Non-perturbative Four-point Scattering from First-quantized Relativistic JWKB

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We apply the quantum mechanical (first-quantized) JWKB approximation to a two-body path integral describing the near-forward scattering of two relativistic, heavy, non-identical, scalar particles in D spacetime dimensions. In contrast to the loop expansion, in $D = 4$ this gives a strong-coupling expansion, and in $D = 3$ a non-perturbative weak-coupling expansion. When the interaction is mediated by massless quanta with spin N, we obtain explicit, relativistic results for the scattering amplitude when $N = 0$, 1 and 2. In $D = 4$ we find a Regge trajectory function that agrees with the usual quantum mechanical spectrum. We also find an exponentiated infrared divergence that becomes a pure phase factor when the Mandelstam invariants s and t are inside of the physical scattering region. In $D = 3$ we find a singularity whose position along the s axis is dependent on t. When the interaction is mediated by a heavy scalar with mass M, in $D = 3$ we find an all-order scattering amplitude where the multi-mass branch points $t = (L + 1)^2 M^2$ appear as Regge poles.

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1. Introduction

Scattering amplitudes are important quantities that bridge theoretical and experimental results. Many tools for the computation of amplitudes have been developed over the years. Exact perturbative amplitudes (i.e. tree-level, one-loop, two-loops, etc.) with generic kinematic data can be computed for many theories (see [1] for a review). There are also tools for the computation of non-perturbative amplitudes, but these typically involve adding an infinite set of perturbative amplitudes with restricted kinematic data (e.g. Regge limit, fixed-angle limit, etc.).

Some years ago, a program was started by Alday & Maldacena [2] to study fixed-angle scattering of gluons in planar $\mathcal{N} = 4$ super Yang-Mills theory at strong coupling. The computation of the scattering amplitude of gluons at strong coupling translates, via the AdS/CFT correspondence, to the computation of a semiclassical amplitude for bosonic strings propagating in AdS_5. The resulting four-point amplitude at strong coupling agrees with an ansatz of Bern, Dixon & Smirnov [3] for the four-point planar MHV scattering amplitude. Other results from this program include dual conformal symmetry, the Yangian and the relation to Wilson loops [4].

It is easy to wonder if analogous results can be obtained for less restrictive theories (i.e. other than planar, superconformal gauge theories with string duals). Halpern & Siegel [5] found that the (semiclassical) JWKB approximation of certain quantum mechanical systems (e.g. systems of particles) leads to a strong-coupling expansion. In nonrelativistic quantum mechanics, the semiclassical limit of the Feynman path integral gives

$$\hbar \to 0 : \qquad \int_{\mathbf{x}_I}^{\mathbf{x}_O} D\mathbf{q}(t) \, \exp\left(-\frac{i}{\hbar}S[\mathbf{q}]\right) \approx \sqrt{\det\left(\mathbf{V}\right)} \exp\left(-\frac{i}{\hbar}\Sigma\right) \tag{1.1}$$

where

$$\Sigma \equiv S[\mathbf{q}_*], \qquad \mathbf{V} \equiv -\frac{i}{\hbar}\frac{\partial^2 \Sigma}{\partial \mathbf{x}_I \partial \mathbf{x}_O} \tag{1.2}$$

Here $\mathbf{q}_*(t)$ is a solution of the classical equations of motion with boundary conditions $\mathbf{q}_*(t_I) = \mathbf{x}_I$ and $\mathbf{q}_*(t_O) = \mathbf{x}_O$. The right-hand side of (1.1) is sometimes known as the Van Vleck-Morette approximation to the path integral [6]. We use the relativistic sister of this approximation to study four-point near-forward scattering (i.e. small-angle) in a system with two non-identical, heavy, scalar particles. When the particles are coupled via the exchange of massless spin N quanta, in D spacetime dimensions the scattering amplitude

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takes the form

$$\mathcal{A} = \delta(P) \left[\frac{\mathcal{K}_N(s)}{\rho_N(s)}\right] \int dB_{12} \exp\left[-iB_{12} \cdot P_{12} + \beta_N \rho_N(s) \Gamma(\Delta - 1) \left(\frac{2}{B_{12}^2}\right)^{(\Delta-1)}\right] \qquad (1.3)$$

where $P_{12}^2 = -t$, $\Delta = (D-2)/2$, β_N is a coupling, and the integral over B_{12} is over a volume in $D-2$ dimensions. This amplitude has the familiar "eikonal" form, which was derived long ago by adding perturbative contributions from all ladder diagrams [7].

In §3 we derive (1.3) without any reference to (perturbative) Feynman diagrams in an attempt to make its non-perturbative nature explicit from the beginning. In §4 and §5 we evaluate this amplitude in $D = 3$ (i.e. $\Delta = 1/2$) and $D = 4$ (i.e. $\Delta = 1$), respectively. For both cases we consider interactions mediated by massless spin 0, 1 and 2 quanta, and we find amplitudes that exhibit bound state singularities. In four spacetime dimensions we find the familiar spectrum with an infinite number of bound states [8], but in three spacetime dimensions we find an amplitude with only one bound-state-like singularity, even for the exchange of non-propagating three-dimensional massless spin 2 quanta.

Then, in §6 we consider the exchange of massive, spin 0 quanta in $D = 3$ and $D = 5$ (such that $D - 2 = 1$ and 3, which are the cases when the long-distance propagator is exact). For the massive exchange in $D = 3$ we find an amplitude that exhibits an infinite number of singularities, but these correspond to the multi-mass branch points (and does not involve the branch cut continuum). Alas, in $D = 5$ it becomes necessary to perform a perturbative expansion in the coupling. We find the expected tree-level amplitude, along with a finite one-loop amplitude and divergent higher-loop amplitudes.

In the next section we begin by introducing the approximation that we use, the forward-JWKB approximation, and contrasting it with another commonly-used approximation, the Regge limit.

2. Forward-JWKB Approximation
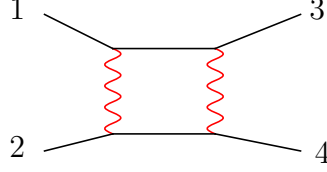
In order to be concrete, we consider a four-point scattering event with two non-identical, massive, scalar particles that exchange massless, scalar quanta. The s-channel process is elastic:

$$A(p_1) + B(p_2) \longrightarrow A(p_3) + B(p_4) \qquad (2.1)$$

In this channel, the center-of-momentum energy is given by $\sqrt{s}$ and the magnitude of the momentum transfer is $\sqrt{-t}$. More details about kinematics can be found in appendix A.

2.1. Regge Limit

Consider the one-loop contribution from the box diagram:

$$(2.2)$$

The scattering amplitude from this contribution can be written as an integral over Feynman variables:

$$\mathcal{A}_{\text{box}}(s,t) \sim g^4 \Gamma\left(\frac{8-D}{2}\right) \int_0^1 \int_0^1 \int_0^1 \int_0^1 \mathrm{d}f_{31}\mathrm{d}f_{42}\mathrm{d}f_{12}\mathrm{d}f_{34} \frac{\delta(1-f_{12}-f_{34}-f_{31}-f_{42})}{[B(s,t|f_{ij})]^{(8-D)/2}} \quad (2.3)$$

where

$$B(s,t|f_{ij}) \equiv m_1^2 f_{31}^2 + m_2^2 f_{42}^2 + (m_1^2 + m_2^2 - s)f_{31}f_{42} - tf_{12}f_{34} \quad (2.4)$$

This expression is valid for generic values of s and t. However, in the Regge limit

$$\frac{t}{m_1 m_2} \to \infty, \qquad \frac{s}{m_1 m_2} \text{ fixed}, \qquad \frac{m_1}{m_2} \text{ fixed}, \quad (2.5)$$

the integrals in (2.3) can be evaluated with asymptotic methods [9]. In $D = 4$ one finds

$$\mathcal{A}_{\text{box}}(s,t) \sim \frac{g^2}{t}\left[g^2\rho(s)\right]\log\left(-\frac{t}{2\mu^2}\right) \quad (2.6)$$

where
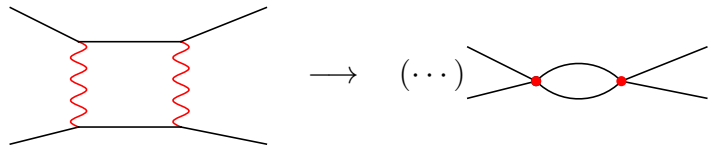
$$\rho(s) \equiv \int_0^1 \frac{\mathrm{d}f}{m_2^2 + (m_1^2 - m_2^2 - s)f + sf^2} = \frac{1}{\sqrt{-\Lambda_{12}(s)}}\log\left[\frac{m_1^2 + m_2^2 - s - \sqrt{\Lambda_{12}(s)}}{m_1^2 + m_2^2 - s + \sqrt{\Lambda_{12}(s)}}\right] \quad (2.7)$$

with $\Lambda_{12}(s)$ the Källén function,

$$\Lambda_{12}(s) \equiv [s - (m_1 - m_2)^2][s - (m_1 + m_2)^2] \quad (2.8)$$

Actually, this result for the one-loop scalar box in the Regge limit agrees with the exact result [10]. The logarithm term in (2.7) can be recognized as twice the sum of the rapidities of the incoming states in the center-of-momentum frame (see appendix A).

Diagrammatically, the Regge limit turns the square polygon in (2.2) into a digon made of matter lines, multiplied by a certain prefactor:

$$(2.9)$$

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Whereas the external lines remain unchanged, the internal lines on the left-hand side are matter propagators in D dimensions, while those on the right-hand side are matter propagators in $D-2$ dimensions. Similarly, the double box diagram becomes the concatenation of two matter digons:

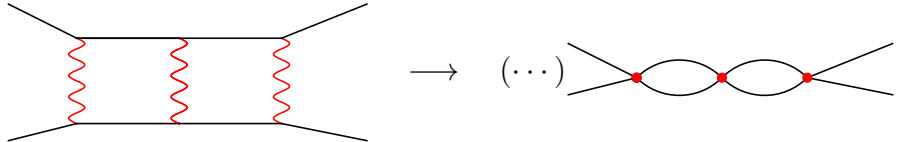In this way the sum over ladder diagrams becomes much simpler in the Regge limit [11]:

$$\mathcal{A}_{\text{ladders}}(s,t) \sim g^2 \Gamma[-R(s)] \left(-\frac{t}{2\mu^2}\right)^{R(s)}, \qquad R(s) = -1 + g^2 \rho(s) \tag{2.11}$$

Note that the Regge limit takes us outside of the physical scattering region of the s-channel process. One way to motivate the use of the Regge limit is to consider the (inelastic) t-channel process:

$$A(p_1) + \bar{A}(\bar{p}_2) \longrightarrow \bar{B}(\bar{p}_3) + B(p_4) \tag{2.12}$$

In this channel, the center-of-momentum energy is $\sqrt{t}$ and the magnitude of the momentum transfer is $\sqrt{-s}$ (after crossing from the s-channel). Thus, the Regge limit (2.5) in the s-channel corresponds to the high-energy and fixed-momentum transfer regime in the t-channel. Note that this regime involves light masses, or via $\lambda_j = \hbar/m_j$, large Compton wavelengths. That is, the Regge limit involves distances that are much smaller than the Compton wavelengths (microscopic regime). This is the same regime as taking $\hbar \to \infty$, so in a way the Regge limit takes us deep into the quantum realm. Indeed, the spectrum of bound states that follows from the poles of the Euler Gamma function in (2.11) involves the exact one-loop (quantum) result (2.7).

Systems where the interaction is mediated by massive quanta can be considered in a similar way, since the Regge limit renders the massive mediator effectively massless, and thus the result is the same. That is, the Regge limit is insensitive to the mediator's mass.

2.2. Forward-JWKB Regime
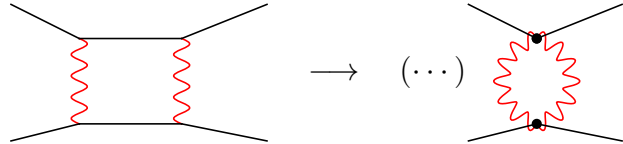
In contrast to the Regge limit (2.5), we *define* the forward-JWKB limit by

$$-\frac{t}{m_1 m_2} \to 0^+, \qquad \frac{s}{m_1 m_2} \text{ fixed}, \qquad \frac{m_1}{m_2} \text{ fixed}. \tag{2.13}$$
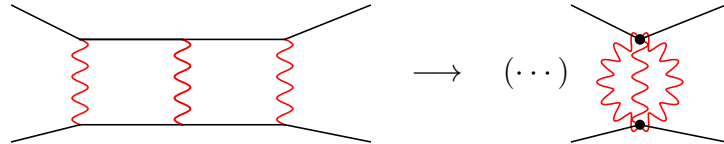
This regime is the same as restricting to small (but physical) scattering angles. Note that the Regge limit corresponds to unphysical scattering angles (i.e. $z_s \to \infty$). From (2.13)

it follows that $-t/s \to 0^+$, meaning that the center-of-momentum energy $\sqrt{s}$ is much larger than the magnitude of the momentum transfer $\sqrt{-t}$. In other words, this is a high-energy approximation (in the s-channel, the Regge limit is a low-energy approximation). Moreover, it also follows that $-t/m_1^2 \to 0^+$ and $-t/m_2^2 \to 0^+$, which mean that the external masses m_j are very large compared to $\sqrt{-t}$. Thus, this regime involves heavy masses, or via $\lambda_j = \hbar/m_j$, short Compton wavelengths. Hence, the forward-JWKB limit involves distances that are much larger than the Compton wavelengths (macroscopic regime), which coincides with the limit $\hbar \to 0$: the (semiclassical) JWKB limit.

