Relativistic Four-Point Kinematics

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Contents

Chapter A

Introduction

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A.1 Mandelstam Invariants

For convenience, one defines the three Mandelstam invariants:

$$s = -|p_1 + p_2|^2, \qquad t = -|p_1 - p_3|^2, \qquad u = -|p_1 - p_4|^2. \tag{A.1}$$

Energy-momentum conservation leads to a constraint,

$$p_1 + p_2 = p_3 + p_4. \tag{A.2}$$

This constraint means that only three out of the four external energy-momentum vectors are linearly independent. Due to the conservation constraint, the three Mandelstam invariants satisfy a linear relation:

$$s + t + u = -|p_1|^2 - |p_2|^2 - |p_3|^2 - |p_4|^2. \tag{A.3}$$

Thus, only two of them are linearly independent.

A.2 Dual Spacetime

You can solve the conservation constraint by introducing dual spacetime coordinates:

$$p_1 = d_1 - d_2, \quad p_2 = d_2 - d_3, \quad p_4 = d_4 - d_3, \quad p_3 = d_1 - d_4. \tag{A.4}$$

Thus, each energy-momentum vector becomes a distance interval in a dual spacetime. However, the solution (A.4) is not the only one allowed. Indeed, two other solutions are allowed:

$$-p_3 = d_1 - d_2, \quad p_1 = d_2 - d_3, \quad p_4 = d_4 - d_3, \quad -p_2 = d_1 - d_4, \tag{A.5}$$

and

$$p_2 = d_1 - d_2, \quad -p_3 = d_2 - d_3, \quad p_4 = d_4 - d_3, \quad -p_1 = d_1 - d_4. \tag{A.6}$$

I will refer to (A.4) as the red planar class, (A.5) as the blue planar class, and (A.6) as the green planar class. In each planar class, the dual spacetime coordinates describe the positions of four points in the dual spacetime. These four points can be taken as the vertices of a (Minkowski) tetrahedron. Many kinematic quantities can be understood in terms of the geometry of these (Minkowski) tetrahedra.

A.3 Simplicial Invariants

A tetrahedron is a 3-simplex. As such, it contains four vertices (0-simplex), six edges (1-simplex), four triangular faces (2-simplex), and one tetrahedron (3-simplex). The n-volume of an n-simplex is found by evaluating an $(n+1)$-point Cayley-Menger determinant. These are useful kinematic invariants.

A.3.1 1-Simplex Invariants

Given two dual spacetime positions, the 2-point Cayley-Menger determinant is given by:

$$C_{IJ} = -\frac{1}{2} \det \begin{pmatrix} 0 & |d_{IJ}|^2 & 1 \\ |d_{IJ}|^2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \qquad d_{IJ} \equiv d_I - d_J. \tag{A.7}$$

Since a tetrahedron has six edges, there are six possible 1-simplex invariants for each tetrahedron. These invariants will be either squared masses or Mandelstam invariants.

A.3.2 2-Simplex Invariants

Given three dual spacetime positions, the 3-point Cayley-Menger determinant is given by:

$$C_{IJK} = \det \begin{pmatrix} 0 & |d_{IJ}|^2 & |d_{IK}|^2 & 1 \\ |d_{IJ}|^2 & 0 & |d_{JK}|^2 & 1 \\ |d_{IK}|^2 & |d_{JK}|^2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \qquad d_{IJ} \equiv d_I - d_J. \tag{A.8}$$

Since a tetrahedron has four triangular faces, there are four possible 2-simplex invariants for each tetrahedron. These invariants take the form of Källén functions. I will introduce three kinds of Källén functions.

Slow Källén Function

The slow Källén function is

$$\Lambda_{ij}(x) = \left[\left(x - m_i^2 - m_j^2\right)^2 - 4m_i^2 m_j^2\right] = \left[x - (m_i - m_j)^2\right]\left[x - (m_i + m_j)^2\right], \tag{A.9}$$

where x is a Mandelstam invariant. Note that $\Lambda_{ij}(x)$ can be either negative, zero, or positive for real values of x.

If the slow Källén function is negative, then

$$\Lambda_{ij}(x) < 0 \quad \Longrightarrow \quad \left|x - m_i^2 - m_j^2\right| < 2m_i m_j, \tag{A.10}$$

which leads to

$$\Lambda_{ij}(x) < 0 \quad \Longrightarrow \quad x > (m_i - m_j)^2 \text{ and } x < (m_i + m_j)^2. \tag{A.11}$$

Similarly, if the slow Källén function is zero, then

$$\Lambda_{ij}(x) = 0 \quad \Longrightarrow \quad \left|x - m_i^2 - m_j^2\right| = 2m_i m_j, \tag{A.12}$$

which leads to

$$\Lambda_{ij}(x) = 0 \quad \Longrightarrow \quad x = (m_i - m_j)^2 \text{ or } x = (m_i + m_j)^2. \tag{A.13}$$

Finally, if the slow Källén function is positive, then

$$\Lambda_{ij}(x) > 0 \quad \Longrightarrow \quad \left|x - m_i^2 - m_j^2\right| > 2m_i m_j, \tag{A.14}$$

which leads to

$$\Lambda_{ij}(x) > 0 \quad \Longrightarrow \quad x < (m_i - m_j)^2 \text{ or } x > (m_i + m_j)^2. \tag{A.15}$$

Fast Källén Function

The fast Källén function is

$$\Upsilon_{ij}(x) = \left[\left(x + w_i^2 + w_j^2\right)^2 - 4w_i^2 w_j^2\right] = \left[x + (w_i - w_j)^2\right]\left[x + (w_i + w_j)^2\right], \tag{A.16}$$

where x is a Mandelstam invariant. Just like $\Lambda_{ij}(x)$, the function Υ_{ij} can be either negative, zero, or positive.

If the fast Källén function is negative, then

$$\Upsilon_{ij}(x) < 0 \quad \Longrightarrow \quad \left|x + w_i^2 + w_j^2\right| < 2w_i w_j, \tag{A.17}$$

which leads to

$$\Upsilon_{ij}(x) < 0 \quad \Longrightarrow \quad x > -\left(w_i + w_j\right)^2 \text{ and } x < -\left(w_i - w_j\right)^2. \tag{A.18}$$

Similarly, if the fast Källén function is zero, then

$$\Upsilon_{ij}(x) = 0 \quad \Longrightarrow \quad \left|x + w_i^2 + w_j^2\right| = 2w_i w_j, \tag{A.19}$$

which leads to

$$\Upsilon_{ij}(x) = 0 \quad \Longrightarrow \quad x = -\left(w_i + w_j\right)^2 \text{ or } x = -\left(w_i - w_j\right)^2. \tag{A.20}$$

Finally, if the fast Källén function is positive, then

$$\Upsilon_{ij}(x) > 0 \quad \Longrightarrow \quad \left|x + w_i^2 + w_j^2\right| > 2w_i w_j, \tag{A.21}$$

which leads to

$$\Upsilon_{ij}(x) > 0 \quad \Longrightarrow \quad x < -\left(w_i + w_j\right)^2 \text{ or } x > -\left(w_i - w_j\right)^2. \tag{A.22}$$

Mixed Källén Function

The mixed Källén function is

$$\Omega_{ij}(x) = \left[\left(x - m_i^2 + w_j^2\right)^2 + 4m_i^2 w_j^2\right], \tag{A.23}$$

where x is a Mandelstam invariant. Note that this function is always positive for x real.

A.3.3 3-Simplex Invariants

Given four dual spacetime positions, the 4-point Cayley-Menger determinant is given by:

$$C_{IJKL} = -\frac{1}{2}\det\begin{pmatrix} 0 & |d_{IJ}|^2 & |d_{IK}|^2 & |d_{IL}|^2 & 1 \\ |d_{IJ}|^2 & 0 & |d_{JK}|^2 & |d_{JL}|^2 & 1 \\ |d_{IK}|^2 & |d_{JK}|^2 & 0 & |d_{KL}|^2 & 1 \\ |d_{IL}|^2 & |d_{JL}|^2 & |d_{KL}|^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}, \qquad d_{IJ} \equiv d_I - d_J. \tag{A.24}$$

A tetrahedron has only one possible 3-simplex invariant.

A.4 Dual Conformal Invariants

The expression $|d_I - d_J|^2$ is not only Lorentz invariant, but dual Poincaré invariant. You can study dual conformal invariants by constructing a dual conformal ratio with a quartet of dual spacetime coordinates:

$$[I, J; K, L] \equiv \frac{|d_{IK}|^2\,|d_{JL}|^2}{|d_{IL}|^2\,|d_{JK}|^2}, \qquad d_{IJ} \equiv d_I - d_J. \tag{A.25}$$

In four-point scattering there is a unique quartet of dual spacetime coordinates. However, there are six inequivalent permutations of these coordinates, and thus, six possible values for the dual conformal ratio.

A.5 Speed and Rapidity

The mass m, spatial momentum $\mathbf{p}$, and energy E of a slow quantum satisfy an on-shell constraint:

$$-m^2 = -E^2 + |\mathbf{p}|^2 . \tag{A.26}$$

This constraint can be solved by writing

$$E = m \cosh \rho, \qquad |\mathbf{p}| = m \sinh \rho. \tag{A.27}$$

Thus,

$$\tanh \rho = \frac{\sqrt{E^2 - m^2}}{E} = \frac{|\mathbf{p}|}{\sqrt{|\mathbf{p}|^2 + m^2}} = \frac{|\mathbf{p}|}{E} \equiv |\mathbf{v}| . \tag{A.28}$$

Here ρ is the (slow) rapidity, and $|\mathbf{v}|$ is the speed of the slow quantum. Note that

$$m \leq E < \infty \text{ or } 0 \leq |\mathbf{p}| < \infty \quad \Longrightarrow \quad 0 \leq |\mathbf{v}| < 1. \tag{A.29}$$

That is, the speed of a slow quantum is bounded from above.

Similarly, the imass w, spatial momentum $\mathbf{p}$, and energy E of a fast quantum satisfy an on-shell constraint:

$$w^2 = -E^2 + |\mathbf{p}|^2 . \tag{A.30}$$

This constraint can be solved by writing

$$E = w \sinh \xi, \qquad |\mathbf{p}| = w \cosh \xi. \tag{A.31}$$

Thus,

$$\coth \xi = \frac{\sqrt{E^2 + w^2}}{E} = \frac{|\mathbf{p}|}{\sqrt{|\mathbf{p}|^2 - w^2}} = \frac{|\mathbf{p}|}{E} \equiv |\mathbf{v}| . \tag{A.32}$$

Here ξ is the (fast) rapidity, and $|\mathbf{v}|$ is the speed of the fast quantum. Note that

$$0 \leq E < \infty \text{ or } w \leq |\mathbf{p}| < \infty \quad \Longrightarrow \quad 1 < |\mathbf{v}| < \infty. \tag{A.33}$$

That is, the speed of a fast quantum is bounded from below.

Recall that

$$\operatorname{argtanh} x = \frac{1}{2} \log \left(\frac{1+x}{1-x} \right), \qquad 0 \leq x < 1, \tag{A.34}$$

and

$$\operatorname{argcoth} x = \frac{1}{2} \log \left(\frac{x+1}{x-1} \right), \qquad 1 < x < \infty. \tag{A.35}$$

Part I

Slow Quanta

Chapter B

Maximal Inelastic Slow Diversity

In this chapter I will consider the kinematics of a 2-to-2 scattering process that has maximal inelastic slow diversity. This means that all external quanta are slow, and all of the corresponding masses are distinct:

$$m_1^2 = -|p_1|^2, \qquad m_2^2 = -|p_2|^2, \qquad m_3^2 = -|p_3|^2, \qquad m_4^2 = -|p_4|^2. \tag{B.1}$$

Thus, this is an inelastic process. For example,

$$A(p_1) + B(p_2) \longrightarrow Y(p_3) + Z(p_4). \tag{B.2}$$

For this process, (A.3) becomes

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2. \tag{B.3}$$

B.1 Simplicial Invariants

Since this scattering process is inelastic, the simplicial invariants take many distinct values.

B.1.1 1-Simplex Invariants

For the red tetrahedron you have

$$C_{12} = m_1^2, \quad C_{13} = s, \quad C_{14} = m_3^2, \quad C_{23} = m_2^2, \quad C_{24} = t, \quad C_{34} = m_4^2. \tag{B.4}$$

Similarly, for the blue tetrahedron you have

$$C_{12} = m_3^2, \quad C_{13} = t, \quad C_{14} = m_2^2, \quad C_{23} = m_1^2, \quad C_{24} = u, \quad C_{34} = m_4^2. \tag{B.5}$$

Finally, for the green tetrahedron you have

$$C_{12} = m_2^2, \quad C_{13} = u, \quad C_{14} = m_1^2, \quad C_{23} = m_3^2, \quad C_{24} = s, \quad C_{34} = m_4^2. \tag{B.6}$$

B.1.2 2-Simplex Invariants

For the red tetrahedron you have

$$\begin{aligned}
C_{123} &= \left[s - (m_1 - m_2)^2\right]\left[s - (m_1 + m_2)^2\right] = \Lambda_{12}(s),\\
C_{124} &= \left[t - (m_1 - m_3)^2\right]\left[t - (m_1 + m_3)^2\right] = \Lambda_{13}(t),\\
C_{134} &= \left[s - (m_3 - m_4)^2\right]\left[s - (m_3 + m_4)^2\right] = \Lambda_{34}(s),\\
C_{234} &= \left[t - (m_2 - m_4)^2\right]\left[t - (m_2 + m_4)^2\right] = \Lambda_{24}(t).
\end{aligned} \tag{B.7}$$

Similarly, for the blue tetrahedron you have

$$\begin{aligned}
C_{123} &= \left[t - (m_1 - m_3)^2\right]\left[t - (m_1 + m_3)^2\right] = \Lambda_{13}(t),\\
C_{124} &= \left[u - (m_2 - m_3)^2\right]\left[u - (m_2 + m_3)^2\right] = \Lambda_{23}(u),\\
C_{134} &= \left[t - (m_2 - m_4)^2\right]\left[t - (m_2 + m_4)^2\right] = \Lambda_{24}(t),\\
C_{234} &= \left[u - (m_1 - m_4)^2\right]\left[u - (m_1 + m_4)^2\right] = \Lambda_{14}(u).
\end{aligned} \tag{B.8}$$

Finally, for the green tetrahedron you have

$$\begin{aligned}
C_{123} &= \left[u-(m_2-m_3)^2\right]\left[u-(m_2+m_3)^2\right] = \Lambda_{23}(u),\\
C_{124} &= \left[s-(m_1-m_2)^2\right]\left[s-(m_1+m_2)^2\right] = \Lambda_{12}(s),\\
C_{134} &= \left[u-(m_1-m_4)^2\right]\left[u-(m_1+m_4)^2\right] = \Lambda_{14}(u),\\
C_{234} &= \left[s-(m_3-m_4)^2\right]\left[s-(m_3+m_4)^2\right] = \Lambda_{34}(s).
\end{aligned} \tag{B.9}$$

B.1.3 3-Simplex Invariants

Each tetrahedron has only one possible 3-simplex invariant. Indeed,

$$C_{1234} = C_{1234} = C_{1234}. \tag{B.10}$$

B.2 Dual Conformal Invariants

For the red tetrahedron, you have

$$\begin{aligned}
[1,2;3,4] &= \frac{st}{m_2^2 m_3^2}, \qquad [1,3;4,2] = \frac{m_2^2 m_3^2}{m_1^2 m_4^2}, \qquad [1,4;2,3] = \frac{m_1^2 m_4^2}{st},\\
[1,2;4,3] &= \frac{m_2^2 m_3^2}{st}, \qquad [1,3;2,4] = \frac{m_1^2 m_4^2}{m_2^2 m_3^2}, \qquad [1,4;3,2] = \frac{st}{m_1^2 m_4^2}.
\end{aligned} \tag{B.11}$$

Similarly, for the blue tetrahedron, you have

$$\begin{aligned}
[1,2;3,4] &= \frac{tu}{m_1^2 m_2^2}, \qquad [1,3;4,2] = \frac{m_1^2 m_2^2}{m_3^2 m_4^2}, \qquad [1,4;2,3] = \frac{m_3^2 m_4^2}{tu},\\
[1,2;4,3] &= \frac{m_1^2 m_2^2}{tu}, \qquad [1,3;2,4] = \frac{m_3^2 m_4^2}{m_1^2 m_2^2}, \qquad [1,4;3,2] = \frac{tu}{m_3^2 m_4^2}.
\end{aligned} \tag{B.12}$$

Finally, for the green tetrahedron, you have

$$\begin{aligned}
[1,2;3,4] &= \frac{su}{m_1^2 m_3^2}, \qquad [1,3;4,2] = \frac{m_1^2 m_3^2}{m_2^2 m_4^2}, \qquad [1,4;2,3] = \frac{m_2^2 m_4^2}{su},\\
[1,2;4,3] &= \frac{m_1^2 m_3^2}{su}, \qquad [1,3;2,4] = \frac{m_2^2 m_4^2}{m_1^2 m_3^2}, \qquad [1,4;3,2] = \frac{su}{m_2^2 m_4^2}.
\end{aligned} \tag{B.13}$$

B.3 Center-of-Momentum Frame

In the center-of-momentum frame you write the energy-momentum vectors as

$$p_1 = \begin{pmatrix} E_1 & \mathbf{p}_1 \end{pmatrix}, \qquad p_2 = \begin{pmatrix} E_2 & -\mathbf{p}_1 \end{pmatrix}, \qquad p_3 = \begin{pmatrix} E_3 & \mathbf{p}_3 \end{pmatrix}, \qquad p_4 = \begin{pmatrix} E_4 & -\mathbf{p}_3 \end{pmatrix}. \tag{B.14}$$

From the on-shell relations (B.1) it follows that

$$|\mathbf{p}_1|^2 = E_1^2 - m_1^2 = E_2^2 - m_2^2, \qquad |\mathbf{p}_3|^2 = E_3^2 - m_3^2 = E_4^2 - m_4^2. \tag{B.15}$$

Using $s = -|p_1+p_2|^2$, it follows that

$$s = (E_1+E_2)^2 \quad \Longrightarrow \quad E_2 = \sqrt{s} - E_1. \tag{B.16}$$

Similarly, using $s = -|p_3+p_4|^2$, it follows that

$$s = (E_3+E_4)^2 \quad \Longrightarrow \quad E_4 = \sqrt{s} - E_3. \tag{B.17}$$

Note that s must be positive.

