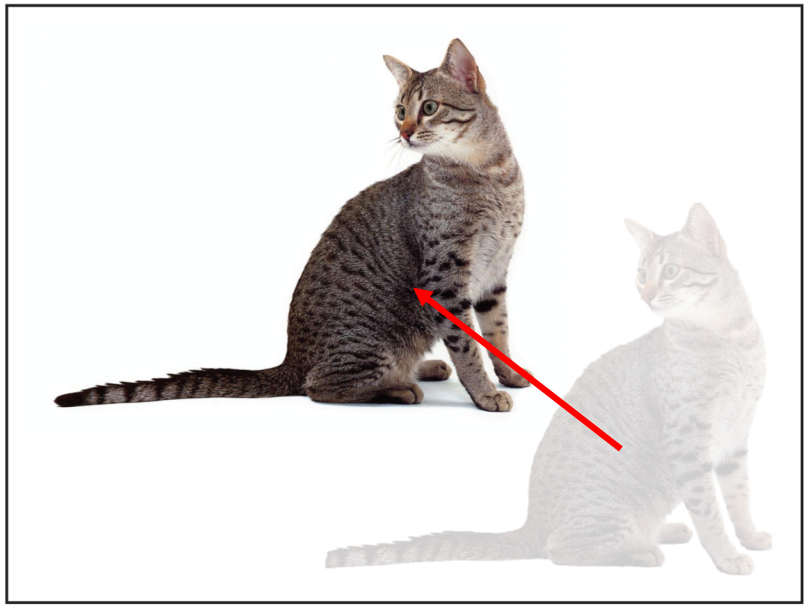
Invariance and equivariance

Shift invariance

Input $\mathbf{x}$

Output $f(\mathbf{x}) = 1$

Shifted input $S_v\mathbf{x}$



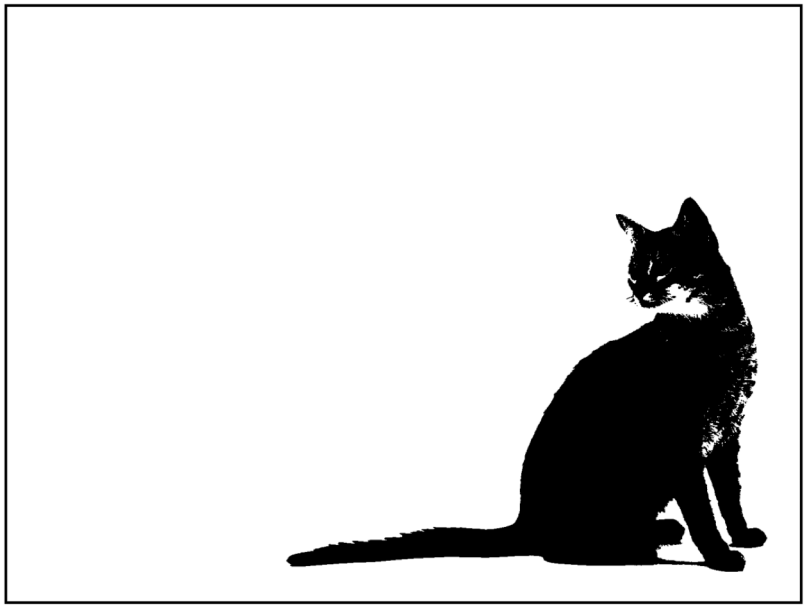
Output $f(S_v\mathbf{x}) = 1$

- **'Cat detector'** $f: \mathbb{R}^d \to \mathbb{R}$
- **Shift operator** $S_v: \mathbb{R}^d \to \mathbb{R}^d$ shifting the image by vector $\boldsymbol{v}$

