

# ELASTIC ONE-LOOP AMPLITUDES

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# Chapter 1

## Introduction

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### 1.1 Euler Gamma Function

The Euler Gamma function is

$$\Gamma(z) = \int_0^{\infty} dx \left(\frac{1}{x}\right)^{1-z} \exp(-x). \quad (1.1)$$

Setting  $x = \kappa^2 w$  with  $\kappa^2 > 0$  leads to

$$\Gamma(z) = (\kappa^2)^z \int_0^{\infty} dw \left(\frac{1}{w}\right)^{1-z} \exp(-\kappa^2 w), \quad (1.2)$$

which allows you to write

$$\left(\frac{1}{\kappa^2}\right)^z = \frac{1}{\Gamma(z)} \int_0^{\infty} dw \left(\frac{1}{w}\right)^{1-z} \exp(-\kappa^2 w). \quad (1.3)$$

Here  $w$  is a Schwinger modulus.

### 1.2 Propagators

In the momentum basis, the propagator for a free quantum with mass  $m$  is given by

$$\widehat{G}_m(p, q) = \left(\frac{2}{|p|^2 + m^2}\right) \delta(p - q). \quad (1.4)$$

Using a Schwinger modulus, this can be re-written as

$$\widehat{G}_m(p, q) = \delta(p - q) \int_0^\infty dT \exp \left[ - \left( \frac{|p|^2 + m^2}{2} \right) T \right]. \quad (1.5)$$

From the momentum basis, you can go to the position basis via a Fourier transform:

$$G_m(x, y) = \int \int dp dq \widehat{G}_m(p, q) \exp (ip \cdot x - iq \cdot y). \quad (1.6)$$

Integration over  $p$  and  $q$  gives

$$G_m(x, y) = \int_0^\infty dT \left( \frac{1}{T} \right)^{D/2} \exp \left[ - \frac{1}{2T} |x - y|^2 - \frac{1}{2} m^2 T \right]. \quad (1.7)$$

As a special case, you can take the  $m \rightarrow 0$  limit to obtain the propagator for a free massless quantum:

$$G_0(x, y) = \int_0^\infty dT \left( \frac{1}{T} \right)^{D/2} \exp \left[ - \frac{1}{2T} |x - y|^2 \right] = \left( \frac{2}{|x - y|^2} \right)^{(D-2)/2} \Gamma \left( \frac{D-2}{2} \right). \quad (1.8)$$

This is valid as long as  $D \neq 2$ .

## 1.3 Kinematics

There are four external quanta; two incoming (labeled 1 and 2) and two outgoing (labeled 3 and 4). In the position basis, each external quantum is associated to a spacetime position. These four spacetime position vectors are independent. Similarly, in the momentum basis, each external quantum is associated to an energy-momentum vector. A priori, these four energy-momentum vectors are independent. But as you will see, due to translation invariance, the four energy-momentum vectors satisfy a linear constraint:

$$p_1 + p_2 = p_3 + p_4. \quad (1.9)$$

There are three Mandelstam invariants:

$$s = -|p_1 + p_2|^2, \quad t = -|p_1 - p_3|^2, \quad u = -|p_1 - p_4|^2. \quad (1.10)$$

Due to the conservation constraint, it follows that

$$s + t + u = 2m_\Phi^2 + 2m_\Psi^2. \quad (1.11)$$

An important function is

$$\Lambda(s) = [s - (m_\Phi - m_\Psi)^2][s - (m_\Phi + m_\Psi)^2]. \quad (1.12)$$

This is known as the Källén function. Note that  $\Lambda(s)$  can also be written as

$$\Lambda(s) = (s - m_\Phi^2 - m_\Psi^2)^2 - 4m_\Phi^2 m_\Psi^2. \quad (1.13)$$

Consider the following expression:

$$\mathbb{F} \equiv x_1 \cdot p_1 + x_2 \cdot p_2 - x_3 \cdot p_3 - x_4 \cdot p_4. \quad (1.14)$$

Now make the change of variables

$$X \equiv \frac{x_1 + x_2 + x_3 + x_4}{4}, \quad x_{12} \equiv x_1 - x_2, \quad x_{31} \equiv x_3 - x_1, \quad x_{42} \equiv x_4 - x_2. \quad (1.15)$$