Using the on-shell relations, you can find the energies:

$$E_1 = \frac{s+m_1^2-m_2^2}{2\sqrt{s}} \quad \Longrightarrow \quad E_2 = \frac{s-m_1^2+m_2^2}{2\sqrt{s}}, \tag{B.18}$$

and

$$E_3 = \frac{s + m_3^2 - m_4^2}{2\sqrt{s}} \quad \Longrightarrow \quad E_4 = \frac{s - m_3^2 + m_4^2}{2\sqrt{s}}. \tag{B.19}$$

These lead to the magnitude of the spatial momenta:

$$|\mathbf{p}_1| = \frac{\sqrt{\Lambda_{12}(s)}}{2\sqrt{s}}, \qquad |\mathbf{p}_3| = \frac{\sqrt{\Lambda_{34}(s)}}{2\sqrt{s}}. \tag{B.20}$$

Using $t = -\left|p_1 - p_3\right|^2$, it follows that

$$t = (E_1 - E_3)^2 - |\mathbf{p}_1 - \mathbf{p}_3|^2 . \tag{B.21}$$

Similarly, using $u = -\left|p_1 - p_4\right|^2$, it follows that

$$u = (E_1 - E_4)^2 - |\mathbf{p}_1 + \mathbf{p}_3|^2 . \tag{B.22}$$

Thus,

$$u - t = (E_1 - E_4)^2 - (E_1 - E_3)^2 - 4\left(\mathbf{p}_1 \cdot \mathbf{p}_3\right), \tag{B.23}$$

and

$$u + t = (E_1 - E_4)^2 + (E_1 - E_3)^2 - 2\left(|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2\right). \tag{B.24}$$

It follows that

$$(\mathbf{p}_1 \cdot \mathbf{p}_3) = \frac{(m_1^2 - m_2^2)(m_3^2 - m_4^2) - s\,(u - t)}{4s}. \tag{B.25}$$

Alternatively, you also find that

$$|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 = \frac{(m_1^2 - m_2^2)^2 + (m_3^2 - m_4^2)^2 - 2s\,(u + t)}{4s}. \tag{B.26}$$

Using

$$\frac{1}{|\mathbf{p}_1|\,|\mathbf{p}_3|} = \frac{1}{|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2}\left(\frac{|\mathbf{p}_1|}{|\mathbf{p}_3|} + \frac{|\mathbf{p}_3|}{|\mathbf{p}_1|}\right). \tag{B.27}$$

From here you can find that

$$z_{13} \equiv \frac{\mathbf{p}_1 \cdot \mathbf{p}_3}{|\mathbf{p}_1|\,|\mathbf{p}_3|} = \left[\frac{\Lambda_{12}(s) + \Lambda_{34}(s)}{\sqrt{\Lambda_{12}(s)}\sqrt{\Lambda_{34}(s)}}\right]\left[\frac{(m_1^2 - m_2^2)(m_3^2 - m_4^2) - s\,(u - t)}{(m_1^2 - m_2^2)^2 + (m_3^2 - m_4^2)^2 - 2s\,(u + t)}\right]. \tag{B.28}$$

Note that

$$\Lambda_{12}(s) + \Lambda_{34}(s) = \left(m_1^2 - m_2^2\right)^2 + \left(m_3^2 - m_4^2\right)^2 - 2s\,(t + u), \tag{B.29}$$

$$\Lambda_{13}(t) + \Lambda_{24}(t) = \left(m_1^2 - m_3^2\right)^2 + \left(m_2^2 - m_4^2\right)^2 - 2t\,(s + u), \tag{B.30}$$

$$\Lambda_{14}(u) + \Lambda_{23}(u) = \left(m_1^2 - m_4^2\right)^2 + \left(m_2^2 - m_3^2\right)^2 - 2u\,(s + t). \tag{B.31}$$

The right-hand side has dependence on one Mandelstam invariant after using (B.3).

B.3.1 Speed and Rapidity

The speed of each external quantum is

$$|\mathbf{v}_1| = \frac{\sqrt{\Lambda_{12}(s)}}{s + m_1^2 - m_2^2}, \tag{B.32}$$

$$|\mathbf{v}_2| = \frac{\sqrt{\Lambda_{12}(s)}}{s - m_1^2 + m_2^2}, \tag{B.33}$$

$$|\mathbf{v}_3| = \frac{\sqrt{\Lambda_{34}(s)}}{s + m_3^2 - m_4^2}, \tag{B.34}$$

$$|\mathbf{v}_4| = \frac{\sqrt{\Lambda_{34}(s)}}{s - m_3^2 + m_4^2}. \tag{B.35}$$

The rapidity of each external quantum is

$$\rho_1 = \frac{1}{2}\log\left[\frac{s+m_1^2-m_2^2+\sqrt{\Lambda_{12}(s)}}{s+m_1^2-m_2^2-\sqrt{\Lambda_{12}(s)}}\right], \tag{B.36}$$

$$\rho_2 = \frac{1}{2}\log\left[\frac{s-m_1^2+m_2^2+\sqrt{\Lambda_{12}(s)}}{s-m_1^2+m_2^2-\sqrt{\Lambda_{12}(s)}}\right], \tag{B.37}$$

$$\rho_3 = \frac{1}{2}\log\left[\frac{s+m_3^2-m_4^2+\sqrt{\Lambda_{34}(s)}}{s+m_3^2-m_4^2-\sqrt{\Lambda_{34}(s)}}\right], \tag{B.38}$$

$$\rho_4 = \frac{1}{2}\log\left[\frac{s-m_3^2+m_4^2+\sqrt{\Lambda_{34}(s)}}{s-m_3^2+m_4^2-\sqrt{\Lambda_{34}(s)}}\right]. \tag{B.39}$$

The sum of the incoming rapidities is

$$\rho_1+\rho_2 = \frac{1}{2}\log\left[\frac{s-m_1^2-m_2^2+\sqrt{\Lambda_{12}(s)}}{s-m_1^2-m_2^2-\sqrt{\Lambda_{12}(s)}}\right]. \tag{B.40}$$

Similarly, the sum of the outgoing rapidities is

$$\rho_3+\rho_4 = \frac{1}{2}\log\left[\frac{s-m_3^2-m_4^2+\sqrt{\Lambda_{34}(s)}}{s-m_3^2-m_4^2-\sqrt{\Lambda_{34}(s)}}\right]. \tag{B.41}$$

B.3.2 Physical Scattering Region

Inside the physical scattering region, you must require that $|\mathbf{p}_1|$ be real and positive. This leads to the condition

$$s > (m_1+m_2)^2. \tag{B.42}$$

Similarly, requiring that $|\mathbf{p}_3|$ be real and positive leads to the condition

$$s > (m_3+m_4)^2. \tag{B.43}$$

Thus, we take the overlap of these two regions:

$$s > \max\left\{(m_1+m_2)^2, (m_3+m_4)^2\right\}. \tag{B.44}$$

B.3.3 Scattering Regimes

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Chapter C

Maximal Elastic Slow Diversity

In this chapter I will consider the kinematics of a 2-to-2 scattering process that has maximal elastic slow diversity. This means that all external quanta are slow, but you have two distinct elastic pairs:

$$m_1^2 = -\left|p_1\right|^2 = -\left|p_3\right|^2, \qquad m_2^2 = -\left|p_2\right|^2 = -\left|p_4\right|^2. \tag{C.1}$$

Thus, this is an elastic process. For example,

$$A(p_1) + B(p_2) \longrightarrow A(p_3) + B(p_4). \tag{C.2}$$

For this process, (A.3) becomes

$$s + t + u = 2m_1^2 + 2m_2^2. \tag{C.3}$$

C.1 Simplicial Invariants

Since this scattering process is elastic, there are fewer distinct values among the simplicial invariants.

C.1.1 1-Simplex Invariants

For the red tetrahedron you have

$$\begin{aligned} C_{12} &= m_1^2, & C_{13} &= s, & C_{14} &= m_1^2, \\ C_{23} &= m_2^2, & C_{24} &= t, & C_{34} &= m_2^2. \end{aligned} \tag{C.4}$$

Similarly, for the blue tetrahedron you have

$$\begin{aligned} C_{12} &= m_1^2, & C_{13} &= t, & C_{14} &= m_2^2, \\ C_{23} &= m_1^2, & C_{24} &= u, & C_{34} &= m_2^2. \end{aligned} \tag{C.5}$$

Finally, for the green tetrahedron you have

$$\begin{aligned} C_{12} &= m_2^2, & C_{13} &= u, & C_{14} &= m_1^2, \\ C_{23} &= m_1^2, & C_{24} &= s, & C_{34} &= m_2^2. \end{aligned} \tag{C.6}$$

C.1.2 2-Simplex Invariants

For the red tetrahedron you have

$$\begin{aligned} C_{123} &= \left[s - (m_1 - m_2)^2\right]\left[s - (m_1 + m_2)^2\right], \\ C_{124} &= t\left[t - 4m_1^2\right], \\ C_{134} &= \left[s - (m_1 - m_2)^2\right]\left[s - (m_1 + m_2)^2\right], \\ C_{234} &= t\left[t - 4m_2^2\right]. \end{aligned} \tag{C.7}$$

Similarly, for the blue tetrahedron you have

$$\begin{aligned}
C_{123} &= t\left[t - 4m_1^2\right], \\
C_{124} &= \left[u - (m_1 - m_2)^2\right]\left[u - (m_1 + m_2)^2\right], \\
C_{134} &= t\left[t - 4m_2^2\right], \\
C_{234} &= \left[u - (m_1 - m_2)^2\right]\left[u - (m_1 + m_2)^2\right].
\end{aligned} \tag{C.8}$$

Finally, for the green tetrahedron you have

$$\begin{aligned}
C_{123} &= \left[u - (m_1 - m_2)^2\right]\left[u - (m_1 + m_2)^2\right], \\
C_{124} &= \left[s - (m_1 - m_2)^2\right]\left[s - (m_1 + m_2)^2\right], \\
C_{134} &= \left[u - (m_1 - m_2)^2\right]\left[u - (m_1 + m_2)^2\right], \\
C_{234} &= \left[s - (m_1 - m_2)^2\right]\left[s - (m_1 + m_2)^2\right].
\end{aligned} \tag{C.9}$$

C.1.3 3-Simplex Invariants

Each tetrahedron has only one possible 3-simplex invariant. Indeed,

$$C_{1234} = C_{1234} = C_{1234} = t\left[su - (m_1 - m_2)^2 (m_1 + m_2)^2\right]. \tag{C.10}$$

C.2 Dual Conformal Invariants

For the red tetrahedron, you have

$$\begin{aligned}
&[1, 2; 3, 4] = \frac{st}{m_1^2 m_2^2}, \qquad [1, 3; 4, 2] = 1, \qquad [1, 4; 2, 3] = \frac{m_1^2 m_2^2}{st}, \\
&[1, 2; 4, 3] = \frac{m_1^2 m_2^2}{st}, \qquad [1, 3; 2, 4] = 1, \qquad [1, 4; 3, 2] = \frac{st}{m_1^2 m_2^2}.
\end{aligned} \tag{C.11}$$

Similarly, for the blue tetrahedron, you have

$$\begin{aligned}
&[1, 2; 3, 4] = \frac{tu}{m_1^2 m_2^2}, \qquad [1, 3; 4, 2] = 1, \qquad [1, 4; 2, 3] = \frac{m_1^2 m_2^2}{tu}, \\
&[1, 2; 4, 3] = \frac{m_1^2 m_2^2}{tu}, \qquad [1, 3; 2, 4] = 1, \qquad [1, 4; 3, 2] = \frac{tu}{m_1^2 m_2^2}.
\end{aligned} \tag{C.12}$$

Finally, for the green tetrahedron, you have

$$\begin{aligned}
&[1, 2; 3, 4] = \frac{su}{m_1^4}, \qquad [1, 3; 4, 2] = \frac{m_1^4}{m_2^4}, \qquad [1, 4; 2, 3] = \frac{m_2^4}{su}, \\
&[1, 2; 4, 3] = \frac{m_1^4}{su}, \qquad [1, 3; 2, 4] = \frac{m_2^4}{m_1^4}, \qquad [1, 4; 3, 2] = \frac{su}{m_2^4}.
\end{aligned} \tag{C.13}$$

Note that

$$[1, 2; 3, 4]\,[1, 3; 4, 2]\,[1, 4; 2, 3] = 1, \tag{C.14}$$

which is analogous to the constraint satisfied by the three Mandelstam invariants.

C.3 Center-of-Momentum Frame

In the center-of-momentum frame you write the energy-momentum vectors as

$$p_1 = \begin{pmatrix} E_1 & \mathbf{p}_1 \end{pmatrix}, \qquad p_2 = \begin{pmatrix} E_2 & -\mathbf{p}_1 \end{pmatrix}, \qquad p_3 = \begin{pmatrix} E_3 & \mathbf{p}_3 \end{pmatrix}, \qquad p_4 = \begin{pmatrix} E_4 & -\mathbf{p}_3 \end{pmatrix}. \tag{C.15}$$

One of the first things to notice is that

$$s = (E_1 + E_2)^2 = (E_3 + E_4)^2. \tag{C.16}$$

Thus, s can be interpreted as the total energy in the center-of-momentum frame. It also follows that s must be positive.

From the on-shell constraints, it follows that

$$\begin{aligned} E_1 &= \sqrt{m_1^2 + |\mathbf{p}_1|^2}, \qquad & E_2 &= \sqrt{m_2^2 + |\mathbf{p}_1|^2}, \\ E_3 &= \sqrt{m_1^2 + |\mathbf{p}_3|^2}, \qquad & E_4 &= \sqrt{m_2^2 + |\mathbf{p}_3|^2}. \end{aligned} \tag{C.17}$$

Using the relations

$$E_2 = \sqrt{s} - E_1, \qquad E_4 = \sqrt{s} - E_3, \tag{C.18}$$

you find that

$$|\mathbf{p}_1| = |\mathbf{p}_3| = \frac{\sqrt{\Lambda_{12}(s)}}{2\sqrt{s}}. \tag{C.19}$$

Thus,

$$E_1 = E_3 = \frac{s + m_1^2 - m_2^2}{2\sqrt{s}}, \qquad E_2 = E_4 = \frac{s - m_1^2 + m_2^2}{2\sqrt{s}}. \tag{C.20}$$

Note that

$$E_1 - E_2 = \frac{m_1^2 - m_2^2}{\sqrt{s}}. \tag{C.21}$$

C.3.1 Speed and Rapidity

The speed of each external quantum is

$$|\mathbf{v}_1| = |\mathbf{v}_3| = \frac{\sqrt{\Lambda_{12}(s)}}{s + m_1^2 - m_2^2}, \tag{C.22}$$

$$|\mathbf{v}_2| = |\mathbf{v}_4| = \frac{\sqrt{\Lambda_{12}(s)}}{s - m_1^2 + m_2^2}. \tag{C.23}$$

The rapidity of each external quantum is

$$\eta_1 = \eta_3 = \frac{1}{2} \log \left[\frac{s + m_1^2 - m_2^2 + \sqrt{\Lambda_{12}(s)}}{s + m_1^2 - m_2^2 - \sqrt{\Lambda_{12}(s)}} \right], \tag{C.24}$$

$$\eta_2 = \eta_4 = \frac{1}{2} \log \left[\frac{s - m_1^2 + m_2^2 + \sqrt{\Lambda_{12}(s)}}{s - m_1^2 + m_2^2 - \sqrt{\Lambda_{12}(s)}} \right]. \tag{C.25}$$

The sum of the incoming rapidites is equal to the sum of the outgoing rapidities:

$$\eta_1 + \eta_2 = \eta_3 + \eta_4 = \frac{1}{2} \log \left[\frac{s - m_1^2 - m_2^2 + \sqrt{\Lambda_{12}(s)}}{s - m_1^2 - m_2^2 - \sqrt{\Lambda_{12}(s)}} \right]. \tag{C.26}$$

Using

$$\frac{s - (m_1 + m_2)^2 + \sqrt{\Lambda_{12}(s)}}{s - (m_1 + m_2)^2 - \sqrt{\Lambda_{12}(s)}} = \frac{2m_1 m_2}{m_1^2 + m_2^2 - s + \sqrt{\Lambda_{12}(s)}} = \frac{m_1^2 + m_2^2 - s - \sqrt{\Lambda_{12}(s)}}{2m_1 m_2}, \tag{C.27}$$

allows you to write

$$\eta_1 + \eta_2 = \log \left[\frac{s - (m_1 + m_2)^2 + \sqrt{\Lambda_{12}(s)}}{s - (m_1 + m_2)^2 - \sqrt{\Lambda_{12}(s)}} \right]. \tag{C.28}$$

C.3.2 Physical Scattering Region

...

C.3.3 Scattering Regimes

...

Slow Small-Speed Scattering

Small slow speeds occur when s is close to the threshold value $(m_1+m_2)^2$.

Slow Fixed-Speed Scattering

The speed of each external quantum can be written as a function of two dimensionless ratios

$$\frac{s}{m_1 m_2}, \qquad \frac{m_1}{m_2}. \tag{C.29}$$

Slow fixed-speed scattering corresponds to the kinematic regime where we keep these ratios fixed:

$$\frac{s}{m_1 m_2} \text{ fixed}, \qquad \frac{m_1}{m_2} \text{ fixed}. \tag{C.30}$$

This regime is appropriate for either large s and large masses, or small s and small masses. Note that fixed-speed is equivalent to fixed-rapidity. By itself, this regime is not very helpful, but when combined with other limits it leads to important approximations.

Slow Large-Speed Scattering

Large-speed scattering involves the regime

$$\frac{s}{m_1 m_2} \to \infty, \qquad \frac{m_1}{m_2} \text{ fixed}. \tag{C.31}$$

That is, s is very large compared to the masses.

Regge Scattering

Regge scattering is the regime of fixed-speed and large (unphysical) z_{13}. This corresponds to

$$\frac{t}{s} \to \infty, \qquad \frac{s}{m_1 m_2} \text{ fixed}, \qquad \frac{m_1}{m_2} \text{ fixed}. \tag{C.32}$$

As a corollary, you have

$$u = 2m_1^2 + 2m_2^2 - s - t \quad \Longrightarrow \quad \frac{u}{s} \to -\infty. \tag{C.33}$$

In this regime all dual conformal invariants become trivial:

$$\begin{aligned} [1,2;3,4] &\to \infty, & [1,3;4,2] &= 1, & [1,4;2,3] &\to 0, \\ [1,2;4,3] &\to 0, & [1,3;2,4] &= 1, & [1,4;3,2] &\to \infty. \end{aligned} \tag{C.34}$$

$$\begin{aligned} [1,2;3,4] &\to -\infty, & [1,3;4,2] &= 1, & [1,4;2,3] &\to 0, \\ [1,2;4,3] &\to 0, & [1,3;2,4] &= 1, & [1,4;3,2] &\to -\infty. \end{aligned} \tag{C.35}$$

$$\begin{aligned} [1,2;3,4] &\to -\infty, & [1,3;4,2] &= \frac{m_1^4}{m_2^4}, & [1,4;2,3] &\to 0, \\ [1,2;4,3] &\to 0, & [1,3;2,4] &= \frac{m_2^4}{m_1^4}, & [1,4;3,2] &\to -\infty. \end{aligned} \tag{C.36}$$

Forward Scattering

Forward scattering is the regime of fixed-speed and small (physical) scattering angles (i.e. $z_{13} \to 1$). This can be stated as

$$\frac{t}{s} \to 0, \qquad \frac{s}{m_1 m_2} \text{ fixed}, \qquad \frac{m_1}{m_2} \text{ fixed}. \tag{C.37}$$

As a corollary, you have

$$u = 2m_1^2 + 2m_2^2 - s - t \quad \Longrightarrow \quad \frac{u}{s} \text{ fixed}. \tag{C.38}$$

Unlike the case of Regge scattering, in the forward regime only some of the dual conformal invariants are trivial:

$$
\begin{aligned}
[1,2;3,4] &\to 0, & [1,3;4,2] &= 1, & [1,4;2,3] &\to \infty, \\
[1,2;4,3] &\to \infty, & [1,3;2,4] &= 1, & [1,4;3,2] &=\to 0.
\end{aligned}
\tag{C.39}
$$

$$
\begin{aligned}
[1,2;3,4] &\to 0, & [1,3;4,2] &= 1, & [1,4;2,3] &=\to \infty, \\
[1,2;4,3] &=\to \infty, & [1,3;2,4] &= 1, & [1,4;3,2] &\to 0.
\end{aligned}
\tag{C.40}
$$

The rest of the dual conformal invariants are fixed and nontrivial:

$$
\begin{aligned}
[1,2;3,4] &= \frac{su}{m_1^4}, & [1,3;4,2] &= \frac{m_1^4}{m_2^4}, & [1,4;2,3] &= \frac{m_2^4}{su}, \\
[1,2;4,3] &= \frac{m_1^4}{su}, & [1,3;2,4] &= \frac{m_2^4}{m_1^4}, & [1,4;3,2] &= \frac{su}{m_2^4}.
\end{aligned}
\tag{C.41}
$$

Backward Scattering

Backward scattering is the regime of fixed-speed and large (physical) scattering angles (i.e. $z_{13} \to -1$).