Although we are not going to use perturbative second-quantized Feynman diagrams, it is somewhat illuminating to see what happens to the ladder series in the forward-JWKB limit. In contrast to (2.9), the one-loop box becomes a digon made with mediator lines,

$$
\longrightarrow \quad (\cdots) \tag{2.14}
$$

and similarly for the two-loop double box:

$$
\longrightarrow \quad (\cdots) \tag{2.15}
$$

We see that the structure of the forward-JWKB ladders is very different from the Regge ladders. Indeed, the forward-JWKB ladders do not involve any internal vertices and only involve internal mediator lines (which, like the internal matter lines in the Regge ladders, live in $D-2$ dimensions). At face value, the forward-JWKB approximation does not make the evaluation of the integrals in (2.3) any easier. This is why we do not explicitly prove (2.14) and (2.15) for generic theories. However, in §6 we obtain a scattering amplitude in $D = 3$ using the forward-JWKB approximation that agrees with the expectations from (2.14) and (2.15).

3. Path Integrals

We treat the matter quanta as particles (i.e. not fields). In the models that we study, each particle couples to a mediating field Φ_N with spin $N \geq 0$. We will consider massless mediating fields with $N = 0$, 1 and 2, and a massive mediating field with $N = 0$. The action functional for the A and B particles in (2.1) has the form

$$
S_{\text{part}}[q_A, q_B, \Phi_N] = S_{\text{free}}[q_A, q_B] + S_{\text{int}}[q_A, q_B, \Phi_N] \tag{3.1}
$$

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Here, S_{free} contains the (free) worldline-gauge-fixed kinetic terms,

$$S_{\text{free}}[q_A, q_B] = \frac{1}{2} \int \mathrm{d}\tau_A \left[-\dot{q}_A^2 + M_A^2 \right] + \frac{1}{2} \int \mathrm{d}\tau_B \left[-\dot{q}_B^2 + M_B^2 \right] \tag{3.2}$$

and S_{int} contains the worldline-gauge-fixed terms with the coupling to the field Φ_N,

$$\begin{aligned}
S_{\text{int}}[q_A, q_B, \Phi_N] &= \frac{g_N}{N!} \int \mathrm{d}\tau_A \, \dot{q}_A^{a_1} \cdots \dot{q}_A^{a_N} \left(\Phi_N[q_A(\tau_A)] \right)_{a_1 \cdots a_N} \\
&+ \frac{g_N}{N!} \int \mathrm{d}\tau_B \, \dot{q}_B^{b_1} \cdots \dot{q}_B^{b_N} \left(\Phi_N[q_B(\tau_B)] \right)_{b_1 \cdots b_N}
\end{aligned} \tag{3.3}$$

where g_N is a dimensionful coupling, and (M_A, M_B) are "internal" worldline masses which are a priori different from the external masses (m_1, m_2). The field Φ_N is totally symmetric in the N spacetime indices.

The mediating field Φ_N is made dynamical by adding a kinetic term to the particle action (3.1),

$$S[q_A, q_B, \Phi_N] = S_{\text{kin}}[\Phi_N] + S_{\text{part}}[q_A, q_B, \Phi_N] \tag{3.4}$$

We will mostly consider the free, massless case. That is, when $N = 0$ we have a free, massless, scalar field Φ_0; when $N = 1$ we have an abelian, vector field $(\Phi_1)_a$ (photon); and when $N = 2$ we have a linearized, symmetric, tensor field $(\Phi_2)_{ab}$ (massless Fierz-Pauli graviton). After an appropriate gauge-fixing (we use the analog of the Fermi-Feynman gauge), the kinetic term in S_{kin} takes the form

$$S_{\text{kin}}[\Phi_N] = \frac{1}{2} \int \int \mathrm{d}x \mathrm{d}y \left([\Phi_N(x)]_{a_1 \cdots a_N} [K_N(x|y)]^{a_1 \cdots a_N b_1 \cdots b_N} [\Phi_N(y)]_{b_1 \cdots b_N} \right) \tag{3.5}$$

where the free, massless, spin N gauge-fixed kinetic operator K_N is given by

$$[K_N(x|y)]^{a_1 \cdots a_N b_1 \cdots b_N} = (\kappa_N)^{a_1 \cdots a_N b_1 \cdots b_N} K_0(x|y) \tag{3.6}$$

with κ_N a constant tensor that is separately totally symmetric in the a_j and b_k indices, and K_0 is the free, massless, scalar kinetic operator,

$$K_0(x|y) = \delta(x - y) \left(-\frac{1}{2} \partial^2 \right) \tag{3.7}$$

3.1. Forward-JWKB Path Integrals

The system $\Phi_N + A + B$ is described by the un-integrated quantum path integral (i.e. a path integral that is dependent on the modulus of each worldline):

$$\mathcal{F}_T(3, 4|1, 2) = \int \widehat{\mathrm{D}} \Phi_N(x) \int_{x_1}^{x_3} \mathrm{D} q_A(\tau_A) \int_{x_2}^{x_4} \mathrm{D} q_B(\tau_B) \exp\left(-iS[q_A, q_B, \Phi_N] \right) \tag{3.8}$$

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The functional measure over Φ_N is normalized such that

$$\int \widehat{\mathrm{D}}\Phi_N(x)\exp\left(-iS_{\text{kin}}[\Phi_N]\right) = 1 \tag{3.9}$$

That is, if $g_N = 0$, then $\mathcal{F}_T$ reduces to the product of two un-integrated free, massive, scalar Green functions.

After writing S_{int} in (3.3) as a spacetime volume integral,

$$S_{\text{int}}[q_A, q_B, \Phi_N] = \int \mathrm{d}x \left[J_A(x) \cdot \Phi_N(x) + J_B(x) \cdot \Phi_N(x)\right] \tag{3.10}$$

with the aid of sources J_A and J_B given by

$$[J_A(x)]^{a_1 \cdots a_N} \equiv \frac{g_N}{N!} \int \mathrm{d}\tau_A \, \dot{q}_A^{a_1} \cdots \dot{q}_A^{a_N} \delta[x - q_A(\tau_A)] \tag{3.11}$$

$$[J_B(x)]^{b_1 \cdots b_N} \equiv \frac{g_N}{N!} \int \mathrm{d}\tau_B \, \dot{q}_B^{b_1} \cdots \dot{q}_B^{b_N} \delta[x - q_B(\tau_B)] \tag{3.12}$$

we perform the functional integral over Φ_N and find

$$\mathcal{F}_T(3, 4|1, 2) = \int\limits_{x_1}^{x_3} \mathrm{D}q_A(\tau_A) \int\limits_{x_2}^{x_4} \mathrm{D}q_B(\tau_B)\exp\left(-iS_{\text{eff}}[q_A, q_B]\right) \tag{3.13}$$

with the functional S_{eff} given by

$$S_{\text{eff}}[q_A, q_B] \equiv S_{\text{free}}[q_A, q_B] - \frac{1}{2}\int\int \mathrm{d}x\mathrm{d}y \left[J_A(x) + J_B(x)\right] \cdot G_N(x|y) \cdot \left[J_A(y) + J_B(y)\right] \tag{3.14}$$

Here $G_N = (K_N)^{-1}$ is the free, massless, spin N gauge-fixed Green function,

$$[G_N(x|y)]_{a_1 \cdots a_N b_1 \cdots b_N} = -i(\nu_N)_{a_1 \cdots a_N b_1 \cdots b_N}\Gamma\left(\Delta\right)\left[\frac{2}{(x - y)^2}\right]^{\Delta}, \qquad \Delta \equiv \frac{D - 2}{2} \tag{3.15}$$

with ν_N a constant tensor that satisfies

$$(\nu_N)_{a_1 \cdots a_N c_1 \cdots c_N}(\kappa_N)^{c_1 \cdots c_N b_1 \cdots b_N} = \frac{1}{N!}\left(\delta_{a_1}{}^{b_1} \cdots \delta_{a_N}{}^{b_N} + \text{ permutations}\right) \tag{3.16}$$

The outcome of integrating over Φ_N is the appearance of "potential" terms that describe the effective interactions between the A and B particles. Indeed, we write S_{eff} in (3.14) as

$$S_{\text{eff}}[q_A, q_B] = S_{\text{free}}[q_A, q_B] - S_1[q_A, q_B] - S_2[q_A, q_B] \tag{3.17}$$

with S_1 containing self-interaction terms,

$$\begin{aligned}
S_1 = {} & \frac{g_N^2}{2(N!)^2}\int\int \mathrm{d}\tau_A \mathrm{d}\sigma_A \left[\dot{q}_A(\tau_A) \cdots \dot{q}_A \cdot G_N[q_A(\tau_A)|q_A(\sigma_A)] \cdot \dot{q}_A(\sigma_A) \cdots \dot{q}_A\right] \\
& + \frac{g_N^2}{2(N!)^2}\int\int \mathrm{d}\tau_B \mathrm{d}\sigma_B \left[\dot{q}_B(\tau_B) \cdots \dot{q}_B \cdot G_N[q_B(\tau_B)|q_B(\sigma_B)] \cdot \dot{q}_B(\sigma_B) \cdots \dot{q}_B\right]
\end{aligned} \tag{3.18}$$

and S_2 containing a two-body interaction term,

$$S_2 = \frac{g_N^2}{(N!)^2} \int \int \mathrm{d}\tau_A \mathrm{d}\tau_B \left[\dot{q}_A(\tau_A) \cdots \dot{q}_A \cdot G_N[q_A(\tau_A)|q_B(\tau_B)] \cdot \dot{q}_B(\tau_B) \cdots \dot{q}_B \right] \tag{3.19}$$

One can think of the first term in S_1 as summing over contributions that involve linking a point $q_A(\tau_A)$ to a point $q_A(\sigma_A)$ with a propagator G_N. Both of these points are on the A worldline. Similarly, the second term in S_1 involves linking two points on the B worldline. On the other hand, S_2 can be understood as summing over contributions that involve linking a point $q_A(\tau_A)$ on the A worldline to a point $q_B(\tau_B)$ on the B worldline.

At this stage, the discussion is general and exact. In what follows **we ignore the self-interaction terms**. When this is done, $\mathcal{F}_T$ in (3.13) takes the form

$$\mathcal{F}_T(3,4|1,2) = \int_{x_1}^{x_3} \mathrm{D}q_A(\tau_A) \int_{x_2}^{x_4} \mathrm{D}q_B(\tau_B) \exp\left(-iS_0\right) \exp\left(iS_2\right) \tag{3.20}$$

We *could* expand the second exponential in (3.20) as a perturbative expansion in g_N,

$$\exp\left(iS_2\right) = 1 + iS_2 + \frac{1}{2!}(iS_2)^2 + \frac{1}{3!}(iS_2)^3 + \cdots \tag{3.21}$$

and evaluate each contribution. The first term in (3.21) is of order g_N^0 and not very interesting. The second term is proportional to S_2 and thus of order g_N^2. It involves a sum over all possible ways to connect a point on the A worldline to a point on the B worldline with a G_N propagator. This is a tree-level contribution. The third term involves

$$(iS_2)^2 \sim g_N^4 \int \int \mathrm{d}\tau_A \mathrm{d}\tau_B \int \int \mathrm{d}\sigma_A \mathrm{d}\sigma_B (\cdots) G_N[q_A(\tau_A)|q_B(\tau_B)] G_N[q_A(\sigma_A)|q_B(\sigma_B)] \tag{3.22}$$

This can be identified with a one-loop contribution, but since the double integration scans all possible orderings of the worldline coordinates, it accounts for both box-like and crossed box-like contributions. Similarly, the fourth term in (3.21) corresponds to the two-loops contribution, and contains the double box, the crossed double box and other non-planar contributions. Thus, we have learned that $\mathcal{F}_T$ in (3.20) contains all perturbative contributions arising from generalized ladder diagrams. Also, it follows that the contributions to the scattering amplitude from $\mathcal{F}_T$ are un-truncated and we need to perform some truncation.

We now incorporate the forward-JWKB approximation into our analysis. Since the forward-JWKB approximation is a combination of the forward approximation and the semi-classical approximation, in the forward-JWKB approximation the un-integrated quantum path integral $\mathcal{F}_T$ takes the form

$$\mathcal{F}_T(3,4|1,2) \longrightarrow \mathcal{G}_T(3,4|1,2) = \sqrt{-\det\left(V\right)} \exp\left(-i\Sigma\right) \tag{3.23}$$

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where the function Σ is the value of S_{eff} evaluated at the forward paths f_A and f_B,

$$\Sigma \equiv S_{\text{eff}}[f_A, f_B] \tag{3.24}$$

and the matrix V is given by

$$V \equiv \begin{pmatrix} V_{13} & V_{23} \\ V_{14} & V_{24} \end{pmatrix}, \qquad V_{jk} \equiv -i\frac{\partial \Sigma}{\partial x_j \partial x_k} \tag{3.25}$$

The forward paths describe particles that are moving along straight paths in spacetime with fixed spacetime speed:

$$\begin{aligned}
f_A(\tau_A) &= \frac{x_1 + x_3}{2} + \left(\frac{\tau_A}{T_A}\right)(x_3 - x_1), & -\frac{T_A}{2} &< \tau_A < \frac{T_A}{2} \\
f_B(\tau_B) &= \frac{x_2 + x_4}{2} + \left(\frac{\tau_B}{T_B}\right)(x_4 - x_2), & -\frac{T_B}{2} &< \tau_B < \frac{T_B}{2}
\end{aligned} \tag{3.26}$$

Here T_A and T_B are the moduli of the A and B worldlines, respectively. The form of $\mathcal{G}_T$ can be recognized as the (relativistic) two-body generalization of the Van Vleck-Morette kernel [6] specialized to the forward paths (i.e. we do not solve for the true classical paths). For this reason we refer to $\mathcal{G}_T$ as the un-integrated forward-JWKB kernel. In the rest of this section we evaluate $\mathcal{G}_T$ and relate it to the scattering amplitude.

