Elastic One-Loop Amplitudes

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Contents

Chapter 1

Introduction

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1.1 Euler Gamma Function

The Euler Gamma function is

$$\Gamma(z) = \int_0^\infty \mathrm{d}x \left(\frac{1}{x}\right)^{1-z} \exp\left(-x\right). \tag{1.1}$$

Setting $x = \kappa^2 w$ with $\kappa^2 > 0$ leads to

$$\Gamma(z) = \left(\kappa^2\right)^z \int_0^\infty \mathrm{d}w \left(\frac{1}{w}\right)^{1-z} \exp\left(-\kappa^2 w\right), \tag{1.2}$$

which allows you to write

$$\left(\frac{1}{\kappa^2}\right)^z = \frac{1}{\Gamma(z)} \int_0^\infty \mathrm{d}w \left(\frac{1}{w}\right)^{1-z} \exp\left(-\kappa^2 w\right). \tag{1.3}$$

Here w is a Schwinger modulus.

1.2 Propagators

In the momentum basis, the propagator for a free quantum with mass m is given by

$$\widehat{G}_m(p,q) = \left(\frac{2}{|p|^2 + m^2}\right) \delta(p-q). \tag{1.4}$$

Using a Schwinger modulus, this can be re-written as

$$\widehat{G}_m(p,q) = \delta(p-q) \int\limits_0^\infty \mathrm{d}T \, \exp\left[-\left(\frac{|p|^2 + m^2}{2}\right) T\right]. \tag{1.5}$$

From the momentum basis, you can go to the position basis via a Fourier transform:

$$G_m(x,y) = \int\int \mathrm{d}p\mathrm{d}q \, \widehat{G}_m(p,q) \exp\left(ip\cdot x - iq\cdot y\right). \tag{1.6}$$

Integration over p and q gives

$$G_m(x,y) = \int\limits_0^\infty \mathrm{d}T \left(\frac{1}{T}\right)^{D/2} \exp\left[-\frac{1}{2T}|x-y|^2 - \frac{1}{2}m^2 T\right]. \tag{1.7}$$

As a special case, you can take the $m \to 0$ limit to obtain the propagator for a free massless quantum:

$$G_0(x,y) = \int\limits_0^\infty \mathrm{d}T \left(\frac{1}{T}\right)^{D/2} \exp\left[-\frac{1}{2T}|x-y|^2\right] = \left(\frac{2}{|x-y|^2}\right)^{(D-2)/2} \Gamma\left(\frac{D-2}{2}\right). \tag{1.8}$$

This is valid as long as $D \neq 2$.

1.3 Kinematics

There are four external quanta; two incoming (labeled 1 and 2) and two outgoing (labeled 3 and 4). In the position basis, each external quantum is associated to a spacetime position. These four spacetime position vectors are independent. Similarly, in the momentum basis, each external quantum is associated to an energy-momentum vector that satisfies an on-shell constraint. Since this process is elastic, you have

$$m_1^2 = -|p_1|^2 = -|p_3|^2, \qquad m_2^2 = -|p_2|^2 = -|p_4|^2. \tag{1.9}$$

A priori, these four energy-momentum vectors are independent. But as you will see, due to translation invariance, the four energy-momentum vectors satisfy a linear constraint:

$$p_1 + p_2 = p_3 + p_4. \tag{1.10}$$

There are three Mandelstam invariants:

$$s = -|p_1+p_2|^2, \qquad t = -|p_1-p_3|^2, \qquad u = -|p_1-p_4|^2. \tag{1.11}$$

Due to the conservation constraint, it follows that

$$s + t + u = 2m_1^2 + 2m_2^2. \tag{1.12}$$

An important function is

$$\Lambda(s) = [s - (m_1 - m_2)^2][s - (m_1 + m_2)^2]. \tag{1.13}$$

This is known as the Källén function. Note that $\Lambda(s)$ can also be written as

$$\Lambda(s) = (s - m_1^2 - m_2^2)^2 - 4m_1^2 m_2^2. \tag{1.14}$$

1.4 Change of Variables

In this section I will discuss different changes of position variables, and the corresponding conjugate momenta.

