This note explains how the *gauge map* and the *generalized gauge map* are used to transport a point onto a polytope.

# 1 The Minkowski (Gauge) Functional

[Minkowski functional] Let $\mathcal{C} \subset \mathbb{R}^n$ be non-empty, convex, and compact, and suppose that the origin $\mathbf{0}$ lies in its algebraic interior. The *Minkowski functional* (or *gauge*) of $\mathcal{C}$ is the mapping

$$\varphi_{\mathcal{C}} : \mathbb{R}^n \longrightarrow [0, \infty], \qquad \varphi_{\mathcal{C}}(\mathbf{x}) \; = \; \inf\{\rho > 0 : \mathbf{x} \in \rho\,\mathcal{C}\}.$$

Thus $\varphi_{\mathcal{C}}(\mathbf{x})$ is the smallest radial stretch factor that brings $\mathbf{x}$ inside $\mathcal{C}$.

**Polytope specialisation.** Suppose

$$\mathcal{C} = \{\mathbf{x} \in \mathbb{R}^n \mid H\mathbf{x} \le h\}, \qquad H \in \mathbb{R}^{m\times n},\; h \in \mathbb{R}^m_{>0}.$$

Because $h_j > 0$ for all $j$, the origin is strictly feasible. For any $\mathbf{x} \in \mathbb{R}^n$ the gauge admits the closed form

$$\boxed{\varphi_{\mathcal{C}}(\mathbf{x}) = \max_{1\le j\le m} \frac{(H\mathbf{x})_j}{h_j}} \tag{1}$$

Scaling $\mathbf{x}$ by $\rho$ modifies the $j$-th constraint to $(H\mathbf{x})_j \le \rho\, h_j$. Each facet therefore imposes $\rho \ge (H\mathbf{x})_j/h_j$, and the minimal admissible stretch is the largest of these ratios. Intuitively, the gauge 'normalizes' each constraint and selects the most limiting—i.e., most violated—facet; that facet dictates the minimal radial factor required for feasibility.

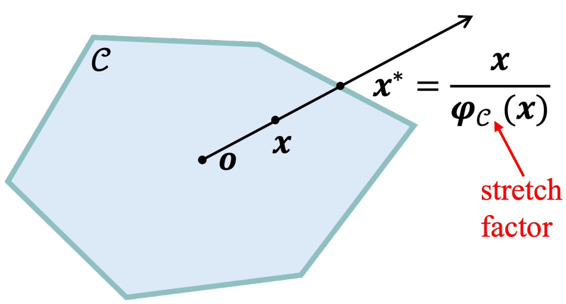


Figure 1: The Minkowski functional $\varphi_{\mathcal{C}}(\mathbf{x})$ is the minimal stretch factor needed to bring $\mathbf{x}$ onto $\mathcal{C}$. The ray through the origin intersects the boundary at $\mathbf{x}^* = \mathbf{x}/\varphi_{\mathcal{C}}(\mathbf{x})$.

# 2 The Gauge Map

[Gauge map] Let $\mathcal{C}, \bar{\mathcal{C}} \subset \mathbb{R}^n$ be convex, compact, and contain $\mathbf{0}$ in their algebraic interiors. Define the *gauge map*

$$G : \mathcal{C} \longrightarrow \bar{\mathcal{C}}, \qquad \boxed{\bar{\mathbf{x}} = G(\mathbf{x}) := \frac{\varphi_{\mathcal{C}}(\mathbf{x})}{\varphi_{\bar{\mathcal{C}}}(\mathbf{x})}\,\mathbf{x}}, \quad \mathbf{x} \in \mathcal{C}.$$

**Why the formula works.** Fix any non-zero $\mathbf{x}$.

1. The ray $\{\alpha\mathbf{x} : \alpha \ge 0\}$ intersects $\mathcal{C}$ and $\bar{\mathcal{C}}$ exactly once, at

$$\mathbf{x}^*_{\mathcal{C}} = \frac{\mathbf{x}}{\varphi_{\mathcal{C}}(\mathbf{x})}, \qquad \mathbf{x}^*_{\bar{\mathcal{C}}} = \frac{\mathbf{x}}{\varphi_{\bar{\mathcal{C}}}(\mathbf{x})}.$$

2. Their radial distances differ by $\varphi_{\mathcal{C}}(\mathbf{x})/\varphi_{\bar{\mathcal{C}}}(\mathbf{x})$. Multiplying $\mathbf{x}$ by this relative gauge ratio slides it along the ray from $\mathcal{C}$ onto $\bar{\mathcal{C}}$.