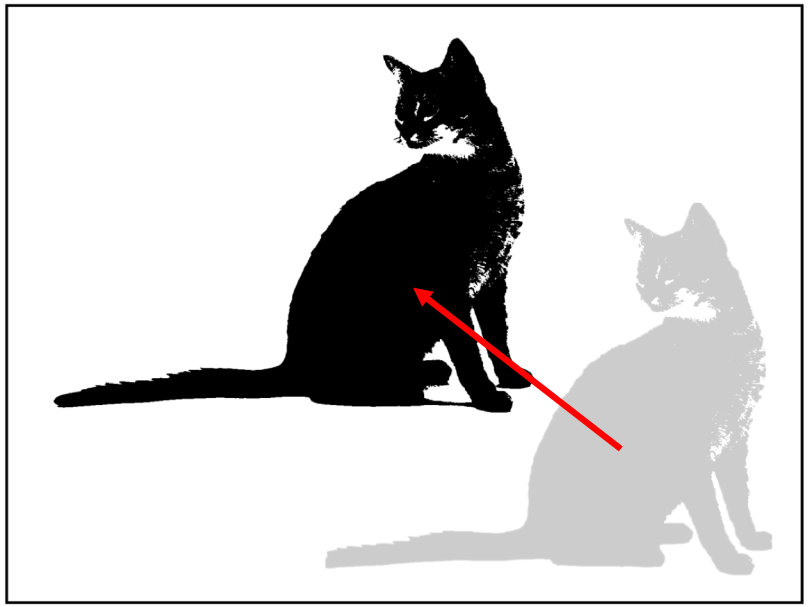
Shift invariance: $f(\mathbf{x}) = f(S_v\mathbf{x})$

Shift equivariance

Input $\mathbf{x}$



Shifted input $S_v\mathbf{x}$



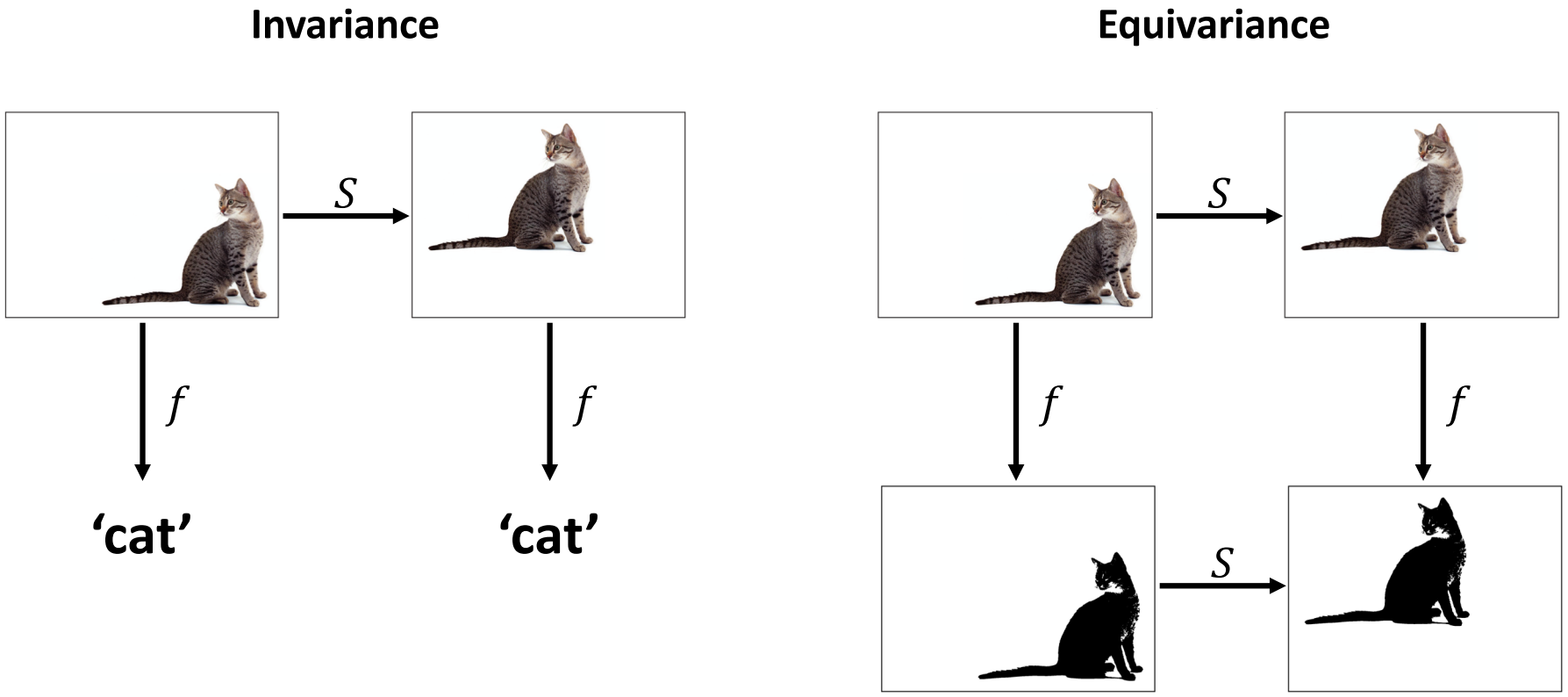
Output $f_i(\mathbf{x}) = \begin{cases} 1 & \text{pixel } i \in \text{cat} \\ 0 & \text{otherwise} \end{cases}$