Fixed-Angle Scattering

Fixed-angle scattering is the regime of large-speed and (physical) fixed-angle. This can be stated as

$$
\frac{t}{s} \text{ fixed}, \qquad \frac{s}{m_1 m_2} \to \infty, \qquad \frac{m_1}{m_2} \text{ fixed}.
\tag{C.42}
$$

As a corollary, you have

$$
u = 2m_1^2 + 2m_2^2 - s - t \quad \Longrightarrow \quad \frac{u}{s} \text{ fixed}.
\tag{C.43}
$$

In this regime all dual conformal invariants become trivial:

$$
\begin{aligned}
[1,2;3,4] &\to -\infty, & [1,3;4,2] &= 1, & [1,4;2,3] &\to 0, \\
[1,2;4,3] &\to 0, & [1,3;2,4] &= 1, & [1,4;3,2] &\to -\infty.
\end{aligned}
\tag{C.44}
$$

$$
\begin{aligned}
[1,2;3,4] &\to \infty, & [1,3;4,2] &= 1, & [1,4;2,3] &\to 0, \\
[1,2;4,3] &\to 0, & [1,3;2,4] &= 1, & [1,4;3,2] &\to \infty.
\end{aligned}
\tag{C.45}
$$

$$
\begin{aligned}
[1,2;3,4] &\to -\infty, & [1,3;4,2] &= \frac{m_1^4}{m_2^4}, & [1,4;2,3] &\to 0, \\
[1,2;4,3] &\to 0, & [1,3;2,4] &= \frac{m_2^4}{m_1^4}, & [1,4;3,2] &\to -\infty.
\end{aligned}
\tag{C.46}
$$

From the point of view of dual conformal invariants, Regge scattering and fixed-angle scattering are very similar.

Chapter D

Minimal Elastic Slow Diversity

In this chapter I will consider the kinematics of a 2-to-2 scattering process that has minimal elastic slow diversity. This means that all external quanta are slow, but you have only one distinct mass:

$$m^2 = -|p_1|^2 = -|p_2|^2 = -|p_3|^2 = -|p_4|^2 . \tag{D.1}$$

Thus, this is an elastic process. For example,

$$A(p_1) + A(p_2) \longrightarrow A(p_3) + A(p_4). \tag{D.2}$$

For this process, (A.3) becomes

$$s + t + u = 4m^2. \tag{D.3}$$

D.1 Simplicial Invariants

A tetrahedron is a 3-simplex. As such, it contains four vertices (0-simplex), six edges (1-simplex), four triangular faces (2-simplex), and one tetrahedron (3-simplex). The n-volume of an n-simplex is found by evaluating an $(n+1)$-point Cayley-Menger determinant.

D.1.1 1-Simplex Invariants

Given two dual spacetime positions, the 2-point Cayley-Menger determinant is given by:

$$C_{ij} = -\frac{1}{2} \det \begin{pmatrix} 0 & |d_{ij}|^2 & 1 \\ |d_{ij}|^2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}. \tag{D.4}$$

Since a tetrahedron has six edges, there are six possible 1-simplex invariants. These invariants will be either squared masses or Mandelstam invariants. For the red tetrahedron you have

$$\begin{aligned} C_{12} &= m^2, & C_{13} &= s, & C_{14} &= m^2, \\ C_{23} &= m^2, & C_{24} &= t, & C_{34} &= m^2. \end{aligned} \tag{D.5}$$

Similarly, for the blue tetrahedron you have

$$\begin{aligned} C_{12} &= m^2, & C_{13} &= t, & C_{14} &= m^2, \\ C_{23} &= m^2, & C_{24} &= u, & C_{34} &= m^2. \end{aligned} \tag{D.6}$$

Finally, for the green tetrahedron you have

$$\begin{aligned} C_{12} &= m^2, & C_{13} &= u, & C_{14} &= m^2, \\ C_{23} &= m^2, & C_{24} &= s, & C_{34} &= m^2. \end{aligned} \tag{D.7}$$

D.1.2 2-Simplex Invariants

Given three dual spacetime positions, the 3-point Cayley-Menger determinant is given by:

$$C_{ijk} = \det \begin{pmatrix} 0 & |d_{ij}|^2 & |d_{ik}|^2 & 1 \\ |d_{ij}|^2 & 0 & |d_{jk}|^2 & 1 \\ |d_{ik}|^2 & |d_{jk}|^2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}. \tag{D.8}$$

Since a tetrahedron has four triangular faces, there are four possible 2-simplex invariants. Each of these invariants has the form

$$\Lambda_{IJ}(x) = \left[x - (m_I - m_J)^2\right] \left[x - (m_I + m_J)^2\right], \tag{D.9}$$

where x is a Mandelstam invariant. This function is also known as the Källén function. For the red tetrahedron you have

$$C_{123} = C_{134} = s\left[s - 4m^2\right], \qquad C_{124} = C_{234} = t\left[t - 4m^2\right]. \tag{D.10}$$

Similarly, for the blue tetrahedron you have

$$C_{123} = C_{134} = t\left[t - 4m^2\right], \qquad C_{124} = C_{234} = u\left[u - 4m^2\right]. \tag{D.11}$$

Finally, for the green tetrahedron you have

$$C_{123} = C_{134} = u\left[u - 4m^2\right], \qquad C_{124} = C_{234} = s\left[s - 4m^2\right]. \tag{D.12}$$

D.1.3 3-Simplex Invariants

Given four dual spacetime positions, the 4-point Cayley-Menger determinant is given by:

$$C_{ijkl} = -\frac{1}{2} \det \begin{pmatrix} 0 & |d_{ij}|^2 & |d_{ik}|^2 & |d_{il}|^2 & 1 \\ |d_{ij}|^2 & 0 & |d_{jk}|^2 & |d_{jl}|^2 & 1 \\ |d_{ik}|^2 & |d_{jk}|^2 & 0 & |d_{kl}|^2 & 1 \\ |d_{il}|^2 & |d_{jl}|^2 & |d_{kl}|^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}. \tag{D.13}$$

A tetrahedron has only one possible 3-simplex invariant. Indeed,

$$C_{1234} = C_{1234} = C_{1234} = stu. \tag{D.14}$$

D.2 Dual Conformal Invariants

Each of the 1-simplex invariants has the form $|d_i - d_j|^2$. This is not only Lorentz invariant, but dual Poincaré invariant. You can study dual conformal invariants by constructing a dual conformal ratio with a quartet of dual spacetime coordinates:

$$[i, j; k, l] \equiv \frac{|d_{ik}|^2 \, |d_{jl}|^2}{|d_{il}|^2 \, |d_{jk}|^2}. \tag{D.15}$$

In four-point scattering there is a unique quartet of dual spacetime coordinates. However, there are six inequivalent permutations of these coordinates, and thus, six possible values for the dual conformal ratio.

For the red tetrahedron, you have

$$\begin{aligned} [1,2;3,4] &= \frac{st}{m^4}, & [1,3;4,2] &= 1, & [1,4;2,3] &= \frac{m^4}{st}, \\ [1,2;4,3] &= \frac{m^4}{st}, & [1,3;2,4] &= 1, & [1,4;3,2] &= \frac{st}{m^4}. \end{aligned} \tag{D.16}$$

Similarly, for the blue tetrahedron, you have

$$\begin{aligned} &[1,2;3,4] = \frac{tu}{m^4}, \qquad &&[1,3;4,2] = 1, \qquad &&[1,4;2,3] = \frac{m^4}{tu}, \\ &[1,2;4,3] = \frac{m^4}{tu}, \qquad &&[1,3;2,4] = 1, \qquad &&[1,4;3,2] = \frac{tu}{m^4}. \end{aligned} \tag{D.17}$$

Finally, for the green tetrahedron, you have

$$\begin{aligned} &[1,2;3,4] = \frac{su}{m^4}, \qquad &&[1,3;4,2] = 1, \qquad &&[1,4;2,3] = \frac{m^4}{su}, \\ &[1,2;4,3] = \frac{m^4}{su}, \qquad &&[1,3;2,4] = 1, \qquad &&[1,4;3,2] = \frac{su}{m^4}. \end{aligned} \tag{D.18}$$

Note that

$$[1,2;3,4]\,[1,3;4,2]\,[1,4;2,3] = 1, \tag{D.19}$$

which is analogous to the constraint satisfied by the three Mandelstam invariants.

D.3 Center-of-Momentum Frame

In the center-of-momentum frame you write the energy-momentum vectors as

$$p_1 = \begin{pmatrix} E_1 & \mathbf{p}_1 \end{pmatrix}, \qquad p_2 = \begin{pmatrix} E_2 & -\mathbf{p}_1 \end{pmatrix}, \qquad p_3 = \begin{pmatrix} E_3 & \mathbf{p}_3 \end{pmatrix}, \qquad p_4 = \begin{pmatrix} E_4 & -\mathbf{p}_3 \end{pmatrix}. \tag{D.20}$$

One of the first things to notice is that

$$s = (E_1 + E_2)^2 = (E_3 + E_4)^2. \tag{D.21}$$

Thus, s can be interpreted as the total energy in the center-of-momentum frame. It also follows that s must be positive.

From the on-shell constraints, it follows that

$$\begin{aligned} E_1 &= \sqrt{m^2 + |\mathbf{p}_1|^2}, \qquad & E_2 &= \sqrt{m^2 + |\mathbf{p}_1|^2}, \\ E_3 &= \sqrt{m^2 + |\mathbf{p}_3|^2}, \qquad & E_4 &= \sqrt{m^2 + |\mathbf{p}_3|^2}. \end{aligned} \tag{D.22}$$

Using the relations

$$E_2 = \sqrt{s} - E_1, \qquad E_4 = \sqrt{s} - E_3, \tag{D.23}$$

you find that

$$|\mathbf{p}_1| = |\mathbf{p}_3| = \frac{\sqrt{s - 4m^2}}{2}. \tag{D.24}$$

Thus,

$$E_1 = E_2 = E_3 = E_4 = \frac{\sqrt{s}}{2}. \tag{D.25}$$

D.3.1 Speed and Rapidity

A quantum with mass m, spatial momentum $\mathbf{p}$, and energy E has a speed $|\mathbf{v}|$ given by

$$|\mathbf{v}| = \frac{\sqrt{E^2 - m^2}}{E^2} = \frac{|\mathbf{p}|}{\sqrt{m^2 + |\mathbf{p}|^2}} = \frac{|\mathbf{p}|}{E}. \tag{D.26}$$

If the quantum is physical, then $E \geq m$, or equivalently, $|\mathbf{p}| < E$. Both of these conditions lead to a bound on speed:

$$0 \leq |\mathbf{v}| < 1. \tag{D.27}$$

In contrast with energy, mass, and spatial momentum, speed and velocity are dimensionless. However, energy and spatial momentum are conserved, but velocity is not. The speed of each external quantum is

$$|\mathbf{v}_1| = |\mathbf{v}_2| = |\mathbf{v}_3| = |\mathbf{v}_4| = \sqrt{1 - \frac{4m^2}{s}}. \tag{D.28}$$

Given the speed of a quantum, you can find its rapidity η via

$$\eta = \operatorname{arctanh}|\mathbf{v}| = \frac{1}{2}\log\left(\frac{1+|\mathbf{v}|}{1-|\mathbf{v}|}\right). \tag{D.29}$$

Since the speed of a physical quantum is bounded, the rapidity of a physical quantum is bounded from below:

$$0 \le \eta < \infty. \tag{D.30}$$

The rapidity of each external quantum is

$$\eta_1 = \eta_2 = \eta_3 = \eta_4 = \frac{1}{2}\log\left[\frac{\sqrt{s}+\sqrt{s-4m^2}}{\sqrt{s}-\sqrt{s-4m^2}}\right]. \tag{D.31}$$

The sum of the incoming rapidites is equal to the sum of the outgoing rapidities:

$$\eta_1 + \eta_2 = \eta_3 + \eta_4 = \log\left[\frac{\sqrt{s}+\sqrt{s-4m^2}}{\sqrt{s}-\sqrt{s-4m^2}}\right]. \tag{D.32}$$

D.3.2 Physical Scattering Region

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Fixed-Speed Scattering

The speed of each external quantum can be written as a function of one dimensionless ratio,

$$\frac{s}{m^2}. \tag{D.33}$$

Fixed-speed scattering corresponds to the kinematic regime where we keep this ratio fixed:

$$\frac{s}{m^2} \text{ fixed.} \tag{D.34}$$

This regime is appropriate for either large s and large mass, or small s and small mass. Note that fixed-speed is equivalent to fixed-rapidity. By itself, this regime is not very helpful, but when combined with other limits it leads to important approximations.

Small-Speed Scattering

Small speeds occur when s is close to the threshold value $4m^2$.

Large-Speed Scattering

Large-speed scattering involves the regime

$$\frac{s}{m^2} \to \infty. \tag{D.35}$$

That is, $\sqrt{s}$ is very large compared to the mass.

Regge Scattering

Regge scattering is the regime of fixed-speed and large (unphysical) z_{13}. This corresponds to

$$\frac{t}{s} \to \infty, \qquad \frac{s}{m^2} \text{ fixed.} \tag{D.36}$$

As a corollary, you have

$$u = 4m^2 - s - t \quad \Longrightarrow \quad \frac{u}{s} \to -\infty. \tag{D.37}$$

In this regime all dual conformal invariants become trivial:

$$\begin{aligned} [1,2;3,4] &\to \infty, & [1,3;4,2] &= 1, & [1,4;2,3] &\to 0, \\ [1,2;4,3] &\to 0, & [1,3;2,4] &= 1, & [1,4;3,2] &\to \infty. \end{aligned} \tag{D.38}$$

$$\begin{aligned} [1,2;3,4] &\to -\infty, & [1,3;4,2] &= 1, & [1,4;2,3] &\to 0, \\ [1,2;4,3] &\to 0, & [1,3;2,4] &= 1, & [1,4;3,2] &\to -\infty. \end{aligned} \tag{D.39}$$

$$\begin{aligned} [1,2;3,4] &\to -\infty, & [1,3;4,2] &= 1, & [1,4;2,3] &\to 0, \\ [1,2;4,3] &\to 0, & [1,3;2,4] &= 1, & [1,4;3,2] &\to -\infty. \end{aligned} \tag{D.40}$$

Forward Scattering

Forward scattering is the regime of fixed-speed and small (physical) scattering angles (i.e. $z_{13} \to 1$). This can be stated as

$$\frac{t}{s} \to 0, \qquad \frac{s}{m^2} \text{ fixed.} \tag{D.41}$$

As a corollary, you have

$$u = 4m^2 - s - t \quad \Longrightarrow \quad \frac{u}{s} \text{ fixed.} \tag{D.42}$$

Unlike the case of Regge scattering, in the forward regime only some of the dual conformal invariants are trivial:

$$\begin{aligned} [1,2;3,4] &\to 0, & [1,3;4,2] &= 1, & [1,4;2,3] &\to \infty, \\ [1,2;4,3] &\to \infty, & [1,3;2,4] &= 1, & [1,4;3,2] &\to 0. \end{aligned} \tag{D.43}$$

$$\begin{aligned} [1,2;3,4] &\to 0, & [1,3;4,2] &= 1, & [1,4;2,3] &\to \infty, \\ [1,2;4,3] &\to \infty, & [1,3;2,4] &= 1, & [1,4;3,2] &\to 0. \end{aligned} \tag{D.44}$$

The rest of the dual conformal invariants are fixed and nontrivial:

$$\begin{aligned} [1,2;3,4] &= \frac{su}{m^4}, & [1,3;4,2] &= 1, & [1,4;2,3] &= \frac{m^4}{su}, \\ [1,2;4,3] &= \frac{m^4}{su}, & [1,3;2,4] &= 1, & [1,4;3,2] &= \frac{su}{m^4}. \end{aligned} \tag{D.45}$$

Backward Scattering

Backward scattering is the regime of fixed-speed and large (physical) scattering angles (i.e. $z_{13} \to -1$). This can be stated as

$$\frac{u}{s} \to 0, \qquad \frac{s}{m^2} \text{ fixed.} \tag{D.46}$$

As a corollary, you have

$$t = 4m^2 - s - u \quad \Longrightarrow \quad \frac{t}{s} \text{ fixed.} \tag{D.47}$$

Similarly to the forward scattering regime, some of the dual conformal invariants are trivial:

$$\begin{aligned} [1,2;3,4] &\to 0, & [1,3;4,2] &= 1, & [1,4;2,3] &\to \infty, \\ [1,2;4,3] &\to \infty, & [1,3;2,4] &= 1, & [1,4;3,2] &\to 0. \end{aligned} \tag{D.48}$$

$$\begin{aligned} [1,2;3,4] &\to 0, & [1,3;4,2] &= 1, & [1,4;2,3] &\to \infty, \\ [1,2;4,3] &\to \infty, & [1,3;2,4] &= 1, & [1,4;3,2] &\to 0. \end{aligned} \tag{D.49}$$

The rest of the dual conformal invariants are fixed and nontrivial:

$$\begin{aligned} [1,2;3,4] &= \frac{st}{m^4}, & [1,3;4,2] &= 1, & [1,4;2,3] &= \frac{m^4}{st}, \\ [1,2;4,3] &= \frac{m^4}{st}, & [1,3;2,4] &= 1, & [1,4;3,2] &= \frac{st}{m^4}. \end{aligned} \tag{D.50}$$

Fixed-Angle Scattering

Fixed-angle scattering is the regime of large-speed and (physical) fixed-angle. This can be stated as

$$\frac{t}{s} \text{ fixed}, \qquad \frac{s}{m^2} \to \infty. \tag{D.51}$$

As a corollary, you have

$$u = 4m^2 - s - t \quad \Longrightarrow \quad \frac{u}{s} \text{ fixed.} \tag{D.52}$$

In this regime all dual conformal invariants become trivial:

$$\begin{aligned} [1,2;3,4] &\to -\infty, & [1,3;4,2] &= 1, & [1,4;2,3] &\to 0, \\ [1,2;4,3] &\to 0, & [1,3;2,4] &= 1, & [1,4;3,2] &\to -\infty. \end{aligned} \tag{D.53}$$

$$\begin{aligned} [1,2;3,4] &\to \infty, \qquad [1,3;4,2] = 1, \qquad [1,4;2,3] \to 0, \\ [1,2;4,3] &\to 0, \qquad [1,3;2,4] = 1, \qquad [1,4;3,2] \to \infty. \end{aligned} \tag{D.54}$$

$$\begin{aligned} [1,2;3,4] &\to -\infty, \qquad [1,3;4,2] = 1, \qquad [1,4;2,3] \to 0, \\ [1,2;4,3] &\to 0, \qquad [1,3;2,4] = 1, \qquad [1,4;3,2] \to -\infty. \end{aligned} \tag{D.55}$$

From the point of view of dual conformal invariants, Regge scattering and fixed-angle scattering are very similar.

