Homework 5

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1 Question 3

1. **Exercise 4.1.3, sections b, c**

   4.1.3 b

   No, $f(x) = \frac{1}{x^2-4}$ is not a function from $\mathbb{R}$ to $\mathbb{R}$.

   $f(x) = \frac{1}{x^2-4}$

   When $x^2 - 4 = 0$, f(x) = $\frac{1}{x^2-4}$ is not defined.

   For x_1 = 2, x_2 = -2, f(x) is not defined.

   x_1 and x_2 all belongs to the set $\mathbb{R}$

   Therefore, f(x) = $\frac{1}{x^2-4}$ is not a function from $\mathbb{R}$ to $\mathbb{R}$

   4.1.3 c

   Yes, it is a function from $\mathbb{R}$ to $\mathbb{R}$.

   It maps elements of $\mathbb{R}$ to elements of $\mathbb{R}$. For every x $\in \mathbb{R}$, there is exactly one y $\in \mathbb{R}$. Therefore, it is a function from $\mathbb{R}$ to $\mathbb{R}$.

   The **range** is $[0, +\infty)$

2. **Exercise 4.1.5**

   b.

   A = $\{2, 3, 4, 5\}$

   f(x) = x^2

   $2^2 = 4$, $3^2 = 9$, $4^2 = 16$, $5^2 = 25$

   Range of f: $\{4, 9, 16, 25\}$

   d.

   According to the problem, x $\in \{0, 1\}^5$, f(x) is the number of 1's that occur in x

   Therefore, x can be 0 to 5 according to different 1's that occurs in x.

   Rangee of f: $\{0, 1, 2, 3, 4, 5\}$

   h. f(x, y) = (y, x)

   x and y all belongs to set A. A = $\{1, 2, 3\}$.

   Range of f: $\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

   i. f(x, y) = (x, y + 1)

   x and y all belongs to set A. A = $\{1, 2, 3\}$.

Therefore, y + 1 $\in \{2, 3, 4\}$

Range of f: $\{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$

l.

A = $\{1, 2, 3\}$

$X \subseteq A$,

X = $\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$

Therefore

f(X) = X - $\{1\}$;

f($\emptyset$) = $\emptyset - \{1\} = \emptyset$;

f($\{1\}$) = $\{1\} - \{1\} = \emptyset$;

f($\{2\}$) = $\{2\} - \{1\} = \{2\}$;

f($\{3\}$) = $\{3\} - \{1\} = \{3\}$;

f($\{1, 2\}$) = $\{1, 2\} - \{1\} = \{2\}$;

f($\{2, 3\}$) = $\{2, 3\} - \{1\} = \{2, 3\}$;

f($\{1, 3\}$) = $\{1, 3\} - \{1\} = \{3\}$;

f($\{1, 2, 3\}$) = $\{1, 2, 3\} - \{1\} = \{2, 3\}$;

Range of f: $\{\emptyset, \{2\}, \{3\}, \{2, 3\}\}$

2 Question 4

1. **I. a. Exercise 4.2.2, sections c, g, k**

   c.

   h: Z $\rightarrow$ Z. h(x) = x^3

   one-to-one and not onto.

   Prove one-to-one:

   For every x, $x_1 \neq\ x_2$ implies that $x_1^3 \neq x_2^3$, $h(x_1) \neq h(x_2)$. Therefore, it is a one-to-one.

   Prove not onto:

   However, it is not onto. For $h(x) = 3 \in Z$, there does not exist an integer x that $x^3 = 3$. Therefore, the function is not onto.

   g.

   f: Z $\times$ Z $\rightarrow$ Z $\times$ Z, f(x, y) = (x+1, 2y)

   one-to-one and not onto.

   Prove one-to-one:

   For every x in f, $x_1 \neq x_2$ implies that $x_1 + 1 \neq x_2 + 1$.

   For every y, $y_1 \neq y_2$ implies that $2 \times y_1 \neq 2 \times y_2$.

   Therefore, or every (x, y) , $(x_1, y_1) \neq (x_2, y_2)$ implies that $f(x_1, y_1) \neq f(x_2, y_2)$.

   Therefore, the function is one-to-one.

   Prove not onto:

   For every odd integer number, there does not exists a integer y that 2y = odd integer number. Therefore, the function is not onto.

   k.

   f: $Z^+Z^+ \rightarrow Z^+$, $f(x, y) = 2^x + y$.

   neither onto nor one-to-one.

   Prove not onto:

   $f(1, 3) = 2^1 + 3 = 5;$

   $f(2, 1) = 2^2 + 1 = 5;$

   f(1, 3) = f(2, 1)

   Therefore, it is not one-to-one.

