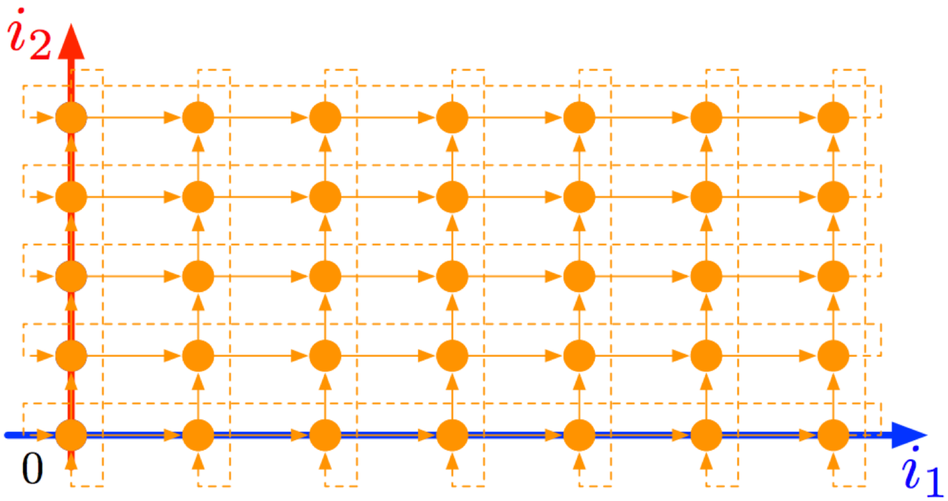
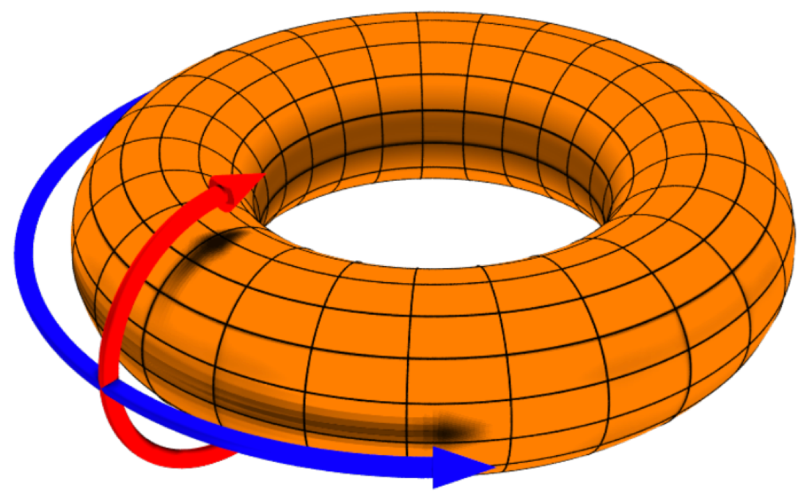
Convolution in 2D