Output $f_i(S_v\mathbf{x}) = \begin{cases} 1 & \text{pixel } i \in \text{cat} \\ 0 & \text{otherwise} \end{cases}$

- **'Cat segmentor' $\mathbf{f}$:** $\mathbb{R}^d \rightarrow \mathbb{R}^d$
- **Shift operator** S_v: $\mathbb{R}^d \rightarrow \mathbb{R}^d$ shifting the image by vector $\boldsymbol{v}$

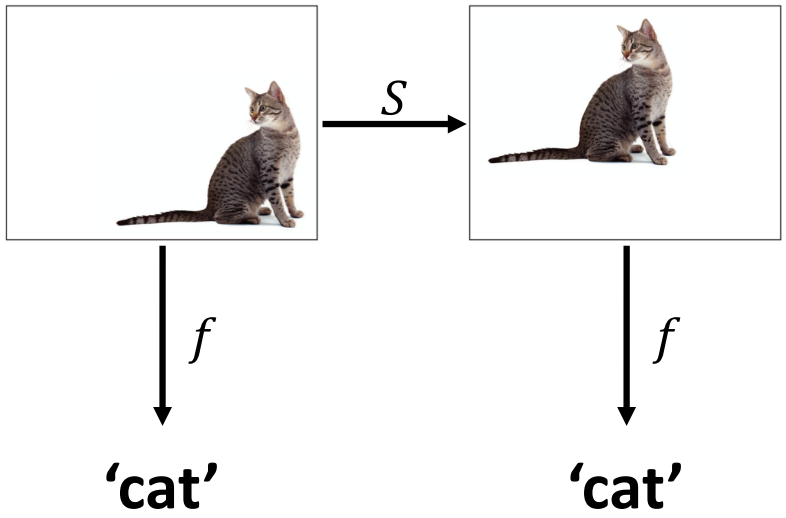
Shift equivariance: $S_v\mathbf{f}(\mathbf{x}) = \mathbf{f}(S_v\mathbf{x})$