   Prove not onto:

   There do not exist two integers a and b that f(a, b) = 1.

2. **I. b. Exercise 4.2.4, sections b, c, d, g**

   b.

   neither one-to-one nor onto

   domain of x: $\{0, 1\}^3 = \{000, 001, 010, 100, 110, 101, 011, 111\}$

   target of y: $\{0, 1\}^3 = \{000, 001, 010, 100, 110, 101, 011, 111\}$

   range of y: $\{100, 101, 110, 111\}$

   Prove not one-to-one:

   When $x_1 = 011$ and $x_2 = 111$, $f(x_1) = f(x_2) = 111$.

   For a one-to-one function, $x_1 \neq x_2$ implies that $f(x_1) \neq f(x_2)$.

   The function is not one-to-one.

   Prove not onto:

   Range of f does not equal to the target Y, therefore, the function is not onto as well based on the onto function definition.

   c.

   both onto and one-to-one.

   X $= \{0, 1\}^3 = \{000, 001, 010, 100, 110, 101, 011, 111\}$;

   Y $= \{0, 1\}^3 = \{000, 001, 010, 100, 110, 101, 011, 111\}$;

   $x_1 \neq x_2$ implies that $f(x_1) \neq f(x_2)$, this function is one-to-one.

   For every y $\in$ Y, there is an x $\in$ X such that f(x) = y. The function f is also onto.

   Additionally, $|X| = |Y|$ is another piece of evidence that f is both onto and one-to-one.

   d.

   one-to-one, not onto

   $domain of y : \{000, 001, 010, 100, 110, 101, 011, 111\}$

   target of y: $\{0, 1\}^4$

   range of y: $\{0000, 0010, 0100, 1000, 1101, 1011, 0110, 1111\}$

   Prove one-to-one:

   for every x_1, x_2 in X, $x_1 \neq x_2$ implies that $f(x_1) \neq f(x_2)$, this function is one-to-one.

   Prove not onto:

   The range of y is less than the target of y. It cannot be an onto.

   g.

   neither one-to-one nor onto

   Prove not one-to-one:

   $\emptyset \in P(A)$, $1 \in P(A)$

$f(\emptyset) = \emptyset - B = \emptyset - \{1\} = \emptyset$

$f(\{1\}) = \{1\} - B = \{1\} - \{1\} = \emptyset$

For $x_1 = \emptyset x_2 = \{1\}$,$x_1 \neq x_2$, $f(x_1) = f(x_2)$.

Therefore, f is not one-to-one.

Prove not onto:

$\{1\} \in$ P(A), but there is no f(x) = X - B = $\{1\}$.

Therefore, f is not onto.

3. **II. Give an example of a function from the set of integers to the set of positive integers that is**

a. One-to-one, but not onto

$$f : Z \to Z^+, f(x) = \begin{cases} -4x + 2 & : x < 0 \\ 0 & : x = 0 \\ 4x + 3 & x > 0 \end{cases}$$

b. onto, but not one-to-one

$h : Z \to Z^+, h(x) = |x| + 1$

c. one-to-one and onto.

$$f : Z \to Z^+, f(x) = \begin{cases} 2x + 1 & : x \geq 0 \\ -2x & : x < 0 \end{cases}$$

d. neither one-to-one nor onto

$g : Z \to Z^+, g(x) = x^2 + 10$

3 Question 5

a) Exercise 4.3.2, sections c, d, g, i

c. Yes, the function has a well-defined inverse. $f^{-1} = \frac{y-3}{2}$
Assumes $x_1 \neq x_2 \in \mathbb{Z}$,
$2x_1 + 3 \neq 2x_1 + 3$
$f(x_1) \neq f(x_2)$
Therefore, f is one-to-one.
The range of f: Z
The target of f: Z
Target of f is equal to range of f. f is onto.
Therefore, f has a well-defined inverse.
When f(x) = 2x + 3 = y, x = $\frac{y-3}{2}$
For all y $\in \mathbb{R}$, $\frac{y-3}{2} \in \mathbb{R}$, the inverse function is well defined. $f^{-1} = \frac{y-3}{2}$ is the inverse function.

d. No, it does not have a well-defined inverse.
Set $\{0, 1\} \subseteq$ A. $f(\{0, 1\}) = |\{0, 1\}| = 2$
Set $\{1, 2\} \subseteq$ A. $f(\{1, 2\}) = |\{1, 2\}| = 2$
$\{0, 1\} \neq \{1, 2\}$, $f(\{0, 1\}) = f(\{1, 2\})$
$\therefore$ f is not one-to-one.
f does not have a well-defined inverse.