1.4.1 Incoming Basis

Consider the following expression:

$$\mathbb{F} \equiv x_1 \cdot p_1 + x_2 \cdot p_2 - x_3 \cdot p_3 - x_4 \cdot p_4. \tag{1.15}$$

Now make the change of variables

$$X \equiv \frac{x_1 + x_2 + x_3 + x_4}{4}, \quad x_{34} \equiv x_3 - x_4, \quad x_{31} \equiv x_3 - x_1, \quad x_{42} \equiv x_4 - x_2. \tag{1.16}$$

The inverse relation is

$$x_1 = \frac{4X + 2x_{34} - 3x_{31} + x_{42}}{4}, \tag{1.17}$$

$$x_2 = \frac{4X + 2x_{34} + x_{31} - 3x_{42}}{4}, \tag{1.18}$$

$$x_3 = \frac{4X + 2x_{34} + x_{31} + x_{42}}{4}, \tag{1.19}$$

$$x_4 = \frac{4X - 2x_{34} + x_{31} + x_{42}}{4}. \tag{1.20}$$

Then $\mathbb{F}$ can be written as

$$\mathbb{F} \equiv X \cdot P + x_{34} \cdot p_{34} - x_{31} \cdot p_{31} - x_{42} \cdot p_{42}, \tag{1.21}$$

where

$$P = p_1 + p_2 - p_3 - p_4, \tag{1.22}$$

$$p_{34} = \frac{p_1 - p_2 - p_3 + p_4}{2}, \tag{1.23}$$

$$p_{31} = \frac{3p_1 - p_2 + p_3 + p_4}{4}, \tag{1.24}$$

$$p_{42} = \frac{-p_1 + 3p_2 + p_3 + p_4}{4}. \tag{1.25}$$

1.4.2 Outgoing Basis

Consider (1.15) and make the change of variables

$$X \equiv \frac{x_1 + x_2 + x_3 + x_4}{4}, \quad x_{12} \equiv x_1 - x_2, \quad x_{31} \equiv x_3 - x_1, \quad x_{42} \equiv x_4 - x_2. \tag{1.26}$$

The inverse relation is

$$x_1 = \frac{4X + 2x_{12} - x_{31} - x_{42}}{4}, \tag{1.27}$$

$$x_2 = \frac{4X - 2x_{12} - x_{31} - x_{42}}{4}, \tag{1.28}$$

$$x_3 = \frac{4X + 2x_{12} + 3x_{31} - x_{42}}{4}, \tag{1.29}$$

$$x_4 = \frac{4X - 2x_{12} - x_{31} + 3x_{42}}{4}. \tag{1.30}$$

Then $\mathbb{F}$ can be written as

$$\mathbb{F} \equiv X \cdot P + x_{12} \cdot p_{12} - x_{31} \cdot p_{31} - x_{42} \cdot p_{42}, \tag{1.31}$$

where

$$P = p_1 + p_2 - p_3 - p_4, \tag{1.32}$$

$$p_{12} = \frac{p_1 - p_2 - p_3 + p_4}{2}, \tag{1.33}$$

$$p_{31} = \frac{p_1 + p_2 + 3p_3 - p_4}{4}, \tag{1.34}$$

$$p_{42} = \frac{p_1 + p_2 - p_3 + 3p_4}{4}. \tag{1.35}$$

1.5 Massless Propagators Between Two Points

Consider the following correlator consisting of $L+1$ massless propagators in $D-2$ spacetime dimensions connecting two points in spacetime:

$$\mathcal{T}_L(x) = \delta(x_3 - x_1)\delta(x_4 - x_2) \prod_{i=1}^{L+1} G_0(x_3|x_4). \tag{1.36}$$

This correlator describes an L-loop level process. In $D-2$ spacetime dimensions, the massless propagator is

$$G_0(x|y) = \left(\frac{2}{|x-y|^2}\right)^{(D-4)/2} \Gamma\left(\frac{D-4}{2}\right). \tag{1.37}$$