Part II

Fast Quanta

Chapter E

Maximal Inelastic Fast Diversity

In this chapter I will consider the kinematics of a 2-to-2 scattering process that has maximal inelastic fast diversity. This means that all external quanta are fast, and all of the corresponding masses are distinct:

$$w_1^2 = |p_1|^2, \qquad w_2^2 = |p_2|^2, \qquad w_3^2 = |p_3|^2, \qquad w_4^2 = |p_4|^2. \tag{E.1}$$

Thus, this is an inelastic process. For example,

$$a(p_1) + b(p_2) \longrightarrow y(p_3) + z(p_4). \tag{E.2}$$

For this process, (A.3) becomes

$$s + t + u = -w_1^2 - w_2^2 - w_3^2 - w_4^2. \tag{E.3}$$

E.1 Simplicial Invariants

The simplicial invariants for fast quanta are slightly different from the slow quanta invariants. Due to diversity, there are many distinct values.

E.1.1 1-Simplex Invariants

For the red tetrahedron you have

$$C_{12} = -w_1^2, \quad C_{13} = s, \quad C_{14} = -w_3^2, \quad C_{23} = -w_2^2, \quad C_{24} = t, \quad C_{34} = -w_4^2. \tag{E.4}$$

Similarly, for the blue tetrahedron you have

$$C_{12} = -w_3^2, \quad C_{13} = t, \quad C_{14} = -w_2^2, \quad C_{23} = -w_1^2, \quad C_{24} = u, \quad C_{34} = -w_4^2. \tag{E.5}$$

Finally, for the green tetrahedron you have

$$C_{12} = -w_2^2, \quad C_{13} = u, \quad C_{14} = -w_1^2, \quad C_{23} = -w_3^2, \quad C_{24} = s, \quad C_{34} = -w_4^2. \tag{E.6}$$

E.1.2 2-Simplex Invariants

For the red tetrahedron you have

$$\begin{aligned} C_{123} &= \left[s + (w_1 - w_2)^2\right]\left[s + (w_1 + w_2)^2\right] = \Upsilon_{12}(s), \\ C_{124} &= \left[t + (w_1 - w_3)^2\right]\left[t + (w_1 + w_3)^2\right] = \Upsilon_{13}(t), \\ C_{134} &= \left[s + (w_3 - w_4)^2\right]\left[s + (w_3 + w_4)^2\right] = \Upsilon_{34}(s), \\ C_{234} &= \left[t + (w_2 - w_4)^2\right]\left[t + (w_2 + w_4)^2\right] = \Upsilon_{24}(t). \end{aligned} \tag{E.7}$$

Similarly, for the blue tetrahedron you have

$$\begin{aligned} C_{123} &= \left[t + (w_1 - w_3)^2\right]\left[t + (w_1 + w_3)^2\right] = \Upsilon_{13}(t), \\ C_{124} &= \left[u + (w_2 - w_3)^2\right]\left[u + (w_2 + w_3)^2\right] = \Upsilon_{23}(u), \\ C_{134} &= \left[t + (w_2 - w_4)^2\right]\left[t + (w_2 + w_4)^2\right] = \Upsilon_{24}(t), \\ C_{234} &= \left[u + (w_1 - w_4)^2\right]\left[u + (w_1 + w_4)^2\right] = \Upsilon_{14}(u). \end{aligned} \tag{E.8}$$

Finally, for the green tetrahedron you have

$$\begin{aligned}
C_{123} &= \left[u + (w_2 - w_3)^2\right]\left[u + (w_2 + w_3)^2\right] = \Upsilon_{23}(u), \\
C_{124} &= \left[s + (w_1 - w_2)^2\right]\left[s + (w_1 + w_2)^2\right] = \Upsilon_{12}(s), \\
C_{134} &= \left[u + (w_1 - w_4)^2\right]\left[u + (w_1 + w_4)^2\right] = \Upsilon_{14}(u), \\
C_{234} &= \left[s + (w_3 - w_4)^2\right]\left[s + (w_3 + w_4)^2\right] = \Upsilon_{34}(s).
\end{aligned} \tag{E.9}$$

E.1.3 3-Simplex Invariants

A tetrahedron has only one possible 3-simplex invariant. Indeed,

$$C_{1234} = C_{1234} = C_{1234}. \tag{E.10}$$

E.2 Dual Conformal Invariants

For the red tetrahedron, you have

$$\begin{aligned}
[1,2;3,4] &= \frac{st}{w_2^2 w_3^2}, & [1,3;4,2] &= \frac{w_2^2 w_3^2}{w_1^2 w_4^2}, & [1,4;2,3] &= \frac{w_1^2 w_4^2}{st}, \\
[1,2;4,3] &= \frac{w_2^2 w_3^2}{st}, & [1,3;2,4] &= \frac{w_1^2 w_4^2}{w_2^2 w_3^2}, & [1,4;3,2] &= \frac{st}{w_1^2 w_4^2}.
\end{aligned} \tag{E.11}$$

Similarly, for the blue tetrahedron, you have

$$\begin{aligned}
[1,2;3,4] &= \frac{tu}{w_1^2 w_2^2}, & [1,3;4,2] &= \frac{w_1^2 w_2^2}{w_3^2 w_4^2}, & [1,4;2,3] &= \frac{w_3^2 w_4^2}{tu}, \\
[1,2;4,3] &= \frac{w_1^2 w_2^2}{tu}, & [1,3;2,4] &= \frac{w_3^2 w_4^2}{w_1^2 w_2^2}, & [1,4;3,2] &= \frac{tu}{w_3^2 w_4^2}.
\end{aligned} \tag{E.12}$$

Finally, for the green tetrahedron, you have

$$\begin{aligned}
[1,2;3,4] &= \frac{su}{w_1^2 w_3^2}, & [1,3;4,2] &= \frac{w_1^2 w_3^2}{w_2^2 w_4^2}, & [1,4;2,3] &= \frac{w_2^2 w_4^2}{su}, \\
[1,2;4,3] &= \frac{w_1^2 w_3^2}{su}, & [1,3;2,4] &= \frac{w_2^2 w_4^2}{w_1^2 w_3^2}, & [1,4;3,2] &= \frac{su}{w_2^2 w_4^2}.
\end{aligned} \tag{E.13}$$

E.3 Center-of-Momentum Frame

In the center-of-momentum frame you write the energy-momentum vectors as

$$p_1 = \begin{pmatrix} E_1 & \mathbf{p}_1 \end{pmatrix}, \qquad p_2 = \begin{pmatrix} E_2 & -\mathbf{p}_1 \end{pmatrix}, \qquad p_3 = \begin{pmatrix} E_3 & \mathbf{p}_3 \end{pmatrix}, \qquad p_4 = \begin{pmatrix} E_4 & -\mathbf{p}_3 \end{pmatrix}. \tag{E.14}$$

Using the on-shell constraints leads to:

$$w^2 = -E^2 + |\mathbf{p}|^2, \quad \Longrightarrow \quad |\mathbf{p}| = \sqrt{w^2 + E^2}. \tag{E.15}$$

Thus,

$$|\mathbf{p}_1| = \sqrt{w_1^2 + E_1^2} = \sqrt{w_2^2 + E_2^2}, \qquad |\mathbf{p}_3| = \sqrt{w_3^2 + E_3^2} = \sqrt{w_4^2 + E_4^2}. \tag{E.16}$$

One of the first things to notice is that

$$s = (E_1 + E_2)^2 = (E_3 + E_4)^2. \tag{E.17}$$

Thus, s can be interpreted as the total energy in the center-of-momentum frame. It also follows that s must be positive. Using

$$E_2 = \sqrt{s} - E_1, \qquad E_4 = \sqrt{s} - E_3, \tag{E.18}$$

leads to

$$E_1 = \frac{s - w_1^2 + w_2^2}{2\sqrt{s}}, \qquad E_3 = \frac{s - w_3^2 + w_4^2}{2\sqrt{s}}. \tag{E.19}$$

Hence

$$E_2 = \frac{s + w_1^2 - w_2^2}{2\sqrt{s}}, \qquad E_4 = \frac{s + w_3^2 - w_4^2}{2\sqrt{s}}. \tag{E.20}$$

Going back to the spatial momentum, you find

$$|\mathbf{p}_1| = \frac{\sqrt{\Upsilon_{12}(s)}}{2\sqrt{s}}, \qquad |\mathbf{p}_3| = \frac{\sqrt{\Upsilon_{34}(s)}}{2\sqrt{s}}. \tag{E.21}$$

E.3.1 Speed and Rapidity

The speed of each external quantum is

$$|\mathbf{v}_1| = \frac{\sqrt{\Upsilon_{12}(s)}}{s - w_1^2 + w_2^2}, \tag{E.22}$$

$$|\mathbf{v}_2| = \frac{\sqrt{\Upsilon_{12}(s)}}{s + w_1^2 - w_2^2}, \tag{E.23}$$

$$|\mathbf{v}_3| = \frac{\sqrt{\Upsilon_{34}(s)}}{s - w_3^2 + w_4^2}, \tag{E.24}$$

$$|\mathbf{v}_4| = \frac{\sqrt{\Upsilon_{34}(s)}}{s + w_3^2 - w_4^2}. \tag{E.25}$$

The (fast) rapidity of each external quantum is

$$\xi_1 = \frac{1}{2}\log\left[\frac{\sqrt{\Upsilon_{12}(s)} + s - w_1^2 + w_2^2}{\sqrt{\Upsilon_{12}(s)} - s + w_1^2 - w_2^2}\right], \tag{E.26}$$

$$\xi_2 = \frac{1}{2}\log\left[\frac{\sqrt{\Upsilon_{12}(s)} + s + w_1^2 - w_2^2}{\sqrt{\Upsilon_{12}(s)} - s - w_1^2 + w_2^2}\right], \tag{E.27}$$

$$\xi_3 = \frac{1}{2}\log\left[\frac{\sqrt{\Upsilon_{34}(s)} + s - w_3^2 + w_4^2}{\sqrt{\Upsilon_{34}(s)} - s + w_3^2 - w_4^2}\right], \tag{E.28}$$

$$\xi_4 = \frac{1}{2}\log\left[\frac{\sqrt{\Upsilon_{34}(s)} + s + w_3^2 - w_4^2}{\sqrt{\Upsilon_{34}(s)} - s - w_3^2 + w_4^2}\right]. \tag{E.29}$$

The sum of the incoming (fast) rapidities is

$$\xi_1 + \xi_2 = \frac{1}{2}\log\left[\frac{s + w_1^2 + w_2^2 + \sqrt{\Upsilon_{12}(s)}}{s + w_1^2 + w_2^2 - \sqrt{\Upsilon_{12}(s)}}\right]. \tag{E.30}$$

Similarly, the sum of the outgoing (fast) rapidities is

$$\xi_3 + \xi_4 = \frac{1}{2}\log\left[\frac{s + w_3^2 + w_4^2 + \sqrt{\Upsilon_{34}(s)}}{s + w_3^2 + w_4^2 - \sqrt{\Upsilon_{34}(s)}}\right]. \tag{E.31}$$

E.3.2 Physical Scattering Region

The on-shell relation for a fast quantum is

$$w^2 = -E^2 + |\mathbf{p}|^2 \quad \Longrightarrow \quad |\mathbf{p}|^2 = w^2 + E^2, \tag{E.32}$$

so requiring $E > 0$ leads to $|\mathbf{p}| > w$. The energies are positive as long as

$$s > |w_1 - w_2|\,(w_1 + w_2)\,, \qquad s > |w_3 - w_4|\,(w_3 + w_4)\,. \tag{E.33}$$

E.3.3 Scattering Regimes

...

Fast Small-Speed Scattering

Fast small-speed scattering is the regime when

$$\frac{s}{w_1 w_2} \to \infty, \qquad \frac{w_1}{w_2} \text{ fixed}, \qquad \frac{w_1}{w_3} \text{ fixed}, \qquad \frac{w_1}{w_4} \text{ fixed}. \tag{E.34}$$

In this regime, all speeds approach unity.

Fast Fixed-Speed Scattering

The speed of each external quantum can be written as a function of four dimensionless ratios

$$\frac{s}{w_1 w_2}, \qquad \frac{w_1}{w_2}, \qquad \frac{w_1}{w_3}, \qquad \frac{w_1}{w_4}. \tag{E.35}$$

Fixed-speed scattering corresponds to the kinematic regime where we keep these ratios fixed:

$$\frac{s}{w_1 w_2} \text{ fixed}, \qquad \frac{w_1}{w_2} \text{ fixed}, \qquad \frac{w_1}{w_3} \text{ fixed}, \qquad \frac{w_1}{w_4} \text{ fixed}. \tag{E.36}$$

This regime is appropriate for either large s and large masses, or small s and small masses. Note that fixed-speed is equivalent to fixed-rapidity. By itself, this regime is not very helpful, but when combined with other limits it leads to important approximations.

Fast Large-Speed Scattering

Fast large speeds occur when

$$s \to \pm (w_1 - w_2)(w_1 + w_2), \qquad s \to \pm (w_3 - w_4)(w_3 + w_4). \tag{E.37}$$

In each of these four limits one of the speeds becomes infinite.

Regge Scattering

Regge scattering is the regime of fixed-speed and large (unphysical) z_{13}. This corresponds to

$$\frac{t}{s} \to \infty, \qquad \frac{s}{m_1 m_2} \text{ fixed}, \qquad \frac{m_1}{m_2} \text{ fixed}. \tag{E.38}$$

As a corollary, you have

$$u = 2m_1^2 + 2m_2^2 - s - t \quad \Longrightarrow \quad \frac{u}{s} \to -\infty. \tag{E.39}$$

In this regime all dual conformal invariants become trivial:

$$\begin{aligned} &[1,2;3,4] \to \infty, && [1,3;4,2] = 1, && [1,4;2,3] \to 0, \\ &[1,2;4,3] \to 0, && [1,3;2,4] = 1, && [1,4;3,2] \to \infty. \end{aligned} \tag{E.40}$$

$$\begin{aligned} &[1,2;3,4] \to -\infty, && [1,3;4,2] = 1, && [1,4;2,3] \to 0, \\ &[1,2;4,3] \to 0, && [1,3;2,4] = 1, && [1,4;3,2] \to -\infty. \end{aligned} \tag{E.41}$$

$$\begin{aligned} &[1,2;3,4] \to -\infty, && [1,3;4,2] = \frac{m_1^4}{m_2^4}, && [1,4;2,3] \to 0, \\ &[1,2;4,3] \to 0, && [1,3;2,4] = \frac{m_2^4}{m_1^4}, && [1,4;3,2] \to -\infty. \end{aligned} \tag{E.42}$$

Forward Scattering

Forward scattering is the regime of fixed-speed and small (physical) scattering angles (i.e. $z_{13} \to 1$). This can be stated as

$$\frac{t}{s} \to 0, \qquad \frac{s}{m_1 m_2} \text{ fixed}, \qquad \frac{m_1}{m_2} \text{ fixed}. \tag{E.43}$$

As a corollary, you have

$$u = 2m_1^2 + 2m_2^2 - s - t \quad \Longrightarrow \quad \frac{u}{s} \text{ fixed}. \tag{E.44}$$

Unlike the case of Regge scattering, in the forward regime only some of the dual conformal invariants are trivial:

$$\begin{aligned} [1,2;3,4] &\to 0, & [1,3;4,2] &= 1, & [1,4;2,3] &\to \infty, \\ [1,2;4,3] &\to \infty, & [1,3;2,4] &= 1, & [1,4;3,2] &=\to 0. \end{aligned} \tag{E.45}$$

$$\begin{aligned} [1,2;3,4] &\to 0, & [1,3;4,2] &= 1, & [1,4;2,3] &=\to \infty, \\ [1,2;4,3] &=\to \infty, & [1,3;2,4] &= 1, & [1,4;3,2] &\to 0. \end{aligned} \tag{E.46}$$

The rest of the dual conformal invariants are fixed and nontrivial:

$$\begin{aligned} [1,2;3,4] &= \frac{su}{m_1^4}, & [1,3;4,2] &= \frac{m_1^4}{m_2^4}, & [1,4;2,3] &= \frac{m_2^4}{su}, \\ [1,2;4,3] &= \frac{m_1^4}{su}, & [1,3;2,4] &= \frac{m_2^4}{m_1^4}, & [1,4;3,2] &= \frac{su}{m_2^4}. \end{aligned} \tag{E.47}$$

Backward Scattering

Backward scattering is the regime of fixed-speed and large (physical) scattering angles (i.e. $z_{13} \to -1$).

Fixed-Angle Scattering

Fixed-angle scattering is the regime of large-speed and (physical) fixed-angle. This can be stated as

$$\frac{t}{s} \text{ fixed}, \qquad \frac{s}{m_1 m_2} \to \infty, \qquad \frac{m_1}{m_2} \text{ fixed}. \tag{E.48}$$

As a corollary, you have

$$u = 2m_1^2 + 2m_2^2 - s - t \quad \Longrightarrow \quad \frac{u}{s} \text{ fixed}. \tag{E.49}$$

In this regime all dual conformal invariants become trivial:

$$\begin{aligned} [1,2;3,4] &\to -\infty, & [1,3;4,2] &= 1, & [1,4;2,3] &\to 0, \\ [1,2;4,3] &\to 0, & [1,3;2,4] &= 1, & [1,4;3,2] &\to -\infty. \end{aligned} \tag{E.50}$$

$$\begin{aligned} [1,2;3,4] &\to \infty, & [1,3;4,2] &= 1, & [1,4;2,3] &\to 0, \\ [1,2;4,3] &\to 0, & [1,3;2,4] &= 1, & [1,4;3,2] &\to \infty. \end{aligned} \tag{E.51}$$

$$\begin{aligned} [1,2;3,4] &\to -\infty, & [1,3;4,2] &= \frac{m_1^4}{m_2^4}, & [1,4;2,3] &\to 0, \\ [1,2;4,3] &\to 0, & [1,3;2,4] &= \frac{m_2^4}{m_1^4}, & [1,4;3,2] &\to -\infty. \end{aligned} \tag{E.52}$$

From the point of view of dual conformal invariants, Regge scattering and fixed-angle scattering are very similar.