Bamieh, arXiv:1805.05533

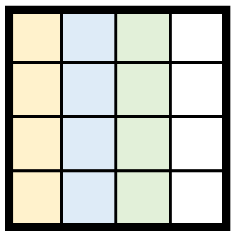
2D convolution





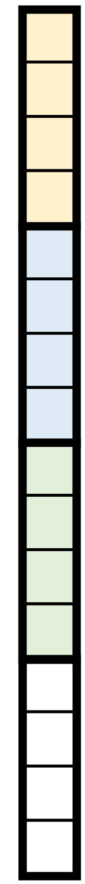
Two views

Matrix



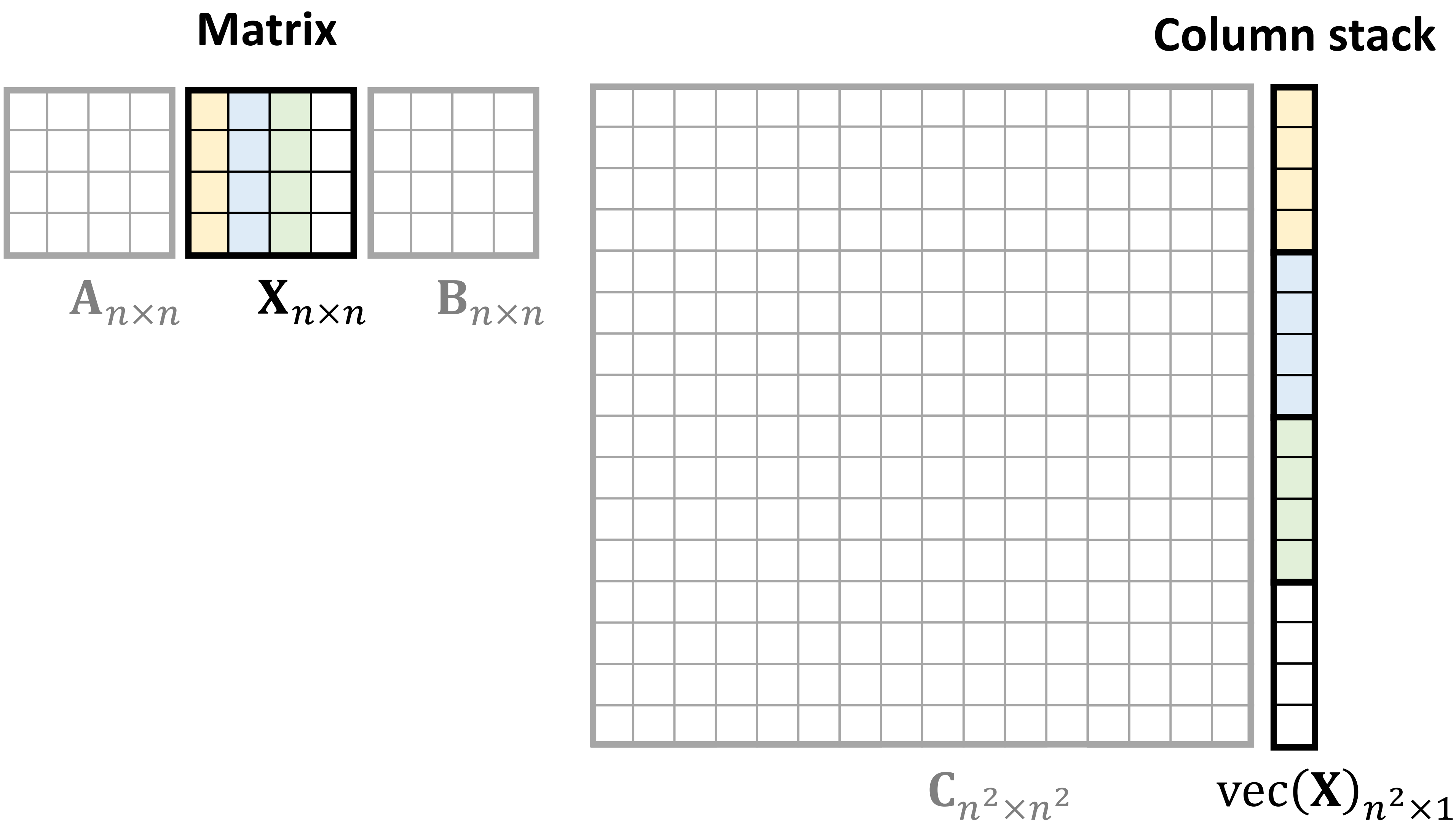
$\mathbf{X}_{n \times n}$

Column stack