3.1.1. Forward Van Vleck Function

At the forward paths (3.26), the free part of S_{eff} gives

$$\Sigma_{\text{free}} \equiv S_{\text{free}}[f_A, f_B] = -\frac{1}{2T_A}x_{31}^2 + \frac{M_A^2 T_A}{2} - \frac{1}{2T_B}x_{42}^2 + \frac{M_B^2 T_B}{2}, \qquad x_{jk} \equiv x_j - x_k \tag{3.27}$$

Recall that we are dropping the self-interactions and keeping the two-body interaction. Evaluating S_2 at the forward paths gives $\Sigma_2 \equiv S_2[f_A, f_B]$, i.e.

$$\Sigma_2 = \frac{g_N^2}{(N!)^2} \int \int \mathrm{d}\tau_A \mathrm{d}\tau_B \left[\dot{f}_A(\tau_A) \cdots \dot{f}_A \cdot G_N[f_A(\tau_A)|f_B(\tau_B)] \cdot \dot{f}_B(\tau_B) \cdots \dot{f}_B \right] \tag{3.28}$$

Note that the forward paths have constant slope:

$$\dot{f}_A = \frac{x_{31}}{T_A} \equiv k_{31}, \qquad \dot{f}_B = \frac{x_{42}}{T_B} \equiv k_{42} \tag{3.29}$$

Using (3.15) and (3.29), we write

$$\dot{f}_A(\tau_A) \cdots \dot{f}_A \cdot G_N[f_A(\tau_A)|f_B(\tau_B)] \cdot \dot{f}_B(\tau_B) \cdots \dot{f}_B = \mathcal{K}_N G_0[f_A(\tau_A)|f_B(\tau_B)] \tag{3.30}$$

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where G_0 is the massless, scalar Green function and the constant, scalar factor $\mathcal{K}_N$ consists of contractions of N copies of k_{31} and k_{42} with the constant tensor ν_N:

$$\mathcal{K}_N = \left(\frac{1}{T_A T_B}\right)^N (x_{31} \cdots x_{31}) \cdot \nu_N \cdot (x_{42} \cdots x_{42}) = (k_{31} \cdots k_{31}) \cdot \nu_N \cdot (k_{42} \cdots k_{42}) \qquad (3.31)$$

With the forward paths (3.26), we have

$$f_A(\tau_A) - f_B(\tau_B) = X_{12} + \left(\frac{\tau_A}{T_A}\right) x_{31} - \left(\frac{\tau_B}{T_B}\right) x_{42} \qquad (3.32)$$

where we have introduced

$$X_{12} \equiv \frac{x_1 - x_2 + x_3 - x_4}{2} \qquad (3.33)$$

Note that X_{12} is the vector average of the separation of the incoming particles (given by the vector $x_1 - x_2$) and the separation of the outgoing particles (given by the vector $x_3 - x_4$). That is, X_{12} is incoming/outgoing symmetric. We can now write Σ_2 as

$$\Sigma_2 = -i\frac{g_N^2}{(N!)^2} T_A T_B \mathcal{K}_N \Upsilon_2 \qquad (3.34)$$

with

$$\Upsilon_2 \equiv \Gamma(\Delta) \int_{-1/2}^{1/2} \mathrm{d}u_A \int_{-1/2}^{1/2} \mathrm{d}u_B \left[\frac{2}{(X_{12} + u_A x_{31} - u_B x_{42})^2}\right]^{\Delta}, \qquad \Delta \equiv \frac{D-2}{2} \qquad (3.35)$$

(We have changed variables from (τ_A, τ_B) to (u_A, u_B), which are dimensionless). It is convenient to introduce a Schwinger parameter T and write

$$\Upsilon_2 = \int_{-1/2}^{1/2} \mathrm{d}u_A \int_{-1/2}^{1/2} \mathrm{d}u_B \int_0^{\infty} \mathrm{d}T \left(\frac{1}{T}\right)^{(\Delta+1)} \exp\left[-\frac{1}{2T}(X_{12} + u_A x_{31} - u_B x_{42})^2\right] \qquad (3.36)$$

The path difference $f_A(u_A) - f_B(u_B) = X_{12} + u_A x_{31} - u_B x_{42}$ describes the separation between the particles during the scattering process. The conjugate momentum to this separation is the momentum transfer. In the forward-JWKB approximation, the momentum transfer is very small compared to the masses or the center-of-momentum energy. By Fourier-Heisenberg conjugacy, this means that the separation between the particles is always very large compared to the displacement of each particle. Thus, in the forward-JWKB approximation the (u_A, u_B) integral in (3.36) is dominated by the contribution from the critical point of the expression in the exponent:

$$\bar{u}_A = -\left[\frac{x_{42}^2(X_{12} \cdot x_{31}) - (X_{12} \cdot x_{42})(x_{31} \cdot x_{42})}{x_{31}^2 x_{42}^2 - (x_{31} \cdot x_{42})^2}\right] \qquad (3.37)$$

$$\bar{u}_B = +\left[\frac{x_{31}^2(X_{12} \cdot x_{42}) - (X_{12} \cdot x_{31})(x_{31} \cdot x_{42})}{x_{31}^2 x_{42}^2 - (x_{31} \cdot x_{42})^2}\right] \qquad (3.38)$$

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At this critical point, we find

$$B_{12} \equiv f_A(\bar{u}_A) - f_B(\bar{u}_B) = X_{12} + \bar{u}_A x_{31} - \bar{u}_B x_{42} \tag{3.39}$$

which satisfies $B_{12} \cdot x_{31} = 0$ and $B_{12} \cdot x_{42} = 0$. That is, B_{12} is the projection of X_{12} to the subspace that is orthogonal to x_{31} and x_{42}. In the forward-JWKB approximation we find

$$\Upsilon_2 \approx \frac{2\pi}{\sqrt{x_{31}^2 x_{42}^2 - (x_{31} \cdot x_{42})^2}} \int_0^\infty dT \left(\frac{1}{T}\right)^\Delta \exp\left[-\frac{1}{2T}B_{12}^2\right]$$

$$= \frac{2\pi}{\sqrt{x_{31}^2 x_{42}^2 - (x_{31} \cdot x_{42})^2}} \Gamma(\Delta - 1) \left(\frac{2}{B_{12}^2}\right)^{(\Delta - 1)} \tag{3.40}$$

and hence Σ_2 gives

$$\Sigma_2 \approx -i\left(\frac{2\pi g_N^2}{(N!)^2}\right) \left[\frac{T_A T_B \mathcal{K}_N}{\sqrt{x_{31}^2 x_{42}^2 - (x_{31} \cdot x_{42})^2}}\right] \Gamma(\Delta - 1) \left(\frac{2}{B_{12}^2}\right)^{(\Delta - 1)} \tag{3.41}$$

Note that this expression is divergent when $\Delta = 1$ (i.e. when $D = 4$). In order to keep things compact, we introduce the coupling

$$\beta_N \equiv \frac{2\pi g_N^2}{(N!)^2} \tag{3.42}$$

and the function

$$\rho_N \equiv \frac{T_A T_B \mathcal{K}_N}{\sqrt{x_{31}^2 x_{42}^2 - (x_{31} \cdot x_{42})^2}} = \frac{\mathcal{K}_N}{\sqrt{k_{31}^2 k_{42}^2 - (k_{31} \cdot k_{42})^2}} \tag{3.43}$$

When ρ_N is written in terms of k_{31} and k_{42}, there are no explicit factors of T_A and T_B.

3.1.2. Forward Van Vleck Matrix

Since the Van Vleck function Σ has the form $\Sigma_{\text{free}} - \Sigma_2$, we write the Van Vleck matrix V also in the form $V_{\text{free}} - V_2$ with

$$V_{\text{free}} = \begin{pmatrix} u_{13} & u_{23} \\ u_{14} & u_{24} \end{pmatrix}, \quad u_{jk} \equiv -i\frac{\partial \Sigma_{\text{free}}}{\partial x_j \partial x_k}, \quad V_2 = \begin{pmatrix} v_{13} & v_{23} \\ v_{14} & v_{24} \end{pmatrix}, \quad v_{jk} \equiv -i\frac{\partial \Sigma_2}{\partial x_j \partial x_k} \tag{3.44}$$

The determinant of V can be written as

$$\det(V) = \det(V_{\text{free}} - V_2) = \det(I - W)\det(V_{\text{free}}), \qquad W \equiv V_2 \cdot (V_{\text{free}})^{-1} \tag{3.45}$$

Hence, the square root of the determinant is

$$\sqrt{-\det(V)} = \sqrt{-\det(V_{\text{free}})} \exp\left[-\sum_{n=1}^\infty \frac{1}{2n} \operatorname{tr}(W^n)\right] \tag{3.46}$$

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Using Σ_{free} from (3.27), it is easy to show that

$$(u_{13})_{ab} = \left(-\frac{i}{T_A}\right)\eta_{ab}, \qquad (u_{23})_{ab} = 0$$

$$(u_{14})_{ab} = 0, \qquad (u_{24})_{ab} = \left(-\frac{i}{T_B}\right)\eta_{ab} \tag{3.47}$$

and thus

$$\sqrt{-\det(V_{\text{free}})} = \left(-\frac{i}{T_A}\right)^{D/2}\left(-\frac{i}{T_B}\right)^{D/2} \tag{3.48}$$

We can think of the terms with traces of W in (3.46) as corrections to Σ, since they appear inside an exponential too. If we compute them, we find that they involve powers of B_{12}^2 that are more negative than the power in Σ_2. In the forward-JWKB approximation B_{12}^2 is large, so we keep the dominant contribution from Σ_2 and drop all terms with traces of W. Hence,

$$\sqrt{-\det(V)} \approx \sqrt{-\det(V_{\text{free}})} \tag{3.49}$$

This step might seem drastic, but as we shall see, the end result will justify our means.

3.2. Integrated Forward-JWKB Scattering Kernel

From $\mathcal{F}_T$ we obtain the un-integrated quantum scattering kernel $\mathcal{S}_T$ via

$$\mathcal{S}_T(3,4|1,2) = \int\int\int\int dx_1 dx_2 dx_3 dx_4\,\overline{\mathcal{W}}_O(3,4)\mathcal{W}_I(1,2)\mathcal{F}_T(3,4|1,2) \tag{3.50}$$

The factors $\mathcal{W}_I$ and $\overline{\mathcal{W}}_O$ account for the asymptotic free massive external states:

$$\mathcal{W}_I(1,2) = \exp\left[\frac{iT_A}{4}\left(p_1^2 + m_1^2\right) + \frac{iT_B}{4}\left(p_2^2 + m_2^2\right) + ix_1\cdot p_1 + ix_2\cdot p_2\right] \tag{3.51}$$

$$\overline{\mathcal{W}}_O(3,4) = \exp\left[\frac{iT_A}{4}\left(p_3^2 + m_3^2\right) + \frac{iT_B}{4}\left(p_4^2 + m_4^2\right) - ix_3\cdot p_3 - ix_4\cdot p_4\right] \tag{3.52}$$

Note that a priori we have $m_1 \neq m_3 \neq M_A$ and $m_2 \neq m_4 \neq M_B$. That is, the external states are off-shell and the external masses m_j are not related to the internal worldline masses M_A and M_B.

In the forward-JWKB approximation, we use $\mathcal{G}_T$ instead of $\mathcal{F}_T$ in (3.50). In order to perform the integration in (3.50), we first make a change of position variables and also

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introduce the corresponding conjugate momenta,

$$X \equiv \frac{x_1 + x_2 + x_3 + x_4}{4} \qquad P \equiv p_3 + p_4 - p_1 - p_2 \tag{3.53}$$

$$X_{12} \equiv \frac{x_1 - x_2 + x_3 - x_4}{2} \qquad P_{12} \equiv \frac{p_3 - p_1 + p_2 - p_4}{2} \tag{3.54}$$

$$x_{31} \equiv x_3 - x_1 \qquad p_{31} \equiv \frac{p_1 + p_3}{2} \tag{3.55}$$

$$x_{42} \equiv x_4 - x_2 \qquad p_{42} \equiv \frac{p_2 + p_4}{2} \tag{3.56}$$

such that

$$x_1 \cdot p_1 + x_2 \cdot p_2 - x_3 \cdot p_3 - x_4 \cdot p_4 = -X \cdot P - X_{12} \cdot P_{12} - x_{31} \cdot p_{31} - x_{42} \cdot p_{42} \tag{3.57}$$

The Jacobian from this change of variables is a constant, which we ignore

$$\mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}x_4 \sim \mathrm{d}X \mathrm{d}X_{12} \mathrm{d}x_{31} \mathrm{d}x_{42} \tag{3.58}$$

In terms of these variables we have

$$\begin{aligned}
\overline{\mathcal{W}}_O(3,4)\mathcal{W}_I(1,2) = {}& \exp\left[-iX \cdot P - iX_{12} \cdot P_{12} - ix_{31} \cdot p_{31} - ix_{42} \cdot p_{42}\right] \\
& \times \exp\left[\frac{iT_A}{2}p_{31}^2 + \frac{iT_A}{32}(2P_{12} + P)^2 + \frac{iT_A}{4}(m_1^2 + m_3^2)\right] \\
& \times \exp\left[\frac{iT_B}{2}p_{42}^2 + \frac{iT_B}{32}(2P_{12} - P)^2 + \frac{iT_B}{4}(m_2^2 + m_4^2)\right]
\end{aligned} \tag{3.59}$$

