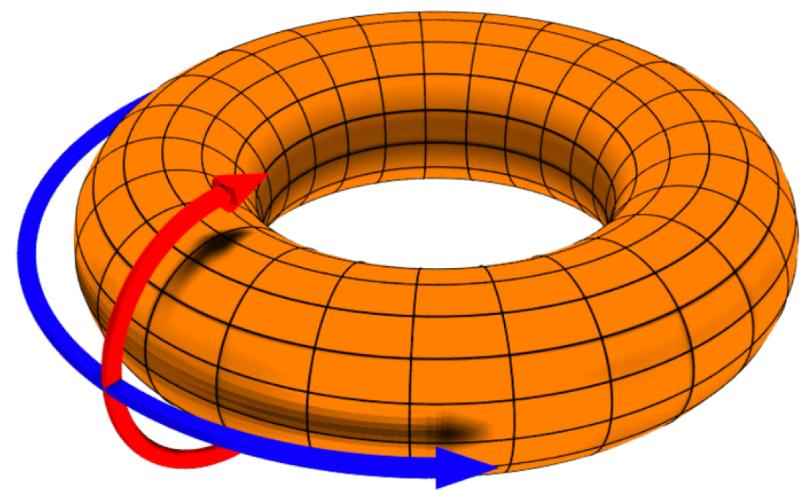
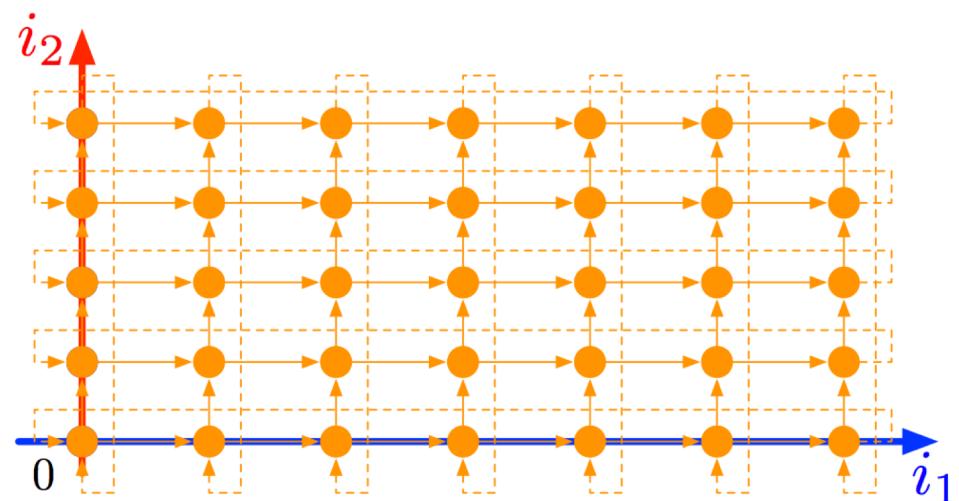


