

## HOMEWORK 1 MATHS 141 WINTER 2018

### 1. INTRODUCTION

In this first assignment we will explore multiple avenues in proof by induction. We will see additive, multiplicative, matrix, divisor, and inequality based proofs by induction.

### 2. PROBLEMS

Problem 1. Prove the following summations by induction. You may use  $n = 0$  as the base case for each of the following:

(a)

$$\sum_{k=0}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$

(b)

$$\sum_{k=0}^n k^3 = \left( \frac{n(n+1)}{2} \right)^2$$

(c)

$$\sum_{j=0}^n 2^j = 2^{n+1} - 1$$

(d)

$$\sum_{j=0}^n r^j = \left( \frac{r^{n+1} - 1}{r - 1} \right)$$

(e)

$$\sum_{k=0}^n (-1)^k = \frac{1}{2}(1 + (-1)^n)$$

Problem 2. Prove the following product formulas by induction. You may assume the lower index to be the base case in all problems.

(a)

$$\prod_{k=2}^n \left(1 - \frac{1}{k^2}\right) = \frac{n+1}{2n}$$

(b)

$$\prod_{j=1}^n (2j-1) = \frac{(2n)!}{2^n n!}$$

Problem 3. For every  $n > 1$  prove the following inequality

$$\sqrt{n} < \sum_{k=1}^n \frac{1}{\sqrt{k}}$$

Problem 4. Let  $x, y$  be nonzero integers and let  $n$  be a positive integer. Prove the following by induction.

$$(x - y) \mid (x^n - y^n)$$

Hint: Consider problem (1d) where  $r = \frac{x}{y}$ .

Problem 5. Prove for the following by induction for  $n > 2$ .

$$\frac{n^n}{3^n} < n! < \frac{n^n}{2^n}$$

Hint: In this problem we're really solving two different problems. Consider each inequality separately. You will find the following inequality to be helpful. You may take this to be true. If you need more documentation or further explanation, read some information about  $e$ .

$$2 \leq \left(1 + \frac{1}{n}\right)^n < 3.$$

Problem 6. Let  $a, b \in \mathbb{R}$  and  $n \in \mathbb{N}$ . Prove the following by induction:

$$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}^n = \begin{bmatrix} a^n & nba^{n-1} \\ 0 & a^n \end{bmatrix}$$

Bonus. In calculus we learn that for any  $x \in \mathbb{R}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + x^2/2 + x^3/6 + \dots$$

We see now that this formula makes sense as long as we can multiply  $x$  by itself and add these powers. In particular, square matrices satisfy these criteria. With that in mind try to exponentiate the following matrices:

In this case the matrix  $I$  represents 1 in the summation.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

In general,  $I$  is the  $n \times n$  matrix with 1 on the diagonal and 0 everywhere else.

(a) Let  $a, b \in \mathbb{R}$

$$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$$

(b) Let  $t \in \mathbb{R}$

$$\begin{bmatrix} t & t \\ t & 0 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Note: This is the Laplacian matrix associated to the graph  $K_3$ .

## 3. NOTES

Problem one gets us exploring the first couple Bernoulli polynomials. In general these are interested in sums of the form

$$\sum_{k=1}^N k^n$$

where  $n$  is some positive integer. We will see these again in greater detail when we get to the analysis of algorithms. For those of you who have studied calculus these are the Riemann sums estimating integrals of the form

$$\int_0^1 x^n dx$$

That is to say, if you are particularly good at discrete mathematics you can derive big chunks of calculus on your own. However, we see this again in problem 3, where  $n = 1/2$ . There is no “nice” closed form solution to this summation, but we have very good estimates. In particular, for some “small” number  $r$  (remainder)

$$\sqrt{n} < \sum_{k=1}^n \frac{1}{\sqrt{k}} < \sqrt{n+r}$$

Problem two gets us thinking about induction in terms of products. Many formulas in number theory involve infinite products. It is of great use to know how to deal with these products, but perhaps it’s a few dozen quarters away to get a lot of utility out of these products in particular.

Problem four introduces to an equation we’ll see quite often in this course. It is particularly useful for factoring numbers and polynomials. In fact, the general equations we’ll see later is

$$\frac{x^n - y^n}{x - y} = x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \dots xy^{n-2} + y^{n-1}$$

Problem five brings us to the weak version of the famed Stirling approximation. It is difficult to overstate exactly how useful this approximation is. This is a weak form of the inequality, but some other versions show us

$$n! \approx \sqrt{2\pi n} \frac{n^n}{e^n}$$

Or the true approximation states

$$\lim_{n \rightarrow \infty} \frac{n!e^n}{n^{n+1/2}} = \sqrt{2\pi}$$

One interesting fact is that using this approximation we can show that when flipping an even number ( $2n$ ) of fair coins the probability that we get exactly  $n$  heads and  $n$  tails is

$$P(H = T = n) = \frac{1}{2^{2n}} \binom{2n}{n} \approx \frac{1}{\sqrt{\pi n}}$$