Invariance vs equivariance

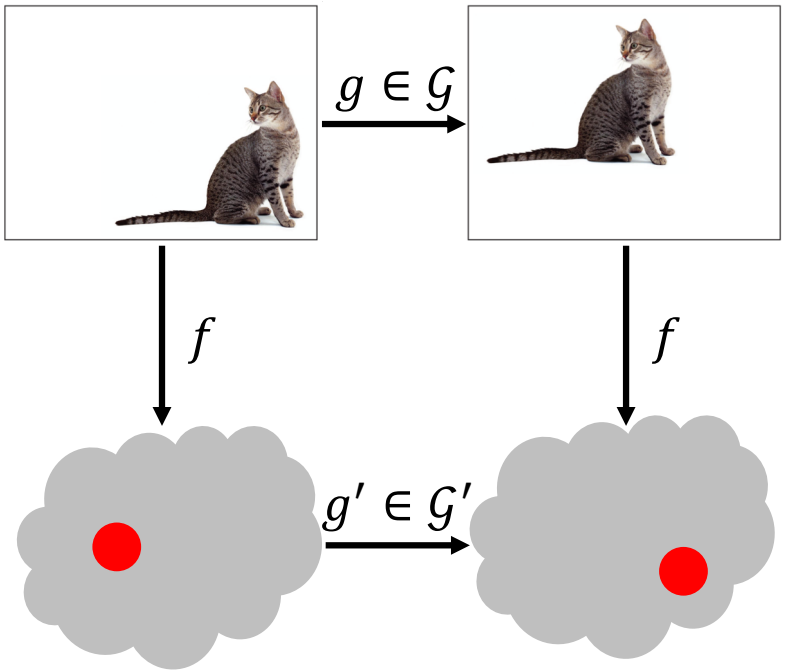


Invariance vs equivariance

Invariance



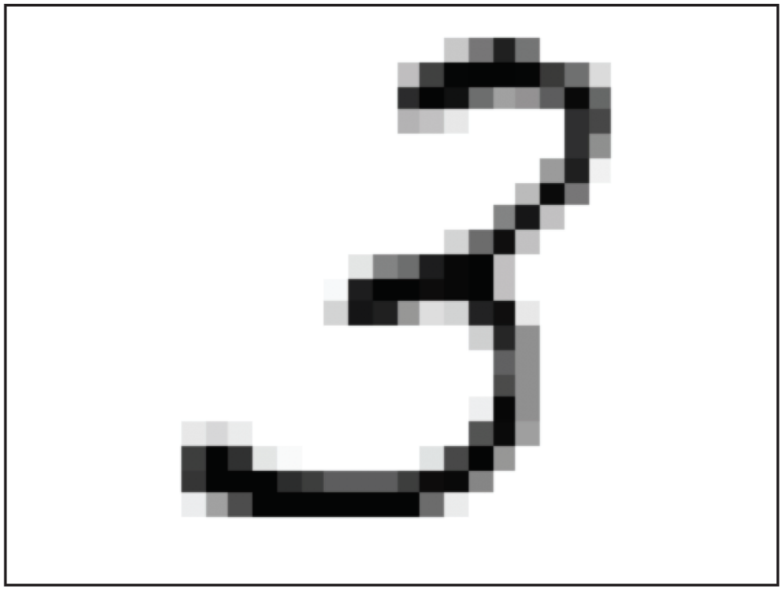
Equivariance



$$f\big(g(\mathbf{x})\big) = g'\big(f(\mathbf{x})\big)$$

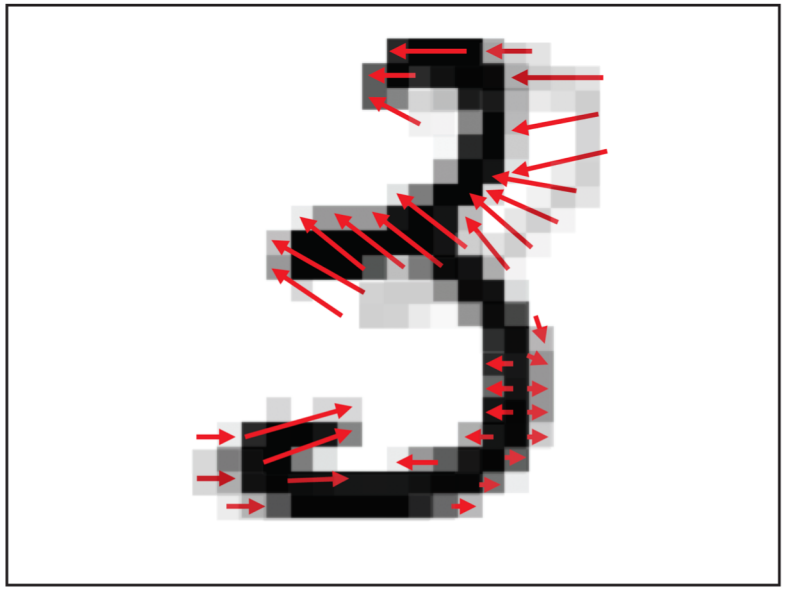
Approximate deformation invariance

Input $\mathbf{x}$



Output $f(\mathbf{x}) = 1$

Shifted input $S_v\mathbf{x}$



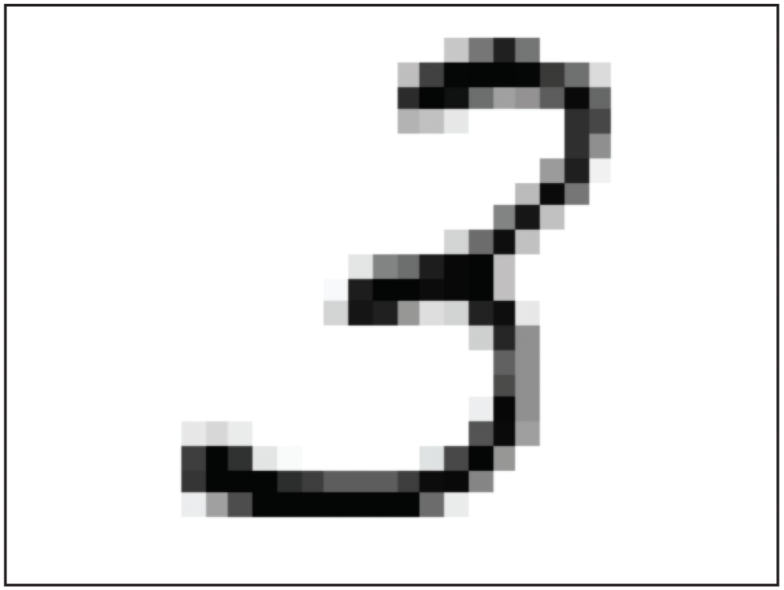
Output $f(D_{\boldsymbol{\tau}}\mathbf{x}) = 1$

- **'Digit 3 detector'** $f: \mathbb{R}^d \rightarrow \mathbb{R}$
- **Warp operator** $D_{\boldsymbol{\tau}}: \mathbb{R}^d \rightarrow \mathbb{R}^d$ warping the image by field $\boldsymbol{\tau}$

Deformation invariance: $f(\mathbf{x}) \approx f(D_{\boldsymbol{\tau}}\mathbf{x})$