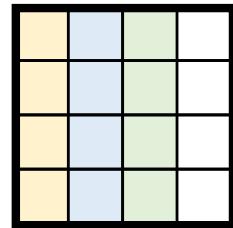
# Convolution in 2D

# 2D convolution



# Two views

**Matrix**



$\mathbf{X}_{n \times n}$

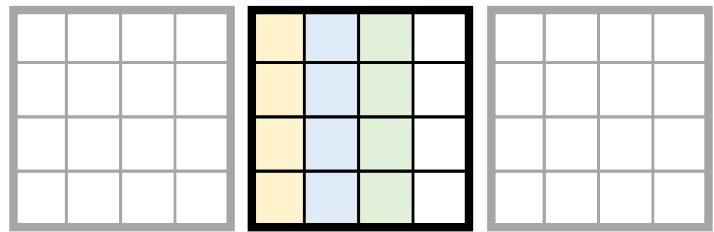
**Column stack**



$\text{vec}(\mathbf{X})_{n^2 \times 1}$

# Two views

**Matrix**

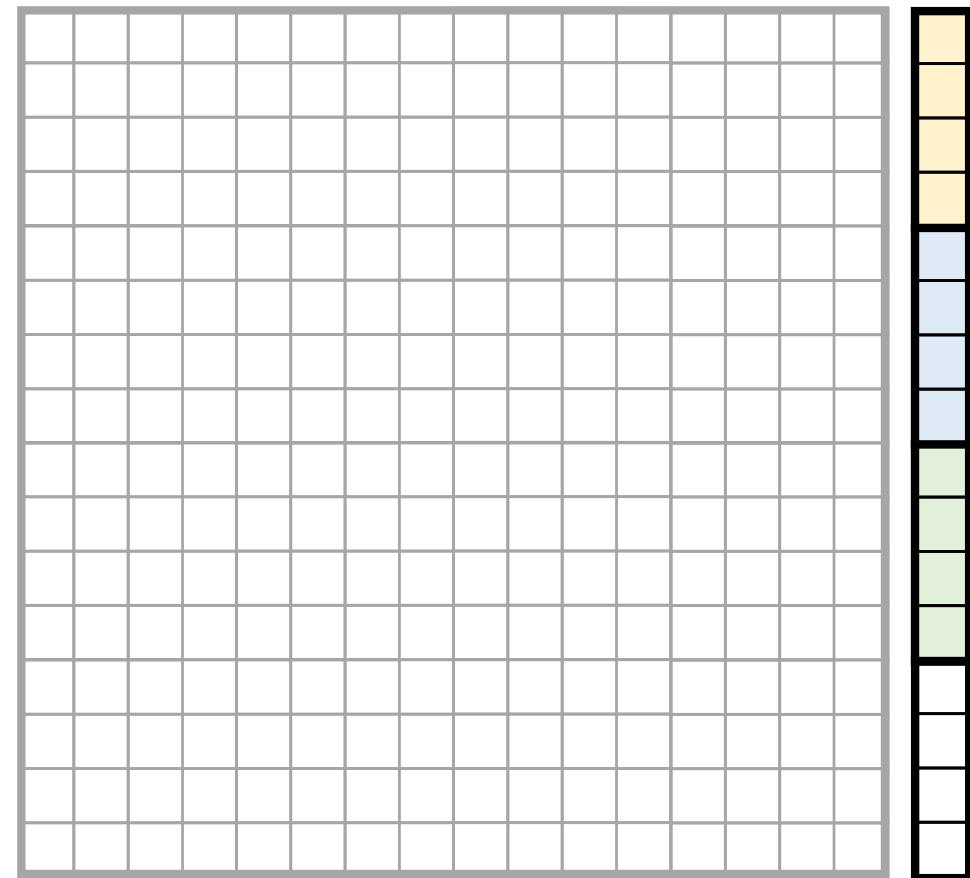


$A_{n \times n}$

$X_{n \times n}$

$B_{n \times n}$

**Column stack**



$C_{n^2 \times n^2}$

$\text{vec}(X)_{n^2 \times 1}$

# 2D shift operators

$$\begin{matrix} \text{Right shift } \mathbf{S}_2 = \mathbf{S} \otimes \mathbf{I} & \text{vec}(\mathbf{X}) \end{matrix}$$

The diagram illustrates the construction of a 2D shift operator  $\mathbf{S}_2$  from a 1D shift operator  $\mathbf{S}$  and a vector  $\mathbf{X}$ . On the left, a 1D shift operator  $\mathbf{S}$  is shown as a 4x4 grid with colored blocks (yellow, light blue, green) and a vertical vector  $\mathbf{X}$  below it. An equals sign follows. In the center, a large 2D shift operator  $\mathbf{S}_2$  is shown as a 16x16 grid. It consists of four 4x4 blocks of red color with the number '1' in each center cell, arranged in a 2x2 pattern. To the right, a vertical vector  $\text{vec}(\mathbf{X})$  is shown as a 16x1 column vector with colored blocks corresponding to the vector  $\mathbf{X}$ .

