Man ermittle die kubische Spline-Interpolierende $S(x)$ für die folgenden Punkte:

x	1	3	4
y	-5	3	4

Allgemein: $\boxed{S_j(x) = a_j + b_j(x-x_j) + c_j(x-x_j)^2 + d_j(x-x_j)^3}$

j stets als Index!

x_i	1	3	4
$f(x_i)$	-5	3	4

$\boxed{h_j = x_{j+1} - x_j \qquad j \in [0, n-1]}$

$h_0 = x_1 - x_0 = 3 - 1 = 2$
$h_1 = x_2 - x_1 = 4 - 3 = 1$

$f(0) = -5 = a_0$
$f(1) = 3 = a_1$
$f(2) = 4 = a_2$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ h_0 & 2(h_0+h_1) & h_1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 6 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$b = \begin{pmatrix} 0 \\ \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -9 \\ 0 \end{pmatrix}$$

$A \cdot x = b$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & 6 & 1 & -9 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{2\mathrm{I}-\mathrm{II}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -6 & -1 & 9 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{\mathrm{II}+\mathrm{III}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\frac{3}{2} \\ 0 & 0 & 1 & 0 \end{array}\right) \Rightarrow x = \begin{pmatrix} 0 \\ -1.5 \\ 0 \end{pmatrix}$$

$\boxed{b_j = \frac{1}{h_j}(a_{j+1} - a_j) - \frac{h_j}{3}(2c_j + c_{j+1})}$

$\boxed{c_{j+1} = c_j + 3d_j h_j} \qquad j \in [0, n-1]$

$$\begin{pmatrix} 0 \\ -1.5 \\ 0 \end{pmatrix} = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix}$$

$b_0 = \frac{1}{2}(3+5) - \frac{2}{3}(2 \cdot 0 + (-1.5)) = 4 + 1 = 5$

$b_1 = \frac{1}{1}(4-3) - \frac{1}{3}(2 \cdot (-1.5) + 0) = 1 + 1 = 2$

$-1.5 = 0 + 3 \cdot d_0 \cdot 2 \Rightarrow d_0 = -\frac{1}{4}$
$0 = -1.5 + 3 \cdot d_1 \cdot 1 \Rightarrow d_1 = \frac{1}{2}$

$\boxed{d_j = \frac{c_{j+1} - c_j}{3h_j}}$

$x_j = x_i$

$S_0(x) = -5 + 5(x-1) + 0(x-1)^2 + (-\frac{1}{4})(x-1)^3$
$= -\frac{1}{4}x^3 + \frac{3}{4}x^2 + 4\frac{1}{4}x - 9\frac{3}{4}$

$S_1(x) = 3 + 2(x-3) + (-1.5)(x-3)^2 + \frac{1}{2}(x-3)^3$
$= \frac{1}{2}x^3 - 6x^2 + 24.5x - \frac{87}{2}$

~~A~~ $\cancel{f'(0.5)} =$

$$S_1(x) = \frac{1}{2}x^3 - 6x^2 + \frac{49}{2}x - \frac{87}{2}$$

$$S_1'(x) = \frac{3}{2}x^2 - 12x + \frac{49}{2}$$

$$\left(f'(0.5) = \frac{3}{2}(0.5)^2 - 12 \cdot 0.5 + \frac{49}{2} = \frac{3}{8} - \frac{48}{8} + \frac{196}{8} = \frac{151}{8} = 18\tfrac{7}{8} \right)$$

$$S(x) = \begin{cases} -\frac{1}{4}x^3 + \frac{3}{4}x^2 + 4\frac{1}{4}x - 9\frac{3}{4} & x \in [1,3] \\ \frac{1}{2}x^3 - 6x^2 + 24.5x - \frac{87}{2} & x \in [3,4] \end{cases}$$