The inverse relation is

$$x_1 = \frac{4X + 2x_{12} - x_{31} - x_{42}}{4}, \quad (1.16)$$

$$x_2 = \frac{4X - 2x_{12} - x_{31} - x_{42}}{4}, \quad (1.17)$$

$$x_3 = \frac{4X + 2x_{12} + 3x_{31} - x_{42}}{4}, \quad (1.18)$$

$$x_4 = \frac{4X - 2x_{12} - x_{31} + 3x_{42}}{4}. \quad (1.19)$$

Then  $\mathbb{F}$  can be written as

$$\mathbb{F} \equiv X \cdot P + x_{12} \cdot p_{12} - x_{31} \cdot p_{31} - x_{42} \cdot p_{42}, \quad (1.20)$$

where

$$P = p_1 + p_2 - p_3 - p_4, \quad (1.21)$$

$$p_{12} = \frac{p_1 - p_2 - p_3 + p_4}{2}, \quad (1.22)$$

$$p_{31} = \frac{p_1 + p_2 + 3p_3 - p_4}{4}, \quad (1.23)$$

$$p_{42} = \frac{p_1 + p_2 - p_3 + 3p_4}{4}. \quad (1.24)$$

# Chapter 2

## Massless Medium

In this chapter we consider one-loop contributions that involve a massless medium.

### 2.1 Box

The box correlator in a massless medium  $A$  is:

$$\mathcal{B}_A(x) = G_A(x_1|x_2)G_\Phi(x_3|x_1)G_A(x_3|x_4)G_\Psi(x_4|x_2). \quad (2.1)$$

In terms of four Schwinger moduli you have

$$\mathcal{B}_A = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty dT_{12}dT_{31}dT_{34}dT_{42} \left( \frac{1}{T_{12}T_{31}T_{34}T_{42}} \right)^{D/2} \exp \left[ -\frac{1}{2}B_A(x, T) \right], \quad (2.2)$$

where

$$B_A = \frac{1}{T_{12}}|x_{12}|^2 + \frac{1}{T_{31}}|x_{31}|^2 + m_\Phi^2 T_{31} + \frac{1}{T_{34}}|x_{34}|^2 + \frac{1}{T_{42}}|x_{42}|^2 + m_\Psi^2 T_{42}. \quad (2.3)$$

The box amplitude follows from the Fourier transform:

$$\widehat{\mathcal{B}}_A(p) = \int \int \int \int dx_1 dx_2 dx_3 dx_4 \mathcal{B}_A(x) \exp [i\mathbb{F}(x, p)], \quad (2.4)$$

with  $\mathbb{F}$  given by (1.14). Note that

$$x_{12} + x_{31} - x_{34} - x_{42} = 0. \quad (2.5)$$

That is,

$$|x_{34}|^2 = |x_{12} + x_{31} - x_{42}|^2. \quad (2.6)$$

We make the change of variables:

$$dx_1 dx_2 dx_3 dx_4 \sim dX dx_{12} dx_{31} dx_{42} = \int dX dx_{12} dx_{31} dx_{34} dx_{42} \delta(x_{12} + x_{31} - x_{34} - x_{42}), \quad (2.7)$$

and use

$$\delta(x_{12} + x_{31} - x_{34} - x_{42}) = \int dq \exp[-iq \cdot (x_{12} + x_{31} - x_{34} - x_{42})], \quad (2.8)$$

to perform the integration over the spacetime positions:

$$\widehat{\mathcal{B}}_A(p) = \delta(P) \int dq \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty dT_{12} dT_{31} dT_{34} dT_{42} \exp\left[-\frac{1}{2}\widehat{B}_A(p, q, T)\right], \quad (2.9)$$

where

$$\widehat{B}_A = |q - p_{12}|^2 T_{12} + (|q + p_{31}|^2 + m_\Phi^2) T_{31} + |q|^2 T_{34} + (|q - p_{42}|^2 + m_\Psi^2) T_{42}. \quad (2.10)$$

This result is kinematically exact.

### 2.1.1 Sudakov Moduli

One can integrate over the Schwinger moduli in (2.9) to obtain:

$$\widehat{\mathcal{B}}_A(p) = \delta(P) \int dq \left(\frac{2}{|q - p_{12}|^2}\right) \left(\frac{2}{|q + p_{31}|^2 + m_\Phi^2}\right) \left(\frac{2}{|q|^2}\right) \left(\frac{2}{|q - p_{42}|^2 + m_\Psi^2}\right). \quad (2.11)$$

In this expression  $q$  plays the role of a (virtual) loop momentum variable.

The Dirac delta enforces the  $P = 0$  constraint. Once this constraint is enforced, it follows that

$$p_{12} = p_1 - p_3 = p_4 - p_2, \quad p_{31} = p_3, \quad p_{42} = p_4. \quad (2.12)$$

Let  $k_3$  and  $k_4$  be null spacetime vectors with units of mass. The Sudakov null decomposition of  $p_3$  and  $p_4$  is as follows:

$$p_3 = k_3 + c_{34}k_4, \quad p_4 = k_4 + c_{43}k_3. \quad (2.13)$$

In terms of  $p_3$  and  $p_4$ , you have

$$k_3 = \frac{p_3 - c_{34}p_4}{1 - c_{34}c_{43}}, \quad k_4 = \frac{p_4 - c_{43}p_3}{1 - c_{34}c_{43}}. \quad (2.14)$$