# Kronecker product

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix}$$

- “Block matrix” operation
- Non-commutative
- Transpose:  $(\mathbf{A} \otimes \mathbf{B})^T = \mathbf{A}^T \otimes \mathbf{B}^T$
- Product:  $(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = (\mathbf{AC}) \otimes (\mathbf{BD})$
- Equivalent vector operation

$$(\mathbf{B}^T \otimes \mathbf{A}) \text{vec}(\mathbf{X}) = \text{vec}(\mathbf{AXB})$$

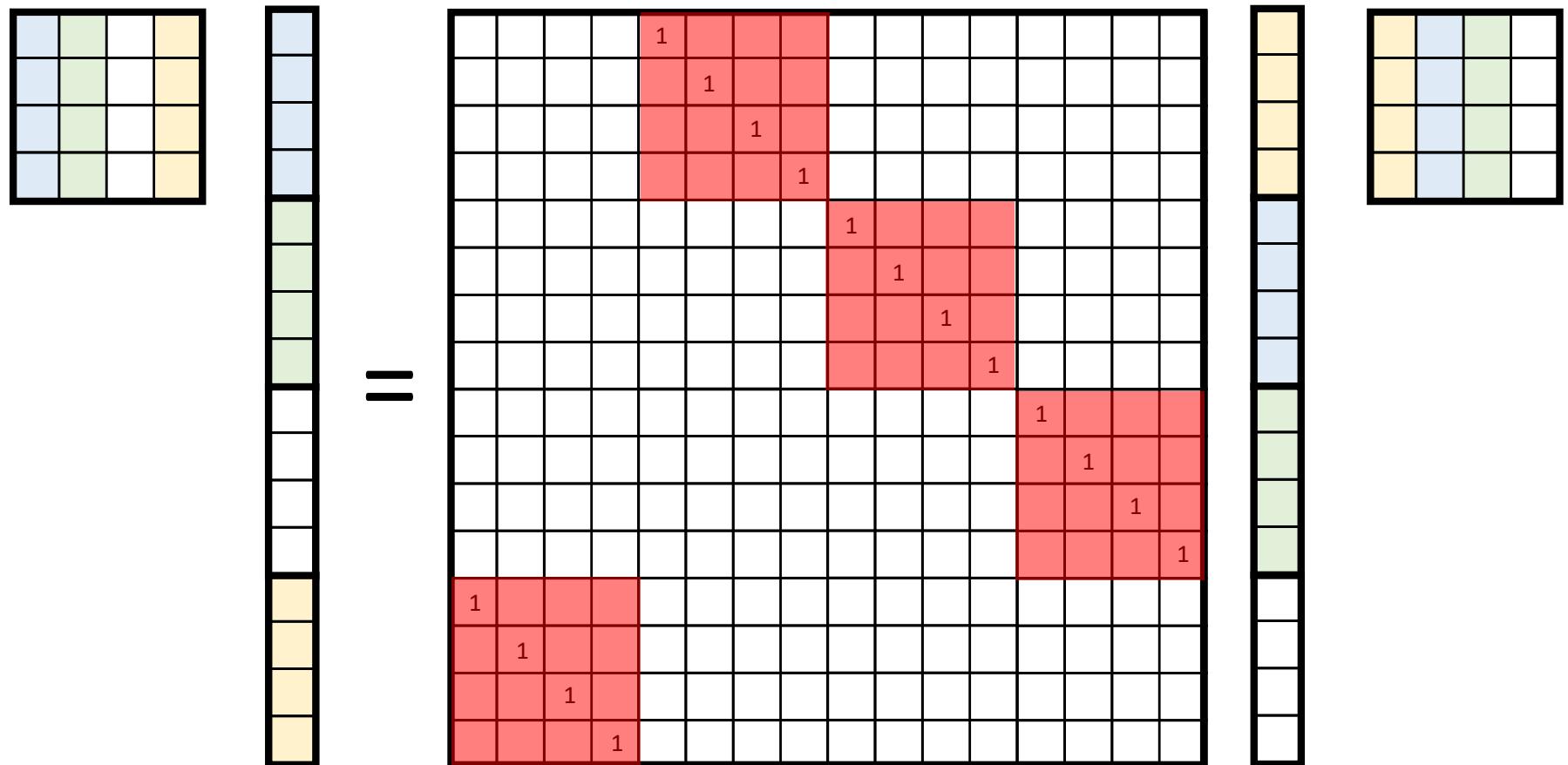
# 2D shift operators

$$\begin{array}{c|c|c|c|c} \textcolor{white}{\cdot} & \textcolor{brown}{\cdot} & \textcolor{blue}{\cdot} & \textcolor{teal}{\cdot} & \textcolor{green}{\cdot} \\ \textcolor{brown}{\cdot} & \textcolor{brown}{\cdot} & \textcolor{white}{\cdot} & \textcolor{white}{\cdot} & \textcolor{white}{\cdot} \\ \textcolor{white}{\cdot} & \textcolor{brown}{\cdot} & \textcolor{blue}{\cdot} & \textcolor{teal}{\cdot} & \textcolor{green}{\cdot} \\ \textcolor{white}{\cdot} & \textcolor{brown}{\cdot} & \textcolor{blue}{\cdot} & \textcolor{teal}{\cdot} & \textcolor{green}{\cdot} \\ \textcolor{white}{\cdot} & \textcolor{brown}{\cdot} & \textcolor{blue}{\cdot} & \textcolor{teal}{\cdot} & \textcolor{green}{\cdot} \end{array} = \begin{array}{c|c|c|c|c} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{array} \begin{array}{c|c|c|c|c} \textcolor{brown}{\cdot} & \textcolor{blue}{\cdot} & \textcolor{teal}{\cdot} & \textcolor{green}{\cdot} & \textcolor{white}{\cdot} \\ \textcolor{brown}{\cdot} & \textcolor{blue}{\cdot} & \textcolor{teal}{\cdot} & \textcolor{green}{\cdot} & \textcolor{white}{\cdot} \\ \textcolor{brown}{\cdot} & \textcolor{blue}{\cdot} & \textcolor{teal}{\cdot} & \textcolor{green}{\cdot} & \textcolor{white}{\cdot} \\ \textcolor{brown}{\cdot} & \textcolor{blue}{\cdot} & \textcolor{teal}{\cdot} & \textcolor{green}{\cdot} & \textcolor{white}{\cdot} \\ \textcolor{brown}{\cdot} & \textcolor{blue}{\cdot} & \textcolor{teal}{\cdot} & \textcolor{green}{\cdot} & \textcolor{white}{\cdot} \end{array} \begin{array}{c|c|c|c|c} & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \\ 1 & & & & \end{array}$$

$$(\mathbf{S} \otimes \mathbf{I}) \operatorname{vec}(\mathbf{X}) = \operatorname{vec}(\mathbf{IXS}^T)$$