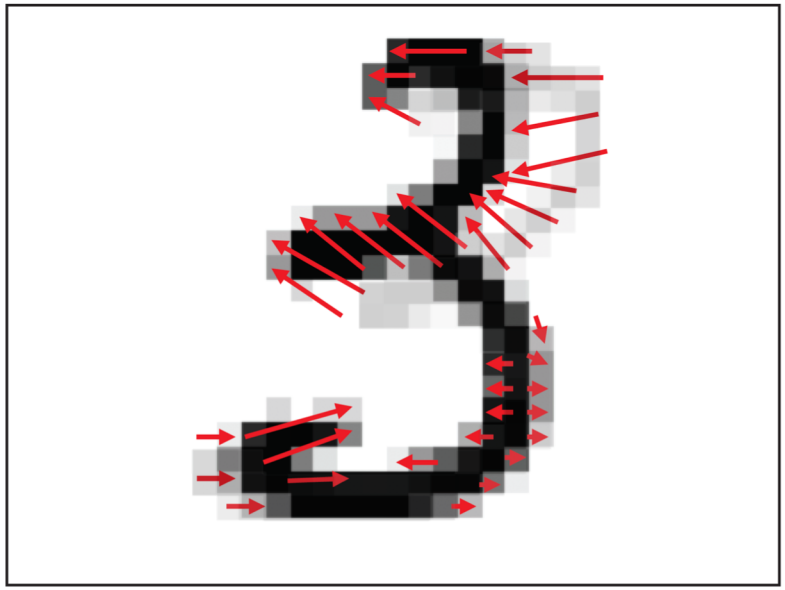
Approximate deformation invariance

Input $\mathbf{x}$



Output $f(\mathbf{x}) = 1$

Shifted input $S_v\mathbf{x}$

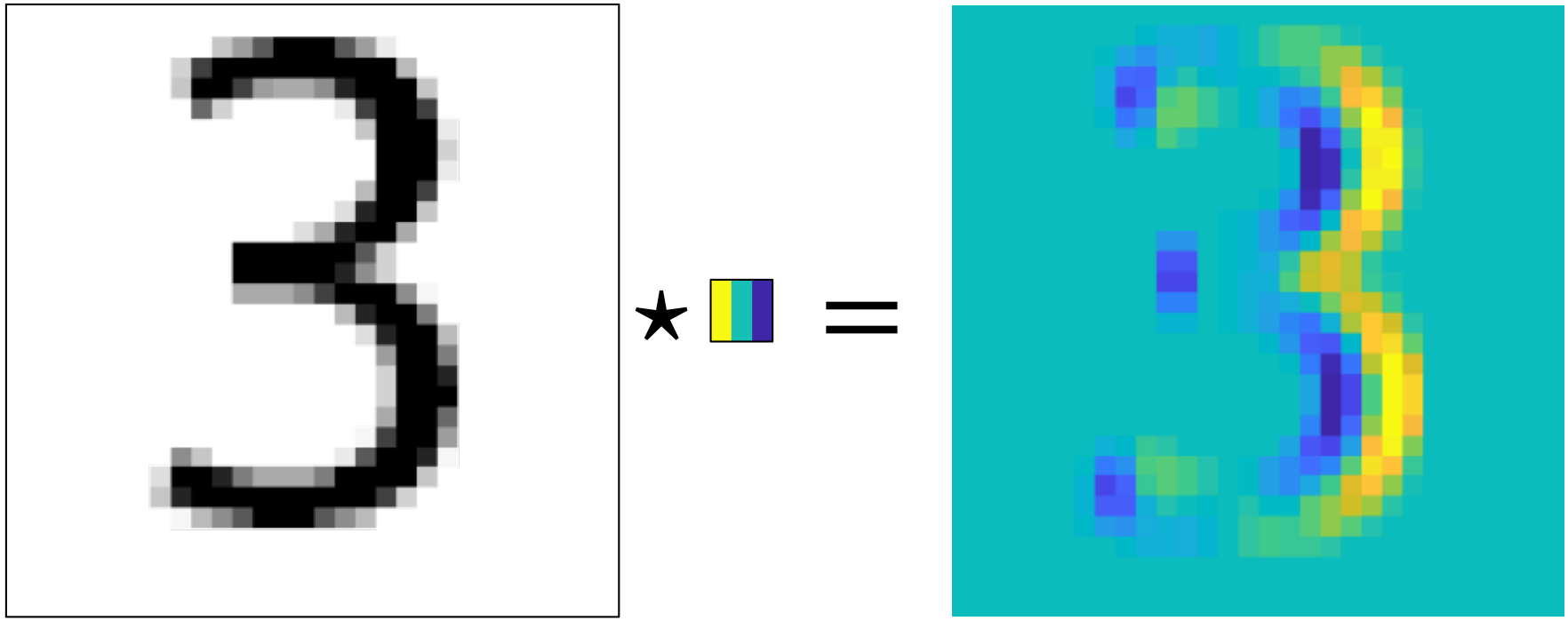


Output $f(D_{\boldsymbol{\tau}}\mathbf{x}) = 1$

- **'Digit 3 detector'** $f: \mathbb{R}^d \rightarrow \mathbb{R}$