Chapter F

Maximal Elastic Fast Diversity

In this chapter I will consider the kinematics of a 2-to-2 scattering process that has maximal elastic fast diversity. This means that all external quanta are fast, but you have two distinct elastic pairs:

$$w_1^2 = |p_1|^2 = |p_3|^2 \,, \qquad w_2^2 = |p_2|^2 = |p_4|^2 \,. \tag{F.1}$$

Thus, this is an elastic process. For example,

$$a(p_1) + b(p_2) \longrightarrow a(p_3) + b(p_4). \tag{F.2}$$

For this process, (A.3) becomes

$$s + t + u = -2w_1^2 - 2w_2^2. \tag{F.3}$$

F.1 Simplicial Invariants

The simplicial invariants for fast quanta are slightly different from the slow quanta invariants. Maximal elastic diversity reduces the amount of distinct values.

F.1.1 1-Simplex Invariants

For the red tetrahedron you have

$$C_{12} = C_{14} = -w_1^2, \qquad C_{13} = s, \qquad C_{23} = C_{34} = -w_2^2, \qquad C_{24} = t. \tag{F.4}$$

Similarly, for the blue tetrahedron you have

$$C_{12} = C_{23} = -w_1^2, \qquad C_{13} = t, \qquad C_{14} = C_{34} = -w_2^2, \qquad C_{24} = u. \tag{F.5}$$

Finally, for the green tetrahedron you have

$$C_{12} = C_{34} = -w_2^2, \qquad C_{13} = u, \qquad C_{14} = C_{23} = -w_1^2, \qquad C_{24} = s. \tag{F.6}$$

F.1.2 2-Simplex Invariants

For the red tetrahedron you have

$$\begin{aligned} C_{123} = C_{134} &= \left[s + (w_1 - w_2)^2\right]\left[s + (w_1 + w_2)^2\right] = \Upsilon_{12}(s), \\ C_{124} &= t\left[t + 4w_1^2\right] = \Upsilon_{13}(t), \\ C_{234} &= t\left[t + 4w_2^2\right] = \Upsilon_{24}(t). \end{aligned} \tag{F.7}$$

Similarly, for the blue tetrahedron you have

$$\begin{aligned} C_{123} &= t\left[t + 4w_1^2\right] = \Upsilon_{13}(t), \\ C_{124} = C_{234} &= \left[u + (w_1 - w_2)^2\right]\left[u + (w_1 + w_2)^2\right] = \Upsilon_{12}(u), \\ C_{134} &= t\left[t + 4w_2^2\right] = \Upsilon_{24}(t). \end{aligned} \tag{F.8}$$

Finally, for the green tetrahedron you have

$$\begin{aligned} C_{123} = C_{134} &= \left[u + (w_1 - w_2)^2\right]\left[u + (w_1 + w_2)^2\right] = \Upsilon_{12}(u), \\ C_{124} = C_{234} &= \left[s + (w_1 - w_2)^2\right]\left[s + (w_1 + w_2)^2\right] = \Upsilon_{12}(s). \end{aligned} \tag{F.9}$$

F.1.3 3-Simplex Invariants

A tetrahedron has only one possible 3-simplex invariant. Indeed,

$$C_{1234} = C_{1234} = C_{1234} = t\left[su - \left(w_1^2 - w_2^2\right)^2\right]. \tag{F.10}$$

F.2 Dual Conformal Invariants

For the red tetrahedron, you have

$$\begin{aligned}
[1,2;3,4] &= \frac{st}{w_1^2 w_2^2}, & [1,3;4,2] &= 1, & [1,4;2,3] &= \frac{w_1^2 w_2^2}{st}, \\
[1,2;4,3] &= \frac{w_1^2 w_2^2}{st}, & [1,3;2,4] &= 1, & [1,4;3,2] &= \frac{st}{w_1^2 w_2^2}.
\end{aligned} \tag{F.11}$$

Similarly, for the blue tetrahedron, you have

$$\begin{aligned}
[1,2;3,4] &= \frac{tu}{w_1^2 w_2^2}, & [1,3;4,2] &= 1, & [1,4;2,3] &= \frac{w_1^2 w_2^2}{tu}, \\
[1,2;4,3] &= \frac{w_1^2 w_2^2}{tu}, & [1,3;2,4] &= 1, & [1,4;3,2] &= \frac{tu}{w_1^2 w_2^2}.
\end{aligned} \tag{F.12}$$

Finally, for the green tetrahedron, you have

$$\begin{aligned}
[1,2;3,4] &= \frac{su}{w_1^4}, & [1,3;4,2] &= \frac{w_1^4}{w_2^4}, & [1,4;2,3] &= \frac{w_2^4}{su}, \\
[1,2;4,3] &= \frac{w_1^4}{su}, & [1,3;2,4] &= \frac{w_2^4}{w_1^4}, & [1,4;3,2] &= \frac{su}{w_2^4}.
\end{aligned} \tag{F.13}$$

F.3 Center-of-Momentum Frame

In the center-of-momentum frame you write the energy-momentum vectors as

$$p_1 = \begin{pmatrix} E_1 & \mathbf{p}_1 \end{pmatrix}, \qquad p_2 = \begin{pmatrix} E_2 & -\mathbf{p}_1 \end{pmatrix}, \qquad p_3 = \begin{pmatrix} E_3 & \mathbf{p}_3 \end{pmatrix}, \qquad p_4 = \begin{pmatrix} E_4 & -\mathbf{p}_3 \end{pmatrix}. \tag{F.14}$$

Using the on-shell constraints leads to:

$$w^2 = -E^2 + |\mathbf{p}|^2, \quad \implies \quad |\mathbf{p}| = \sqrt{w^2 + E^2}. \tag{F.15}$$

Thus,

$$|\mathbf{p}_1| = \sqrt{w_1^2 + E_1^2} = \sqrt{w_2^2 + E_2^2}, \qquad |\mathbf{p}_3| = \sqrt{w_1^2 + E_3^2} = \sqrt{w_2^2 + E_4^2}. \tag{F.16}$$

One of the first things to notice is that

$$s = (E_1 + E_2)^2 = (E_3 + E_4)^2. \tag{F.17}$$

Thus, s can be interpreted as the total energy in the center-of-momentum frame. It also follows that s must be positive. Using

$$E_2 = \sqrt{s} - E_1, \qquad E_4 = \sqrt{s} - E_3, \tag{F.18}$$

leads to

$$E_1 = E_3 = \frac{s - w_1^2 + w_2^2}{2\sqrt{s}}, \tag{F.19}$$

Hence

$$E_2 = E_4 = \frac{s + w_1^2 - w_2^2}{2\sqrt{s}}, \tag{F.20}$$

Going back to the spatial momentum, you find

$$|\mathbf{p}_1| = |\mathbf{p}_3| = \frac{\sqrt{\Upsilon_{12}(s)}}{2\sqrt{s}}. \tag{F.21}$$

F.3.1 Speed and Rapidity

The speed of each external quantum is

$$|\mathbf{v}_1| = |\mathbf{v}_3| = \frac{\sqrt{\Upsilon_{12}(s)}}{s - w_1^2 + w_2^2}, \tag{F.22}$$

$$|\mathbf{v}_2| = |\mathbf{v}_4| = \frac{\sqrt{\Upsilon_{12}(s)}}{s + w_1^2 - w_2^2}. \tag{F.23}$$

The (fast) rapidity of each external quantum is

$$\xi_1 = \xi_3 = \frac{1}{2}\log\left[\frac{\sqrt{\Upsilon_{12}(s)} + s - w_1^2 + w_2^2}{\sqrt{\Upsilon_{12}(s)} - s + w_1^2 - w_2^2}\right], \tag{F.24}$$

$$\xi_2 = \xi_4 = \frac{1}{2}\log\left[\frac{\sqrt{\Upsilon_{12}(s)} + s + w_1^2 - w_2^2}{\sqrt{\Upsilon_{12}(s)} - s - w_1^2 + w_2^2}\right]. \tag{F.25}$$

The sum of the incoming (fast) rapidities is equal to the sum of outgoing (fast) rapidities:

$$\xi_1 + \xi_2 = \xi_3 + \xi_4 = \frac{1}{2}\log\left[\frac{s + w_1^2 + w_2^2 + \sqrt{\Upsilon_{12}(s)}}{s + w_1^2 + w_2^2 - \sqrt{\Upsilon_{12}(s)}}\right]. \tag{F.26}$$

F.3.2 Physical Scattering Region

The on-shell relation for a fast quantum is

$$w^2 = -E^2 + |\mathbf{p}|^2 \quad \Longrightarrow \quad |\mathbf{p}|^2 = w^2 + E^2, \tag{F.27}$$

so requiring $E > 0$ leads to $|\mathbf{p}| > w$. The energies are positive as long as

$$s > |w_1 - w_2|\,(w_1 + w_2)\,. \tag{F.28}$$

F.3.3 Scattering Regimes

...

Fast Small-Speed Scattering

Fast small-speed scattering is the regime when

$$\frac{s}{w_1 w_2} \to \infty, \qquad \frac{w_1}{w_2} \text{ fixed.} \tag{F.29}$$

In this regime, all speeds approach unity.

Fast Fixed-Speed Scattering

The speed of each external quantum can be written as a function of two dimensionless ratios

$$\frac{s}{w_1 w_2}, \qquad \frac{w_1}{w_2}. \tag{F.30}$$

Fixed-speed scattering corresponds to the kinematic regime where we keep these ratios fixed:

$$\frac{s}{w_1 w_2} \text{ fixed,} \qquad \frac{w_1}{w_2} \text{ fixed.} \tag{F.31}$$

This regime is appropriate for either large s and large masses, or small s and small masses. Note that fixed-speed is equivalent to fixed-rapidity. By itself, this regime is not very helpful, but when combined with other limits it leads to important approximations.

Fast Large-Speed Scattering

Fast large speeds occur when

$$s \to \pm (w_1 - w_2)(w_1 + w_2)\,. \tag{F.32}$$

In each of these two limits a pair of the speeds becomes infinite.

Regge Scattering

Regge scattering is the regime of fixed-speed and large (unphysical) z_{13}. This corresponds to

$$\frac{t}{s} \to \infty, \qquad \frac{s}{w_1 w_2} \text{ fixed}, \qquad \frac{w_1}{w_2} \text{ fixed}. \tag{F.33}$$

As a corollary, you have

$$u = -2w_1^2 - 2w_2^2 - s - t \quad \Longrightarrow \quad \frac{u}{s} \to -\infty. \tag{F.34}$$

In this regime all dual conformal invariants become trivial:

$$\begin{aligned} [1,2;3,4] &\to \infty, & [1,3;4,2] &= 1, & [1,4;2,3] &\to 0, \\ [1,2;4,3] &\to 0, & [1,3;2,4] &= 1, & [1,4;3,2] &\to \infty. \end{aligned} \tag{F.35}$$

$$\begin{aligned} [1,2;3,4] &\to -\infty, & [1,3;4,2] &= 1, & [1,4;2,3] &\to 0, \\ [1,2;4,3] &\to 0, & [1,3;2,4] &= 1, & [1,4;3,2] &\to -\infty. \end{aligned} \tag{F.36}$$

$$\begin{aligned} [1,2;3,4] &\to -\infty, & [1,3;4,2] &= \frac{w_1^4}{w_2^4}, & [1,4;2,3] &\to 0, \\ [1,2;4,3] &\to 0, & [1,3;2,4] &= \frac{w_2^4}{w_1^4}, & [1,4;3,2] &\to -\infty. \end{aligned} \tag{F.37}$$

Forward Scattering

Forward scattering is the regime of fixed-speed and small (physical) scattering angles (i.e. $z_{13} \to 1$). This can be stated as

$$\frac{t}{s} \to 0, \qquad \frac{s}{w_1 w_2} \text{ fixed}, \qquad \frac{w_1}{w_2} \text{ fixed}. \tag{F.38}$$

As a corollary, you have

$$u = -2w_1^2 - 2w_2^2 - s - t \quad \Longrightarrow \quad \frac{u}{s} \text{ fixed}. \tag{F.39}$$

Unlike the case of Regge scattering, in the forward regime only some of the dual conformal invariants are trivial:

$$\begin{aligned} [1,2;3,4] &\to 0, & [1,3;4,2] &= 1, & [1,4;2,3] &\to \infty, \\ [1,2;4,3] &\to \infty, & [1,3;2,4] &= 1, & [1,4;3,2] &=\to 0. \end{aligned} \tag{F.40}$$

$$\begin{aligned} [1,2;3,4] &\to 0, & [1,3;4,2] &= 1, & [1,4;2,3] &=\to \infty, \\ [1,2;4,3] &=\to \infty, & [1,3;2,4] &= 1, & [1,4;3,2] &\to 0. \end{aligned} \tag{F.41}$$

The rest of the dual conformal invariants are fixed and nontrivial:

$$\begin{aligned} [1,2;3,4] &= \frac{su}{w_1^4}, & [1,3;4,2] &= \frac{w_1^4}{w_2^4}, & [1,4;2,3] &= \frac{w_2^4}{su}, \\ [1,2;4,3] &= \frac{w_1^4}{su}, & [1,3;2,4] &= \frac{w_2^4}{w_1^4}, & [1,4;3,2] &= \frac{su}{w_2^4}. \end{aligned} \tag{F.42}$$

Backward Scattering

Backward scattering is the regime of fixed-speed and large (physical) scattering angles (i.e. $z_{13} \to -1$).

Fixed-Angle Scattering

Fixed-angle scattering is the regime of large-speed and (physical) fixed-angle. This can be stated as

$$\frac{t}{s}\ \text{fixed}, \qquad \frac{s}{m_1 m_2} \to \infty, \qquad \frac{m_1}{m_2}\ \text{fixed}. \tag{F.43}$$

As a corollary, you have

$$u = 2m_1^2 + 2m_2^2 - s - t \quad \Longrightarrow \quad \frac{u}{s}\ \text{fixed}. \tag{F.44}$$

In this regime all dual conformal invariants become trivial:

$$\begin{aligned} [1,2;3,4] &\to -\infty, \qquad [1,3;4,2] = 1, \qquad [1,4;2,3] \to 0, \\ [1,2;4,3] &\to 0, \qquad [1,3;2,4] = 1, \qquad [1,4;3,2] \to -\infty. \end{aligned} \tag{F.45}$$

$$\begin{aligned} [1,2;3,4] &\to \infty, \qquad [1,3;4,2] = 1, \qquad [1,4;2,3] \to 0, \\ [1,2;4,3] &\to 0, \qquad [1,3;2,4] = 1, \qquad [1,4;3,2] \to \infty. \end{aligned} \tag{F.46}$$

$$\begin{aligned} [1,2;3,4] &\to -\infty, \qquad [1,3;4,2] = \frac{m_1^4}{m_2^4}, \qquad [1,4;2,3] \to 0, \\ [1,2;4,3] &\to 0, \qquad [1,3;2,4] = \frac{m_2^4}{m_1^4}, \qquad [1,4;3,2] \to -\infty. \end{aligned} \tag{F.47}$$

From the point of view of dual conformal invariants, Regge scattering and fixed-angle scattering are very similar.

Chapter G

Minimal Elastic Fast Diversity

In this chapter I will consider the kinematics of a 2-to-2 scattering process that has minimal elastic fast diversity. This means that all external quanta are fast, and all imasses are identical:

$$w^2 = |p_1|^2 = |p_3|^2 = |p_2|^2 = |p_4|^2 . \tag{G.1}$$

Thus, this is an elastic process. For example,

$$a(p_1) + a(p_2) \longrightarrow a(p_3) + a(p_4). \tag{G.2}$$

For this process, (A.3) becomes

$$s + t + u = -4w^2. \tag{G.3}$$

G.1 Simplicial Invariants

The simplicial invariants for fast quanta are slightly different from the slow quanta invariants. Minimal elastic diversity reduces the amount of distinct values.