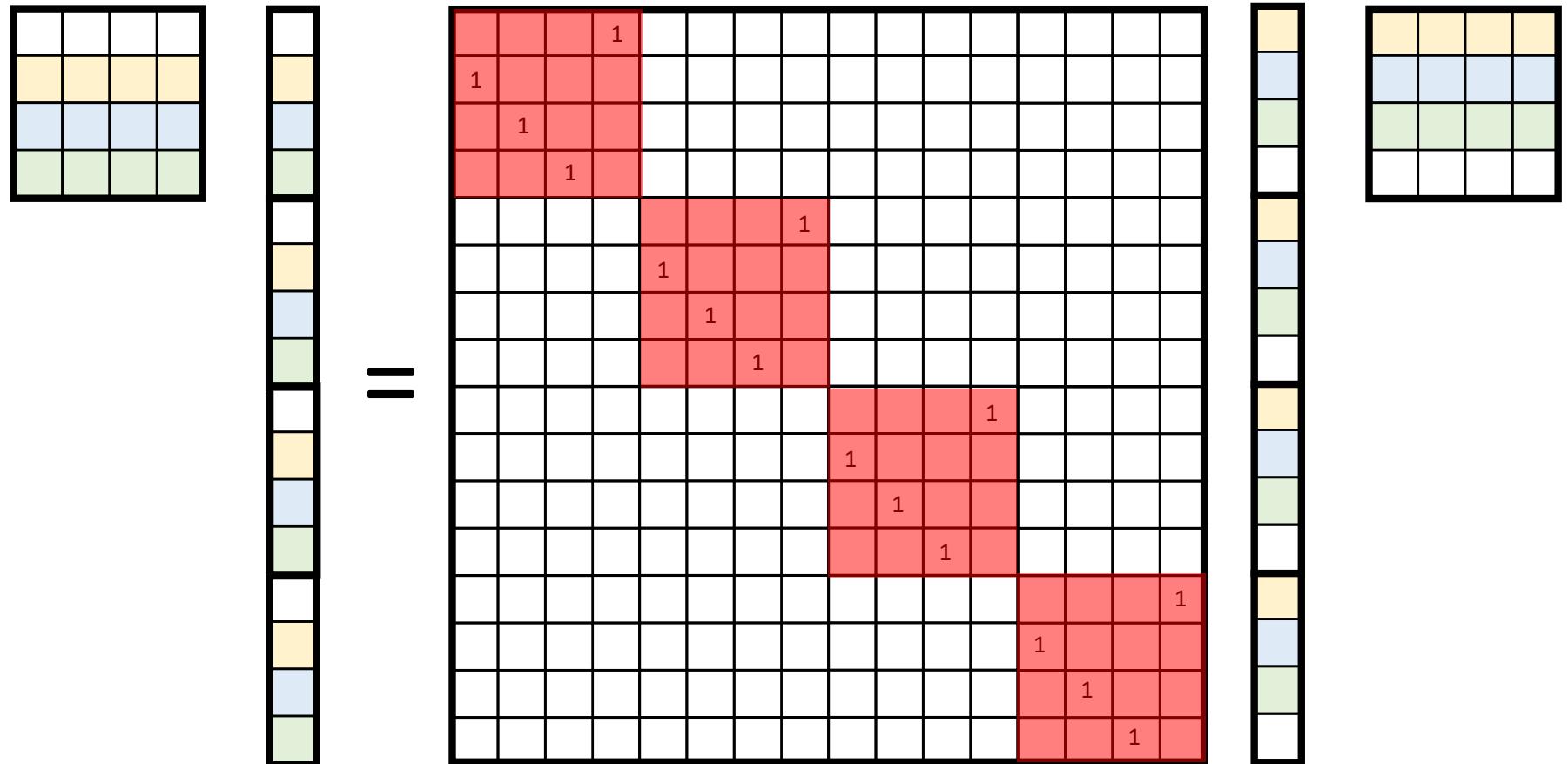
$$\begin{matrix} \text{I} & = & \begin{matrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{matrix} & \begin{matrix} \text{X} \\ \text{S}^T \end{matrix} \\ & & \begin{matrix} \text{I} & \text{X} & \text{S}^T \end{matrix} & \end{matrix}$$