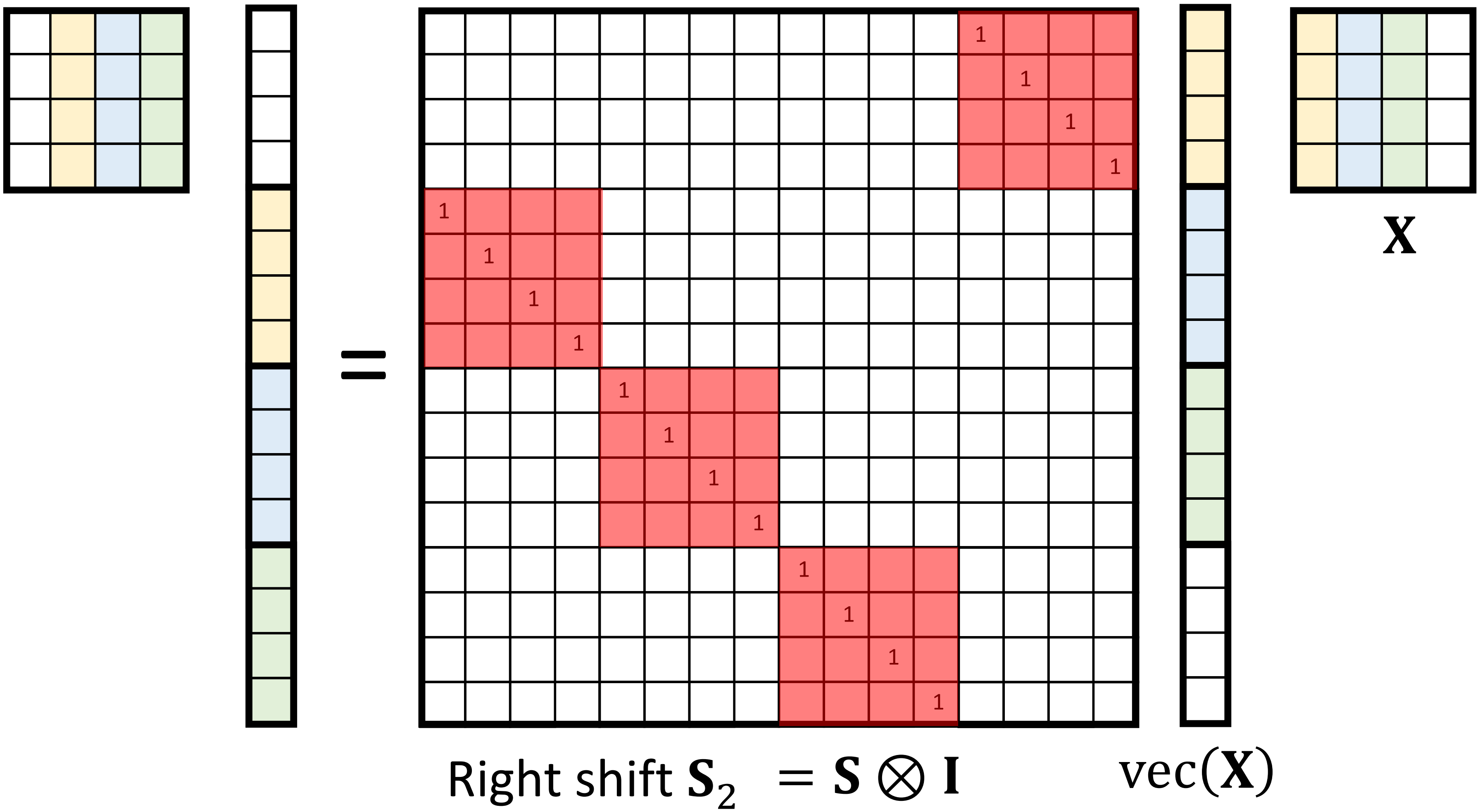


$\mathrm{vec}(\mathbf{X})_{n^2 \times 1}$

Two views


Matrix
Column stack
$\mathbf{A}_{n\times n}$
$\mathbf{X}_{n\times n}$
$\mathbf{B}_{n\times n}$
$\mathbf{C}_{n^2\times n^2}$
$\mathrm{vec}(\mathbf{X})_{n^2\times 1}$