G.1.1 1-Simplex Invariants

For the red tetrahedron you have

$$C_{12} = C_{14} = C_{23} = C_{34} = -w^2, \qquad C_{13} = s, \qquad C_{24} = t. \tag{G.4}$$

Similarly, for the blue tetrahedron you have

$$C_{12} = C_{23} = C_{14} = C_{34} = -w^2, \qquad C_{13} = t, \qquad C_{24} = u. \tag{G.5}$$

Finally, for the green tetrahedron you have

$$C_{12} = C_{34} = C_{14} = C_{23} = -w^2, \qquad C_{13} = u, \qquad C_{24} = s. \tag{G.6}$$

G.1.2 2-Simplex Invariants

For the red tetrahedron you have

$$\begin{aligned} C_{123} = C_{134} = s\left[s + 4w^2\right] = \Upsilon_{12}(s), \\ C_{124} = C_{234} = t\left[t + 4w^2\right] = \Upsilon_{13}(t). \end{aligned} \tag{G.7}$$

Similarly, for the blue tetrahedron you have

$$\begin{aligned} C_{123} = C_{134} = t\left[t + 4w^2\right] = \Upsilon_{13}(t), \\ C_{124} = C_{234} = u\left[u + 4w^2\right] = \Upsilon_{14}(u). \end{aligned} \tag{G.8}$$

Finally, for the green tetrahedron you have

$$\begin{aligned} C_{123} = C_{134} = u\left[u + 4w^2\right] = \Upsilon_{14}(u), \\ C_{124} = C_{234} = s\left[s + 4w^2\right] = \Upsilon_{12}(s). \end{aligned} \tag{G.9}$$

G.1.3 3-Simplex Invariants

A tetrahedron has only one possible 3-simplex invariant. Indeed,

$$C_{1234} = C_{1234} = C_{1234} = stu. \tag{G.10}$$

G.2 Dual Conformal Invariants

For the red tetrahedron, you have

$$\begin{aligned} [1,2;3,4] &= \frac{st}{w^4}, \qquad [1,3;4,2] = 1, \qquad [1,4;2,3] = \frac{w^4}{st}, \\ [1,2;4,3] &= \frac{w^4}{st}, \qquad [1,3;2,4] = 1, \qquad [1,4;3,2] = \frac{st}{w^4}. \end{aligned} \tag{G.11}$$

Similarly, for the blue tetrahedron, you have

$$\begin{aligned} [1,2;3,4] &= \frac{tu}{w^4}, \qquad [1,3;4,2] = 1, \qquad [1,4;2,3] = \frac{w^4}{tu}, \\ [1,2;4,3] &= \frac{w^4}{tu}, \qquad [1,3;2,4] = 1, \qquad [1,4;3,2] = \frac{tu}{w^4}. \end{aligned} \tag{G.12}$$

Finally, for the green tetrahedron, you have

$$\begin{aligned} [1,2;3,4] &= \frac{su}{w^4}, \qquad [1,3;4,2] = 1, \qquad [1,4;2,3] = \frac{w^4}{su}, \\ [1,2;4,3] &= \frac{w^4}{su}, \qquad [1,3;2,4] = 1, \qquad [1,4;3,2] = \frac{su}{w^4}. \end{aligned} \tag{G.13}$$

G.3 Center-of-Momentum Frame

In the center-of-momentum frame you write the energy-momentum vectors as

$$p_1 = \begin{pmatrix} E_1 & \mathbf{p}_1 \end{pmatrix}, \qquad p_2 = \begin{pmatrix} E_2 & -\mathbf{p}_1 \end{pmatrix}, \qquad p_3 = \begin{pmatrix} E_3 & \mathbf{p}_3 \end{pmatrix}, \qquad p_4 = \begin{pmatrix} E_4 & -\mathbf{p}_3 \end{pmatrix}. \tag{G.14}$$

Using the on-shell constraints leads to:

$$w^2 = -E^2 + |\mathbf{p}|^2, \qquad \Longrightarrow \qquad |\mathbf{p}| = \sqrt{w^2 + E^2}. \tag{G.15}$$

Thus,

$$|\mathbf{p}_1| = \sqrt{w^2 + E_1^2} = \sqrt{w^2 + E_2^2}, \qquad |\mathbf{p}_3| = \sqrt{w^2 + E_3^2} = \sqrt{w^2 + E_4^2}. \tag{G.16}$$

One of the first things to notice is that

$$s = (E_1 + E_2)^2 = (E_3 + E_4)^2. \tag{G.17}$$

Thus, s can be interpreted as the total energy in the center-of-momentum frame. It also follows that s must be positive. Using

$$E_2 = \sqrt{s} - E_1, \qquad E_4 = \sqrt{s} - E_3, \tag{G.18}$$

leads to

$$E_1 = E_2 = E_3 = E_4 = \frac{\sqrt{s}}{2}. \tag{G.19}$$

Going back to the spatial momentum, you find

$$|\mathbf{p}_1| = |\mathbf{p}_3| = \frac{\sqrt{\Upsilon_{12}(s)}}{2\sqrt{s}} = \frac{\sqrt{s + 4w^2}}{2}. \tag{G.20}$$

G.3.1 Speed and Rapidity

The speed of each external quantum is

$$|\mathbf{v}_1| = |\mathbf{v}_2| = |\mathbf{v}_3| = |\mathbf{v}_4| = \frac{\sqrt{\Upsilon_{12}(s)}}{s} = \sqrt{1 + \frac{4w^2}{s}}. \tag{G.21}$$

The (fast) rapidity of each external quantum is

$$\xi_1 = \xi_2 = \xi_3 = \xi_4 = \frac{1}{2}\log\left[\frac{\sqrt{\Upsilon_{12}(s)} + s}{\sqrt{\Upsilon_{12}(s)} - s}\right]. \tag{G.22}$$

The sum of the incoming (fast) rapidities is equal to the sum of outgoing (fast) rapidities:

$$\xi_1 + \xi_2 = \xi_3 + \xi_4 = \frac{1}{2}\log\left[\frac{s + 2w^2 + \sqrt{\Upsilon_{12}(s)}}{s + 2w^2 - \sqrt{\Upsilon_{12}(s)}}\right]. \tag{G.23}$$

G.3.2 Physical Scattering Region

The on-shell relation for a fast quantum is

$$w^2 = -E^2 + |\mathbf{p}|^2 \quad \Longrightarrow \quad |\mathbf{p}|^2 = w^2 + E^2, \tag{G.24}$$

so requiring $E > 0$ leads to $|\mathbf{p}| > w$. The energies are positive as long as

$$s > 0. \tag{G.25}$$

G.3.3 Scattering Regimes

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Fast Small-Speed Scattering

Fast small-speed scattering is the regime when

$$\frac{s}{w^2} \to \infty. \tag{G.26}$$

In this regime, all speeds approach unity.

Fast Fixed-Speed Scattering

The speed of each external quantum can be written as a function of one dimensionless ratio,

$$\frac{s}{w^2}. \tag{G.27}$$

Fixed-speed scattering corresponds to the kinematic regime where we keep this ratio fixed:

$$\frac{s}{w^2} \text{ fixed.} \tag{G.28}$$

This regime is appropriate for either large s and large imasses, or small s and small imasses. Note that fixed-speed is equivalent to fixed-rapidity. By itself, this regime is not very helpful, but when combined with other limits it leads to important approximations.

Fast Large-Speed Scattering

Fast large speeds occur when

$$\frac{s}{w^2} \to 0. \tag{G.29}$$

In this limit all speeds become infinite.

Regge Scattering

Regge scattering is the regime of fixed-speed and large (unphysical) z_{13}. This corresponds to

$$\frac{t}{s} \to \infty, \qquad \frac{s}{w^2} \text{ fixed.} \tag{G.30}$$

As a corollary, you have

$$u = -4w^2 - s - t \quad \Longrightarrow \quad \frac{u}{s} \to -\infty. \tag{G.31}$$

In this regime all dual conformal invariants become trivial:

$$\begin{array}{rrr} [1,2;3,4] \to \infty, & [1,3;4,2] = 1, & [1,4;2,3] \to 0, \\ [1,2;4,3] \to 0, & [1,3;2,4] = 1, & [1,4;3,2] \to \infty. \end{array} \tag{G.32}$$

$$\begin{array}{rrr} [1,2;3,4] \to -\infty, & [1,3;4,2] = 1, & [1,4;2,3] \to 0, \\ [1,2;4,3] \to 0, & [1,3;2,4] = 1, & [1,4;3,2] \to -\infty. \end{array} \tag{G.33}$$

$$\begin{array}{rrr} [1,2;3,4] \to -\infty, & [1,3;4,2] = 1, & [1,4;2,3] \to 0, \\ [1,2;4,3] \to 0, & [1,3;2,4] = 1, & [1,4;3,2] \to -\infty. \end{array} \tag{G.34}$$

Forward Scattering

Forward scattering is the regime of fixed-speed and small (physical) scattering angles (i.e. $z_{13} \to 1$). This can be stated as

$$\frac{t}{s} \to 0, \qquad \frac{s}{w^2} \text{ fixed.} \tag{G.35}$$

As a corollary, you have

$$u = -4w^2 - s - t \quad \Longrightarrow \quad \frac{u}{s} \text{ fixed.} \tag{G.36}$$

Unlike the case of Regge scattering, in the forward regime only some of the dual conformal invariants are trivial:

$$\begin{array}{rrr} [1,2;3,4] \to 0, & [1,3;4,2] = 1, & [1,4;2,3] \to \infty, \\ [1,2;4,3] \to \infty, & [1,3;2,4] = 1, & [1,4;3,2] =\to 0. \end{array} \tag{G.37}$$

$$\begin{array}{rrr} [1,2;3,4] \to 0, & [1,3;4,2] = 1, & [1,4;2,3] =\to \infty, \\ [1,2;4,3] =\to \infty, & [1,3;2,4] = 1, & [1,4;3,2] \to 0. \end{array} \tag{G.38}$$

The rest of the dual conformal invariants are fixed and nontrivial:

$$\begin{array}{rrr} [1,2;3,4] = \dfrac{su}{w^4}, & [1,3;4,2] = 1, & [1,4;2,3] = \dfrac{w^4}{su}, \\ [1,2;4,3] = \dfrac{w^4}{su}, & [1,3;2,4] = 1, & [1,4;3,2] = \dfrac{su}{w^4}. \end{array} \tag{G.39}$$

Backward Scattering

Backward scattering is the regime of fixed-speed and large (physical) scattering angles (i.e. $z_{13} \to -1$).

Fixed-Angle Scattering

Fixed-angle scattering is the regime of small-speed and (physical) fixed-angle. This can be stated as

$$\frac{t}{s} \text{ fixed}, \qquad \frac{s}{w^2} \to \infty. \tag{G.40}$$

As a corollary, you have

$$u = -4w^2 - s - t \quad \Longrightarrow \quad \frac{u}{s} \text{ fixed.} \tag{G.41}$$

In this regime all dual conformal invariants become trivial:

$$\begin{aligned} [1,2;3,4] &\to -\infty, & [1,3;4,2] &= 1, & [1,4;2,3] &\to 0, \\ [1,2;4,3] &\to 0, & [1,3;2,4] &= 1, & [1,4;3,2] &\to -\infty. \end{aligned} \tag{G.42}$$

$$\begin{aligned} [1,2;3,4] &\to \infty, & [1,3;4,2] &= 1, & [1,4;2,3] &\to 0, \\ [1,2;4,3] &\to 0, & [1,3;2,4] &= 1, & [1,4;3,2] &\to \infty. \end{aligned} \tag{G.43}$$

$$\begin{aligned} [1,2;3,4] &\to -\infty, & [1,3;4,2] &= 1, & [1,4;2,3] &\to 0, \\ [1,2;4,3] &\to 0, & [1,3;2,4] &= 1, & [1,4;3,2] &\to -\infty. \end{aligned} \tag{G.44}$$

From the point of view of dual conformal invariants, Regge scattering and fixed-angle scattering are very similar.

Part III

Null Quanta

Chapter H

Null Kinematics

In this chapter I will consider the kinematics of a 2-to-2 scattering process that involves only identical massless quanta. Thus, this is an elastic process. For example,

$$\gamma(p_1) + \gamma(p_2) \longrightarrow \gamma(p_3) + \gamma(p_4). \tag{H.1}$$

Energy-momentum conservation leads to the constraint

$$p_1 + p_2 = p_3 + p_4. \tag{H.2}$$

This constraint means that only three out of the four external energy-momentum vectors are linearly independent. Due to this process being elastic, there are now two elasticity constraints:

$$|p_1|^2 = |p_2|^2 = |p_3|^2 = |p_4|^2 = 0. \tag{H.3}$$

H.1 Mandelstam Invariants

For convenience, one defines the three Mandelstam invariants:

$$s = -|p_1 + p_2|^2, \qquad t = -|p_1 - p_3|^2, \qquad u = -|p_1 - p_4|^2. \tag{H.4}$$

Due to the conservation constraint, these three invariants satisfy a linear relation:

$$s + t + u = 0. \tag{H.5}$$

H.2 Dual Spacetime

You can solve the conservation constraint by introducing dual spacetime coordinates:

$$p_1 = d_1 - d_2, \quad p_2 = d_2 - d_3, \quad p_4 = d_4 - d_3, \quad p_3 = d_1 - d_4. \tag{H.6}$$

Thus, each energy-momentum vector becomes a distance interval in a dual spacetime. However, the solution (H.6) is not the only one allowed. Indeed, two other solutions are allowed:

$$-p_3 = d_1 - d_2, \quad p_1 = d_2 - d_3, \quad p_4 = d_4 - d_3, \quad -p_2 = d_1 - d_4, \tag{H.7}$$

and

$$p_2 = d_1 - d_2, \quad -p_3 = d_2 - d_3, \quad p_4 = d_4 - d_3, \quad -p_1 = d_1 - d_4. \tag{H.8}$$

I will refer to (H.6) as the red planar class, (H.7) as the blue planar class, and (H.8) as the green planar class. In each planar class, the dual spacetime coordinates describe the positions of four points in the dual spacetime. These four points can be taken as the vertices of a tetrahedron. Many kinematic quantities can be understood in terms of the geometry of these tetrahedra.

H.3 Simplicial Invariants

A tetrahedron is a 3-simplex. As such, it contains four vertices (0-simplex), six edges (1-simplex), four triangular faces (2-simplex), and one tetrahedron (3-simplex). The n-volume of an n-simplex is found by evaluating an $(n+1)$-point Cayley-Menger determinant.

H.3.1 1-Simplex Invariants

Given two dual spacetime positions, the 2-point Cayley-Menger determinant is given by:

$$C_{ij} = -\frac{1}{2}\det\begin{pmatrix} 0 & |d_{ij}|^2 & 1 \\ |d_{ij}|^2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}. \tag{H.9}$$

Since a tetrahedron has six edges, there are six possible 1-simplex invariants. These invariants will be either squared masses or Mandelstam invariants. For the red tetrahedron you have

$$\begin{aligned} C_{12} &= 0, & C_{13} &= s, & C_{14} &= 0, \\ C_{23} &= 0, & C_{24} &= t, & C_{34} &= 0. \end{aligned} \tag{H.10}$$

Similarly, for the blue tetrahedron you have

$$\begin{aligned} C_{12} &= 0, & C_{13} &= t, & C_{14} &= 0, \\ C_{23} &= 0, & C_{24} &= u, & C_{34} &= 0. \end{aligned} \tag{H.11}$$

Finally, for the green tetrahedron you have

$$\begin{aligned} C_{12} &= 0, & C_{13} &= u, & C_{14} &= 0, \\ C_{23} &= 0, & C_{24} &= s, & C_{34} &= 0. \end{aligned} \tag{H.12}$$

H.3.2 2-Simplex Invariants

Given three dual spacetime positions, the 3-point Cayley-Menger determinant is given by:

$$C_{ijk} = \det\begin{pmatrix} 0 & |d_{ij}|^2 & |d_{ik}|^2 & 1 \\ |d_{ij}|^2 & 0 & |d_{jk}|^2 & 1 \\ |d_{ik}|^2 & |d_{jk}|^2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}. \tag{H.13}$$

Since a tetrahedron has four triangular faces, there are four possible 2-simplex invariants. Each of these invariants has the form

$$\Lambda_{IJ}(x) = \left[x - (m_I - m_J)^2\right]\left[x - (m_I + m_J)^2\right], \tag{H.14}$$

where x is a Mandelstam invariant. This function is also known as the Källén function. For the red tetrahedron you have

$$C_{123} = C_{134} = s^2, \qquad C_{124} = C_{234} = t^2. \tag{H.15}$$

Similarly, for the blue tetrahedron you have

$$C_{123} = C_{134} = t^2, \qquad C_{124} = C_{234} = u^2. \tag{H.16}$$

Finally, for the green tetrahedron you have

$$C_{123} = C_{134} = u^2, \qquad C_{124} = C_{234} = s^2. \tag{H.17}$$

Note that all 2-simplex invariants are strictly positive by definition.

H.3.3 3-Simplex Invariants

Given four dual spacetime positions, the 4-point Cayley-Menger determinant is given by:

$$C_{ijkl} = -\frac{1}{2}\det\begin{pmatrix} 0 & |d_{ij}|^2 & |d_{ik}|^2 & |d_{il}|^2 & 1 \\ |d_{ij}|^2 & 0 & |d_{jk}|^2 & |d_{jl}|^2 & 1 \\ |d_{ik}|^2 & |d_{jk}|^2 & 0 & |d_{kl}|^2 & 1 \\ |d_{il}|^2 & |d_{jl}|^2 & |d_{kl}|^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}. \tag{H.18}$$

A tetrahedron has only one possible 3-simplex invariant. Indeed,

$$C_{1234} = C_{1234} = C_{1234} = stu. \tag{H.19}$$

H.4 Dual Conformal Invariants

Each of the 1-simplex invariants has the form $|d_i - d_j|^2$. This is not only Lorentz invariant, but dual Poincaré invariant. You can study dual conformal invariants by constructing a dual conformal ratio with a quartet of dual spacetime coordinates:

$$[i, j; k, l] \equiv \frac{|d_{ik}|^2 \, |d_{jl}|^2}{|d_{il}|^2 \, |d_{jk}|^2}. \tag{H.20}$$

In four-point scattering there is a unique quartet of dual spacetime coordinates. However, there are six inequivalent permutations of these coordinates, and thus, six possible values for the dual conformal ratio.

Due to the appearance of only massless quanta, all dual conformal invariants are trivial. For the red tetrahedron, you have

$$\begin{aligned} [1,2;3,4] &\to \infty, & [1,3;4,2] &= 1, & [1,4;2,3] &\to 0, \\ [1,2;4,3] &\to 0, & [1,3;2,4] &= 1, & [1,4;3,2] &\to \infty. \end{aligned} \tag{H.21}$$

Similarly, for the blue tetrahedron, you have

$$\begin{aligned} [1,2;3,4] &\to \infty, & [1,3;4,2] &= 1, & [1,4;2,3] &\to 0, \\ [1,2;4,3] &\to 0, & [1,3;2,4] &= 1, & [1,4;3,2] &\to \infty. \end{aligned} \tag{H.22}$$

Finally, for the green tetrahedron, you have

$$\begin{aligned} [1,2;3,4] &\to \infty, & [1,3;4,2] &= 1, & [1,4;2,3] &\to 0, \\ [1,2;4,3] &\to 0, & [1,3;2,4] &= 1, & [1,4;3,2] &\to \infty. \end{aligned} \tag{H.23}$$

Note that

$$[1,2;3,4]\,[1,3;4,2]\,[1,4;2,3] = 1, \tag{H.24}$$

which is analogous to the constraint satisfied by the three Mandelstam invariants.

H.5 Center-of-Momentum Frame

In the center-of-momentum frame you write the energy-momentum vectors as

$$p_1 = \begin{pmatrix} E_1 & \mathbf{p}_1 \end{pmatrix}, \qquad p_2 = \begin{pmatrix} E_2 & -\mathbf{p}_1 \end{pmatrix}, \qquad p_3 = \begin{pmatrix} E_3 & \mathbf{p}_3 \end{pmatrix}, \qquad p_4 = \begin{pmatrix} E_4 & -\mathbf{p}_3 \end{pmatrix}. \tag{H.25}$$

One of the first things to notice is that

$$s = (E_1 + E_2)^2 = (E_3 + E_4)^2. \tag{H.26}$$

Thus, s can be interpreted as the total energy in the center-of-momentum frame. It also follows that s must be positive.