# 2D shift operators



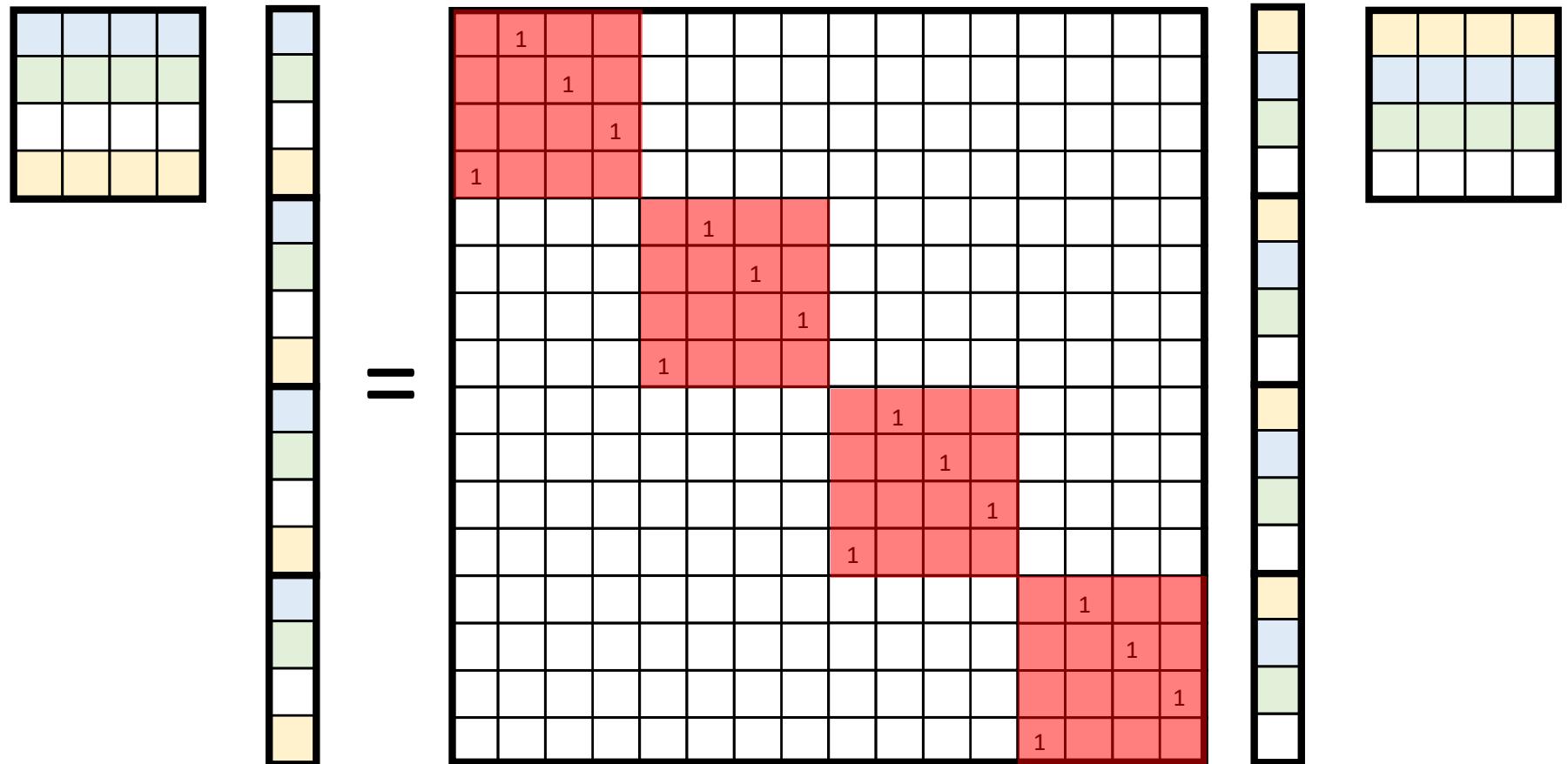
$$\text{Left shift } S_2^T = S^T \otimes I$$

# 2D shift operators



$$\text{Up shift } S_1 = I \otimes S$$

# 2D shift operators



$$\text{Down shift } S^T_1 = I \otimes S^T$$

## 2D Fourier transform

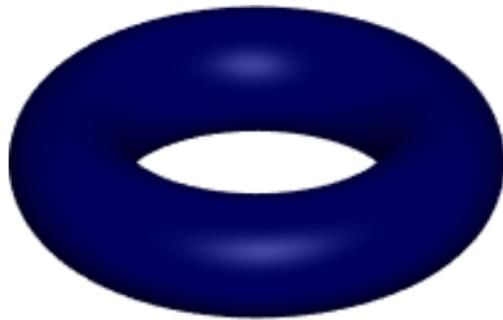
$$(\mathbf{V}^* \otimes \mathbf{V}) \text{vec}(\mathbf{X}) = \text{vec}(\mathbf{V}^* \mathbf{X} \mathbf{V})$$