2D shift operators



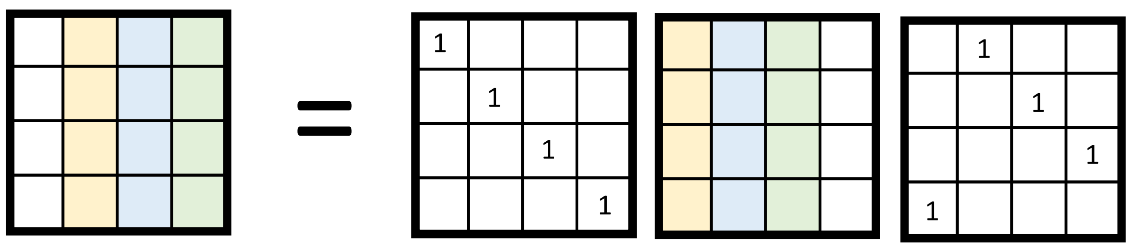
Kronecker product

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix}$$

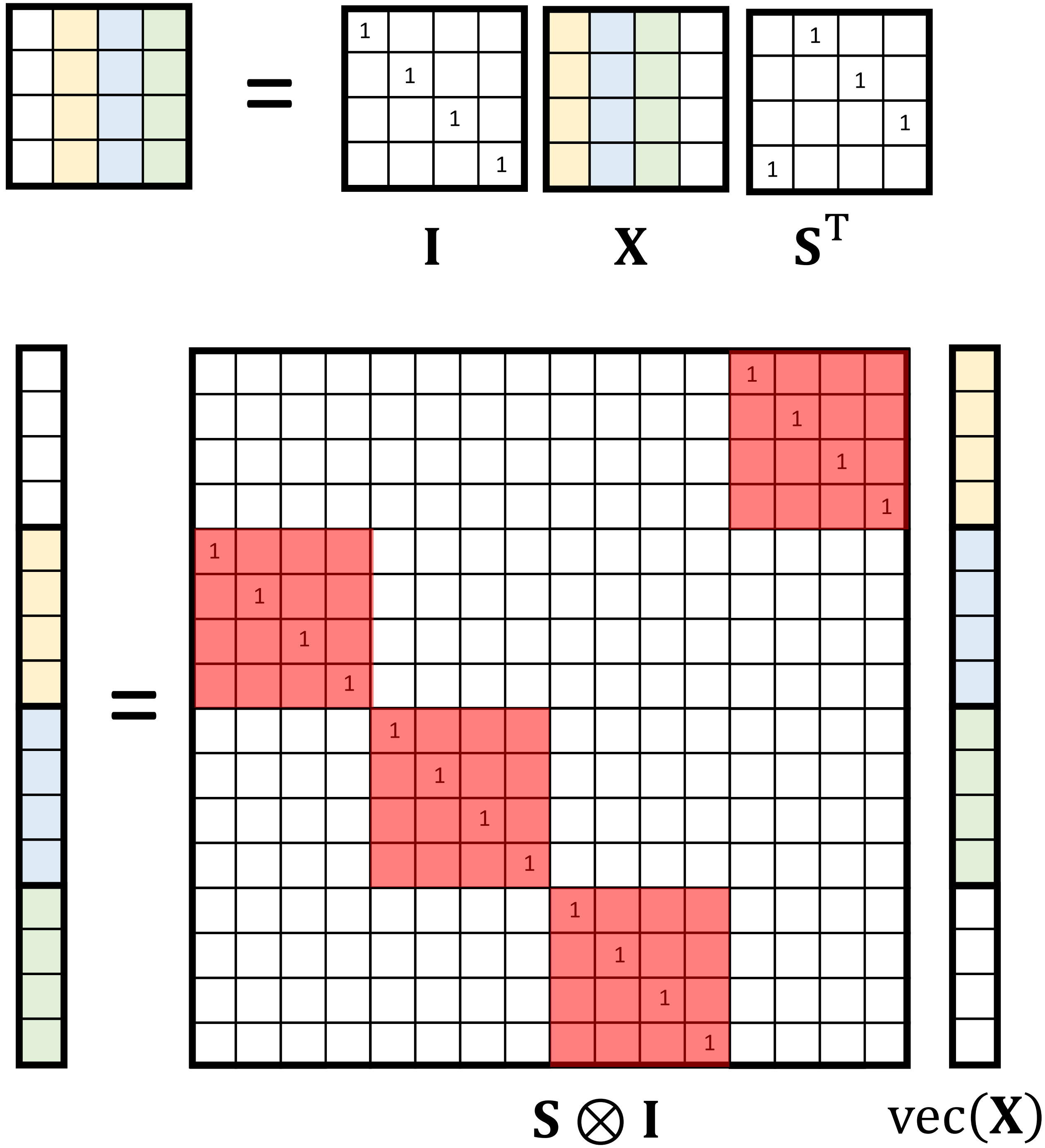
- “Block matrix” operation
- Non-commutative
- Transpose: $(\mathbf{A} \otimes \mathbf{B})^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}} \otimes \mathbf{B}^{\mathrm{T}}$
- Product: $(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = (\mathbf{AC}) \otimes (\mathbf{BD})$
- Equivalent vector operation

$$\left(\mathbf{B}^{\mathrm{T}} \otimes \mathbf{A}\right) \mathrm{vec}(\mathbf{X}) = \mathrm{vec}(\mathbf{AXB})$$