- **Warp operator** $D_{\boldsymbol{\tau}}: \mathbb{R}^d \rightarrow \mathbb{R}^d$ warping the image by field $\boldsymbol{\tau}$

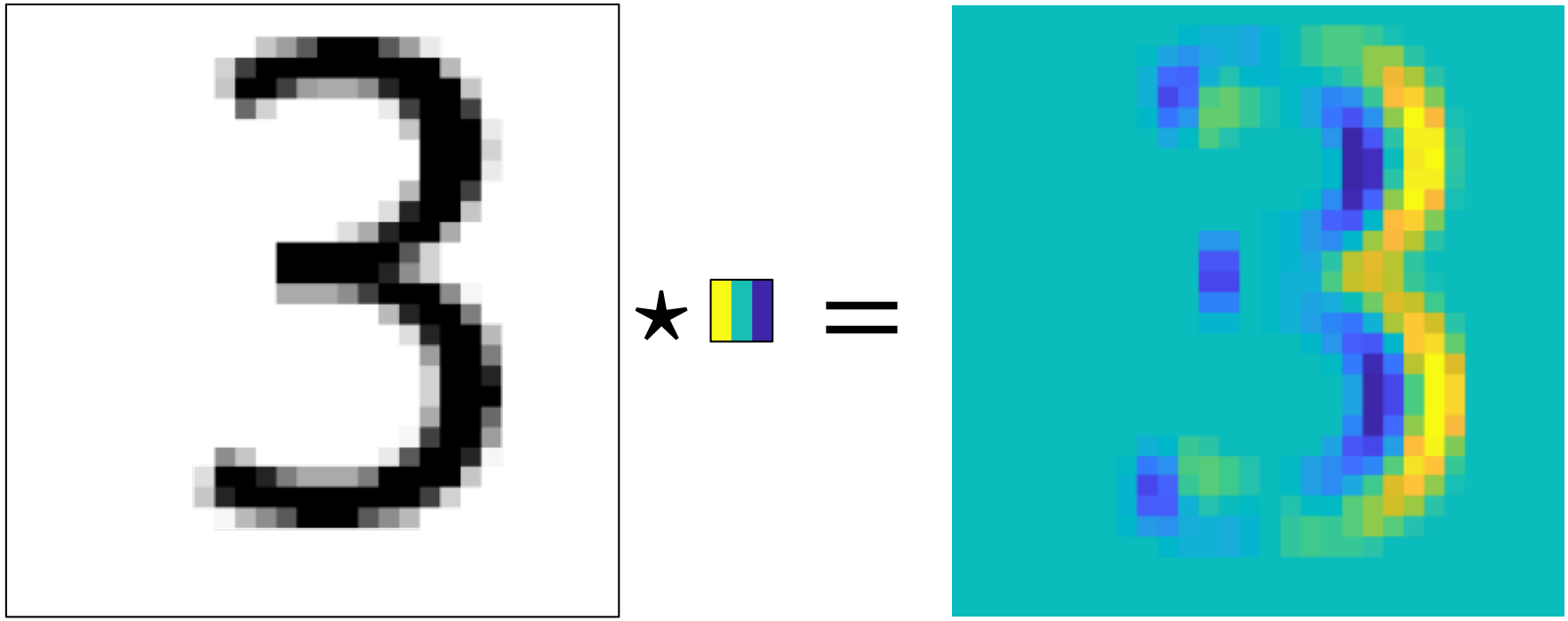
$$\|f(\mathbf{x}) - f(D_{\boldsymbol{\tau}}\mathbf{x})\| \approx \|\nabla\boldsymbol{\tau}\|$$