**apply FFT row-wise then inverse FFT column-wise**

# 2D Fourier basis

$$\mathbf{W} = \mathbf{V}^* \otimes \mathbf{V}$$

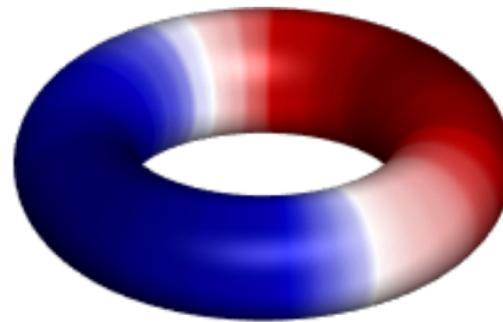
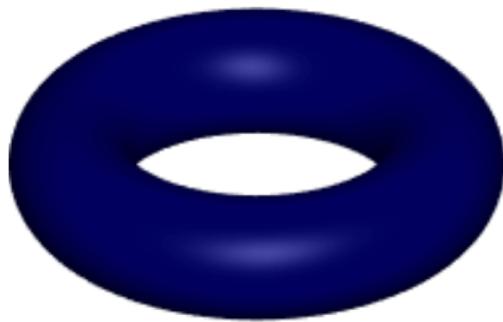
$$\mathbf{V} = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w^1 & w^2 & \cdots & w^{n-1} \\ 1 & w^2 & w^4 & & w^{2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & w^{n-1} & w^{n-1} & \cdots & w^{(n-1)^2} \end{bmatrix} \quad w = e^{i \frac{2\pi}{n}}$$



# 2D Fourier basis

$$\mathbf{W} = \mathbf{V}^* \otimes \mathbf{V}$$

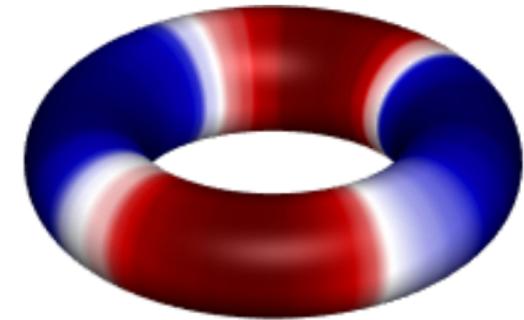
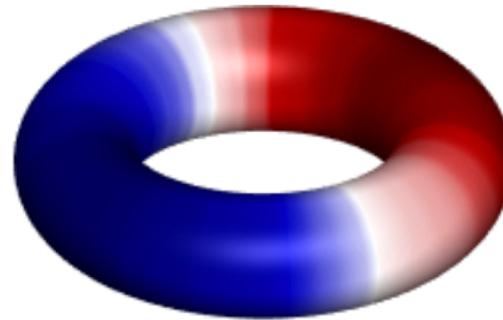
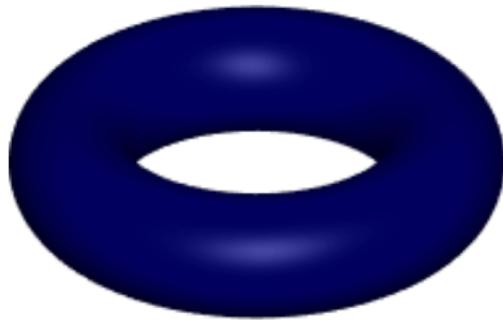
$$\mathbf{V} = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 \\ 1 & w^1 \\ 1 & w^2 \\ \vdots & \vdots \\ 1 & w^{n-1} \end{bmatrix} \begin{bmatrix} 1 & & & 1 \\ w^2 & \cdots & & w^{n-1} \\ w^4 & & & w^{2(n-1)} \\ \vdots & & & \vdots \\ w^{n-1} & \cdots & & w^{(n-1)^2} \end{bmatrix}$$
$$w = e^{i \frac{2\pi}{n}}$$



