}\mathrm{d}T_{42}\mathrm{d}T_{34}}{(T_{13} + T_{21} + T_{42} + T_{34})^{D/2}}\exp\left[\frac{1}{2}\tilde{B}_A(p, T)\right], \tag{2.39}$$

where

$$\tilde{B}_A = tT_{21} + \frac{\left|T_{13}p_{13} - T_{21}p_{21} - T_{42}p_{42}\right|^2}{T_{13} + T_{21} + T_{42} + T_{34}}. \tag{2.40}$$

2.1.3 Regge Limit

...

2.1.4 Forward-JWKB Limit

...

2.2 Crossed Box

The cross-box correlator is given by

$$\mathcal{C}_A(x) = G_\Phi(x_1|x_3)G_A(x_4|x_1)G_\Psi(x_2|x_4)G_A(x_3|x_2), \tag{2.41}$$

but this expression is related to the box correlator (2.1) by swapping $x_2 \longleftrightarrow x_4$.

2.2.1 Regge Limit

...

2.2.2 Forward-JWKB Limit

...

2.3 Vertex Corrections

There are two one-loop vertex corrections:

$$\mathcal{V}_\Phi(x) = \delta(x_2 - x_4)G_A(x_1|x_3)\int \mathrm{d}y\, G_A(y|x_2)G_\Phi(y|x_1)G_\Phi(y|x_3), \tag{2.42}$$

$$\mathcal{V}_\Psi(x) = \delta(x_1 - x_3)G_A(x_2|x_4)\int \mathrm{d}y\, G_A(y|x_1)G_\Psi(y|x_2)G_\Psi(y|x_4). \tag{2.43}$$

2.3.1 Regge Limit

...

2.3.2 Forward-JWKB Limit

...

2.4 Vaccum Polarizations

There are two one-loop vacuum polarizations:

$$\mathcal{W}_\Phi(x) = \delta(x_1 - x_3)\delta(x_2 - x_4)\int\int \mathrm{d}y_1\mathrm{d}y_2\, G_A(x_1|y_1)G_A(x_2|y_2)G_\Phi(y_1|y_2)G_\Phi(y_2|y_1), \tag{2.44}$$

$$\mathcal{W}_\Psi(x) = \delta(x_1 - x_3)\delta(x_2 - x_4)\int\int \mathrm{d}y_1\mathrm{d}y_2\, G_A(x_1|y_1)G_A(x_2|y_2)G_\Psi(y_1|y_2)G_\Psi(y_2|y_1). \tag{2.45}$$

2.4.1 Regge Limit

...

2.4.2 Forward-JWKB Limit

...

Chapter 3

Massive Medium

The box correlator in a massive medium Y is:

$$\mathcal{B}_Y(x) = G_\Phi(x_1, x_3)G_\Psi(x_2, x_4)G_Y(x_1, x_2)G_Y(x_3, x_4). \tag{3.1}$$