Since Σ and V have no dependence on X, the un-integrated forward-JWKB kernel $\mathcal{G}_T$ does not depend on X. Thus, the integral over X yields a Dirac delta:

$$\int \mathrm{d}X \exp\left(-iX \cdot P\right) = \delta(P) \tag{3.60}$$

This Dirac delta imposes the constraint $P = 0$, which leads to

$$p_1 + p_2 = p_3 + p_4 \tag{3.61}$$

That is, the total external momentum is conserved, as expected from translation invariance. After enforcing $P = 0$, we find

$$P_{12} = p_3 - p_1 = p_2 - p_4 \quad \implies \quad P_{12}^2 = -t \tag{3.62}$$

Next, we tackle the integration over the x_{ij}. The exact integration is nontrivial because of the way that ρ_N in Σ_2 depends on these variables. We make another change of variables:

$$x_{31} = T_A k_{31}, \qquad x_{42} = T_B k_{42} \quad \implies \quad \mathrm{d}x_{31}\mathrm{d}x_{42} = (T_A T_B)^D \mathrm{d}k_{31}\mathrm{d}k_{42} \tag{3.63}$$

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In terms of the k_{ij} we have

$$\Sigma_{\text{free}} = -\frac{T_A}{2}\left(k_{31}^2 - M_A^2\right) - \frac{T_B}{2}\left(k_{42}^2 - M_B^2\right) \tag{3.64}$$

and thus, after integrating over X and enforcing $P = 0$, we find

$$\overline{\mathcal{W}}_O \mathcal{W}_I \exp\left(-i\Sigma_{\text{free}}\right) = \exp\left[\frac{iT_A}{2}(k_{31} - p_{31})^2 + \frac{iT_B}{2}(k_{42} - p_{42})^2 - iX_{12} \cdot P_{12}\right]$$

$$\times \exp\left[-\frac{iT_A}{4}\left(\frac{t}{2} - m_1^2 - m_3^2 + 2M_A^2\right)\right] \tag{3.65}$$

$$\times \exp\left[-\frac{iT_B}{4}\left(\frac{t}{2} - m_2^2 - m_4^2 + 2M_B^2\right)\right]$$

This expression is Gaussian in the k_{ij}. The full integrand has the form "Gaussian $\times$ function". We resort to stationary methods to approximate the integral over the k_{ij}. The stationary point is

$$\bar{k}_{31} = p_{31}, \qquad \bar{k}_{42} = p_{42} \tag{3.66}$$

At this stationary point, ρ_N becomes a function of the (off-shell) external momenta p_{ij},

$$\rho_N = \frac{\mathcal{K}_N}{\sqrt{p_{31}^2 p_{42}^2 - (p_{31} \cdot p_{42})^2}}, \qquad \mathcal{K}_N = (p_{31} \cdots p_{31}) \cdot \nu_N \cdot (p_{42} \cdots p_{42}) \tag{3.67}$$

So far, the un-integrated forward-JWKB scattering kernel looks like

$$\mathcal{S}_T(3,4|1,2) \approx \delta(P) \int dX_{12} \exp\left[-iX_{12} \cdot P_{12} + \beta_N \rho_N \Gamma(\Delta - 1)\left(\frac{2}{B_{12}^2}\right)^{(\Delta-1)}\right]$$

$$\times \exp\left[-\frac{iT_A}{4}\left(\frac{t}{2} - m_1^2 - m_3^2 + 2M_A^2\right)\right] \tag{3.68}$$

$$\times \exp\left[-\frac{iT_B}{4}\left(\frac{t}{2} - m_2^2 - m_4^2 + 2M_B^2\right)\right]$$

We defined B_{12} as the part of X_{12} that is orthogonal to any linear combination of the x_{ij}. But the effect of integration over the x_{ij} was to replace (x_{31}, x_{42}) with $(T_A p_{31}, T_B p_{42})$. So now we have the decomposition

$$X_{12} = B_{12} + T_A b_{31} p_{31} + T_B b_{42} p_{42}, \qquad B_{12} \cdot p_{31} = 0, \qquad B_{12} \cdot p_{42} = 0 \tag{3.69}$$

The X_{12} volume element becomes

$$dX_{12} = T_A T_B \sqrt{p_{31}^2 p_{42}^2 - (p_{31} \cdot p_{42})^2} \, dB_{12} db_{31} db_{42} \tag{3.70}$$

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which we can write in terms of ρ_N,

$$\mathrm{d}X_{12} = T_A T_B \left(\frac{\mathcal{K}_N}{\rho_N}\right) \mathrm{d}B_{12}\mathrm{d}b_{31}\mathrm{d}b_{42} \tag{3.71}$$

Note that

$$X_{12} \cdot P_{12} = B_{12} \cdot P_{12} + T_A b_{31}(p_{31} \cdot P_{12}) + T_B b_{42}(p_{42} \cdot P_{12}) \tag{3.72}$$

Since Σ_2 has no dependence on the b_{ij}, integration yields two one-dimensional Dirac deltas:

$$\int \mathrm{d}b_{31} \, \exp\left[-iT_A b_{31}(p_{31} \cdot P_{12})\right] = \frac{1}{T_A}\delta(p_{31} \cdot P_{12}) \tag{3.73}$$

$$\int \mathrm{d}b_{42} \, \exp\left[-iT_B b_{42}(p_{42} \cdot P_{12})\right] = \frac{1}{T_B}\delta(p_{42} \cdot P_{12}) \tag{3.74}$$

We will examine later the constraints that these two Dirac deltas impose. Now the only part of the amplitude that depends on (T_A, T_B) are the second and third lines of (3.68). We must integrate over the moduli in order to obtain the integrated scattering kernel:

$$\widehat{\mathcal{A}}(3, 4|1, 2) \equiv \int\limits_0^\infty \mathrm{d}T_A \int\limits_0^\infty \mathrm{d}T_B \, \mathcal{S}_T(3, 4|1, 2) \tag{3.75}$$

Performing the integration over (T_A, T_B) yields

$$\int\limits_0^\infty \mathrm{d}T_A \, \exp\left[-\frac{iT_A}{4}\left(\frac{t}{2} - m_1^2 - m_3^2 + 2M_A^2\right)\right] = \frac{8i}{t - 2m_1^2 - 2m_3^2 + 4M_A^2} \tag{3.76}$$

$$\int\limits_0^\infty \mathrm{d}T_B \, \exp\left[-\frac{iT_B}{4}\left(\frac{t}{2} - m_2^2 - m_4^2 + 2M_B^2\right)\right] = \frac{8i}{t - 2m_2^2 - 2m_4^2 + 4M_B^2} \tag{3.77}$$

The integral over B_{12} remains:

$$\widehat{\mathcal{A}} = \mathcal{N}\delta(P)\left(\frac{\mathcal{K}_N}{\rho_N}\right)\int \mathrm{d}B_{12} \, \exp\left[-iB_{12} \cdot P_{12} + \beta_N \rho_N \Gamma(\Delta - 1)\left(\frac{2}{B_{12}^2}\right)^{(\Delta-1)}\right] \tag{3.78}$$

where we have collected some terms into a prefactor:

$$\mathcal{N} \equiv \frac{(8i)^2 \delta(p_{31} \cdot P_{12})\delta(p_{42} \cdot P_{12})}{(t - 2m_1^2 - 2m_3^2 + 4M_A^2)(t - 2m_2^2 - 2m_4^2 + 4M_B^2)} \tag{3.79}$$

Before we put the external momenta on-shell, we need to truncate from $\widehat{\mathcal{A}}$ the part that is divergent on-shell.

3.2.1. Truncation of External On-shell States

In quantum field theory, truncation typically involves multiplying the scattering amplitude by a product of inverse propagators $(p_j^2 + m_j^2)$, and taking the limit $p_j^2 \to -m_j^2$. This is done in order to remove the part that is divergent on-shell from the scattering amplitude. Since we have four external states, we need to remove from $\widehat{\mathcal{A}}$ four factors.

From the relations

$$p_{31} \cdot P_{12} = \frac{p_3^2 - p_1^2}{2}, \qquad p_{42} \cdot P_{12} = \frac{p_2^2 - p_4^2}{2} \tag{3.80}$$

we see that the two Dirac deltas in $\mathcal{N}$ enforce the elasticity constraints

$$p_1^2 = p_3^2, \qquad p_2^2 = p_4^2 \tag{3.81}$$

Furthermore, on-shell we have

$$p_{31}^2 = \frac{t - 2m_1^2 - 2m_3^2}{4}, \qquad p_{42}^2 = \frac{t - 2m_2^2 - 2m_4^2}{4} \tag{3.82}$$

so then the denominators in $\mathcal{N}$ become

$$\frac{8i}{t - 2m_1^2 - 2m_3^2 + 4M_A^2} = \frac{2i}{p_{31}^2 + M_A^2}, \qquad \frac{8i}{t - 2m_2^2 - 2m_4^2 + 4M_B^2} = \frac{2i}{p_{42}^2 + M_B^2} \tag{3.83}$$

These two factors have the form of (free) Feynman propagators for particles with momenta (p_{31}, p_{42}) and masses (M_A, M_B); they diverge when $p_{31}^2 \to -M_A^2$ and $p_{42}^2 \to -M_B^2$. We can think of these limits as ways to relate the external on-shell momenta to the "internal" masses (M_A, M_B):

$$M_A^2 = \frac{2m_1^2 + 2m_3^2 - t}{4}, \qquad M_B^2 = \frac{2m_2^2 + 2m_4^2 - t}{4} \tag{3.84}$$

Note that these relations satisfy

$$2M_A^2 + 2M_B^2 = m_1^2 + m_2^2 + m_3^2 + m_4^2 - t = s + u \tag{3.85}$$

After we enforce the elasticity constraints, we find

$$M_A^2 = m_1^2 \left(1 - \frac{t}{4m_1^2}\right), \qquad M_B^2 = m_2^2 \left(1 - \frac{t}{4m_2^2}\right) \tag{3.86}$$

and thus, in the forward-JWKB approximation (2.13) we have $M_A \approx m_1$ and $M_B \approx m_2$. The upshot of this discussion is that we can think of the factors in $\mathcal{N}$ as restricting the

external momenta to be on-shell. Truncation is achieved by simply dropping $\mathcal{N}$ from (3.78). Thus, the *truncated, on-shell, forward-JWKB scattering amplitude* $\mathcal{A}$ is given by

$$\mathcal{A} = \delta(P) \left(\frac{\mathcal{K}_N}{\rho_N} \right) \int \mathrm{d}B_{12} \, \exp \left[-iB_{12} \cdot P_{12} + \beta_N \rho_N \Gamma(\Delta - 1) \left(\frac{2}{B_{12}^2} \right)^{(\Delta - 1)} \right] \tag{3.87}$$

Recall that B_{12} is a vector in D dimensions subjected to two orthogonality constraints. Thus, the B_{12} integral is over a $(D-2)$-dimensional volume. In §4 and §5 we evaluate this integral in $D = 3$ and $D = 4$.

4. Forward-JWKB Amplitudes in $D = 3$

We now consider scattering in three spacetime dimensions. The coupling g_N has units

$$D = 3: \qquad [g_N] = \left(\frac{3 - 2N}{2} \right) [\mathrm{mass}] \tag{4.1}$$

and thus the coupling β_N introduced in (3.42) has units

$$D = 3: \qquad [\beta_N] = 2[g_N] = (3 - 2N) \, [\mathrm{mass}] \tag{4.2}$$

For future reference we record the units of $\mathcal{K}_N$ and ρ_N (in any number of dimensions):

$$[\mathcal{K}_N] = 2N[\mathrm{mass}], \qquad [\rho_N] = 2(N - 1)[\mathrm{mass}] \tag{4.3}$$

When $D = 3$ we have $\Delta = 1/2$, and thus (3.87) takes the form

$$\mathcal{A} = \beta_N \mathcal{K}_N \delta(P) \left(\frac{\sqrt{2\pi}}{\sqrt{2\pi} \beta_N \rho_N} \right) \int \mathrm{d}B_{12} \, \exp \left[-iB_{12} \cdot P_{12} - \sqrt{2\pi} \beta_N \rho_N |B_{12}| \right] \tag{4.4}$$

We recognize this as the Fourier transform of a massive, scalar propagator in one spacetime dimension along a spacelike coordinate with "mass" given by $\sqrt{2\pi} \beta_N \rho_N$. The Fourier transform is just the familiar massive Feynman propagator:

$$\mathcal{A}(s, t) = \delta(P) \left[\frac{2\beta_N \mathcal{K}_N(s)}{2\pi \beta_N^2 \rho_N^2(s) - t} \right] = \delta(P) \left[-\frac{2\beta_N \mathcal{K}_N(s)}{t} \right] \left[1 - \frac{2\pi \beta_N^2 \rho_N^2(s)}{t} \right]^{-1} \tag{4.5}$$

In the second step we have extracted the expected tree-level massless singularity. Besides this singularity, the amplitude has a simple pole at $t = 2\pi \beta_N^2 \rho_N^2$. We now specialize to particular values of N in order to study this singularity further.

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4.1. Exchange of Massless Scalar

With $N = 0$, the matter particles exchange a massless scalar. The coupling β_0 has units

$$[\beta_0] = 3[\text{mass}] \tag{4.6}$$

For the tensor in the kinetic operator, we have $\kappa_0 = 1$, which leads to $\nu_0 = 1$, and thus $\mathcal{K}_0 = 1$. Hence,

$$\rho_0 = \frac{1}{\sqrt{p_{31}^2 p_{42}^2 - (p_{31} \cdot p_{42})^2}} \tag{4.7}$$

Using the on-shell identities

$$p_{31}^2 = -M_A^2, \qquad p_{42}^2 = -M_B^2, \qquad p_{31} \cdot p_{42} = \frac{M_A^2 + M_B^2 - s}{2} \tag{4.8}$$

leads to

$$\rho_0(s) = \frac{2}{\sqrt{-\Lambda_{AB}(s)}}, \qquad \Lambda_{AB}(s) \equiv [s - (M_A - M_B)^2][s - (M_A + M_B)^2] \tag{4.9}$$

The singularity $t_* = 2\pi\beta_0^2 \rho_0^2(s_*)$ leads to

$$s_* = M_A^2 + M_B^2 + 2M_A M_B \left(1 - \frac{2\pi\beta_0^2}{M_A^2 M_B^2 t_*}\right)^{1/2} \tag{4.10}$$

Using $u_* = 2m_1^2 + 2m_2^2 - t_* - s_* = 2M_A^2 + 2M_B^2 - s_*$ leads to

$$u_* = M_A^2 + M_B^2 - 2M_A M_B \left(1 - \frac{2\pi\beta_0^2}{M_A^2 M_B^2 t_*}\right)^{1/2} \tag{4.11}$$

The product $s_* u_*$ gives

$$s_* u_* = (M_A - M_B)^2 (M_A + M_B)^2 + \frac{8\pi\beta_0^2}{t_*} = (m_1 - m_2)^2 (m_1 + m_2)^2 + \frac{8\pi\beta_0^2}{t_*} \tag{4.12}$$

Thus, if $t_* \leq 0$ then s_* and u_* are inside of the physical scattering region. However, continuation to $t_* > 0$ allows a window with real values of s_* and u_* as long as

$$t_* > \frac{2\pi\beta_0^2}{M_A^2 M_B^2} \tag{4.13}$$

This lies outside of the physical scattering region and suggests a bound state.