# 2D Fourier basis

$$\mathbf{W} = \mathbf{V}^* \otimes \mathbf{V}$$

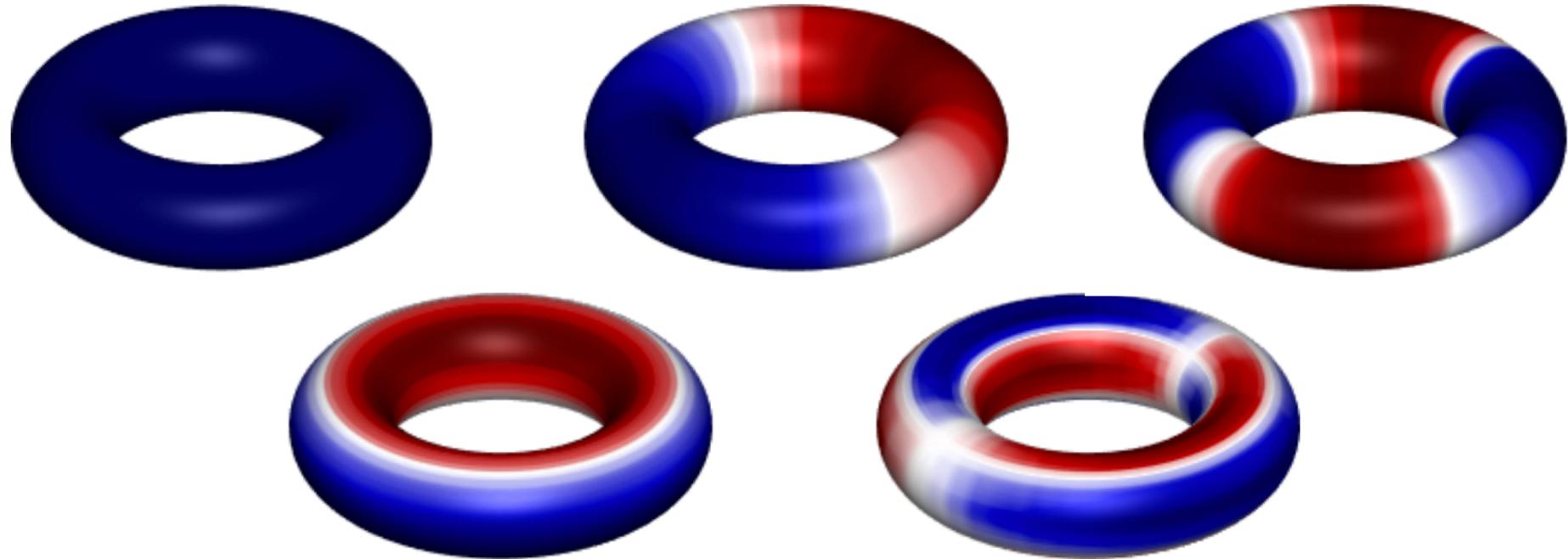
$$\mathbf{V} = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & w^1 & w^2 & \dots & w^{n-1} \\ 1 & w^2 & w^4 & \dots & w^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w^{n-1} & w^{n-1} & \dots & w^{(n-1)^2} \end{bmatrix} \quad w = e^{i \frac{2\pi}{n}}$$



# 2D Fourier basis

$$\mathbf{W} = \mathbf{V}^* \otimes \mathbf{V}$$

$$\mathbf{V} = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & w^1 & w^2 & \dots & w^{n-1} \\ 1 & w^2 & w^4 & & w^{2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & w^{n-1} & w^{n-1} & \dots & w^{(n-1)^2} \end{bmatrix} \quad w = e^{i \frac{2\pi}{n}}$$



# 2D Fourier transform

$\mathbf{W} = \mathbf{V}^* \otimes \mathbf{V}$  diagonalizes the 2D shift operator

$$\begin{aligned}(\mathbf{V}^* \otimes \mathbf{V})(\mathbf{S} \otimes \mathbf{I})(\mathbf{V}^* \otimes \mathbf{V})^* &= ((\mathbf{V}^* \mathbf{S}) \otimes \mathbf{V})(\mathbf{V} \otimes \mathbf{V}^*) \\&= (\mathbf{V}^* \mathbf{S} \mathbf{V}) \otimes (\mathbf{V} \mathbf{V}^*) \\&= \Lambda \otimes \mathbf{I}\end{aligned}$$