Let $D = 4 + 2\epsilon$. Then $\mathcal{T}_L$ becomes

$$\mathcal{T}_L(x) = \delta(x_3 - x_1)\delta(x_4 - x_2) \left(\frac{2}{|x_{34}|^2}\right)^{(L+1)\epsilon} [\Gamma(\epsilon)]^{L+1}. \tag{1.38}$$

Taking the Fourier transform, leads to

$$\widehat{\mathcal{T}}_L(p) = [\Gamma(\epsilon)]^{L+1}\,\delta(P)\int \mathrm{d}x_{34}\left(\frac{2}{|x_{34}|^2}\right)^{(L+1)\epsilon}\exp\left(ix_{34}\cdot p_{34}\right). \tag{1.39}$$

Using a Schwinger modulus leads to

$$\widehat{\mathcal{T}}_L = \frac{[\Gamma(\epsilon)]^{L+1}}{\Gamma\left[(L+1)\epsilon\right]}\delta(P)\int_0^\infty \mathrm{d}T\left(\frac{1}{T}\right)^{1-(L+1)\epsilon}\exp\left[-\frac{1}{2}|x_{34}|^2\,T + ix_{34}\cdot p_{34}\right]. \tag{1.40}$$

Integration over x_{34} gives

$$\widehat{\mathcal{T}}_L = \frac{[\Gamma(\epsilon)]^{L+1}}{\Gamma\left[(L+1)\epsilon\right]}\delta(P)\int_0^\infty \mathrm{d}T\left(\frac{1}{T}\right)^{2-L\epsilon}\exp\left[-\frac{1}{2T}|p_{34}|^2\right]. \tag{1.41}$$

Finally, integrating over T gives

$$\widehat{\mathcal{T}}_L = \frac{[\Gamma(\epsilon)]^{L+1}\,\Gamma(1-L\epsilon)}{\Gamma\left[(L+1)\epsilon\right]}\delta(P)\left(-\frac{2}{t}\right)\left(-\frac{t}{2}\right)^{L\epsilon}, \tag{1.42}$$

where I have used $|p_{34}|^2 = -t$.

1.6 Massive Propagators Between Two Points

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Chapter 2

Massless Medium

In this chapter I will consider one-loop contributions that involve a massless medium.

2.1 Box

The box correlator in a massless medium is:

$$\mathcal{B}_0(x) = G_0(x_1|x_2)G_\Phi(x_3|x_1)G_0(x_3|x_4)G_\Psi(x_4|x_2). \tag{2.1}$$

In terms of four Schwinger moduli you have

$$\mathcal{B}_0 = \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \mathrm{d}T_{12}\mathrm{d}T_{31}\mathrm{d}T_{34}\mathrm{d}T_{42}\left(\frac{1}{T_{12}T_{31}T_{34}T_{42}}\right)^{D/2}\exp\left[-\frac{1}{2}B_0(x,T)\right], \tag{2.2}$$

where

$$B_0 = \frac{1}{T_{12}}\left|x_{12}\right|^2 + \frac{1}{T_{31}}\left|x_{31}\right|^2 + m_1^2T_{31} + \frac{1}{T_{34}}\left|x_{34}\right|^2 + \frac{1}{T_{42}}\left|x_{42}\right|^2 + m_2^2T_{42}. \tag{2.3}$$

The box amplitude follows from the Fourier transform:

$$\widehat{\mathcal{B}}_0(p) = \int\int\int\int \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4\,\mathcal{B}_0(x)\exp\left[i\mathbb{F}(x,p)\right], \tag{2.4}$$

with $\mathbb{F}$ given by (1.15). Note that

$$x_{34} - x_{31} - x_{12} + x_{42} = 0. \tag{2.5}$$

That is,

$$|x_{12}|^2 = |x_{34} - x_{31} + x_{42}|^2. \tag{2.6}$$

We make a change of variables:

$$\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4 \sim \mathrm{d}X\mathrm{d}x_{34}\mathrm{d}x_{31}\mathrm{d}x_{42} = \int \mathrm{d}X\mathrm{d}x_{34}\mathrm{d}x_{31}\mathrm{d}x_{12}\mathrm{d}x_{42}\delta(x_{34} - x_{31} - x_{12} + x_{42}), \tag{2.7}$$

and use

$$\delta(x_{34}-x_{31}-x_{12}+x_{42})=\int \mathrm{d}q\, \exp\left[-iq\cdot(x_{34}-x_{31}-x_{12}+x_{42})\right], \tag{2.8}$$

to perform the integration over the spacetime positions:

$$\widehat{\mathcal{B}}_0(p)=\delta(P)\int \mathrm{d}q \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \mathrm{d}T_{12}\mathrm{d}T_{31}\mathrm{d}T_{34}\mathrm{d}T_{42}\, \exp\left[-\frac{1}{2}\widehat{B}_0(p,q,T)\right], \tag{2.9}$$

where

$$\widehat{B}_0=|q|^2\,T_{12}+\left(|q-p_{31}|^2+m_1^2\right)T_{31}+|q-p_{34}|^2\,T_{34}+\left(|q+p_{42}|^2+m_2^2\right)T_{42}. \tag{2.10}$$

This result is kinematic-exact.

2.1.1 Sudakov Moduli

One can integrate over the Schwinger moduli in (2.9) to obtain:

$$\widehat{\mathcal{B}}_A(p)=\delta(P)\int \mathrm{d}q\left(\frac{2}{|q|^2}\right)\left(\frac{2}{|q-p_{31}|^2+m_1^2}\right)\left(\frac{2}{|q-p_{34}|^2}\right)\left(\frac{2}{|q+p_{42}|^2+m_2^2}\right). \tag{2.11}$$

In this expression q plays the role of a (virtual) loop momentum variable.

The Dirac delta enforces the $P=0$ constraint. Enforcing this constraint leads to

$$p_{34}=p_1-p_3=p_4-p_2,\qquad p_{31}=p_1,\qquad p_{42}=p_2. \tag{2.12}$$

Thus,

$$|p_{34}|^2=-t,\qquad |p_{31}|^2=-m_1^2,\qquad |p_{42}|^2=-m_2^2. \tag{2.13}$$

Let k_1 and k_2 be null spacetime vectors with units of mass. The Sudakov null decomposition of p_1 and p_2 is as follows:

$$p_1=k_1+c_{12}k_2,\qquad p_2=k_2+c_{21}k_1. \tag{2.14}$$

In terms of p_1 and p_2, you have

$$k_1=\frac{p_1-c_{12}p_2}{1-c_{12}c_{21}},\qquad k_2=\frac{p_2-c_{21}p_1}{1-c_{12}c_{21}}. \tag{2.15}$$

From $|p_1|^2=-m_1^2$ and $|p_2|^2=-m_2^2$ it follows that

$$c_{12}=-\frac{m_1^2}{2(k_1\cdot k_2)},\qquad c_{21}=-\frac{m_2^2}{2(k_1\cdot k_2)}\qquad\Longrightarrow\qquad \frac{c_{12}}{c_{21}}=\frac{m_1^2}{m_2^2}. \tag{2.16}$$

Using $|k_1|^2=0$ and $|k_2|^2=0$ you find quadratic equations for c_{12} and c_{21}:

$$m_2^2c_{12}^2+\left(m_1^2+m_2^2-s\right)c_{12}+m_1^2=0,\qquad m_1^2c_{21}^2+\left(m_1^2+m_2^2-s\right)c_{21}+m_2^2=0. \tag{2.17}$$

Solving each quadratic equation yields

$$c_{21} = \left(\frac{m_2^2}{m_1^2}\right) c_{12}, \qquad c_{12} = \frac{s - m_1^2 - m_2^2 \pm \sqrt{\Lambda(s)}}{2m_2^2}. \tag{2.18}$$