In the forward-JWKB approximation (2.13), we have $M_A \approx m_1$ and $M_B \approx m_2$. When $t_* \leq 0$, we expect

$$\frac{s_*}{m_1 m_2} \text{ fixed}, \qquad \frac{u_*}{m_1 m_2} \text{ fixed.} \tag{4.14}$$

In order for this to hold, in $D = 3$ we must supplement (2.13) with

$$\frac{\beta_0^2}{m_1^2 m_2^2 t_*} \text{ fixed} \quad \Longrightarrow \quad \frac{\beta_0^2}{m_1^3 m_2^3} \to 0^+ \tag{4.15}$$

which suggest *weak-coupling* in the $D = 3$ version of the forward-JWKB approximation.

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4.2. Exchange of Massless Vector

With $N = 1$, the particles exchange a massless vector. The coupling β_1 now has units

$$[\beta_1] = [\text{mass}] \tag{4.16}$$

The tensor in the gauge-fixed kinetic operator is $(\kappa_1)^{ab} = \eta^{ab}$, which leads to $(\nu_1)_{ab} = \eta_{ab}$ and thus

$$\mathcal{K}_1 = p_{31} \cdot p_{42} = \frac{M_A^2 + M_B^2 - s}{2} \tag{4.17}$$

Hence,

$$\rho_1(s) = Z_A Z_B \left[\frac{M_A^2 + M_B^2 - s}{\sqrt{-\Lambda_{AB}(s)}} \right], \qquad \Lambda_{AB}(s) \equiv [s - (M_A - M_B)^2][s - (M_A + M_B)^2] \tag{4.18}$$

where we have included dimensionless charges Z_A and Z_B for each particle. The singularity $t_* = 2\pi \beta_1^2 \rho_1^2(s_*)$ now leads to

$$s_* = M_A^2 + M_B^2 + 2 M_A M_B \left(1 + \frac{2\pi Z_A^2 Z_B^2 \beta_1^2}{t_*} \right)^{-1/2} \tag{4.19}$$

and thus

$$u_* = M_A^2 + M_B^2 - 2 M_A M_B \left(1 + \frac{2\pi Z_A^2 Z_B^2 \beta_1^2}{t_*} \right)^{-1/2} \tag{4.20}$$

The product $s_* u_*$ now gives

$$s_* u_* = (M_A - M_B)^2 (M_A + M_B)^2 + 4 M_A^2 M_B^2 \left(\frac{2\pi Z_A^2 Z_B^2 \beta_1^2}{2\pi Z_A^2 Z_B^2 \beta_1^2 + t_*} \right)$$
$$= (m_1 - m_2)^2 (m_1 + m_2)^2 + 4 m_1^2 m_2^2 \left(1 - \frac{t_*}{4m_1^2} \right) \left(1 - \frac{t_*}{4m_2^2} \right) \left(\frac{2\pi Z_A^2 Z_B^2 \beta_1^2}{2\pi Z_A^2 Z_B^2 \beta_1^2 + t_*} \right) \tag{4.21}$$

Again, inside of the physical scattering region we have $t_* \leq 0$ and hence s_* and u_* are also inside of the physical scattering region. However, in order for s_* and u_* to be real and finite, we must require

$$-t_* > 2\pi Z_A^2 Z_B^2 \beta_1^2 \tag{4.22}$$

If we analytically continue to $t_* > 0$, we find s_* and u_* real for any value of t_*.

With the massless vector exchange, in $D = 3$ we must supplement (2.13) with

$$\frac{\beta_1^2}{t_*} \text{ fixed} \quad \Longrightarrow \quad \frac{\beta_1^2}{m_1 m_2} \to 0^+ \tag{4.23}$$

which also suggest *weak-coupling*.

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4.3. Exchange of Massless Symmetric Tensor

A massless symmetric (traceless) tensor has no propagating degrees of freedom in $D = 3$. Setting $N = 2$ corresponds to a sort of analytic continuation. Whatever the result gives, it should not have an interpretation in terms of propagating gravitons. The coupling β_2 has units

$$[\beta_2] = -[\text{mass}] \tag{4.24}$$

When $N = 2$, the tensor in the gauge-fixed kinetic operator is

$$(\kappa_2)^{a_1 b_1 a_2 b_2} = \frac{1}{2} \left(\eta^{a_1 a_2} \eta^{b_1 b_2} + \eta^{a_1 b_2} \eta^{b_1 a_2} - \eta^{a_1 b_1} \eta^{a_2 b_2} \right) \tag{4.25}$$

In $D = 3$, we find

$$D = 3: \qquad (\nu_2)_{a_1 b_1 a_2 b_2} = \frac{1}{2} \left(\eta_{a_1 a_2} \eta_{b_1 b_2} + \eta_{a_1 b_2} \eta_{b_1 a_2} - 2\eta_{a_1 b_1} \eta_{a_2 b_2} \right) \tag{4.26}$$

Thus, in $D = 3$ we have

$$D = 3: \qquad \mathcal{K}_2 = (p_{31} \cdot p_{42})^2 - p_{31}^2 p_{42}^2 = \frac{1}{4} \Lambda_{AB}(s) \tag{4.27}$$

and hence

$$D = 3: \qquad \rho_2(s) = -\frac{1}{2} \sqrt{-\Lambda_{AB}(s)} \tag{4.28}$$

The singularity $t_* = 2\pi \beta_2^2 \rho_2^2(s_*)$ leads to

$$s_* = M_A^2 + M_B^2 + 2M_A M_B \left(1 - \frac{t_*}{2\pi M_A^2 M_B^2 \beta_2^2} \right)^{1/2} \tag{4.29}$$

and thus

$$u_* = M_A^2 + M_B^2 - 2M_A M_B \left(1 - \frac{t_*}{2\pi M_A^2 M_B^2 \beta_2^2} \right)^{1/2} \tag{4.30}$$

We look at the product $s_* u_*$:

$$s_* u_* = (M_A - M_B)^2 (M_A + M_B)^2 + \frac{2t_*}{\pi \beta_2^2} = (m_1 - m_2)^2 (m_1 + m_2)^2 + \frac{2t_*}{\pi \beta_2^2} \tag{4.31}$$

Note the similarity between (4.12) and (4.31). Indeed, under the replacement

$$\frac{2\pi \beta_0^2}{t_*} \longleftrightarrow \frac{t_*}{2\pi \beta_2^2} \tag{4.32}$$

we find complete agreement. It follows that if $t_* \leq 0$, then s_* and u_* are inside of the physical scattering region. If we let $t_* > 0$, then we must have

$$t_* \leq 2\pi M_A^2 M_B^2 \beta_2^2 \tag{4.33}$$

in order for s_* and u_* to be real.

Looking at (4.29) and enforcing the forward-JWKB approximation (2.13), it is easy to conclude that $s_* \approx (m_1 + m_2)^2$. This assumes that $m_1 m_2 \beta_2^2$ is kept fixed in the forward-JWKB approximation. A more correct statement is that $s_*/(m_1 m_2)$ is kept fixed, which leads to

$$\frac{t_*}{m_1^2 m_2^2 \beta_2^2} \text{ fixed} \quad \Longrightarrow \quad m_1 m_2 \beta_2^2 \to 0^+ \tag{4.34}$$

which, yet again, suggests *weak-coupling*. However, since the coupling β_2 appears in denominators in (4.29) and (4.31), it seems that this weak-coupling phenomenon does not arises from perturbative contributions involving the exchange of propagating quanta.

5. Forward-JWKB Amplitudes in $D = 4$

In $D = 3$ we found that the forward-JWKB scattering amplitude has one extra singularity. As we will see shortly, in $D = 4$ we have an infinite number of singularities. For $N = 0$ and $N = 1$ the structure of these singularities in $D = 4$ is related to the singularity in $D = 3$. When $N = 2$ we find an infinite number of singularities that have a very different interpretation from the singularity in $D = 3$.

But as soon as we set $D = 4$ (i.e. $\Delta = 1$), we find trouble inside of the exponential in (3.87). To get around this issue, we work instead in $D = 4 + 2\epsilon$ with $\epsilon > 0$. The coupling parameter g_N and β_N have units

$$D = 4 + 2\epsilon : \qquad [g_N] = (1 - \epsilon - N) \, [\text{mass}], \qquad [\beta_N] = 2 \, (1 - \epsilon - N) \, [\text{mass}] \tag{5.1}$$

It is convenient to extract from β_N the four-dimensional coupling α_N by introducing a constant μ with units of mass:

$$\beta_N = \alpha_N \left(\frac{1}{\mu^2} \right)^\epsilon \tag{5.2}$$

Inside the exponential in (3.87), we have

$$\beta_N \Gamma(\Delta - 1) \left(\frac{2}{B_{12}^2} \right)^{(\Delta - 1)} = \alpha_N \Gamma(\epsilon) \left(\frac{2}{\mu^2 B_{12}^2} \right)^\epsilon$$

$$\approx \alpha_N \Gamma(\epsilon) + \alpha_N \log \left(\frac{2}{\mu^2 B_{12}^2} \right) + O(\epsilon) \tag{5.3}$$

where in the second line we have expanded near $\epsilon = 0$ and kept the leading logarithm. The amplitude near four-dimensions becomes

$$\mathcal{A}(s, t) \approx \delta(P) \left[\frac{\mathcal{K}_N}{\rho_N} \right] \exp \left[\alpha_N \rho_N \Gamma(\epsilon) \right] \int dB_{12} \left(\frac{2}{\mu^2 B_{12}^2} \right)^{\alpha_N \rho_N} \exp \left(-i B_{12} \cdot P_{12} \right) \tag{5.4}$$

Note that the divergent part has factored out and appears inside an exponential. The integral over B_{12} is now over a $D - 2 \approx 2$ dimensional volume. Integration yields

$$\mathcal{A}(s,t) \approx \delta(P) \left[\frac{\alpha_N \mathcal{K}_N(s)}{\mu^2} \right] \exp \left[\alpha_N \rho_N(s) \Gamma(\epsilon) \right] \frac{\Gamma[1 - \alpha_N \rho_N(s)]}{\Gamma[1 + \alpha_N \rho_N(s)]} \left(-\frac{t}{2\mu^2} \right)^{(\alpha_N \rho_N(s) - 1)} \tag{5.5}$$

A convenient way to write this result is

$$\mathcal{A}(s,t) \approx \mathcal{A}_{\text{tree}}(s,t) \exp \left[\alpha_N \rho_N(s) \Gamma(\epsilon) \right] \frac{\Gamma[1 - \alpha_N \rho_N(s)]}{\Gamma[1 + \alpha_N \rho_N(s)]} \left(-\frac{t}{2\mu^2} \right)^{\alpha_N \rho_N(s)} \tag{5.6}$$

where we have collected the tree-level contribution into a prefactor,

$$\mathcal{A}_{\text{tree}}(s,t) = \delta(P) \left[-\frac{2\alpha_N \mathcal{K}_N(s)}{t} \right] \tag{5.7}$$

which exhibits the familiar massless pole at $t = 0$. Indeed, the result (5.6) exhibits an infinite number of singularities from the poles of the Euler Gamma function in the numerator. In order to make all of the singularities manifest, it is useful to decompose the amplitude (5.5) into partial waves:

$$\mathcal{A}(s,t) = \sum_{l=0}^{\infty} (2l + 1) \mathcal{A}_l(s) P_l(z_s), \qquad z_s = \cos(\theta_s) \tag{5.8}$$

where P_l is a Legendre polynomial, and the partial amplitude $\mathcal{A}_l$ is given by

$$\mathcal{A}_l(s) = \frac{1}{2} \int_{-1}^{1} \mathrm{d}z_s \, \mathcal{A}(s,t) P_l(z_s) \tag{5.9}$$

Using

$$-t = \frac{\Lambda_{12}(s)}{s} \left(\frac{1 - z_s}{2} \right) \tag{5.10}$$

and

$$P_l(z_s) = \sum_{k=0}^{l} \frac{\Gamma(1+l)}{\Gamma(1+k)\Gamma(1+l-k)} \frac{\Gamma(-l)}{\Gamma(1+k)\Gamma(-l-k)} \left(\frac{1 - z_s}{2} \right)^k \tag{5.11}$$

leads to the partial amplitude

$$\mathcal{A}_l(s) = \delta(P) \frac{\sqrt{-\Lambda_{AB}(s)}}{2\mu^2} \exp \left[\alpha_N \rho_N(s) \Gamma(\epsilon) \right] \left[\frac{\Lambda_{12}(s)}{2\mu^2 s} \right]^{(\alpha_N \rho_N - 1)} \frac{\Gamma[1 - \alpha_N \rho_N(s) + l]}{\Gamma[1 + \alpha_N \rho_N(s) + l]} \tag{5.12}$$

This partial amplitude has a singularity whenever

$$1 - \alpha_N \rho_N(s_{nl}) + l = -n, \qquad n = 0, 1, 2, \ldots \qquad l = 0, 1, 2, \ldots \tag{5.13}$$

At first glance, we see a close similarity between this singularity condition in $D = 4$ and the condition $t = 2\pi \beta_N^2 \rho_N^2(s_*)$ in $D = 3$. We now consider specific values of N.