From  $|p_3|^2 = -m_3^2$  and  $|p_4|^2 = -m_4^2$  it follows that

$$c_{34} = -\frac{m_3^2}{2(k_3 \cdot k_4)}, \quad c_{43} = -\frac{m_4^2}{2(k_3 \cdot k_4)} \implies \frac{c_{34}}{c_{43}} = \frac{m_3^2}{m_4^2}. \quad (2.15)$$

Using  $|k_3|^2 = 0$  and  $|k_4|^2 = 0$  you find quadratic equations for  $c_{34}$  and  $c_{43}$ :

$$m_4^2 c_{34}^2 + (m_3^2 + m_4^2 - s)c_{34} + m_3^2 = 0, \quad m_3^2 c_{43}^2 + (m_3^2 + m_4^2 - s)c_{43} + m_4^2 = 0. \quad (2.16)$$

Solving each quadratic equation yields

$$c_{43} = \left(\frac{m_4^2}{m_3^2}\right) c_{34}, \quad c_{34} = \frac{s - m_3^2 - m_4^2 \pm \sqrt{\Lambda(s)}}{2m_4^2}. \quad (2.17)$$

Note that  $c_{34}$  and  $c_{43}$  are (dimensionless) functions that can be written in terms of two (dimensionless) ratios

$$\frac{s}{m_3 m_4}, \quad \frac{m_3}{m_4}. \quad (2.18)$$

From  $s = -|p_3 + p_4|^2$  it follows that

$$2(k_3 \cdot k_4) = -\frac{s}{(1 + c_{34})(1 + c_{43})} = -m_3 m_4 \left[ \frac{2m_3 m_4}{s - m_3^2 - m_4^2 \pm \sqrt{\Lambda(s)}} \right]. \quad (2.19)$$

This can also be written as

$$2(k_3 \cdot k_4) = -m_3 m_4 \left[ \frac{s - m_3^2 - m_4^2 \mp \sqrt{\Lambda(s)}}{2m_3 m_4} \right]. \quad (2.20)$$

Next you decompose the loop momentum  $q$  as

$$q = a_q k_3 + b_q k_4 + q_\perp. \quad (2.21)$$

Here  $a_q$  and  $b_q$  are Sudakov moduli. The integration measure over  $q$  becomes

$$dq = \sqrt{|k_3|^2 |k_4|^2 - (k_3 \cdot k_4)^2} da_q db_q dq_\perp. \quad (2.22)$$

Note that the volume measure for  $q_\perp$  is in  $D - 2$  spacetime dimensions. Since  $|k_3|^2 = 0$  and  $|k_4|^2 = 0$ , the overall factor becomes  $\sqrt{-(k_3 \cdot k_4)^2}$ .

Now you write each of the factors in the denominator in (2.11) in terms of the Sudakov moduli and the transversal momentum  $q_\perp$ . First write

$$p_{12} = a_{12} k_3 + b_{12} k_4 + p_\perp. \quad (2.23)$$

Since  $p_{12}$  is known,  $a_{12}$ ,  $b_{12}$  and  $p_\perp$  are also known. From  $k_3 \cdot p_{12}$  and  $k_4 \cdot p_{12}$  it follows that

$$b_{12} = \frac{k_3 \cdot p_{12}}{k_3 \cdot k_4}, \quad a_{12} = \frac{k_4 \cdot p_{12}}{k_3 \cdot k_4}. \quad (2.24)$$

Using (2.14) leads to:

$$k_3 \cdot p_{12} = \frac{(p_3 - c_{34} p_4) \cdot (p_1 - p_3)}{1 - c_{34} c_{43}}, \quad k_4 \cdot p_{12} = \frac{(p_4 - c_{43} p_3) \cdot (p_1 - p_3)}{1 - c_{34} c_{43}}. \quad (2.25)$$



Recall that

$$s = -|p_3 + p_4|^2 \Rightarrow p_3 \cdot p_4 = \frac{m_3^2 + m_4^2 - s}{2}, \quad (2.26)$$

$$t = -|p_1 - p_3|^2 \Rightarrow p_1 \cdot p_3 = \frac{t - m_1^2 - m_3^2}{2}, \quad (2.27)$$

$$u = -|p_1 - p_4|^2 \Rightarrow p_1 \cdot p_4 = \frac{u - m_1^2 - m_4^2}{2}. \quad (2.28)$$

$$(2.29)$$

Thus,

$$k_3 \cdot p_{12} = \frac{t}{2} \left( \frac{1 + c_{34}}{1 - c_{34}c_{43}} \right), \quad k_4 \cdot p_{12} = -\frac{t}{2} \left( \frac{1 + c_{43}}{1 - c_{34}c_{43}} \right). \quad (2.30)$$