Equivariance in CNNs



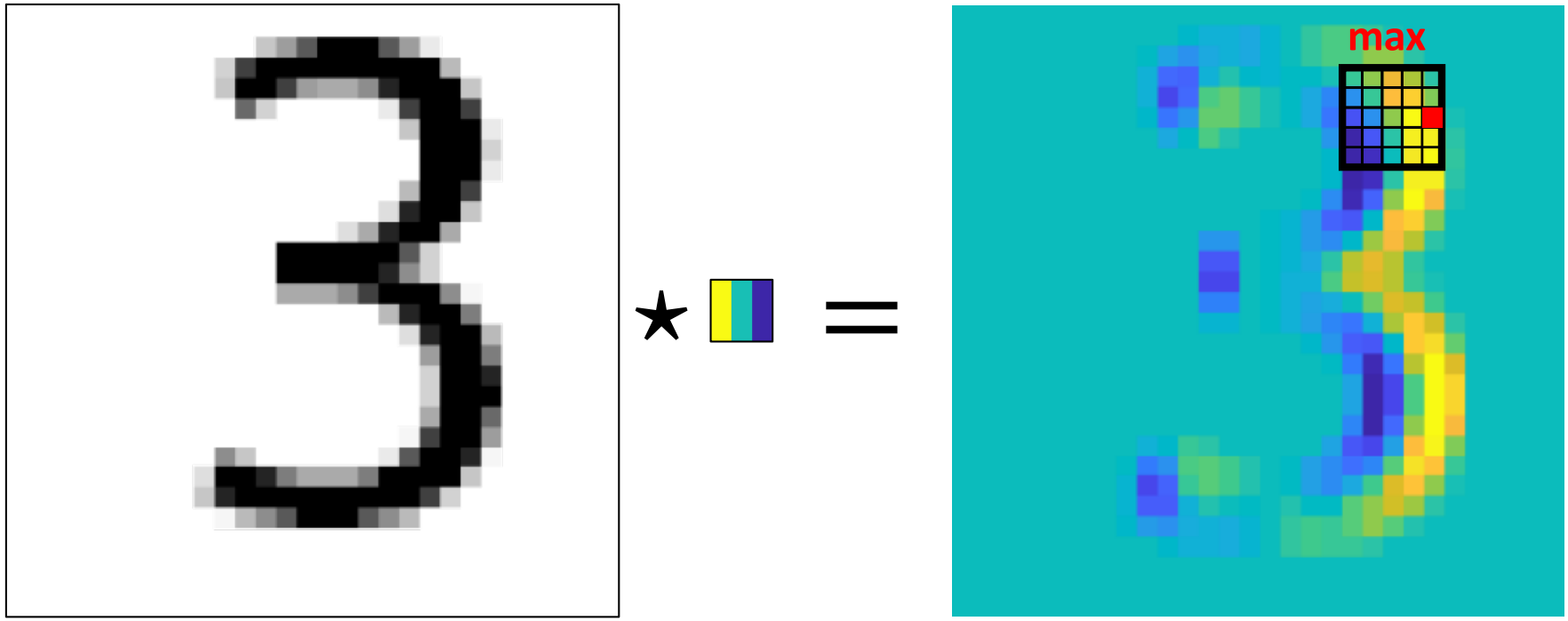
Output of convolutional layer (shift equivariant)