Note that c_{12} and c_{21} are (dimensionless) functions that can be written in terms of two (dimensionless) ratios

$$\frac{s}{m_1 m_2}, \qquad \frac{m_1}{m_2}. \tag{2.19}$$

From $s = -|p_1 + p_2|^2$ it follows that

$$2(k_1 \cdot k_2) = -\frac{s}{(1+c_{12})(1+c_{21})} = -m_1 m_2 \left[\frac{2m_1 m_2}{s - m_1^2 - m_2^2 \pm \sqrt{\Lambda(s)}}\right]. \tag{2.20}$$

This can also be written as

$$2(k_1 \cdot k_2) = -m_1 m_2 \left[\frac{s - m_1^2 - m_2^2 \mp \sqrt{\Lambda(s)}}{2m_1 m_2}\right]. \tag{2.21}$$

Next you decompose the loop momentum q as

$$q = Q + a_q k_1 + b_q k_2. \tag{2.22}$$

Here a_q and b_q are Sudakov moduli. The integration measure over q becomes

$$\mathrm{d}q = \sqrt{|k_1|^2 |k_2|^2 - (k_1 \cdot k_2)^2}\,\mathrm{d}a_q \mathrm{d}b_q \mathrm{d}Q. \tag{2.23}$$

Note that the volume measure for Q is in $D-2$ spacetime dimensions.

Now you write each of the factors in the denominator in (2.11) in terms of the Sudakov moduli and the transversal momentum Q. First write

$$p_{34} = P_{34} + a_{34} k_1 + b_{34} k_2. \tag{2.24}$$

Since p_{34} is known, a_{34}, b_{34}, and P_{34} are also known. From $k_1 \cdot p_{34}$ and $k_2 \cdot p_{34}$ it follows that

$$b_{34} = \frac{k_1 \cdot p_{34}}{k_1 \cdot k_2}, \qquad a_{34} = \frac{k_2 \cdot p_{34}}{k_1 \cdot k_2}. \tag{2.25}$$

Using (2.15) leads to:

$$k_1 \cdot p_{34} = \frac{(p_1 - c_{12} p_2) \cdot (p_1 - p_3)}{1 - c_{12} c_{21}}, \qquad k_2 \cdot p_{34} = \frac{(p_2 - c_{21} p_1) \cdot (p_1 - p_3)}{1 - c_{12} c_{21}}. \tag{2.26}$$

Recall that

$$s = -|p_1 + p_2|^2 \quad \Rightarrow \quad p_1 \cdot p_2 = \frac{m_1^2 + m_2^2 - s}{2}, \tag{2.27}$$

$$t = -|p_1 - p_3|^2 \quad \Rightarrow \quad p_1 \cdot p_3 = \frac{t - 2m_1^2}{2}, \tag{2.28}$$

$$u = -|p_2 - p_3|^2 \quad \Rightarrow \quad p_2 \cdot p_3 = \frac{u - m_1^2 - m_2^2}{2}. \tag{2.29}$$

$$\tag{2.30}$$

Thus,

$$k_1 \cdot p_{34} = \frac{t}{2}\left(\frac{1 + c_{12}}{1 - c_{12}c_{21}}\right), \qquad k_2 \cdot p_{34} = -\frac{t}{2}\left(\frac{1 + c_{21}}{1 - c_{12}c_{21}}\right). \tag{2.31}$$

Hence,

$$a_{34} = \frac{t}{s}\left[\frac{(1 + c_{12})(1 + c_{21})^2}{1 - c_{12}c_{21}}\right], \qquad b_{34} = -\frac{t}{s}\left[\frac{(1 + c_{12})^2(1 + c_{21})}{1 - c_{12}c_{21}}\right]. \tag{2.32}$$

Using $|p_{12}|^2 = -t$ it follows that

$$|P_{34}|^2 = -t - 2a_{34}b_{34}\left(k_1 \cdot k_2\right) = -t - 2\left[\frac{\left(k_1 \cdot p_{34}\right)\left(k_2 \cdot p_{34}\right)}{\left(k_1 \cdot k_2\right)}\right], \tag{2.33}$$

which can be written as

$$|P_{34}|^2 = -t\left(1 + \frac{t}{s}\frac{(1 + c_{12})^2(1 + c_{21})^2}{(1 - c_{12}c_{21})^2}\right). \tag{2.34}$$