Hence,

$$a_{12} = \frac{t}{s} \left[ \frac{(1 + c_{34})(1 + c_{43})^2}{1 - c_{34}c_{43}} \right], \quad b_{12} = -\frac{t}{s} \left[ \frac{(1 + c_{34})^2(1 + c_{43})}{1 - c_{34}c_{43}} \right]. \quad (2.31)$$

Using  $|p_{12}|^2 = -t$  it follows that

$$|p_\perp|^2 = -t - 2a_{12}b_{12}(k_3 \cdot k_4) = -t - 2 \left[ \frac{(k_3 \cdot p_{12})(k_4 \cdot p_{12})}{(k_3 \cdot k_4)} \right], \quad (2.32)$$

which can be written as

$$|p_\perp|^2 = -t \left( 1 + \frac{t}{s} \frac{(1 + c_{34})^2(1 + c_{43})^2}{(1 - c_{34}c_{43})^2} \right). \quad (2.33)$$

The terms in the denominator become:

$$|q - p_{12}|^2 = 2(a_q - a_{12})(b_q - b_{12})(k_3 \cdot k_4) + |q_\perp - p_\perp|^2, \quad (2.34)$$

$$|q + p_3|^2 + m_\Phi^2 = 2(a_q + 1)(b_q + c_{34})(k_3 \cdot k_4) + |q_\perp|^2 + m_\Phi^2, \quad (2.35)$$

$$|q|^2 = 2a_q b_q (k_3 \cdot k_4) + |q_\perp|^2, \quad (2.36)$$

$$|q - p_4|^2 + m_\Psi^2 = 2(a_q - c_{43})(b_q - 1)(k_3 \cdot k_4) + |q_\perp|^2 + m_\Psi^2. \quad (2.37)$$

### 2.1.2 Feynman Moduli

After integrating over  $q$  in (2.9), you find:

$$\widehat{\mathcal{B}}_A(p) = \delta(P) \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \frac{dT_{13}dT_{21}dT_{42}dT_{34}}{(T_{13} + T_{21} + T_{42} + T_{34})^{D/2}} \exp \left[ \frac{1}{2} \tilde{B}_A(p, T) \right], \quad (2.38)$$

where

$$\tilde{B}_A = tT_{21} + \frac{|T_{13}p_{13} - T_{21}p_{21} - T_{42}p_{42}|^2}{T_{13} + T_{21} + T_{42} + T_{34}}. \quad (2.39)$$

### 2.1.3 Regge Limit

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### 2.1.4 Forward-JWKB Limit

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## 2.2 Crossed Box

The cross-box correlator is given by

$$\mathcal{C}_A(x) = G_\Phi(x_1|x_3)G_A(x_4|x_1)G_\Psi(x_2|x_4)G_A(x_3|x_2), \quad (2.40)$$

but this expression is related to the box correlator (2.1) by swapping  $x_2 \longleftrightarrow x_4$ .

### 2.2.1 Regge Limit

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### 2.2.2 Forward-JWKB Limit

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## 2.3 Vertex Corrections

There are two one-loop vertex corrections:

$$\mathcal{V}_\Phi(x) = \delta(x_2 - x_4)G_A(x_1|x_3) \int dy G_A(y|x_2)G_\Phi(y|x_1)G_\Phi(y|x_3), \quad (2.41)$$

$$\mathcal{V}_\Psi(x) = \delta(x_1 - x_3)G_A(x_2|x_4) \int dy G_A(y|x_1)G_\Psi(y|x_2)G_\Psi(y|x_4). \quad (2.42)$$

### 2.3.1 Regge Limit

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### 2.3.2 Forward-JWKB Limit

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## 2.4 Vacuum Polarizations

There are two one-loop vacuum polarizations:

$$\mathcal{W}_\Phi(x) = \delta(x_1 - x_3)\delta(x_2 - x_4) \int \int dy_1 dy_2 G_A(x_1|y_1)G_A(x_2|y_2)G_\Phi(y_1|y_2)G_\Phi(y_2|y_1), \quad (2.43)$$

$$\mathcal{W}_\Psi(x) = \delta(x_1 - x_3)\delta(x_2 - x_4) \int \int dy_1 dy_2 G_A(x_1|y_1)G_A(x_2|y_2)G_\Psi(y_1|y_2)G_\Psi(y_2|y_1). \quad (2.44)$$

### 2.4.1 Regge Limit

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### 2.4.2 Forward-JWKB Limit

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# Chapter 3

## Massive Medium

The box correlator in a massive medium  $Y$  is:

$$\mathcal{B}_Y(x) = G_\Phi(x_1, x_3)G_\Psi(x_2, x_4)G_Y(x_1, x_2)G_Y(x_3, x_4). \quad (3.1)$$