HOMEWORK 2 MATHS 141 WINTER 2018

1. Introduction

We continue our look at induction, but now with an eye to solving recurrence relations and using recursion for the purposes of approximation rather than explicit calculations. We will also explore a few mathematical niceties.

2. Problems

Problem 1: (a) Prove DeMoivre's theorem.

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$$

Where $i = \sqrt{-1}$, $n \in \mathbb{N}_0$ and we have the old trigonometric identities:

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$
$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$

(b) Using this information write the formulas for

$$\cos(5\theta) \text{ and } \sin(5\theta)$$

in terms of simple $\cos(\theta)$ and $\sin(\theta)$ by expanding the left side by the binomial theorem and collecting real and imaginary terms.

(c) The Chebyshev polynomials (of the first kind) are defined by

$$T_n(\cos(\theta)) = \cos(n\theta)$$

For example $\cos(2\theta) = 2\cos^2(\theta) - 1$ this yields:

$$T_2(x) = 2x^2 - 1.$$

Write down the Chebyshev polynomials $T_1(x), T_2(x), T_3(x), T_4(x)$.

(bonus) What is the recurrence relation between the polynomials T_n?

Problem 2. Pell's equation is defined by considering a square free integer n. The equation is

$$x^2 - ny^2 = 1$$

Geometrically this defines a hyperbola on which there are infinitely many points. We, however, are interested in only the points (x, y) where both are integers. Additionally, since a hyperbola is symmetric about both major axes in the plane each solution (x, y) actually defines 4 solutions. For example:

$$x^2 - 2y^2 = 1$$

Has solution $3^2 - 2(2^2) = 1$. This could represent the solutions $(3, 2), (-3, 2), (3, -2), (-3, -2)$. We'll only consider the solutions in the first quadrant for simplicity. If Pell's equation has a solution then it has infinitely many solutions. The solution which is closest to the origin is called the fundamental solution denoted (x_1, y_1) (this will serve as our base case). Consider then,

$$(x^2 - ny^2)^N = 1^N = 1$$

So we will factor via a difference of squares.

$$x^2 - ny^2 = (x + \sqrt{n}y)(x - \sqrt{n}y) = 1$$

(a) Prove by induction that Pell's equation has infinitely many solutions defined by:

$$x_k + \sqrt{n}y_k = (x_1 + \sqrt{n}y_1)^k$$

Hint: You will have to do some factoring here.

(b) We can convert this into a matrix equation:

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_1 & ny_1 \\ y_1 & x_1 \end{bmatrix} \begin{bmatrix} x_k \\ y_k \end{bmatrix}$$

Prove by induction:

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_1 & ny_1 \\ y_1 & x_1 \end{bmatrix}^k \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

(c) The fundamental solution for $n = 2$ is $(3, 2)$. Compute the first 5 solutions for $n = 2$. What happens to the ratios

$$\frac{x_1}{y_1}, \frac{x_2}{y_2}, \frac{x_3}{y_3}, \frac{x_4}{y_4}, \frac{x_5}{y_5}, \ldots ?$$

(d) the fundamental solution for $n = 7$ is $(8, 3)$. Compute the first 5 solutions for $n = 7$. What happens to the ratios

$$\frac{x_1}{y_1}, \frac{x_2}{y_2}, \frac{x_3}{y_3}, \frac{x_4}{y_4}, \frac{x_5}{y_5}, \ldots ?$$

(e) What is the fundamental solution for $n = m^2 - 1$, for some integer m?

(f) Why do these ratios seem to converge? Can you prove it?

(bonus): How fast do these ratios converge?

Problem 3. Solve the recurrence relation with intial conditions $a_0 = 0, a_1 = 2$

$$a_{n+2} - 3a_{n+1} + 3a_n = 0$$

Problem 4. So far we've only studied recurrence relations with distinct real roots. If we have a repeated real root, this is called resonance (feedback for guitarists, busted shock absorbers for the motor fiends). The general solution looks

$$a_n = (C + Dn)r^n.$$

(a) Given this, solve the following recurrence relation with initial conditions $a_0 = 2$ and $a_1 = 21$.

$$a_{n+2} - 6a_{n+1} + 9a_n = 0.$$

(b) Show by induction that $a_n < (r + 1)^n$ for all $n \geq 16$. Note the funny base case here. You'll have to do a little algebra here to get this into a nice form. It may take two or three (dozen) tries. Don't worry. Plug these numbers into a calculator to convince yourself if you must.