Because direction is preserved and the rescaling is strictly monotone, $G$ is a bijection.

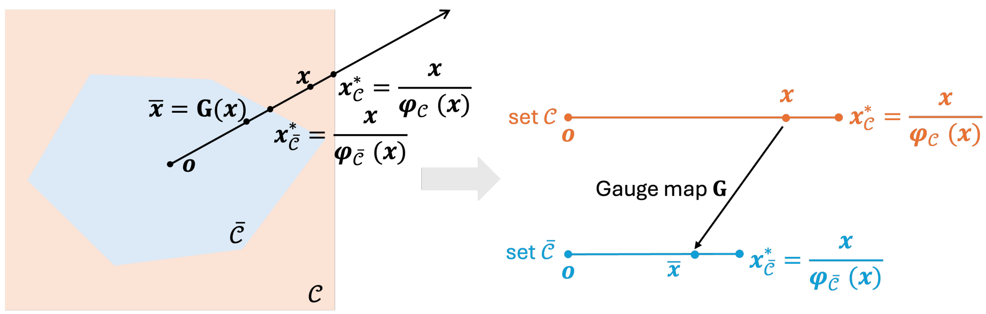


Figure 2: The gauge map rescales a point $\mathbf{x} \in \mathcal{C}$ along its ray to reach the corresponding point $\bar{\mathbf{x}} \in \bar{\mathcal{C}}$, using the relative gauge ratio $\frac{\varphi_{\mathcal{C}}(\mathbf{x})}{\varphi_{\bar{\mathcal{C}}}(\mathbf{x})}$. This transformation preserves direction and is bijective.

# 3 Application 1 Gauge Map: Mapping the $\ell_\infty$ Unit Ball to a Polytope

**High-level idea.** We want a one-to-one map $\Psi : \mathcal{B} \to \mathcal{S}$ so that machine learning models only need to output predictions within the set $\mathcal{B} = [-1, 1]^n$, while $\Psi$ automatically converts them into points that satisfy the original polyhedral constraints $H\mathbf{x} \leq h$. The map is the composition of three elementary operations: 1) shift $\mathcal{S}$ so that the origin becomes interior; 2) apply the gauge map to align radial distances between the cube and the shifted polytope; 3) shift back. Since each step is bijective, the overall map $\Psi$ is also a bijection and admits a closed-form inverse.

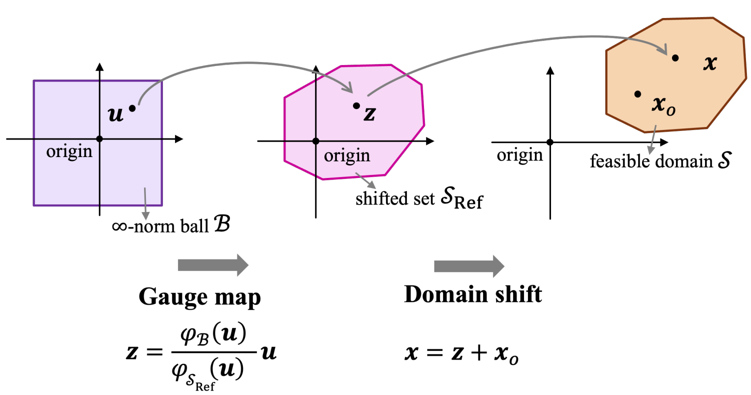


Figure 3: Three–step transformation from the $\ell_\infty$ unit ball $\mathcal{B}$ to the target polytope $\mathcal{S}$: (i) Preparation: Shift $\mathcal{S}$ by $-\mathbf{x}_0$ to obtain the reference set $\mathcal{S}_{\text{ref}}$; (ii) Gauge map: Apply the gauge map $G_{\mathcal{B}\to\mathcal{S}_{\text{ref}}}$ to move a point $\mathbf{u} \in \mathcal{B}$ onto $\mathcal{S}_{\text{ref}}$ along its ray; (iii) Inverse shift: Shift back by $+\mathbf{x}_0$ to recover the corresponding feasible point $\mathbf{x} \in \mathcal{S}$.