The terms in the denominator in (2.11) become:

$$|q|^2 = |Q|^2 + 2a_q b_q\left(k_1 \cdot k_2\right), \tag{2.35}$$

$$|q - p_{31}|^2 + m_1^2 = |Q|^2 + m_1^2 + 2\left(a_q + 1\right)\left(b_q + c_{12}\right)\left(k_1 \cdot k_2\right), \tag{2.36}$$

$$|q - p_{34}|^2 = |Q - P_{34}|^2 + 2\left(a_q - a_{34}\right)\left(b_q - b_{34}\right)\left(k_1 \cdot k_2\right), \tag{2.37}$$

$$|q + p_{42}|^2 + m_2^2 = |Q|^2 + m_2^2 + 2\left(a_q - c_{21}\right)\left(b_q - 1\right)\left(k_1 \cdot k_2\right). \tag{2.38}$$

2.1.2 Feynman Moduli

After integrating over q in (2.9), you find:

$$\widehat{\mathcal{B}}_A(p) = \delta(P)\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{\mathrm{d}T_{13}\mathrm{d}T_{21}\mathrm{d}T_{42}\mathrm{d}T_{34}}{(T_{13} + T_{21} + T_{42} + T_{34})^{D/2}} \exp\left[\frac{1}{2}\tilde{B}_A(p, T)\right], \tag{2.39}$$

where

$$\tilde{B}_A = tT_{21} + \frac{|T_{13}p_{13} - T_{21}p_{21} - T_{42}p_{42}|^2}{T_{13} + T_{21} + T_{42} + T_{34}}. \tag{2.40}$$

2.1.3 Regge Limit

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2.1.4 Forward-JWKB Limit

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2.2 Crossed Box

The cross-box correlator is given by

$$\mathcal{C}_A(x) = G_\Phi(x_1|x_3)G_A(x_4|x_1)G_\Psi(x_2|x_4)G_A(x_3|x_2), \tag{2.41}$$

but this expression is related to the box correlator (2.1) by swapping $x_2 \longleftrightarrow x_4$.

2.2.1 Regge Limit

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2.2.2 Forward-JWKB Limit

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2.3 Vertex Corrections

There are two one-loop vertex corrections:

$$\mathcal{V}_\Phi(x) = \delta(x_2 - x_4)G_A(x_1|x_3)\int \mathrm{d}y\, G_A(y|x_2)G_\Phi(y|x_1)G_\Phi(y|x_3), \tag{2.42}$$

$$\mathcal{V}_\Psi(x) = \delta(x_1 - x_3)G_A(x_2|x_4)\int \mathrm{d}y\, G_A(y|x_1)G_\Psi(y|x_2)G_\Psi(y|x_4). \tag{2.43}$$

2.3.1 Regge Limit

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2.3.2 Forward-JWKB Limit

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2.4 Vaccum Polarizations

There are two one-loop vacuum polarizations:

$$\mathcal{W}_\Phi(x) = \delta(x_1 - x_3)\delta(x_2 - x_4) \int\int \mathrm{d}y_1 \mathrm{d}y_2\, G_A(x_1|y_1) G_A(x_2|y_2) G_\Phi(y_1|y_2) G_\Phi(y_2|y_1), \quad (2.44)$$

$$\mathcal{W}_\Psi(x) = \delta(x_1 - x_3)\delta(x_2 - x_4) \int\int \mathrm{d}y_1 \mathrm{d}y_2\, G_A(x_1|y_1) G_A(x_2|y_2) G_\Psi(y_1|y_2) G_\Psi(y_2|y_1). \quad (2.45)$$

2.4.1 Regge Limit

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2.4.2 Forward-JWKB Limit

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Chapter 3

Massive Medium

The box correlator in a massive medium Y is:

$$\mathcal{B}_Y(x) = G_\Phi(x_1, x_3) G_\Psi(x_2, x_4) G_Y(x_1, x_2) G_Y(x_3, x_4). \tag{3.1}$$