Equivariance in CNNs



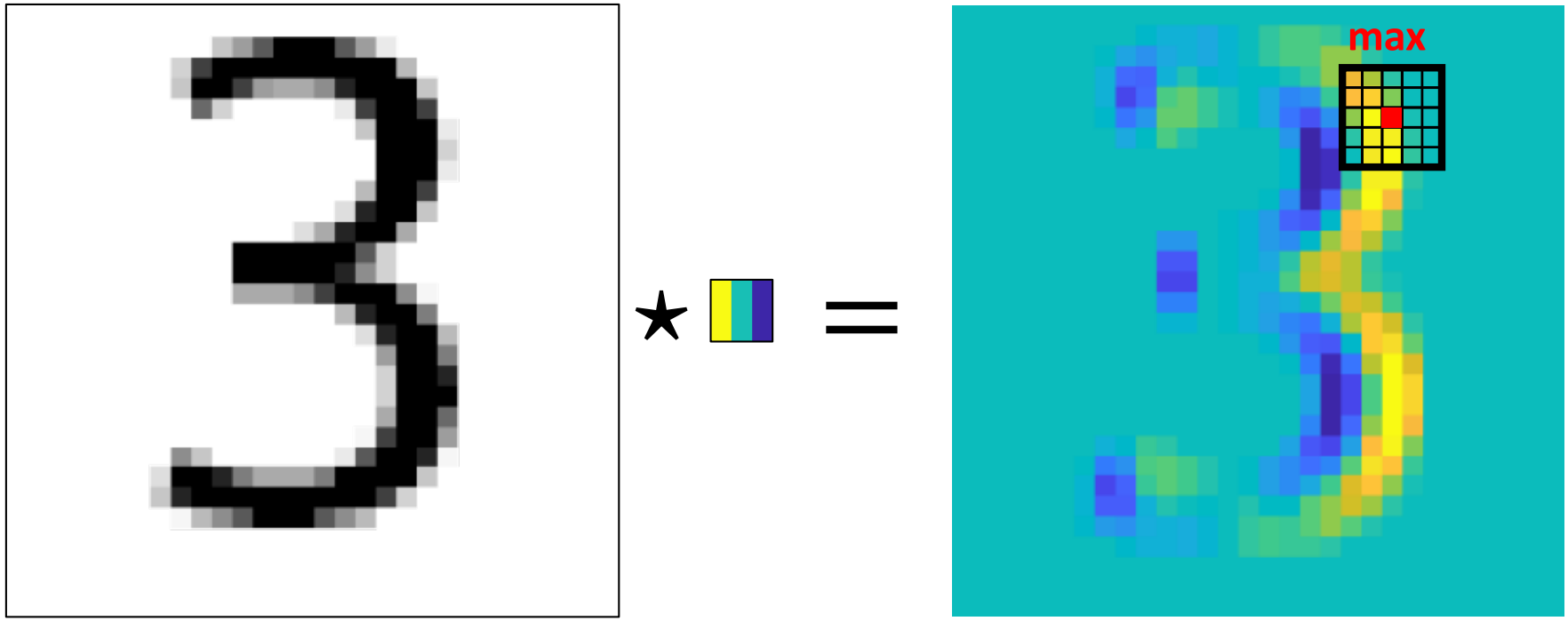
Output of convolutional layer (shift equivariant)

Approximate invariance in CNNs with pooling



Output of convolutional layer+max pooling (~shift invariant)

Approximate invariance in CNNs with pooling



Output of convolutional layer+max pooling (~shift invariant)