2D shift operators

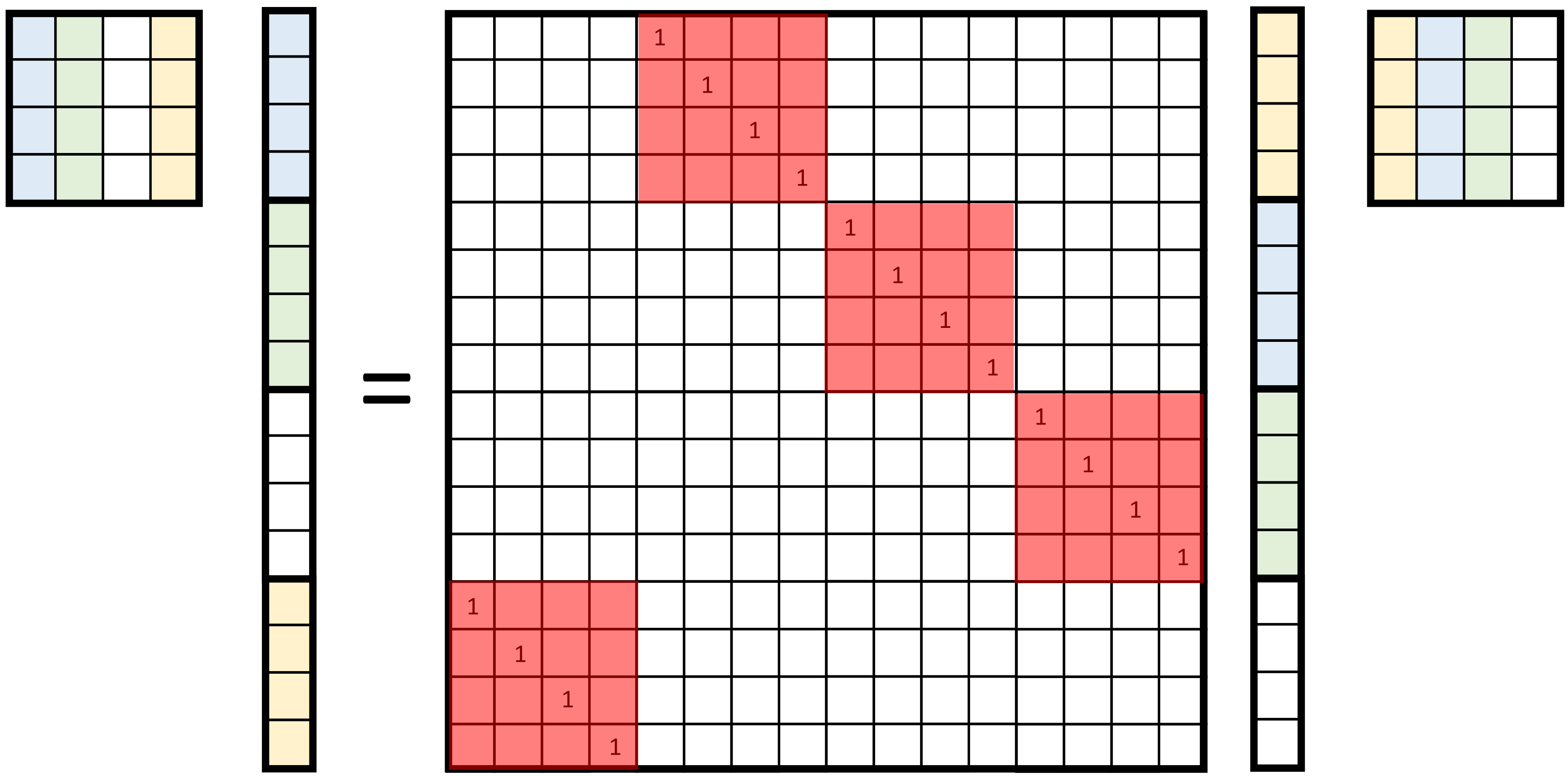


$$(\mathbf{S} \otimes \mathbf{I})\ \mathrm{vec}(\mathbf{X}) = \mathrm{vec}\left(\mathbf{I}\mathbf{X}\mathbf{S}^{\mathrm{T}}\right)$$