From the on-shell constraints, it follows that

$$\begin{aligned} E_1 &= \sqrt{m^2 + |\mathbf{p}_1|^2}, & E_2 &= \sqrt{m^2 + |\mathbf{p}_1|^2}, \\ E_3 &= \sqrt{m^2 + |\mathbf{p}_3|^2}, & E_4 &= \sqrt{m^2 + |\mathbf{p}_3|^2}. \end{aligned} \tag{H.27}$$

Using the relations

$$E_2 = \sqrt{s} - E_1, \qquad E_4 = \sqrt{s} - E_3, \tag{H.28}$$

you find that

$$|\mathbf{p}_1| = |\mathbf{p}_3| = \frac{\sqrt{s - 4m^2}}{2}. \tag{H.29}$$

Thus,

$$E_1 = E_2 = E_3 = E_4 = \frac{\sqrt{s}}{2}. \tag{H.30}$$

H.5.1 Speed and Rapidity

A quantum with mass m, spatial momentum $\mathbf{p}$, and energy E has a speed $|\mathbf{v}|$ given by

$$|\mathbf{v}| = \frac{\sqrt{E^2 - m^2}}{E^2} = \frac{|\mathbf{p}|}{\sqrt{m^2 + |\mathbf{p}|^2}} = \frac{|\mathbf{p}|}{E}. \tag{H.31}$$

If the quantum is physical, then $E \geq m$, or equivalently, $|\mathbf{p}| < E$. Both of these conditions lead to a bound on speed:

$$0 \leq |\mathbf{v}| < 1. \tag{H.32}$$

In contrast with energy, mass, and spatial momentum, speed and velocity are dimensionless. However, energy and spatial momentum are conserved, but velocity is not. The speed of each external quantum is

$$|\mathbf{v}_1| = |\mathbf{v}_2| = |\mathbf{v}_3| = |\mathbf{v}_4| = \sqrt{1 - \frac{4m^2}{s}}. \tag{H.33}$$

Given the speed of a quantum, you can find its rapidity η via

$$\eta = \operatorname{arctanh} |\mathbf{v}| = \frac{1}{2} \log \left(\frac{1 + |\mathbf{v}|}{1 - |\mathbf{v}|} \right). \tag{H.34}$$

Since the speed of a physical quantum is bounded, the rapidity of a physical quantum is bounded from below:

$$0 \leq \eta < \infty. \tag{H.35}$$

The rapidity of each external quantum is

$$\eta_1 = \eta_2 = \eta_3 = \eta_4 = \frac{1}{2} \log \left[\frac{\sqrt{s} + \sqrt{s - 4m^2}}{\sqrt{s} - \sqrt{s - 4m^2}} \right]. \tag{H.36}$$

The sum of the incoming rapidites is equal to the sum of the outgoing rapidities:

$$\eta_1 + \eta_2 = \eta_3 + \eta_4 = \log \left[\frac{\sqrt{s} + \sqrt{s - 4m^2}}{\sqrt{s} - \sqrt{s - 4m^2}} \right]. \tag{H.37}$$

H.5.2 Physical Scattering Region

...

Regge Scattering

Regge scattering is the regime of large (unphysical) z_{13}. This corresponds to

$$\frac{t}{s} \to \infty. \tag{H.38}$$

As a corollary, you have

$$u = -s - t \quad \Longrightarrow \quad \frac{u}{s} \to -\infty. \tag{H.39}$$

Forward Scattering

Forward scattering is the regime of small (physical) scattering angles (i.e. $z_{13} \to 1$). This can be stated as

$$\frac{t}{s} \to 0. \tag{H.40}$$

As a corollary, you have

$$u = -s - t \quad \Longrightarrow \quad \frac{u}{s} \text{ fixed}. \tag{H.41}$$

Backward Scattering

Backward scattering is the regime of large (physical) scattering angles (i.e. $z_{13} \to -1$). This can be stated as

$$\frac{u}{s} \to 0. \tag{H.42}$$

As a corollary, you have

$$t = -s - u \quad \Longrightarrow \quad \frac{t}{s} \text{ fixed.} \tag{H.43}$$

Fixed-Angle Scattering

Fixed-angle scattering is the regime of (physical) fixed-angle. This can be stated as

$$\frac{t}{s} \text{ fixed.} \tag{H.44}$$

As a corollary, you have

$$u = -s - t \quad \Longrightarrow \quad \frac{u}{s} \text{ fixed.} \tag{H.45}$$

Part IV

Mixed Quanta

Chapter I

Slow Compton Kinematics

In this chapter I will consider the kinematics of a 2-to-2 Compton scattering process. This means that you have a massive elastic pair, and a massless elastic pair. Thus, this is an elastic process. For example,

$$e^-(p_1) + \gamma(p_2) \longrightarrow e^-(p_3) + \gamma(p_4). \tag{I.1}$$

Energy-momentum conservation leads to the constraint

$$p_1 + p_2 = p_3 + p_4. \tag{I.2}$$

This constraint means that only three out of the four external energy-momentum vectors are linearly independent. Due to this process being elastic, there are now two elasticity constraints:

$$|p_1|^2 = |p_3|^2, \qquad |p_2|^2 = |p_4|^2 = 0. \tag{I.3}$$

I.1 Simplicial Invariants

A tetrahedron is a 3-simplex. As such, it contains four vertices (0-simplex), six edges (1-simplex), four triangular faces (2-simplex), and one tetrahedron (3-simplex). The n-volume of an n-simplex is found by evaluating an $(n+1)$-point Cayley-Menger determinant.

I.1.1 1-Simplex Invariants

Given two dual spacetime positions, the 2-point Cayley-Menger determinant is given by:

$$C_{ij} = -\frac{1}{2}\det\begin{pmatrix} 0 & |d_{ij}|^2 & 1 \\ |d_{ij}|^2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}. \tag{I.4}$$

Since a tetrahedron has six edges, there are six possible 1-simplex invariants. These invariants will be either squared masses or Mandelstam invariants. For the red tetrahedron you have

$$\begin{aligned} C_{12} &= m^2, & C_{13} &= s, & C_{14} &= m^2, \\ C_{23} &= m^2, & C_{24} &= t, & C_{34} &= m^2. \end{aligned} \tag{I.5}$$

Similarly, for the blue tetrahedron you have

$$\begin{aligned} C_{12} &= m^2, & C_{13} &= t, & C_{14} &= m^2, \\ C_{23} &= m^2, & C_{24} &= u, & C_{34} &= m^2. \end{aligned} \tag{I.6}$$

Finally, for the green tetrahedron you have

$$\begin{aligned} C_{12} &= m^2, & C_{13} &= u, & C_{14} &= m^2, \\ C_{23} &= m^2, & C_{24} &= s, & C_{34} &= m^2. \end{aligned} \tag{I.7}$$

I.1.2 2-Simplex Invariants

Given three dual spacetime positions, the 3-point Cayley-Menger determinant is given by:

$$C_{ijk} = \det \begin{pmatrix} 0 & |d_{ij}|^2 & |d_{ik}|^2 & 1 \\ |d_{ij}|^2 & 0 & |d_{jk}|^2 & 1 \\ |d_{ik}|^2 & |d_{jk}|^2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}. \tag{I.8}$$

Since a tetrahedron has four triangular faces, there are four possible 2-simplex invariants. Each of these invariants has the form

$$\Lambda_{IJ}(x) = \left[x - (m_I - m_J)^2\right]\left[x - (m_I + m_J)^2\right], \tag{I.9}$$

where x is a Mandelstam invariant. This function is also known as the Källén function. For the red tetrahedron you have

$$C_{123} = C_{134} = \left(s - m^2\right)^2, \qquad C_{124} = C_{234} = \left(t - m^2\right)^2. \tag{I.10}$$

Similarly, for the blue tetrahedron you have

$$C_{123} = C_{134} = \left(t - m^2\right)^2, \qquad C_{124} = C_{234} = \left(u - m^2\right)^2. \tag{I.11}$$

Finally, for the green tetrahedron you have

$$C_{123} = C_{134} = \left(u - m^2\right)^2, \qquad C_{124} = C_{234} = \left(s - m^2\right)^2. \tag{I.12}$$

Note that all 2-simplex invariants are strictly positive by definition.

I.1.3 3-Simplex Invariants

Given four dual spacetime positions, the 4-point Cayley-Menger determinant is given by:

$$C_{ijkl} = -\frac{1}{2} \det \begin{pmatrix} 0 & |d_{ij}|^2 & |d_{ik}|^2 & |d_{il}|^2 & 1 \\ |d_{ij}|^2 & 0 & |d_{jk}|^2 & |d_{jl}|^2 & 1 \\ |d_{ik}|^2 & |d_{jk}|^2 & 0 & |d_{kl}|^2 & 1 \\ |d_{il}|^2 & |d_{jl}|^2 & |d_{kl}|^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}. \tag{I.13}$$

A tetrahedron has only one possible 3-simplex invariant. Indeed,

$$C_{1234} = C_{1234} = C_{1234} = t\left(su - m^4\right). \tag{I.14}$$

I.2 Dual Conformal Invariants

Each of the 1-simplex invariants has the form $|d_i - d_j|^2$. This is not only Lorentz invariant, but dual Poincaré invariant. You can study dual conformal invariants by constructing a dual conformal ratio with a quartet of dual spacetime coordinates:

$$[i, j; k, l] \equiv \frac{|d_{ik}|^2 |d_{jl}|^2}{|d_{il}|^2 |d_{jk}|^2}. \tag{I.15}$$

In four-point scattering there is a unique quartet of dual spacetime coordinates. However, there are six inequivalent permutations of these coordinates, and thus, six possible values for the dual conformal ratio.

For the red tetrahedron, you have

$$\begin{aligned} [1,2;3,4] &= \frac{st}{m^4}, & [1,3;4,2] &= 1, & [1,4;2,3] &= \frac{m^4}{st}, \\ [1,2;4,3] &= \frac{m^4}{st}, & [1,3;2,4] &= 1, & [1,4;3,2] &= \frac{st}{m^4}. \end{aligned} \tag{I.16}$$

Similarly, for the blue tetrahedron, you have

$$\begin{aligned} [1,2;3,4] &= \frac{tu}{m^4}, & [1,3;4,2] &= 1, & [1,4;2,3] &= \frac{m^4}{tu}, \\ [1,2;4,3] &= \frac{m^4}{tu}, & [1,3;2,4] &= 1, & [1,4;3,2] &= \frac{tu}{m^4}. \end{aligned} \tag{I.17}$$

Finally, for the green tetrahedron, you have

$$\begin{aligned} [1,2;3,4] &= \frac{su}{m^4}, & [1,3;4,2] &= 1, & [1,4;2,3] &= \frac{m^4}{su}, \\ [1,2;4,3] &= \frac{m^4}{su}, & [1,3;2,4] &= 1, & [1,4;3,2] &= \frac{su}{m^4}. \end{aligned} \tag{I.18}$$

Note that

$$[1,2;3,4]\,[1,3;4,2]\,[1,4;2,3] = 1, \tag{I.19}$$

which is analogous to the constraint satisfied by the three Mandelstam invariants.

I.3 Center-of-Momentum Frame

In the center-of-momentum frame you write the energy-momentum vectors as

$$p_1 = \begin{pmatrix} E_1 & \mathbf{p}_1 \end{pmatrix}, \qquad p_2 = \begin{pmatrix} E_2 & -\mathbf{p}_1 \end{pmatrix}, \qquad p_3 = \begin{pmatrix} E_3 & \mathbf{p}_3 \end{pmatrix}, \qquad p_4 = \begin{pmatrix} E_4 & -\mathbf{p}_3 \end{pmatrix}. \tag{I.20}$$

One of the first things to notice is that

$$s = (E_1 + E_2)^2 = (E_3 + E_4)^2. \tag{I.21}$$

Thus, s can be interpreted as the total energy in the center-of-momentum frame. It also follows that s must be positive.

From the on-shell constraints, it follows that

$$\begin{aligned} E_1 &= \sqrt{m^2 + |\mathbf{p}_1|^2}, & E_2 &= \sqrt{m^2 + |\mathbf{p}_1|^2}, \\ E_3 &= \sqrt{m^2 + |\mathbf{p}_3|^2}, & E_4 &= \sqrt{m^2 + |\mathbf{p}_3|^2}. \end{aligned} \tag{I.22}$$

Using the relations

$$E_2 = \sqrt{s} - E_1, \qquad E_4 = \sqrt{s} - E_3, \tag{I.23}$$

you find that

$$|\mathbf{p}_1| = |\mathbf{p}_3| = \frac{\sqrt{s - 4m^2}}{2}. \tag{I.24}$$

Thus,

$$E_1 = E_2 = E_3 = E_4 = \frac{\sqrt{s}}{2}. \tag{I.25}$$

I.3.1 Speed and Rapidity

A quantum with mass m, spatial momentum $\mathbf{p}$, and energy E has a speed $|\mathbf{v}|$ given by

$$|\mathbf{v}| = \frac{\sqrt{E^2 - m^2}}{E^2} = \frac{|\mathbf{p}|}{\sqrt{m^2 + |\mathbf{p}|^2}} = \frac{|\mathbf{p}|}{E}. \tag{I.26}$$

If the quantum is physical, then $E \geq m$, or equivalently, $|\mathbf{p}| < E$. Both of these conditions lead to a bound on speed:

$$0 \leq |\mathbf{v}| < 1. \tag{I.27}$$

In contrast with energy, mass, and spatial momentum, speed and velocity are dimensionless. However, energy and spatial momentum are conserved, but velocity is not. The speed of each external quantum is

$$|\mathbf{v}_1| = |\mathbf{v}_2| = |\mathbf{v}_3| = |\mathbf{v}_4| = \sqrt{1 - \frac{4m^2}{s}}. \tag{I.28}$$

Given the speed of a quantum, you can find its rapidity η via

$$\eta = \operatorname{arctanh}|\mathbf{v}| = \frac{1}{2}\log\left(\frac{1+|\mathbf{v}|}{1-|\mathbf{v}|}\right). \tag{I.29}$$

Since the speed of a physical quantum is bounded, the rapidity of a physical quantum is bounded from below:

$$0 \leq \eta < \infty. \tag{I.30}$$

The rapidity of each external quantum is

$$\eta_1 = \eta_2 = \eta_3 = \eta_4 = \frac{1}{2}\log\left[\frac{\sqrt{s}+\sqrt{s-4m^2}}{\sqrt{s}-\sqrt{s-4m^2}}\right]. \tag{I.31}$$

The sum of the incoming rapidites is equal to the sum of the outgoing rapidities:

$$\eta_1 + \eta_2 = \eta_3 + \eta_4 = \log\left[\frac{\sqrt{s}+\sqrt{s-4m^2}}{\sqrt{s}-\sqrt{s-4m^2}}\right]. \tag{I.32}$$

I.3.2 Physical Scattering Region

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Fixed-Speed Scattering

The speed of each external quantum can be written as a function of one dimensionless ratio,

$$\frac{s}{m^2}. \tag{I.33}$$

Fixed-speed scattering corresponds to the kinematic regime where we keep this ratio fixed:

$$\frac{s}{m^2} \text{ fixed.} \tag{I.34}$$

This regime is appropriate for either large s and large mass, or small s and small mass. Note that fixed-speed is equivalent to fixed-rapidity. By itself, this regime is not very helpful, but when combined with other limits it leads to important approximations.

Small-Speed Scattering

Small speeds occur when s is close to the threshold value $4m^2$.

Large-Speed Scattering

Large-speed scattering involves the regime

$$\frac{s}{m^2} \to \infty. \tag{I.35}$$

That is, $\sqrt{s}$ is very large compared to the mass.

Regge Scattering

Regge scattering is the regime of fixed-speed and large (unphysical) z_{13}. This corresponds to

$$\frac{t}{s} \to \infty, \qquad \frac{s}{m^2} \text{ fixed.} \tag{I.36}$$

As a corollary, you have

$$u = 4m^2 - s - t \quad \Longrightarrow \quad \frac{u}{s} \to -\infty. \tag{I.37}$$

In this regime all dual conformal invariants become trivial:

$$\begin{aligned} [1,2;3,4] &\to \infty, & [1,3;4,2] &= 1, & [1,4;2,3] &\to 0, \\ [1,2;4,3] &\to 0, & [1,3;2,4] &= 1, & [1,4;3,2] &\to \infty. \end{aligned} \tag{I.38}$$

$$\begin{aligned} [1,2;3,4] &\to -\infty, & [1,3;4,2] &= 1, & [1,4;2,3] &\to 0, \\ [1,2;4,3] &\to 0, & [1,3;2,4] &= 1, & [1,4;3,2] &\to -\infty. \end{aligned} \tag{I.39}$$

$$\begin{aligned} [1,2;3,4] &\to -\infty, & [1,3;4,2] &= 1, & [1,4;2,3] &\to 0, \\ [1,2;4,3] &\to 0, & [1,3;2,4] &= 1, & [1,4;3,2] &\to -\infty. \end{aligned} \tag{I.40}$$

Forward Scattering

Forward scattering is the regime of fixed-speed and small (physical) scattering angles (i.e. $z_{13} \to 1$). This can be stated as

$$\frac{t}{s} \to 0, \qquad \frac{s}{m^2} \text{ fixed.} \tag{I.41}$$

As a corollary, you have

$$u = 4m^2 - s - t \quad \Longrightarrow \quad \frac{u}{s} \text{ fixed.} \tag{I.42}$$

Unlike the case of Regge scattering, in the forward regime only some of the dual conformal invariants are trivial:

$$\begin{aligned} [1,2;3,4] &\to 0, & [1,3;4,2] &= 1, & [1,4;2,3] &\to \infty, \\ [1,2;4,3] &\to \infty, & [1,3;2,4] &= 1, & [1,4;3,2] &\to 0. \end{aligned} \tag{I.43}$$

$$\begin{aligned} [1,2;3,4] &\to 0, & [1,3;4,2] &= 1, & [1,4;2,3] &\to \infty, \\ [1,2;4,3] &\to \infty, & [1,3;2,4] &= 1, & [1,4;3,2] &\to 0. \end{aligned} \tag{I.44}$$

The rest of the dual conformal invariants are fixed and nontrivial:

$$\begin{aligned} [1,2;3,4] &= \frac{su}{m^4}, & [1,3;4,2] &= 1, & [1,4;2,3] &= \frac{m^4}{su}, \\ [1,2;4,3] &= \frac{m^4}{su}, & [1,3;2,4] &= 1, & [1,4;3,2] &= \frac{su}{m^4}. \end{aligned} \tag{I.45}$$

Backward Scattering

Backward scattering is the regime of fixed-speed and large (physical) scattering angles (i.e. $z_{13} \to -1$). This can be stated as

$$\frac{u}{s} \to 0, \qquad \frac{s}{m^2} \text{ fixed.} \tag{I.46}$$

As a corollary, you have

$$t = 4m^2 - s - u \quad \Longrightarrow \quad \frac{t}{s} \text{ fixed.} \tag{I.47}$$

Similarly to the forward scattering regime, some of the dual conformal invariants are trivial:

$$\begin{aligned} [1,2;3,4] &\to 0, & [1,3;4,2] &= 1, & [1,4;2,3] &\to \infty, \\ [1,2;4,3] &\to \infty, & [1,3;2,4] &= 1, & [1,4;3,2] &\to 0. \end{aligned} \tag{I.48}$$

$$\begin{aligned} [1,2;3,4] &\to 0, & [1,3;4,2] &= 1, & [1,4;2,3] &\to \infty, \\ [1,2;4,3] &\to \infty, & [1,3;2,4] &= 1, & [1,4;3,2] &\to 0. \end{aligned} \tag{I.49}$$

The rest of the dual conformal invariants are fixed and nontrivial:

$$\begin{aligned} [1,2;3,4] &= \frac{st}{m^4}, & [1,3;4,2] &= 1, & [1,4;2,3] &= \frac{m^4}{st}, \\ [1,2;4,3] &= \frac{m^4}{st}, & [1,3;2,4] &= 1, & [1,4;3,2] &= \frac{st}{m^4}. \end{aligned} \tag{I.50}$$

Fixed-Angle Scattering

Fixed-angle scattering is the regime of large-speed and (physical) fixed-angle. This can be stated as

$$\frac{t}{s} \text{ fixed}, \qquad \frac{s}{m^2} \to \infty. \tag{I.51}$$

As a corollary, you have

$$u = 4m^2 - s - t \quad \Longrightarrow \quad \frac{u}{s} \text{ fixed.} \tag{I.52}$$

In this regime all dual conformal invariants become trivial:

$$\begin{aligned} [1,2;3,4] &\to -\infty, & [1,3;4,2] &= 1, & [1,4;2,3] &\to 0, \\ [1,2;4,3] &\to 0, & [1,3;2,4] &= 1, & [1,4;3,2] &\to -\infty. \end{aligned} \tag{I.53}$$

$$\begin{aligned} &[1,2;3,4] \to \infty, \qquad [1,3;4,2] = 1, \qquad [1,4;2,3] \to 0, \\ &[1,2;4,3] \to 0, \qquad [1,3;2,4] = 1, \qquad [1,4;3,2] \to \infty. \end{aligned} \tag{I.54}$$

$$\begin{aligned} &[1,2;3,4] \to -\infty, \qquad [1,3;4,2] = 1, \qquad [1,4;2,3] \to 0, \\ &[1,2;4,3] \to 0, \qquad [1,3;2,4] = 1, \qquad [1,4;3,2] \to -\infty. \end{aligned} \tag{I.55}$$

From the point of view of dual conformal invariants, Regge scattering and fixed-angle scattering are very similar.