g. Yes, the function has a well-defined inverse. The inverse function is itself. $f^{-1} : \{0, 1\}^3 \rightarrow \{0, 1\}^3$. The output of f is obtained by taking the input string and reversing the bits.
Range of f: $\{000, 001, 010, 100, 011, 101, 110, 111\}$
Target of f: $\{000, 001, 010, 100, 011, 101, 110, 111\}$
Target of f = range of f
f is onto.
For $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$
f is one-to-one.
Therefore, f has a well-defined inverse. The inverse function is itself.

i. Yes, the function has a well-defined inverse. $f^{-1} : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}, f(x, y) = (x - 5, y + 2)$
For $x_1 \neq x_2$, $y_1 \neq y_2$,
$x_1 + 5 \neq x_2 + 5$, $y_1 - 2 \neq y_2 - 2$
f is one-to-one.
For every x + 5 $\in \mathbb{Z}$, $y - 2 \in \mathbb{Z}$, there is an x $\in \mathbb{Z}$, a y $\in \mathbb{Z}$ such that f(x, y) = (x + 5, y -2).
f is onto.
x + 5 = k, x = k - 5; y -2 = d, y = d + 2
$\therefore$, $f^{-1} : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}, f(x, y) = (x - 5, y + 2)$

b) Exercise 4.4.8, sections c, d
f(x) = 2x + 3; g(x) = 5x + 7; h(x) = x^2 + 1

c. $\underline{\text{f o h} = 2 \times x^2 + 5}$
f o h
= 2 × h(x) + 3
= 2×($x^2 + 1$) + 3
= $2 \times x^2 + 5$

d. $\underline{\text{h o f} = 4x^2 + 12\text{x} + 10}$
h o f
= $f(x)^2 + 1$
= $(2x + 3)^2$ + 1
= $4x^2$ + 12x + 9 + 1
= $4x^2$ + 12x + 10

c) Exercise 4.4.2, sections b-d

b. <u>(f o h)(52) = 121</u>
h(52) = 11
(f o h)(52) = f(h(52))
= f(11)
= 121

c. <u>(g o h o f)(4) = 16</u>
f(4) = 4^2 = 16
h(f(4)) = h(16) = 4
(g o h o f)(4) = g(h(f(4)))
= g(h(16))
= g(4)
= 2^4 = 16

d. $\underline{\text{h o f} = \left\lceil \frac{x^2}{5} \right\rceil}$
Give a mathematical expression for h o f.
h o f
= h(f(x)) = $\left\lceil \frac{x^2}{5} \right\rceil$

Exercise 4.4.6, sections c-e

c. <u>(h o f)(010) = 111</u>
f(010) = 110
(h o f)(010)
= h(f(010)) = h(110) = 111

d. <u>range of h o f: {101, 111}</u>

Range of f: {100, 101, 110, 111}
Range of h o f: {101, 111}
Therefore, the range of h o f: {101, 111}

e. range of g o f: {001, 101, 011, 111}
Range of f: {100, 101, 110, 111}
the output of g o f: {001, 101, 011, 111}
Therefore, the range of g o f: {001, 101, 011, 111}

Extra Credit: Exercise 4.4.4, sections c, d

c. No, it is not possible.
Assume f and g are two functions.
If f is not one-to-one, it implies that there exists x_1, x_2, $x_1 \neq x_2$and $f(x_1) = f(x_2)$
Let k = $f(x_1) = f(x_2)$.
Since g o f is one-to-one, then $g(f(x_1)) \neq g(f(x_2))$, which indicates that g(k) has two different results.
For a function, each x ∈ X should have exactly one y ∈ Y such that (x, y) ∈ f.
g(k) apparently does not follow this rule.
We can conclude that g(x) is not a function, but this obeys our assumption.
Therefore, it is not possible that f is not one-to-one and g o f is one-to-one.

d. No, it is not possible
Because f is not one-to-one, there exists $x_1 \neq x_2 . f(x_1) = f(x_2)$
Let k = $f(x_1) = f(x_2)$
If g o f is one-to-one, then $g(f(x_1)) \neq g(f(x_2))$
However, k = $f(x_1) = f(x_2)$
$g(k) \neq g(k)$
For a function, each x ∈ X should have exactly one y ∈ Y such that (x, y) ∈ f.
g(k) apparently does not follow this rule.
We can conclude that g(x) is not a function, but this obeys our assumption.
Therefore, it is not possible that g is not one-to-one and g f is one-to-one.