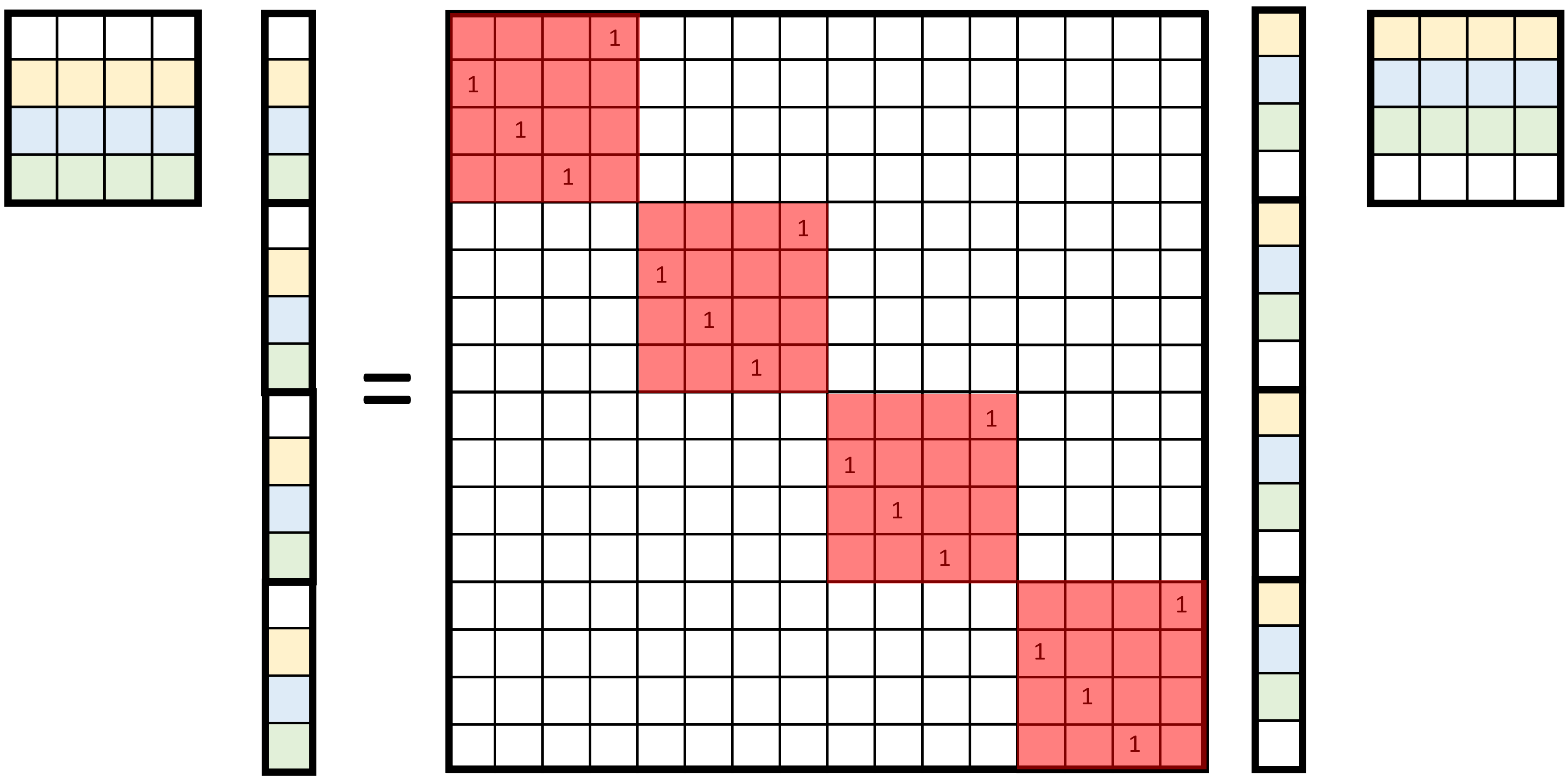
=
I
X
$\mathbf{S}^{\mathrm{T}}$
=
$\mathbf{S} \otimes \mathbf{I}$
vec(**X**)