Chapter J

Fast Compton Kinematics

In this chapter I will consider the kinematics of a 2-to-2 Compton scattering process. This means that you have a massive elastic pair, and a massless elastic pair. Thus, this is an elastic process. For example,

$$e^{-}(p_1) + \gamma(p_2) \longrightarrow e^{-}(p_3) + \gamma(p_4). \tag{J.1}$$

Energy-momentum conservation leads to the constraint

$$p_1 + p_2 = p_3 + p_4. \tag{J.2}$$

This constraint means that only three out of the four external energy-momentum vectors are linearly independent. Due to this process being elastic, there are now two elasticity constraints:

$$|p_1|^2 = |p_3|^2, \qquad |p_2|^2 = |p_4|^2 = 0. \tag{J.3}$$

J.1 Simplicial Invariants

A tetrahedron is a 3-simplex. As such, it contains four vertices (0-simplex), six edges (1-simplex), four triangular faces (2-simplex), and one tetrahedron (3-simplex). The n-volume of an n-simplex is found by evaluating an $(n+1)$-point Cayley-Menger determinant.

J.1.1 1-Simplex Invariants

Given two dual spacetime positions, the 2-point Cayley-Menger determinant is given by:

$$C_{ij} = -\frac{1}{2}\det\begin{pmatrix} 0 & |d_{ij}|^2 & 1 \\ |d_{ij}|^2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}. \tag{J.4}$$

Since a tetrahedron has six edges, there are six possible 1-simplex invariants. These invariants will be either squared masses or Mandelstam invariants. For the red tetrahedron you have

$$\begin{aligned} C_{12} &= m^2, & C_{13} &= s, & C_{14} &= m^2, \\ C_{23} &= m^2, & C_{24} &= t, & C_{34} &= m^2. \end{aligned} \tag{J.5}$$

Similarly, for the blue tetrahedron you have

$$\begin{aligned} C_{12} &= m^2, & C_{13} &= t, & C_{14} &= m^2, \\ C_{23} &= m^2, & C_{24} &= u, & C_{34} &= m^2. \end{aligned} \tag{J.6}$$

Finally, for the green tetrahedron you have

$$\begin{aligned} C_{12} &= m^2, & C_{13} &= u, & C_{14} &= m^2, \\ C_{23} &= m^2, & C_{24} &= s, & C_{34} &= m^2. \end{aligned} \tag{J.7}$$

J.1.2 2-Simplex Invariants

Given three dual spacetime positions, the 3-point Cayley-Menger determinant is given by:

$$C_{ijk} = \det \begin{pmatrix} 0 & |d_{ij}|^2 & |d_{ik}|^2 & 1 \\ |d_{ij}|^2 & 0 & |d_{jk}|^2 & 1 \\ |d_{ik}|^2 & |d_{jk}|^2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}. \tag{J.8}$$

Since a tetrahedron has four triangular faces, there are four possible 2-simplex invariants. Each of these invariants has the form

$$\Lambda_{IJ}(x) = \left[x - (m_I - m_J)^2\right]\left[x - (m_I + m_J)^2\right], \tag{J.9}$$

where x is a Mandelstam invariant. This function is also known as the Källén function. For the red tetrahedron you have

$$C_{123} = C_{134} = \left(s - m^2\right)^2, \qquad C_{124} = C_{234} = \left(t - m^2\right)^2. \tag{J.10}$$

Similarly, for the blue tetrahedron you have

$$C_{123} = C_{134} = \left(t - m^2\right)^2, \qquad C_{124} = C_{234} = \left(u - m^2\right)^2. \tag{J.11}$$

Finally, for the green tetrahedron you have

$$C_{123} = C_{134} = \left(u - m^2\right)^2, \qquad C_{124} = C_{234} = \left(s - m^2\right)^2. \tag{J.12}$$

Note that all 2-simplex invariants are strictly positive by definition.

J.1.3 3-Simplex Invariants

Given four dual spacetime positions, the 4-point Cayley-Menger determinant is given by:

$$C_{ijkl} = -\frac{1}{2} \det \begin{pmatrix} 0 & |d_{ij}|^2 & |d_{ik}|^2 & |d_{il}|^2 & 1 \\ |d_{ij}|^2 & 0 & |d_{jk}|^2 & |d_{jl}|^2 & 1 \\ |d_{ik}|^2 & |d_{jk}|^2 & 0 & |d_{kl}|^2 & 1 \\ |d_{il}|^2 & |d_{jl}|^2 & |d_{kl}|^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}. \tag{J.13}$$

A tetrahedron has only one possible 3-simplex invariant. Indeed,

$$C_{1234} = C_{1234} = C_{1234} = t\left(su - m^4\right). \tag{J.14}$$

J.2 Dual Conformal Invariants

Each of the 1-simplex invariants has the form $|d_i - d_j|^2$. This is not only Lorentz invariant, but dual Poincaré invariant. You can study dual conformal invariants by constructing a dual conformal ratio with a quartet of dual spacetime coordinates:

$$[i, j; k, l] \equiv \frac{|d_{ik}|^2 \, |d_{jl}|^2}{|d_{il}|^2 \, |d_{jk}|^2}. \tag{J.15}$$

In four-point scattering there is a unique quartet of dual spacetime coordinates. However, there are six inequivalent permutations of these coordinates, and thus, six possible values for the dual conformal ratio.

For the red tetrahedron, you have

$$\begin{aligned} &[1, 2; 3, 4] = \frac{st}{m^4}, \qquad [1, 3; 4, 2] = 1, \qquad [1, 4; 2, 3] = \frac{m^4}{st}, \\ &[1, 2; 4, 3] = \frac{m^4}{st}, \qquad [1, 3; 2, 4] = 1, \qquad [1, 4; 3, 2] = \frac{st}{m^4}. \end{aligned} \tag{J.16}$$

Similarly, for the blue tetrahedron, you have

$$\begin{aligned}
[1,2;3,4] &= \frac{tu}{m^4}, \qquad [1,3;4,2] = 1, \qquad [1,4;2,3] = \frac{m^4}{tu}, \\
[1,2;4,3] &= \frac{m^4}{tu}, \qquad [1,3;2,4] = 1, \qquad [1,4;3,2] = \frac{tu}{m^4}.
\end{aligned} \tag{J.17}$$

Finally, for the green tetrahedron, you have

$$\begin{aligned}
[1,2;3,4] &= \frac{su}{m^4}, \qquad [1,3;4,2] = 1, \qquad [1,4;2,3] = \frac{m^4}{su}, \\
[1,2;4,3] &= \frac{m^4}{su}, \qquad [1,3;2,4] = 1, \qquad [1,4;3,2] = \frac{su}{m^4}.
\end{aligned} \tag{J.18}$$

Note that

$$[1,2;3,4]\,[1,3;4,2]\,[1,4;2,3] = 1, \tag{J.19}$$

which is analogous to the constraint satisfied by the three Mandelstam invariants.

J.3 Center-of-Momentum Frame

In the center-of-momentum frame you write the energy-momentum vectors as

$$p_1 = \begin{pmatrix} E_1 & \mathbf{p}_1 \end{pmatrix}, \qquad p_2 = \begin{pmatrix} E_2 & -\mathbf{p}_1 \end{pmatrix}, \qquad p_3 = \begin{pmatrix} E_3 & \mathbf{p}_3 \end{pmatrix}, \qquad p_4 = \begin{pmatrix} E_4 & -\mathbf{p}_3 \end{pmatrix}. \tag{J.20}$$

One of the first things to notice is that

$$s = (E_1 + E_2)^2 = (E_3 + E_4)^2. \tag{J.21}$$

Thus, s can be interpreted as the total energy in the center-of-momentum frame. It also follows that s must be positive.

From the on-shell constraints, it follows that

$$\begin{aligned}
E_1 &= \sqrt{m^2 + |\mathbf{p}_1|^2}, \qquad E_2 = \sqrt{m^2 + |\mathbf{p}_1|^2}, \\
E_3 &= \sqrt{m^2 + |\mathbf{p}_3|^2}, \qquad E_4 = \sqrt{m^2 + |\mathbf{p}_3|^2}.
\end{aligned} \tag{J.22}$$

Using the relations

$$E_2 = \sqrt{s} - E_1, \qquad E_4 = \sqrt{s} - E_3, \tag{J.23}$$

you find that

$$|\mathbf{p}_1| = |\mathbf{p}_3| = \frac{\sqrt{s - 4m^2}}{2}. \tag{J.24}$$

Thus,

$$E_1 = E_2 = E_3 = E_4 = \frac{\sqrt{s}}{2}. \tag{J.25}$$

J.3.1 Speed and Rapidity

A quantum with mass m, spatial momentum $\mathbf{p}$, and energy E has a speed $|\mathbf{v}|$ given by

$$|\mathbf{v}| = \frac{\sqrt{E^2 - m^2}}{E^2} = \frac{|\mathbf{p}|}{\sqrt{m^2 + |\mathbf{p}|^2}} = \frac{|\mathbf{p}|}{E}. \tag{J.26}$$

If the quantum is physical, then $E \geq m$, or equivalently, $|\mathbf{p}| < E$. Both of these conditions lead to a bound on speed:

$$0 \leq |\mathbf{v}| < 1. \tag{J.27}$$

In contrast with energy, mass, and spatial momentum, speed and velocity are dimensionless. However, energy and spatial momentum are conserved, but velocity is not. The speed of each external quantum is

$$|\mathbf{v}_1| = |\mathbf{v}_2| = |\mathbf{v}_3| = |\mathbf{v}_4| = \sqrt{1 - \frac{4m^2}{s}}. \tag{J.28}$$

Given the speed of a quantum, you can find its rapidity η via

$$\eta = \operatorname{arctanh}|\mathbf{v}| = \frac{1}{2}\log\left(\frac{1+|\mathbf{v}|}{1-|\mathbf{v}|}\right). \tag{J.29}$$

Since the speed of a physical quantum is bounded, the rapidity of a physical quantum is bounded from below:

$$0 \le \eta < \infty. \tag{J.30}$$

The rapidity of each external quantum is

$$\eta_1 = \eta_2 = \eta_3 = \eta_4 = \frac{1}{2}\log\left[\frac{\sqrt{s}+\sqrt{s-4m^2}}{\sqrt{s}-\sqrt{s-4m^2}}\right]. \tag{J.31}$$

The sum of the incoming rapidites is equal to the sum of the outgoing rapidities:

$$\eta_1 + \eta_2 = \eta_3 + \eta_4 = \log\left[\frac{\sqrt{s}+\sqrt{s-4m^2}}{\sqrt{s}-\sqrt{s-4m^2}}\right]. \tag{J.32}$$

J.3.2 Physical Scattering Region

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Fixed-Speed Scattering

The speed of each external quantum can be written as a function of one dimensionless ratio,

$$\frac{s}{m^2}. \tag{J.33}$$

Fixed-speed scattering corresponds to the kinematic regime where we keep this ratio fixed:

$$\frac{s}{m^2} \text{ fixed.} \tag{J.34}$$

This regime is appropriate for either large s and large mass, or small s and small mass. Note that fixed-speed is equivalent to fixed-rapidity. By itself, this regime is not very helpful, but when combined with other limits it leads to important approximations.

Small-Speed Scattering

Small speeds occur when s is close to the threshold value $4m^2$.

Large-Speed Scattering

Large-speed scattering involves the regime

$$\frac{s}{m^2} \to \infty. \tag{J.35}$$

That is, $\sqrt{s}$ is very large compared to the mass.

Regge Scattering

Regge scattering is the regime of fixed-speed and large (unphysical) z_{13}. This corresponds to

$$\frac{t}{s} \to \infty, \qquad \frac{s}{m^2} \text{ fixed.} \tag{J.36}$$

As a corollary, you have

$$u = 4m^2 - s - t \quad \Longrightarrow \quad \frac{u}{s} \to -\infty. \tag{J.37}$$

In this regime all dual conformal invariants become trivial:

$$\begin{aligned} [1,2;3,4] &\to \infty, & [1,3;4,2] &= 1, & [1,4;2,3] &\to 0, \\ [1,2;4,3] &\to 0, & [1,3;2,4] &= 1, & [1,4;3,2] &\to \infty. \end{aligned} \tag{J.38}$$

$$\begin{aligned} [1,2;3,4] &\to -\infty, & [1,3;4,2] &= 1, & [1,4;2,3] &\to 0, \\ [1,2;4,3] &\to 0, & [1,3;2,4] &= 1, & [1,4;3,2] &\to -\infty. \end{aligned} \tag{J.39}$$

$$\begin{aligned} [1,2;3,4] &\to -\infty, & [1,3;4,2] &= 1, & [1,4;2,3] &\to 0, \\ [1,2;4,3] &\to 0, & [1,3;2,4] &= 1, & [1,4;3,2] &\to -\infty. \end{aligned} \tag{J.40}$$

Forward Scattering

Forward scattering is the regime of fixed-speed and small (physical) scattering angles (i.e. $z_{13} \to 1$). This can be stated as

$$\frac{t}{s} \to 0, \qquad \frac{s}{m^2} \text{ fixed.} \tag{J.41}$$

As a corollary, you have

$$u = 4m^2 - s - t \quad \Longrightarrow \quad \frac{u}{s} \text{ fixed.} \tag{J.42}$$

Unlike the case of Regge scattering, in the forward regime only some of the dual conformal invariants are trivial:

$$\begin{aligned} [1,2;3,4] &\to 0, & [1,3;4,2] &= 1, & [1,4;2,3] &\to \infty, \\ [1,2;4,3] &\to \infty, & [1,3;2,4] &= 1, & [1,4;3,2] &\to 0. \end{aligned} \tag{J.43}$$

$$\begin{aligned} [1,2;3,4] &\to 0, & [1,3;4,2] &= 1, & [1,4;2,3] &\to \infty, \\ [1,2;4,3] &\to \infty, & [1,3;2,4] &= 1, & [1,4;3,2] &\to 0. \end{aligned} \tag{J.44}$$

The rest of the dual conformal invariants are fixed and nontrivial:

$$\begin{aligned} [1,2;3,4] &= \frac{su}{m^4}, & [1,3;4,2] &= 1, & [1,4;2,3] &= \frac{m^4}{su}, \\ [1,2;4,3] &= \frac{m^4}{su}, & [1,3;2,4] &= 1, & [1,4;3,2] &= \frac{su}{m^4}. \end{aligned} \tag{J.45}$$

Backward Scattering

Backward scattering is the regime of fixed-speed and large (physical) scattering angles (i.e. $z_{13} \to -1$). This can be stated as

$$\frac{u}{s} \to 0, \qquad \frac{s}{m^2} \text{ fixed.} \tag{J.46}$$

As a corollary, you have

$$t = 4m^2 - s - u \quad \Longrightarrow \quad \frac{t}{s} \text{ fixed.} \tag{J.47}$$

Similarly to the forward scattering regime, some of the dual conformal invariants are trivial:

$$\begin{aligned} [1,2;3,4] &\to 0, & [1,3;4,2] &= 1, & [1,4;2,3] &\to \infty, \\ [1,2;4,3] &\to \infty, & [1,3;2,4] &= 1, & [1,4;3,2] &\to 0. \end{aligned} \tag{J.48}$$

$$\begin{aligned} [1,2;3,4] &\to 0, & [1,3;4,2] &= 1, & [1,4;2,3] &\to \infty, \\ [1,2;4,3] &\to \infty, & [1,3;2,4] &= 1, & [1,4;3,2] &\to 0. \end{aligned} \tag{J.49}$$

The rest of the dual conformal invariants are fixed and nontrivial:

$$\begin{aligned} [1,2;3,4] &= \frac{st}{m^4}, & [1,3;4,2] &= 1, & [1,4;2,3] &= \frac{m^4}{st}, \\ [1,2;4,3] &= \frac{m^4}{st}, & [1,3;2,4] &= 1, & [1,4;3,2] &= \frac{st}{m^4}. \end{aligned} \tag{J.50}$$

Fixed-Angle Scattering

Fixed-angle scattering is the regime of large-speed and (physical) fixed-angle. This can be stated as

$$\frac{t}{s} \text{ fixed}, \qquad \frac{s}{m^2} \to \infty. \tag{J.51}$$

As a corollary, you have

$$u = 4m^2 - s - t \quad \Longrightarrow \quad \frac{u}{s} \text{ fixed.} \tag{J.52}$$

In this regime all dual conformal invariants become trivial:

$$\begin{aligned} [1,2;3,4] &\to -\infty, & [1,3;4,2] &= 1, & [1,4;2,3] &\to 0, \\ [1,2;4,3] &\to 0, & [1,3;2,4] &= 1, & [1,4;3,2] &\to -\infty. \end{aligned} \tag{J.53}$$

$$
\begin{aligned}
&[1,2;3,4] \to \infty, \qquad [1,3;4,2] = 1, \qquad [1,4;2,3] \to 0, \\
&[1,2;4,3] \to 0, \qquad [1,3;2,4] = 1, \qquad [1,4;3,2] \to \infty.
\end{aligned}
\tag{J.54}
$$

$$
\begin{aligned}
&[1,2;3,4] \to -\infty, \qquad [1,3;4,2] = 1, \qquad [1,4;2,3] \to 0, \\
&[1,2;4,3] \to 0, \qquad [1,3;2,4] = 1, \qquad [1,4;3,2] \to -\infty.
\end{aligned}
\tag{J.55}
$$

From the point of view of dual conformal invariants, Regge scattering and fixed-angle scattering are very similar.

Chapter K

Maximal Inelastic Mixed Diversity

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Chapter L

Maximal Elastic Mixed Diversity

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Chapter M

Minimal Elastic Mixed Diversity

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