5.1. Exchange of Massless Scalar

A massless scalar field has the same dynamics in $D = 3$ and $D = 4$. Thus, we again have

$$\rho_0(s) = \frac{2}{\sqrt{-\Lambda_{AB}(s)}}, \qquad \Lambda_{AB}(s) \equiv [s - (M_A - M_B)^2][s - (M_A + M_B)^2] \qquad (5.14)$$

In $D = 4$, the coupling α_0 has units

$$[\alpha_0] = 2[\text{mass}] \qquad (5.15)$$

The singularity condition becomes $1 - \alpha_0\rho_0(s_{nl}) + l = -n$. Comparing this to $t_* = 2\pi\beta_0^2\rho_0^2(s_*)$ in $D = 3$, we can find the solution for s_{nl} by using (4.10) with the replacement

$$\frac{2\pi\beta_0^2}{t_*} \longrightarrow \frac{\alpha_0^2}{(n + l + 1)^2} \qquad (5.16)$$

Note that this replacement is only valid when $t_* > 0$ (i.e. outside of the physical scattering region), since the right-hand side is always positive. Thus, in $D = 4$ we have the infinite sequence

$$s_{nl} = M_A^2 + M_B^2 + 2M_A M_B \left(1 - \frac{\alpha_0^2}{M_A^2 M_B^2 (n + l + 1)^2}\right)^{1/2} \qquad (5.17)$$

Using $s_{nl} + u_{nl} = 2M_A^2 + 2M_B^2$ leads to

$$u_{nl} = M_A^2 + M_B^2 - 2M_A M_B \left(1 - \frac{\alpha_0^2}{M_A^2 M_B^2 (n + l + 1)^2}\right)^{1/2} \qquad (5.18)$$

Unlike in $D = 3$, the infinite sequences (s_{nl}, u_{nl}) always lie outside of the physical scattering region. This suggest an interpretation as bound state singularities.

The singularities (s_{nl}, u_{nl}) lie outside of the physical scattering region, but we can still look for the requirement that $s_{nl}/(m_1 m_2)$ is fixed in the forward-JWKB approximation (2.13):

$$\frac{s_{nl}}{m_1 m_2} \text{ fixed} \implies \frac{\alpha_0^2}{M_A^2 M_B^2 (n + l + 1)^2} \text{ fixed} \qquad (5.19)$$

Since the semiclassical approximation involves large quantum numbers, we have

$$n + l \to \infty \implies \frac{\alpha_0}{M_A M_B} \to \infty \qquad (5.20)$$

which suggests *strong-coupling* in the forward-JWKB regime in $D = 4$.

5.2. Exchange of Massless Vector

A massless vector field has two physical polarizations in $D = 4$, and one in $D = 3$. However, the gauge-fixed kinetic operators are the same in any number of dimensions. Again, we have

$$\rho_1(s) = Z_A Z_B \left[\frac{M_A^2 + M_B^2 - s}{\sqrt{-\Lambda_{AB}(s)}} \right] \tag{5.21}$$

where, again, we have introduced dimensionless charges Z_A and Z_B. The singularity condition becomes $1 - \alpha_1 \rho_1(s_{nl}) + l = -n$, which is again analogous to the singularity condition in $D = 3$ with the replacement

$$\frac{2\pi \beta_1^2}{t_*} \longrightarrow \frac{\alpha_1^2}{(n + l + 1)^2} \tag{5.22}$$

Hence, in $D = 4$ we have

$$s_{nl} = M_A^2 + M_B^2 + 2 M_A M_B \left(1 + \frac{Z_A^2 Z_B^2 \alpha_1^2}{(n + l + 1)^2} \right)^{-1/2} \tag{5.23}$$

We find that s_{nl} is always outside of the physical scattering region.

Keeping $s_{nl}/(m_1 m_2)$ fixed in the forward-JWKB approximation requires

$$\frac{\alpha_1^2}{(n + l + 1)^2} \text{ fixed} \tag{5.24}$$

If $n + l \to \infty$, we must also have $\alpha_1 \to \infty$. Thus, in $D = 4$ we again find *strong-coupling*.

5.3. Exchange of Massless Symmetric Tensor

A massless symmetric (traceless) tensor has two physical polarizations in $D = 4$. The kinetic operator (and thus the tensor κ_2) are the same as in $D = 3$. However, in $D = 4$, the ν_2 tensor gives

$$D = 4: \qquad (\nu_2)_{a_1 b_1 a_2 b_2} = \frac{1}{2} \left(\eta_{a_1 a_2} \eta_{b_1 b_2} + \eta_{a_1 b_2} \eta_{b_1 a_2} - \eta_{a_1 b_1} \eta_{a_2 b_2} \right) \tag{5.25}$$

This leads to a different $\mathcal{K}_2$ from (4.27),

$$D = 4: \qquad \mathcal{K}_2 = (p_{31} \cdot p_{42})^2 - \frac{1}{2} p_{31}^2 p_{42}^2 = \frac{1}{4} \left[(M_A^2 + M_B^2 - s)^2 - 2 M_A^2 M_B^2 \right] \tag{5.26}$$

and thus, a different $\rho_2(s)$ from (4.28),

$$D = 4: \qquad \rho_2(s) = \frac{1}{2} \left[\frac{(M_A^2 + M_B^2 - s)^2 - 2 M_A^2 M_B^2}{\sqrt{-\Lambda_{AB}(s)}} \right] \tag{5.27}$$

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In $D = 4$ the coupling α_2 has units

$$[\alpha_2] = -2[\text{mass}] \tag{5.28}$$

Solving the singularity condition $1 - \alpha_2 \rho_2(s_{nl}) + l = -n$ leads to

$$s_{nl} = M_A^2 + M_B^2 + 2M_A M_B \left[\frac{1}{2} + \left(1 + \sqrt{1 + \frac{2M_A^2 M_B^2 \alpha_2^2}{(n+l+1)^2}} \right)^{-1} \right]^{1/2} \tag{5.29}$$

Again, we find that the infinite sequence of singularities s_{nl} lie outside of the physical scattering region.

Keeping $s_{nl}/(m_1 m_2)$ fixed in the forward-JWKB approximation requires

$$\frac{M_A^2 M_B^2 \alpha_2^2}{(n+l+1)^2} \text{ fixed} \tag{5.30}$$

If $n + l \to \infty$, then also $M_A M_B \alpha_2 \to \infty$, which yet again suggest *strong-coupling* in $D = 4$. Moreover, for spin 2 interactions the product $m_i \alpha_2$ corresponds to the Schwarzschild radius r_i of a particle with mass m_i. The product $M_A M_B \alpha_2 \approx m_1 m_2 \alpha_2$ can be interpreted as the ratio of the Schwarzschild radius of one particle to the Compton wavelength of the other:

$$m_1 m_2 \alpha_2 = \frac{r_1}{\lambda_2} = \frac{r_2}{\lambda_1} \tag{5.31}$$

Thus, $m_1 m_2 \alpha_2 \to \infty$ is also the regime of large Schwarzschild radii.

6. Exchange of Heavy Scalar

In the previous two sections we studied a system with two non-identical heavy scalar particles interacting via the exchange of massless quanta with spin 0, 1 or 2. For these three cases, the truncated forward-JWKB scattering amplitude has the same general form (see (4.5) and (5.6)). We now consider a system with two non-identical heavy scalar particles that exchange heavy scalar quanta.

We can repeat most of the steps as before to derive the path integral analogous to (3.13). The tensor κ_0 for a massive scalar is the same as for a massless scalar, so again we have $\kappa_0 = 1$, $\nu_0 = 1$ and thus $\mathcal{K}_0 = 1$. Thus, the kinetic term for the mediating field φ is

$$S_{\text{kin}}[\varphi] = \frac{1}{2} \int \int dx dy \, \varphi(x) K_M(x|y) \varphi(y), \qquad K_M(x|y) \equiv \delta(x-y) \left(-\frac{1}{2}\partial^2 + \frac{1}{2}M^2 \right) \tag{6.1}$$

Here M is a constant with units of mass. The functional integral over φ gives rise to interaction terms analogous to (3.18) and (3.19). Just as before, we will **ignore the contributions from the self-interactions**. The two-body interaction term is

$$S_2[q_A, q_B] = g_0^2 \int \int \mathrm{d}\tau_A \mathrm{d}\tau_B \, G_M[q_A(\tau_A)|q_B(\tau_B)] \tag{6.2}$$

where $G_M = (K_M)^{-1}$ is the free massive scalar Green function.

In the forward-JWKB approximation, we evaluate S_2 at the forward paths (3.26):

$$\Sigma_2 = S_2[f_A, f_B] = g_0^2 \int \int \mathrm{d}\tau_A \mathrm{d}\tau_B \, G_M[f_A(\tau_A)|f_B(\tau_B)] \tag{6.3}$$

Recall the expression for G_M in terms of a Schwinger integral:

$$G_M(x|y) = M^{(D-2)} \int\limits_0^\infty \mathrm{d}T \left(-\frac{i}{T}\right)^{D/2} \exp\left[\frac{i}{2T}M^2(y-x)^2 - \frac{iT}{2}\right] \tag{6.4}$$

Since the separation of the particles is very large in the forward-JWKB approximation, we can still integrate over (τ_A, τ_B) with stationary methods. The result gives

$$\Sigma_2 \approx 2\pi g_0^2 \left[\frac{T_A T_B}{\sqrt{x_{31}^2 x_{42}^2 - (x_{31} \cdot x_{42})^2}}\right] M^{(D-4)} \int\limits_0^\infty \mathrm{d}T \left(-\frac{i}{T}\right)^\Delta \exp\left[\frac{i}{2T}M^2 B_{12}^2 - \frac{iT}{2}\right] \tag{6.5}$$

with $\Delta = (D-2)/2$. In the regime $M^2 B_{12}^2 \to \infty$ (heavy messenger and large separations), the integral over T is also done with stationary methods:

$$\Sigma_2 \approx -i\sqrt{2\pi} \beta_0 \rho_0 M^{(D-4)} \left(iM\sqrt{-B_{12}^2}\right)^{(3-D)/2} \exp\left(-iM\sqrt{-B_{12}^2}\right) \tag{6.6}$$

where we have used

$$\beta_0 = 2\pi g_0^2, \qquad \rho_0 = \frac{T_A T_B}{\sqrt{x_{31}^2 x_{42}^2 - (x_{31} \cdot x_{42})^2}} \tag{6.7}$$

One can recognize Σ_2 in (6.6) as being proportional to a long-distance massive propagator in $D-2$ dimensions (i.e. given by the asymptotic expansion of the familiar Bessel function). The truncated on-shell forward-JWKB scattering amplitude gives

$$\mathcal{A} = \delta(P) \left(\frac{1}{\rho_0}\right) \int \mathrm{d}B_{12} \, \exp\left(-iB_{12} \cdot P_{12}\right) \left[-1 + \exp\left(i\Sigma_2\right)\right] \tag{6.8}$$

with Σ_2 given by (6.6) and we have subtracted the disconnected part. Note that unlike the massless exchange, with the massive exchange Σ_2 is not explicitly divergent when $D = 4$.

6.1. Three Spacetime Dimensions

When $D = 3$, we have

$$\Sigma_2 \approx -i \left(\frac{\sqrt{2\pi} \beta_0 \rho_0}{M} \right) \exp \left(-iM\sqrt{-B_{12}^2} \right) \tag{6.9}$$

So then

$$-1 + \exp\left(i\Sigma_2 \right) = \sum_{L=0}^{\infty} \frac{1}{L!(L+1)} \left(\frac{\sqrt{2\pi} \beta_0 \rho_0}{M} \right)^{(L+1)} \exp \left(-i(L+1)M\sqrt{-B_{12}^2} \right) \tag{6.10}$$

which we recognize as the sum of one-dimensional massive scalar propagators with mass $(L+1)M$. Thus, after integration over B_{12} we find

$$\mathcal{A}(s,t) = \beta_0 \delta(P) \sum_{L=0}^{\infty} \frac{1}{L!} \left[\frac{\sqrt{2\pi} \beta_0 \rho_0(s)}{M} \right]^L \left[\frac{2}{(L+1)^2 M^2 - t} \right] \tag{6.11}$$

This result has an infinite number of singularities given by $t = (L+1)^2 M^2$, which can be recognized as the branch points of the $(L+1)$-mass branch cuts. We also have the singularities whenever $\rho_0(s_*) \to \infty$, which correspond to $s_* = (m_A \pm m_B)^2$. It is interesting that instead of getting a whole branch cut (a continuum of singularities), in the forward-JWKB approximation we seem to only get the branch point (a single singularity).

Using the relation

$$\frac{2}{(L+1)^2 M^2 - t} = \frac{1}{\sqrt{t}} \left[\frac{1}{(L+1)M - \sqrt{t}} - \frac{1}{(L+1)M + \sqrt{t}} \right] \tag{6.12}$$

and the incomplete Euler Gamma function,

$$\Gamma_{\text{inc}}(z,a) = \int_0^a dT \left(\frac{1}{T} \right)^{1-z} \exp\left(-T \right) = a^z \sum_{n=0}^{\infty} \frac{1}{n!} \frac{(-a)^n}{(n+z)} \tag{6.13}$$

we can rewrite (6.11) as

$$\mathcal{A}(s,t) = \delta(P) \left(\frac{\beta_0}{M\sqrt{t}} \right) \left[\mathcal{R}_+(s,t) + \mathcal{R}_-(s,t) \right] \tag{6.14}$$

where

$$\mathcal{R}_\pm(s,t) \equiv \pm \left[-\frac{\sqrt{2\pi} \beta_0 \rho_0(s)}{M} \right]^{R_\pm(t)} \Gamma_{\text{inc}} \left[-R_\pm(t), -\frac{\sqrt{2\pi} \beta_0 \rho_0(s)}{M} \right] \tag{6.15}$$

with

$$R_\pm(t) \equiv -1 \pm \sqrt{\frac{t}{M^2}} \tag{6.16}$$

This form of the amplitude is akin to Regge behavior, with the Regge poles being the multi-mass branch points.