2D shift operators



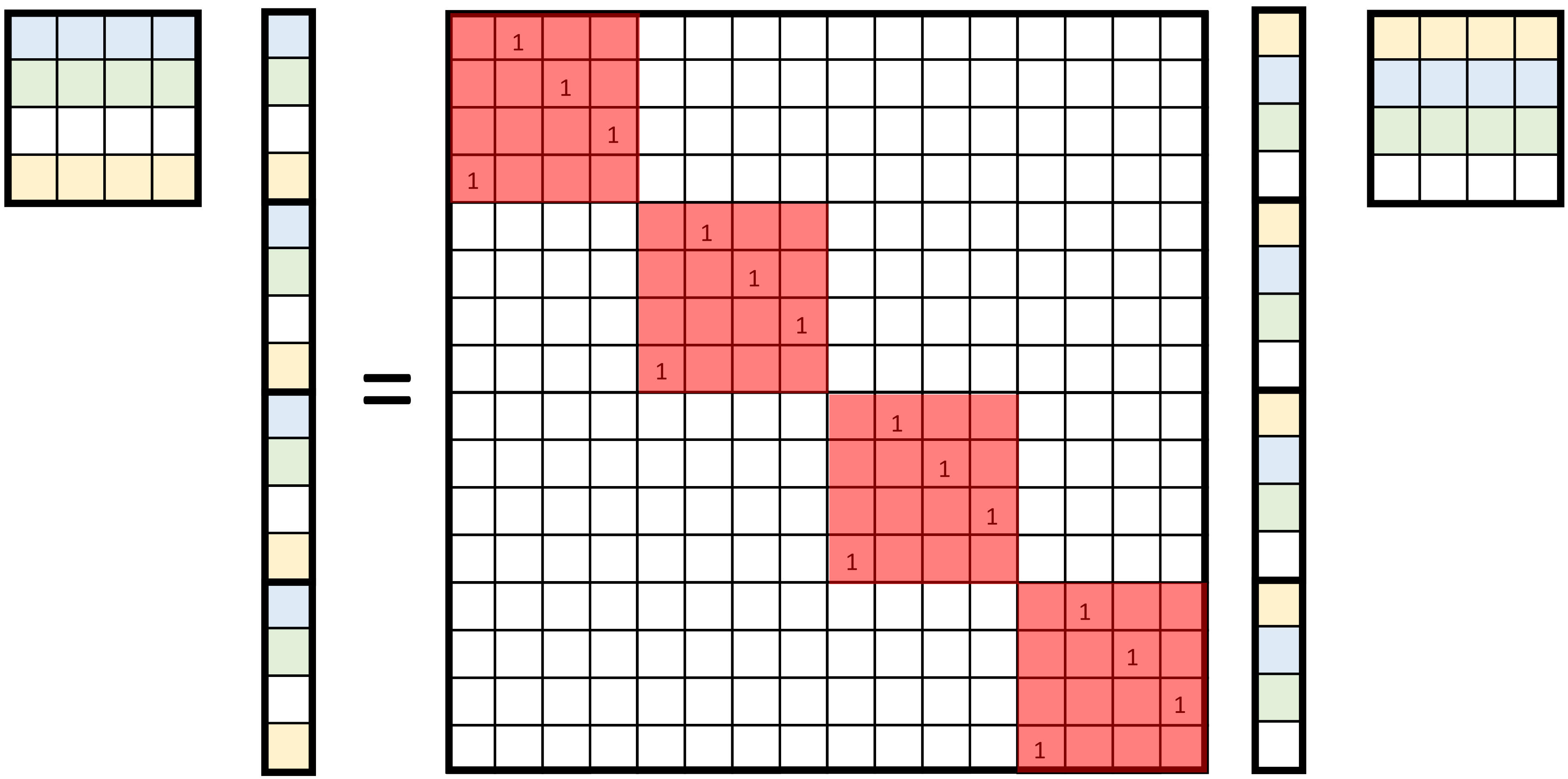
Left shift $\mathbf{S}_2^{\mathrm{T}} = \mathbf{S}^{\mathrm{T}} \otimes \mathbf{I}$

2D shift operators



Up shift $\mathbf{S}_1 = \mathbf{I} \otimes \boldsymbol{S}$

2D shift operators



Down shift $\mathbf{S}_1^{\mathbf{T}} = \mathbf{I} \otimes \mathbf{S}^{\mathbf{T}}$

2D Fourier transform

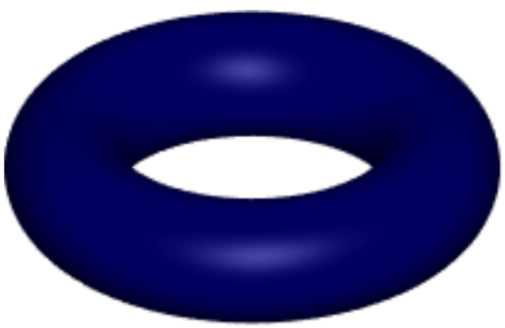
$$(\mathbf{V}^* \otimes \mathbf{V})\,\mathrm{vec}(\mathbf{X}) = \mathrm{vec}(\mathbf{V}^*\mathbf{X}\mathbf{V})$$

apply FFT row-wise then inverse FFT column-wise

2D Fourier basis

$$\mathbf{W} = \mathbf{V}^* \otimes \mathbf{V}$$

$$\mathbf{V} = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w^1 & w^2 & & w^{n-1} \\ 1 & w^2 & w^4 & & w^{2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & w^{n-1} & w^{n-1} & \cdots & w^{(n-1)^2} \end{bmatrix} \qquad w = e^{\iota \frac{2\pi}{n}}$$