**Detailed construction.**

1. *Preparation.* Choose an interior point $\mathbf{x}_0 \in \text{int}(\mathcal{S})$ and shift the set $\mathcal{S}$ as:

$$\mathcal{S}_{\text{ref}} := \{\mathbf{x} - \mathbf{x}_0 \mid \mathbf{x} \in \mathcal{S}\}.$$

The set $\mathcal{S}_{\text{ref}}$ is convex, compact, and contains the origin in its interior, meeting the gauge-map assumptions.

2. *Apply the gauge map.* For any $\mathbf{u} \in \mathcal{B}$, map it to

$$\mathbf{z} \ = \ G_{\mathcal{B}\to\mathcal{S}_{\mathrm{ref}}}(\mathbf{u}) := \frac{\varphi_{\mathcal{B}}(\mathbf{u})}{\varphi_{\mathcal{S}_{\mathrm{ref}}}(\mathbf{u})}\,\mathbf{u} \ \in \ \mathcal{S}_{\mathrm{ref}}.$$

3. *Inverse shift.* Shift back to the original coordinates:

$$\mathbf{x} \ = \ \mathbf{z} + \mathbf{x}_0 \ \in \ \mathcal{S}.$$

Since the gauge map (together with the shift) is bijective, each point in $\mathcal{B}$ corresponds to a unique feasible point in $\mathcal{S}$, and vice versa.

Refer to "Li, Meiyi, Soheil Kolouri, and Javad Mohammadi. "Learning to solve optimization problems with hard linear constraints." IEEE Access 11 (2023): 59995-60004." for more details.

# 4 Application 2 Generalized Gauge Map: Radial projection of an arbitrary point onto a polytope

**High-level idea** Given any predicted point $\mathbf{u} \in \mathbb{R}^n$ from a machine learning model, we construct a mapping $\Pi : \mathbb{R}^n \to \mathcal{S}$ that (1) leaves feasible points unchanged, and (2) projects infeasible points *radially*—i.e., along the line connecting an interior point $\mathbf{x}_0 \in \mathrm{int}(\mathcal{S})$ and $\mathbf{u}$—onto the boundary of $\mathcal{S}$.

**Detailed construction.**

1. *Reference shift.* Pick an interior point $\mathbf{x}_0 \in \mathrm{int}(\mathcal{S})$ and form $\mathcal{S}_{\mathrm{ref}} := \{\mathbf{x} - \mathbf{x}_0 : \mathbf{x} \in \mathcal{S}\}$. Then $\mathbf{0} \in \mathrm{int}(\mathcal{S}_{\mathrm{ref}})$.

2. *Radial scaling*

   $\varphi_{\mathcal{S}_{\mathrm{ref}}}(\mathbf{u})$ quantifies the feasibility of $\mathbf{u}$ in $\mathcal{S}_{\mathrm{ref}}$: it is less than or equal to 1 if $\mathbf{u}$ lies inside the set, and greater than 1 if $\mathbf{u}$ is outside. Accordingly, define the projected point as

   $$\mathbf{z} := \frac{\mathbf{u}}{\max\{1, \varphi_{\mathcal{S}_{\mathrm{ref}}}(\mathbf{u})\}} \in \mathcal{S}_{\mathrm{ref}}.$$

   If $\mathbf{u}$ is already feasible, then $\mathbf{z} = \mathbf{u}$; otherwise, $\mathbf{u}$ is rescaled onto the boundary along the ray from the origin.

3. *Inverse shift.* Finally, output $\mathbf{x} := \mathbf{z} + \mathbf{x}_0 \in \mathcal{S}$.

Unlike the bijective gauge map, $\Pi$ is a *many-to-one* projection. This loss of injectivity is acceptable in contexts where the primary goal is to enforce feasibility.

Refer to "Li, Meiyi, and Javad Mohammadi. "Toward rapid, optimal, and feasible power dispatch through generalized neural mapping." 2024 IEEE Power and Energy Society General Meeting (PESGM). IEEE, 2024." for more details.