6.2. Five Spacetime Dimensions

We can set $D = 5$ in (6.6) and find no explicit divergences. However, as a precaution we work in $D = 5 - 4\varepsilon$ with $\varepsilon > 0$. Just like we did before in $D = 4 + 2\epsilon$, we extract the $D = 5$ coupling γ_0 by introducing a constant μ with units of mass:

$$\beta_0 = \gamma_0 \mu^{4\varepsilon} \tag{6.17}$$

Note that γ_0 has units of mass. Then, Σ_2 becomes

$$\Sigma_2 \approx -i\sqrt{2\pi} M \gamma_0 \rho_0 \left(\frac{\mu}{M}\right)^{4\varepsilon} \left(iM\sqrt{-B_{12}^2}\right)^{(2\varepsilon - 1)} \exp\left(-iM\sqrt{-B_{12}^2}\right) \tag{6.18}$$

and thus

$$-1 + \exp\left(i\Sigma_2\right) = \sum_{L=0}^{\infty} \frac{1}{\Gamma(L+2)} \left[\sqrt{2\pi} M \gamma_0 \rho_0 \left(\frac{\mu}{M}\right)^{4\varepsilon}\right]^{(L+1)} \left(iM\sqrt{-B_{12}^2}\right)^{(L+1)(2\varepsilon - 1)}$$
$$\times \exp\left(-i(L+1)M\sqrt{-B_{12}^2}\right) \tag{6.19}$$

In order to perform the B_{12} integral in (6.8), we write the exponential in the second line as an infinite sum too. This leads to a double sum involving powers of B_{12}:

$$-1 + \exp\left(i\Sigma_2\right) = \sum_{L=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n (L+1)^{(n-1)}}{\Gamma(L+1)\Gamma(n+1)} \left[\sqrt{2\pi} M \gamma_0 \rho_0 \left(\frac{\mu}{M}\right)^{4\varepsilon}\right]^{(L+1)}$$
$$\times \left(\frac{1}{2}\right)^{\theta_{nL}} \left(\frac{2}{M^2 B_{12}^2}\right)^{\theta_{nL}} \tag{6.20}$$

with

$$\theta_{nL} \equiv \frac{(L+1)(1-2\varepsilon)}{2} - \frac{n}{2} \tag{6.21}$$

Taking the Fourier transform of each power yields

$$\mathcal{A} = \frac{1}{\rho_0} \left(\frac{1}{M^2}\right)^{(3-4\varepsilon)/2} \delta(P) \sum_{L=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n (L+1)^{(n-1)}}{\Gamma(L+1)\Gamma(n+1)} \left[\sqrt{2\pi} M \gamma_0 \rho_0 \left(\frac{\mu}{M}\right)^{4\varepsilon}\right]^{(L+1)}$$
$$\times \left(\frac{1}{2}\right)^{\theta_{nL}} \frac{\Gamma(\omega_{nL})}{\Gamma(\theta_{nL})} \left(-\frac{2M^2}{t}\right)^{\omega_{nL}} \tag{6.22}$$

with

$$\omega_{nL} \equiv \frac{3}{2} - 2\varepsilon - \theta_{nL} \tag{6.23}$$

From this result, we anticipate a divergence whenever $\omega_{nL} = -l$ with $l = 0, 1, 2, \ldots$ or

$$L = \frac{2 - 2\varepsilon + 2l + n}{1 - 2\varepsilon}, \qquad l = 0, 1, 2, \ldots \qquad n = 0, 1, 2, \ldots \tag{6.24}$$

Hence, if we take $\varepsilon \to 0$ we expect to find a divergence when $L \geq 2$. To explicitly see this, we keep L fixed and split the sum over n into even and odd parts:

$$\mathcal{A} = \frac{1}{M^3 \rho_0} M^{4\varepsilon} \delta(P) \sum_{L=0}^{\infty} \frac{1}{\Gamma(L+1)} \left[\sqrt{2\pi} M \gamma_0 \rho_0 \left(\frac{\mu}{M} \right)^{4\varepsilon} \right]^{(L+1)} [\mathcal{E}_L(t) + \mathcal{O}_L(t)] \tag{6.25}$$

We find

$$\mathcal{E}_L(t) = \frac{2^{(2\varepsilon-1)(L+1)/2}}{(L+1)} \frac{\Gamma\left(\frac{2-2\varepsilon-(1-2\varepsilon)L}{2} \right)}{\Gamma\left(\frac{(L+1)(1-2\varepsilon)}{2} \right)} \left(-\frac{2M^2}{t} \right)^{(2-2\varepsilon-(1-2\varepsilon)L)/2}$$

$$\times \, _2F_1 \left(\frac{2-2\varepsilon-(1-2\varepsilon)L}{2}, \frac{1+2\varepsilon-(1-2\varepsilon)L}{2}; \frac{1}{2}; \frac{(L+1)^2 M^2}{t} \right) \tag{6.26}$$

$$\mathcal{O}_L(t) = -\, 2^{[(2\varepsilon-1)(L+1)+1]/2} \frac{\Gamma\left(\frac{3-2\varepsilon-(1-2\varepsilon)L}{2} \right)}{\Gamma\left(\frac{L(1-2\varepsilon)-2\varepsilon}{2} \right)} \left(-\frac{2M^2}{t} \right)^{(3-2\varepsilon-(1-2\varepsilon)L)/2}$$

$$\times \, _2F_1 \left(\frac{3-2\varepsilon-(1-2\varepsilon)L}{2}, \frac{2+2\varepsilon-(1-2\varepsilon)L}{2}; \frac{3}{2}; \frac{(L+1)^2 M^2}{t} \right) \tag{6.27}$$

When $L = 0$ we find

$$\mathcal{E}_0(t) = \sqrt{2} \left(-\frac{M^2}{t} \right)^{(1-\varepsilon)} \frac{\Gamma(1-\varepsilon)}{\Gamma\left(\frac{1-2\varepsilon}{2} \right)} \, _2F_1 \left(1-\varepsilon, \frac{1+2\varepsilon}{2}; \frac{1}{2}; \frac{M^2}{t} \right) \tag{6.28}$$

$$\mathcal{O}_0(t) = -\sqrt{8} \left(-\frac{M^2}{t} \right)^{(3-2\varepsilon)/2} \frac{\Gamma\left(\frac{3-2\varepsilon}{2} \right)}{\Gamma(-\varepsilon)} \, _2F_1 \left(\frac{3-2\varepsilon}{2}, 1+\varepsilon; \frac{3}{2}; \frac{M^2}{t} \right) \tag{6.29}$$

which are well-behaved in the $\varepsilon \to 0$ limit:

$$\mathcal{E}_0(t) \to \sqrt{\frac{2}{\pi}} \frac{M^2}{M^2 - t}, \qquad \mathcal{O}_0(t) \to 0 \tag{6.30}$$

Indeed, we find the familiar tree-level contribution. For $L = 1$ and $\varepsilon \to 0$ we find

$$\mathcal{E}_1(t) = \frac{\sqrt{\pi}}{4} \sqrt{-\frac{2M^2}{t}} \tag{6.31}$$

$$\mathcal{O}_1(t) = \frac{1}{2\sqrt{2\pi}} \sqrt{\frac{4M^2}{t}} \operatorname{arctanh} \left(\sqrt{\frac{4M^2}{t}} \right) \tag{6.32}$$

However, starting with $L = 2$ we find a divergence near $\varepsilon = 0$:

$$\mathcal{E}_2(t) = \frac{2^{(8\varepsilon-3)/2}}{3} \left(-\frac{M^2}{t} \right)^{\varepsilon} \frac{\Gamma(\varepsilon)}{\Gamma\left(\frac{3-6\varepsilon}{2} \right)} \, _2F_1 \left(\varepsilon, \frac{6\varepsilon-1}{2}; \frac{1}{2}; \frac{9M^2}{t} \right) \tag{6.33}$$

$$\mathcal{O}_2(t) = -2^{(8\varepsilon-1)/2} \left(-\frac{M^2}{t} \right)^{(1+2\varepsilon)/2} \frac{\Gamma\left(\frac{1+2\varepsilon}{2} \right)}{\Gamma(1-3\varepsilon)} \, _2F_1 \left(3\varepsilon, \frac{1+2\varepsilon}{2}; \frac{3}{2}; \frac{9M^2}{t} \right) \tag{6.34}$$

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Taking the $\varepsilon \to 0$ limit leads to

$$\mathcal{O}_2(t) = -\frac{\sqrt{\pi}}{2}\sqrt{-\frac{2M^2}{t}} \tag{6.35}$$

but $\mathcal{E}_2$ has a divergent part. Using the identity

$$\Gamma(a)\,_2F_1(a,b;c;z) = \Gamma(a) + \frac{\Gamma(c)}{\Gamma(b)}\sum_{n=1}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{z^n}{n!} \tag{6.36}$$

leads to

$$\mathcal{E}_2(t) \approx \frac{1}{3\sqrt{2\pi}}\left[\left(-\frac{2M^2}{t}\right)^{\varepsilon}\Gamma(\varepsilon) - 2\sqrt{\frac{9M^2}{t}}\,\text{arctanh}\left(\sqrt{\frac{9M^2}{t}}\right) - \log\left(1 - \frac{9M^2}{t}\right)\right] \tag{6.37}$$

Similarly with $L = 3$: Taking the $\varepsilon \to 0$ limit leads to

$$\mathcal{E}_3(t) = \sqrt{\pi}\sqrt{-\frac{2M^2}{t}}\left(1 + \frac{t}{16M^2}\right) \tag{6.38}$$

but $\mathcal{O}_3(t)$ has a divergent part,

$$\mathcal{O}_3(t) \approx \frac{1}{3\sqrt{2\pi}}\left[\left(-\frac{2M^2}{t}\right)^{2\varepsilon}\Gamma(2\varepsilon) - \left(\sqrt{\frac{16M^2}{t}} + \sqrt{\frac{t}{16M^2}}\right)\text{arctanh}\left(\sqrt{\frac{16M^2}{t}}\right)\right.$$
$$\left. - \log\left(1 - \frac{16M^2}{t}\right) + 1\right] \tag{6.39}$$

In general, for L even and $L \geq 2$ we find that $\mathcal{E}_L(t)$ has a divergent part, and for L odd and $L \geq 3$ we find that $\mathcal{O}_L(t)$ has a divergent part. Each of these divergent terms consist of a simple pole at $\varepsilon = 0$ (i.e. a term proportional to $\Gamma(n\varepsilon)$).

7. Discussion

Using the forward-JWKB approximation we obtained three sets of results: amplitudes in $D = 3$ with massless mediation, amplitudes in $D = 4$ with massless mediation, and amplitudes in $D = 3$ and $D = 5$ with massive mediation.

In $D = 4$ with massless mediation, we found forward-JWKB amplitudes (5.6) for mediating quanta with spin 0, 1 and 2. Each of these amplitudes exhibits an infinite number of singularities that lie outside of the physical scattering region. Indeed, the singularities (5.17), (5.23) and (5.29) agree with the two-body bound-state energies found in [8]. Upon taking the static limit $(m_1/m_2 \to 0)$, the resulting one-body amplitudes agree with those
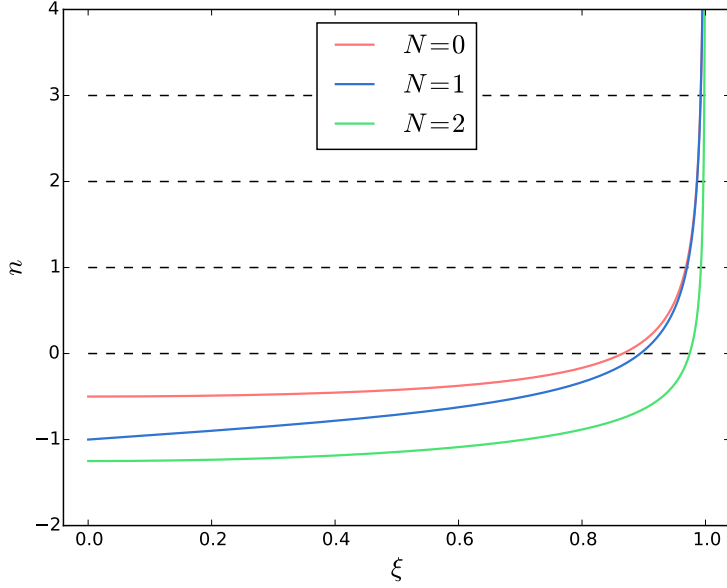
Fig. 1: The leading Regge trajectory functions $R_N(\xi)$ for spin $N = 0$, 1 and 2 in the forward-JWKB regime. The dashed lines correspond to integer values of n. For the couplings we have used $\alpha_0/(M_A M_B) = 0.5$, $Z_A Z_B \alpha_1 = -0.5$, and $M_A M_B \alpha_2 = 0.5$.

found in [12] by solving one-body field equations. All of these amplitudes display an exponentiated divergent contribution, the kinematic dependence of which agrees with those found in [13] due to infrared photons and gravitons. Inside the physical scattering region, the coefficient of this divergence is imaginary and thus the whole divergence appears as a pure phase factor. This guarantees that physical observables are finite.