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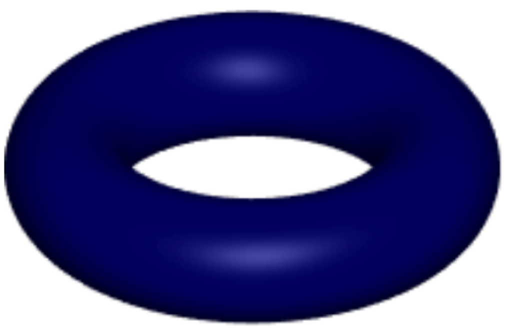

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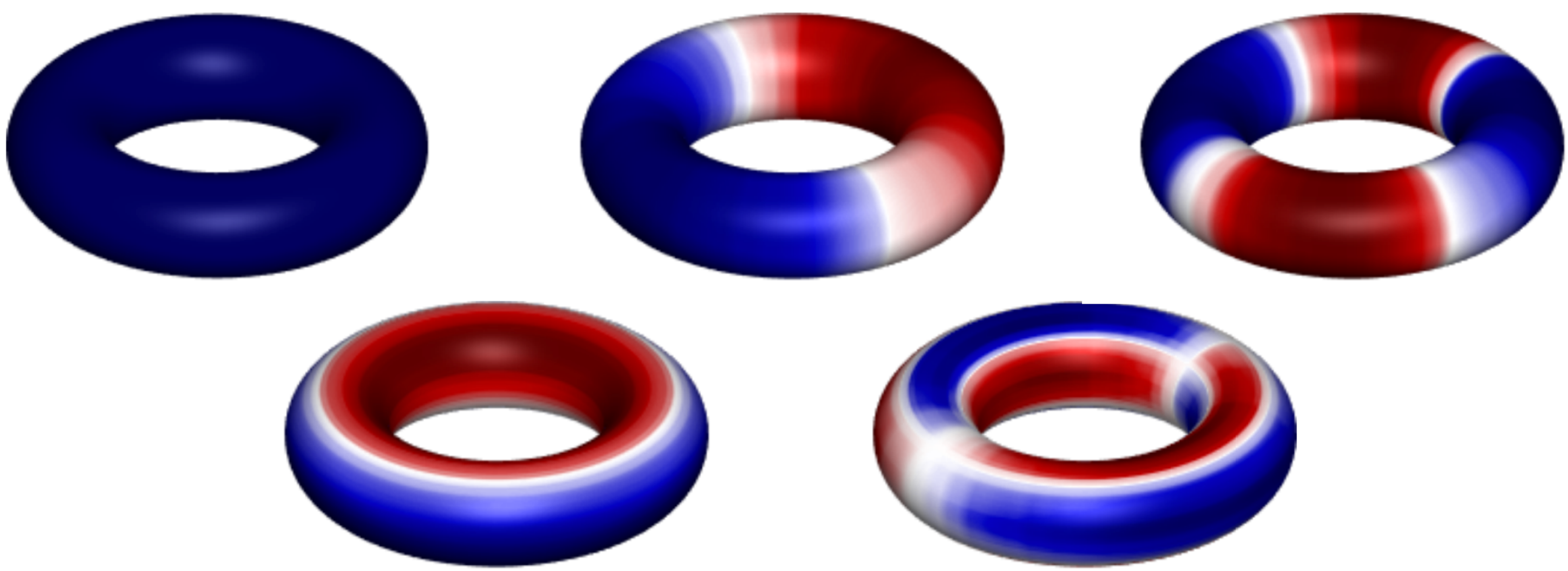

2D Fourier transform

$\boxed{\mathbf{W} = \mathbf{V}^* \otimes \mathbf{V} \text{ diagonalizes the 2D shift operator}}$

$$
\begin{aligned}
(\mathbf{V}^* \otimes \mathbf{V})(\mathbf{S} \otimes \mathbf{I})(\mathbf{V}^* \otimes \mathbf{V})^* &= \big((\mathbf{V}^*\mathbf{S}) \otimes \mathbf{V}\big)(\mathbf{V} \otimes \mathbf{V}^*) \\
&= (\mathbf{V}^*\mathbf{S}\mathbf{V}) \otimes (\mathbf{V}\mathbf{V}^*) \\
&= \boldsymbol{\Lambda} \otimes \mathbf{I}
\end{aligned}
$$