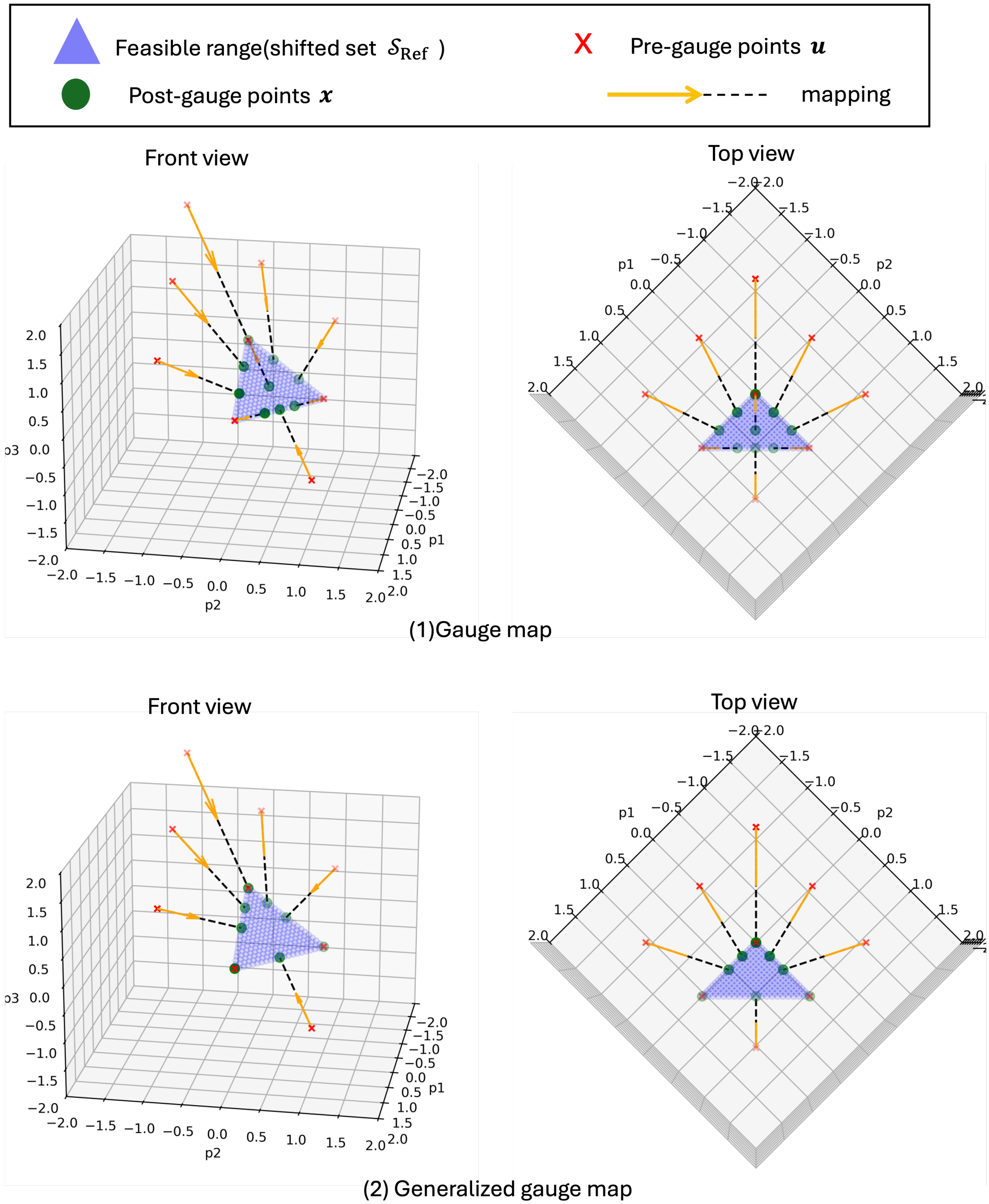


Figure 4: Given a point $\mathbf{u}$, the gauge map rescales it by the relative gauge ratio—regardless of whether it is feasible. In contrast, the generalized gauge map leaves feasible points unchanged and only rescales infeasible points onto the boundary of the polytope. For example, three pre-gauge points (shown as red "×" marks at the intersections within the triangle) are already feasible. The gauge map still rescales them (to the green circles, via yellow arrows), while the generalized gauge map keeps them unchanged.