From the discussion in §2 it should be clear that the Regge regime and the forward-JWKB regime are very different. The amplitude (5.6) in $D = 4$ exhibits Regge behavior with leading Regge trajectory function

$$R_N(s) = -1 + \alpha_N \rho_N(s) \tag{7.1}$$

and daughter trajectories $R_N(s) - l$. In Figure 1 we plot the forward-JWKB leading Regge trajectories for $N = 0$, 1 and 2 as functions of the dimensionless variable

$$\xi(s) \equiv \frac{s - M_A^2 - M_B^2}{2 M_A M_B} \approx \frac{s - m_1^2 - m_2^2}{2 m_1 m_2} \tag{7.2}$$

When $N = 0$ the forward-JWKB leading Regge trajectory R_0 is qualitatively different from the leading Regge trajectory function R found in the Regge limit (2.11). In Figure 2 we compare the leading trajectory in these two regimes. This comparison is only qualitative,
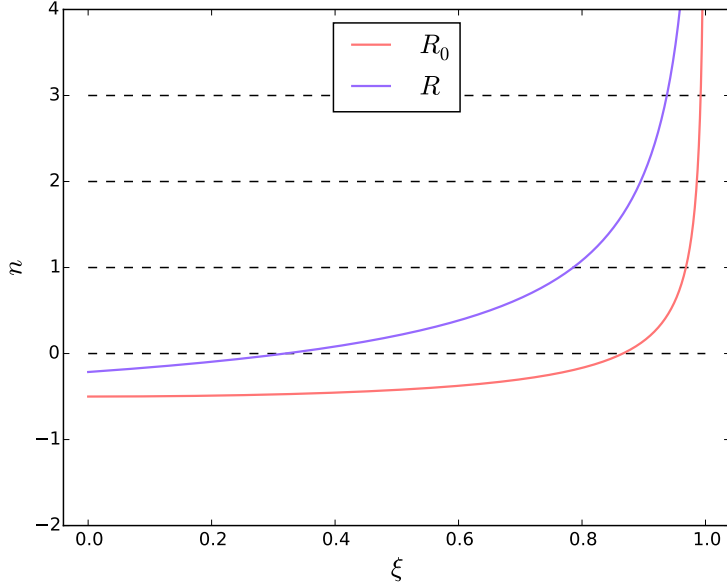
Fig. 2: The leading Regge trajectory for an interaction mediated by a massless scalar in the Regge limit (R) and the forward-JWKB regime (R_0).

as the normalization of the coupling strength g in the Regge ladder sum (2.11) is not necessarily the same as the one we used in the forward-JWKB amplitude. One major difference is that the real part of the Regge limit trajectory R is non-vanishing in the region $\xi < 1$, while the real part of the forward-JWKB trajectory R_0 is non-vanishing in the smaller interval $-1 < \xi < 1$. However, near the threshold value both trajectories agree. Indeed, the discrepancy between these two trajectories was already noticed in [14], where diagrammatic methods were used to sum over ladder contributions. Looking at the equations for ρ in (2.7) and ρ_0 in (5.14) we see that the logarithm term in ρ is missing from ρ_0. This logarithm term is proportional to the sum of the rapidities of the incoming states (A.17), and is kept fixed in both the Regge and forward-JWKB limits.

The forward-JWKB amplitude in $D = 3$ with massless mediation exhibits the familiar tree-level singularity near $t = 0$, and also an "extra" non-perturbative singularity at a particular value $s = s_*(t_*)$ that depends explicitly on t_*. For mediating scalars and photons, the location of the extra singularity is very similar to the location of the corresponding two-body bound state energies s_{nl} in $D = 4$. Indeed, one can "switch" between the many singularities s_{nl} and the one singularity s_* with the replacement

$$\frac{2\pi\beta_N^2}{t_*} \longleftrightarrow \frac{\alpha_N^2}{(n+l+1)^2} \tag{7.3}$$

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where β_N is the coupling in $D = 3$, and α_N is the coupling in $D = 4$. Upon analytic continuation to spin 2, we also find a forward-JWKB amplitude with two singularities. However, unlike the scalar or photon cases, the s_* singularity does not seem to correspond to anything that can be built with perturbative contributions (i.e. from contributions that are proportional to positive integer powers of the coupling). Unlike the $D = 4$ singularities s_{nl}, we seem to always be able to move the $D = 3$ singularity s_* to the interior of the physical scattering region by allowing t_* to be negative. Of course, the relation (7.3) only holds when t_* is positive.

We also studied the mediation of a heavy scalar in $D = 3$. In this case we found a non-perturbative expression that agrees with the expected result from adding forward-JWKB ladder diagrams of the form (2.14) and (2.15). Due to the particular details of the long-distance propagator in $D - 2 = 1$ dimensions, the result can be written as a sum of simple poles at the multi-mass values $(L+1)^2 M^2$. The end result (6.15) takes a very simple form that is analogous to Regge behavior, but instead of Regge poles along the s axis, we find poles along the t axis.

Then, instead of turning to $D = 4$ with a heavy mediator, we turned to $D = 5$. The reason for this is that when $D = 5$ we have $D - 2 = 3$ which is the other instance when the long-distance propagator is exact. Although we cannot evaluate the "eikonal" integral (6.8) exactly, after a series expansion we recover the familiar tree-level amplitude, a finite one-loop contribution and divergent higher-loop contributions. This system in $D = 5$ was considered mostly to illustrate that for higher-dimensional systems one can apply the forward-JWKB approximation and obtain limited results for all-order amplitudes.

In our calculation of forward-JWKB amplitudes we neglected the part of the forward Van Vleck matrix with derivatives of Σ_2 (the term with two-body interactions). The Van Vleck determinant is the first-order JWKB contribution. When written in the form (3.46), it is clear that this contribution involves an infinite set of perturbative contributions (i.e. involving powers of the coupling). This demonstrates nicely the non-perturbative nature of the JWKB approximation. In principle, incorporating the contributions from W in (3.46) offers a way to go beyond the familiar $D = 4$ results obtained in this work. However, in practice it is not clear how to later evaluate the integrals in $\mathcal{S}_T$ and find the amplitude.

At first glance, the statement of the forward-JWKB approximation (2.13) amounts to a restriction on kinematics. But as was found in [5], the semiclassical approximation has dynamical consequences that depend on the number of spacetime dimensions. Indeed, we found that in $D = 3$ the forward-JWKB approximation was consistent with weak couplings β_N, but in $D = 4$ we require strong couplings α_N when the quantum number becomes large. This strong-coupling feature is attractive and one hopes to be able to extend it to other theories (or at least to higher-point forward-JWKB scattering). It is also one of the reasons why we believe semiclassical first-quantized methods are important and useful to obtain non-perturbative results.

Acknowledgments

Appendix A. Four-point Kinematics

We study the scattering of two non-identical massive scalar particles. The external states are labeled such that the s-channel process is elastic:

$$A(p_1) + B(p_2) \longrightarrow A(p_3) + B(p_4) \tag{A.1}$$

Note that the u-channel process is also elastic:

$$A(p_1) + \bar{B}(\bar{p}_2) \longrightarrow A(p_3) + \bar{B}(\bar{p}_4) \tag{A.2}$$

but the t-channel process is inelastic:

$$A(p_1) + \bar{A}(\bar{p}_2) \longrightarrow \bar{B}(\bar{p}_3) + B(p_4) \tag{A.3}$$

We will only consider the amplitude for the (A.1) process. The amplitude for the (A.2) process follows after setting $\bar{p}_2 = -p_4$ and $\bar{p}_4 = -p_2$, and the amplitude for the (A.3) process follows after setting $\bar{p}_2 = -p_3$ and $\bar{p}_3 = -p_2$.

The four external energy-momentum vectors satisfy the on-shell constraints

$$p_j^2 = -m_j^2; \tag{A.4}$$

the elasticity constraints

$$p_1^2 = p_3^2, \qquad p_2^2 = p_4^2 \quad \Longrightarrow \quad m_1 = m_3, \qquad m_2 = m_4; \tag{A.5}$$

and also the conservation constraint

$$p_1 + p_2 = p_1 + p_4 \tag{A.6}$$

We use the familiar Mandelstam energy-momentum invariants

$$s = -(p_1 + p_2)^2, \qquad t = -(p_1 - p_3)^2, \qquad u = -(p_1 - p_4)^2 \tag{A.7}$$

which satisfy

$$s + t + u = 2m_1^2 + 2m_2^2 \tag{A.8}$$

Note that s and u carry data from both particles, while t carries data from a single particle.

In the center-of-momentum frame, we write the energy-momentum vectors as

$$p_1 = (E_1, \mathbf{p}_1), \qquad p_2 = (E_2, -\mathbf{p}_1), \qquad p_3 = (E_3, \mathbf{p}_3), \qquad p_4 = (E_4, -\mathbf{p}_3) \tag{A.9}$$

It is easy to show that, in terms of s and the masses (m_1, m_2), the magnitude of the spatial vectors $(\mathbf{p}_1, \mathbf{p}_3)$ are given by

$$|\mathbf{p}_1| = |\mathbf{p}_3| = \frac{\sqrt{\Lambda_{12}(s)}}{2\sqrt{s}}, \qquad \Lambda_{12}(s) \equiv \left[s - (m_1 - m_2)^2 \right] \left[s - (m_1 + m_2)^2 \right] \qquad (A.10)$$

and the energies of the external states are given by

$$E_1 = E_3 = \frac{s + (m_1 - m_2)(m_1 + m_2)}{2\sqrt{s}}, \qquad E_2 = E_4 = \frac{s - (m_1 - m_2)(m_1 + m_2)}{2\sqrt{s}} \qquad (A.11)$$

A relativistic particle with mass $m \geq 0$ and energy $E \geq m$ has a speed $|\mathbf{v}| \leq 1$ given by

$$|\mathbf{v}| = \frac{\sqrt{E^2 - m^2}}{E} \qquad (A.12)$$

Thus, in the center-of-momentum frame the external states have speeds

$$|\mathbf{v}_1| = |\mathbf{v}_3| = \frac{\sqrt{\Lambda_{12}(s)}}{s + (m_1 - m_2)(m_1 + m_2)}$$

$$|\mathbf{v}_2| = |\mathbf{v}_4| = \frac{\sqrt{\Lambda_{12}(s)}}{s - (m_1 - m_2)(m_1 + m_2)} \qquad (A.13)$$

and rapidities $\varphi_j \equiv \operatorname{artanh}(|\mathbf{v}_j|)$ given by

$$\varphi_1 = \varphi_3 = \frac{1}{2} \log \left[\frac{(m_1 - m_2)(m_1 + m_2) + s + \sqrt{\Lambda_{12}(s)}}{(m_1 - m_2)(m_1 + m_2) + s - \sqrt{\Lambda_{12}(s)}} \right]$$

$$\varphi_2 = \varphi_4 = \frac{1}{2} \log \left[\frac{(m_1 - m_2)(m_1 + m_2) - s - \sqrt{\Lambda_{12}(s)}}{(m_1 - m_2)(m_1 + m_2) - s + \sqrt{\Lambda_{12}(s)}} \right] \qquad (A.14)$$

Note that the sum of the incoming rapidities gives

$$\varphi_1 + \varphi_2 = \frac{1}{2} \log \left[\frac{m_1^2 + m_2^2 - s - \sqrt{\Lambda_{12}(s)}}{m_1^2 + m_2^2 - s + \sqrt{\Lambda_{12}(s)}} \right] \qquad (A.15)$$

Using

$$\frac{(m_1 + m_2)^2 - s + \sqrt{\Lambda_{12}(s)}}{(m_1 + m_2)^2 - s - \sqrt{\Lambda_{12}(s)}} = \frac{2m_1 m_2}{m_1^2 + m_2^2 - s + \sqrt{\Lambda_{12}(s)}} = \frac{m_1^2 + m_2^2 - s - \sqrt{\Lambda_{12}(s)}}{2m_1 m_2} \qquad (A.16)$$

we can also write

$$\varphi_1 + \varphi_2 = \log \left[\frac{(m_1 + m_2)^2 - s + \sqrt{\Lambda_{12}(s)}}{(m_1 + m_2)^2 - s - \sqrt{\Lambda_{12}(s)}} \right] \qquad (A.17)$$

The cosine of the angle between $\mathbf{p}_1$ and $\mathbf{p}_3$ is given by

$$z_s \equiv \cos(\theta_s) = \frac{\mathbf{p}_1 \cdot \mathbf{p}_3}{|\mathbf{p}_1||\mathbf{p}_3|} = \frac{(m_1 - m_2)^2 (m_1 + m_2)^2 - s(u - t)}{(m_1 - m_2)^2 (m_1 + m_2)^2 - s(u + t)} \qquad (A.18)$$

The angle θ_s is known as the scattering angle.

A.1. Physical Scattering Region

Requiring each of the energies in (A.11) to be real and non-negative leads to the condition

$$s \geq |m_1 - m_2|(m_1 + m_2) \tag{A.19}$$

Indeed, the same requirements on the magnitudes in (A.10) leads to a stronger condition:

$$\Lambda_{12}(s) \geq 0 \quad \implies \quad s \geq (m_1 + m_2)^2 > |m_1 - m_2|(m_1 + m_2) \tag{A.20}$$

We must also require z_s in (A.18) to satisfy

$$-1 \leq z_s \leq 1 \tag{A.21}$$

This is equivalent to the conditions

$$t \leq 0, \qquad su \leq (m_1 - m_2)^2(m_1 + m_2)^2 \tag{A.22}$$

Thus, the physical scattering region is defined by

$$s \geq (m_1 + m_2)^2, \qquad t \leq 0, \qquad su \leq (m_1 - m_2)^2(m_1 + m_2)^2 \tag{A.23}$$

Since these relations involve Lorentz invariants, they hold on any reference frame related to the center-of-momentum frame by a Lorentz transformation.

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