Problem 5. Prove the binomial theorem by induction. Here we'll take a slightly easier approach. Consider the fact that we know the relationship between binomial coefficients

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

This is a fact we showed in discrete math 1. If you want to look it up, go ahead. What we want to show in this problem is simply

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Prove this simplified formula by induction.

Problem 6. (a) Prove by induction:

$$\begin{bmatrix} 5 & -6 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} (3^{n+1} - 2^{n+1}) & 6 \cdot (2^n - 3^n) \\ (3^n - 2^n) & 6 \cdot (2^{n-1} - 3^{n-1}) \end{bmatrix}$$

(b) Now consider the simultaneous recurrence relations

$$\begin{aligned} b_{n+1} &= 5b_n - 6a_n \\ a_{n+1} &= b_n \end{aligned}$$

Where $a_0 = 1, b_0 = 5$

Verify (by plugging in directly) that the solution is

$$a_n = 2^n + 3^n$$

and

$$b_n = 2^{n+1} + 3^{n+1}$$

(c) Consider the second order homogeneous recurrence relation

$$a_{n+2} - 5a_{n+1} + 6a_n = 0$$

with $a_0 = 1, a_1 = 5$. Solve this recurrence relation.

(d) By making the substitution $a_{n+1} = b_n$ Show that we can transform part (c) into

$$\begin{bmatrix} b_{n+1} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 5 & -6 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b_n \\ a_n \end{bmatrix}$$

(e) Show that a first order recurrence relation

$$x_{n+1} = Ax_n, \text{ where } x_0 \text{ is given.}$$

has the solution $x_n = A^n x_0$ where A is a matrix and x_n is a sequence of vectors.

Conclude that

$$\begin{bmatrix} b_{n+1} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 5 & -6 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} b_0 \\ a_0 \end{bmatrix}$$

3. Notes

In this homework we begin exploring some well-known and highly useful theorems. Most of these theorems come from either physics or computer science where some mathematician was trying to compute something and needed a new formula. Problem one explores three theorems in one way or another. DeMoivre's theorem is really just the idea that Euler's formula makes sense if one writes it in trigonometric form

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

In that context

$$(e^{i\theta})^n = e^{in\theta}$$

which is exactly the statement in part (a). Part c explored Pafnuty Chebyshev's work. A quick note of personal interest: Chebyshev is my mathematical great great great great great grandfather via the advisor-student relationships:

Alexander, Tsygan, Manin, Shafarevich, Delone, Voronoy, Markov, Chebyshev.

If you like, you can look through the mathematics genealogy project.

Back to math: Chebyshev worked in probability and differential equations. You may know of his work regarding standard deviations where he proved:

$$P(|z| < k) > \left(1 - \frac{1}{k^2}\right)$$

That is (in English) at most $1/k^2$ amount of observations are more than k standard deviations away from center, regardless of probability distribution. You may recall this term $\left(1 - \frac{1}{k^2}\right)$ from homework 1.

Problem two brings us to Pell's equation. This is a celebrated theorem which allows us to get arbitrarily close to squareroots of (nonsquare) integers as rational approximations. The reasoning behind this is as n grows large $x + \sqrt{n}y$ grows very large. So we have the approximation

$$x - \sqrt{n}y = \frac{1}{x + \sqrt{n}y} \approx 0 \implies \frac{x}{y} \approx \sqrt{n}$$

Problems three and four and straight forward solutions, just getting us into the mechanics of solving recurrence relations, nothing special here, save the repeated root in problem four. Repeated roots are somewhat rare, but when they occur, we see and additional linear term show up in the growth of the sequence. We'll discuss this at some length during the analysis of algorithms.

Problem five goes right back to the simplest form of the binomial theorem. I cannot overstate the utility in the binomial theorem. Here we will separate $(1 + x)^{n+1} = (1 + x)^n(1 + x)$ expand, collect terms relative to the powers of x and use our combinatorial identity. It takes probably four lines, but is amongst the most useful theorems in mathematics.

Problem six shows a common technique that we can convert any linear recurrence relation into a first order matrix equation. Again, this is useful as we use matrices to compute walks on graphs, which at the most basic level is how neural networks work. The technique of converting more complex equations into simple matrix equations shows us fairly often in physics and engineering.