Homework 2

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1 Question Five

a)

1. Exercise 1.12.2, sections b,e

   (b.)

1.	$p \to (q \land r)$	Hypothesis
2.	$p \to q$	Simplification, 1
3.	$\neg q$	Hypothesis
4.	$\neg p$	Modus tollens,2,3

1. $p \to (q \land r)$ is a hypothesis.
2. Simplification says that if $p \to (q \land r)$ (line 1)is true, then $p \to q$ must also be true.
3. $\neg q$ is a hypothesis.
4. Modus tollens says that if $p \to q$(line 2) is true and $\neg q$ (line 3) is true, then $\neg p$ must also be true.

Therefore, the argument is valid.

(e.)

1.	$p \lor q$	Hypothesis
2.	$\neg p \lor r$	Hypothesis
3.	$q \lor r$	Resolution, 1, 2
4.	$\neg q$	Hypothesis
5.	r	Disjunctive syllogism, 3,4

1. $p \lor q$ is a hypothesis.
2. $\neg p \lor r$ is a hypothesis.
3. Resolution says that if $p \lor q$ (line 1)is true and $\neg p \lor r$ (line 2)is true, then $q \lor r$ must be true.
4. $\neg q$ is a hypothesis.
5. Disjunctive syllogism says that if $q \lor r$ (line 3)is true and $\neg q$ (line 4)is true, then r is true.

Therefore, the argument is valid.

2. Exercise 1.12.3, section c

1.	$p \lor q$	Hypothesis
2.	$\neg(\neg p) \lor q$	Double negation law, 1
3.	$\neg p \to q$	Conditional Identity, 2
4.	$\neg p$	Hypothesis
5.	q	Modus ponens, 3, 4

1. $p \lor q$ is a hypothesis.
2. Double negation law says $p \lor q \equiv \neg(\neg p) \lor q$.
3. Conditional identity says $\neg(\neg p) \lor q \equiv \neg p \to q$.
4. $\neg p$ is a hypothesis.

5. Modus ponens says that if $\neg p \to q$ is true and $\neg p$ is true, then q is true.

3. Exercise 1.12.5, sections c,d

   (c.)

   Replace the English phrase with variable:

   c: I will buy a new car.
   h: I will buy a new house.
   j: I will get a job.

   The form of the argument is:

   $$\begin{array}{c} (c \wedge h) \to j \\ \neg j \\ \hline \therefore \neg c \end{array}$$

   The argument is invalid.

   For the scenario like this $\to$:

c	h	j	$\neg c$	$\neg j$	$c \wedge h$	$(c \wedge h) \to j$
T	F	F	F	T	F	T

   The two hypotheses are True, but the conclusion is False.
   Therefore, the argument is invalid.

   (d.)

   Replace the English phrase with variable:

   c: I will buy a new car.
   h: I will buy a new house.
   j: I will get a job.

   The form of the argument is:

   $$\begin{array}{c} (c \wedge h) \to j \\ \neg j \\ \text{h} \\ \hline \therefore \neg c \end{array}$$

   The argument is valid.

1.	$(c \wedge h) \to j$	Hypothesis
2.	$\neg j$	Hypothesis
3.	$\neg(c \wedge h)$	Modus Tollen, 1, 2
4.	$\neg c \vee \neg h$	De Morgan's law, 3
5.	h	Hypothesis
6.	$\neg(\neg h)$	Double negation law, 5
7.	$\neg c$	Disjunctive syllogism, 4, 6

   1. $(c \wedge h) \to j$ is a hypothesis.
   2. $\neg j$ is a hypothesis.

3. Modus tollen says if $(c \wedge h) \rightarrow j$ (Line 1) is true and $\neg j$ (Line 2) is true, then $\neg(c \wedge h)$ is true.
4. De Morgan's law says $\neg(c \wedge h) \equiv \neg c \vee \neg h$.
5. h is a hypothesis.
6. Double negation law says $h \equiv \neg(\neg h)$
7. Disjunctive syllogism says if $\neg c \vee \neg h$ (Line 4)is true and $\neg(\neg h)$ (Line 6)is true, then $\neg c$ is true.
Therefore, the argument is valid.

b)

1. Exercise 1.13.3, section b

	P(x)	Q(x)
a	F	T
b	F	F

2. Exercise 1.13.5, sections d,e

(d.)
Define the following two predicates:
M(x) : x missed the class.
D(x) : x got a detention.

The form of the argument is:

$$\begin{array}{c} \forall x(M(x) \rightarrow D(x)) \\ \text{Penelope, a student in the class} \\ \neg M(Penelope) \\ \hline \therefore \neg D(x) \end{array}$$

The argument is invalid.

For the situation like below:

	M(x)	D(x)	$M(x) \rightarrow D(x)$	$\neg M(x)$	$\neg D(x)$
Penelope	F	T	T	T	F

The hypotheses are true while the conclusion is false. Therefore, the argument is invalid.

(e.)
Define the following three predicates:
M(x) : x missed the class.
D(x) : x got a detention.
A(x) : x got an A.

The form of the argument is:

$$\begin{array}{c} \forall x((M(x) \vee D(x)) \rightarrow \neg A(x)) \\ \text{Penelope, a student in the class.} \\ \text{A(Penelope)} \\ \hline \therefore \neg D(Penelope) \end{array}$$

The argument is valid.

1.	$\forall((M(x) \vee D(x)) \rightarrow \neg A(x))$	Hypothesis
2.	Penelope, a student in the class.	Hypothesis
3.	$M(Penelope) \vee D(Penelope) \rightarrow \neg A(Penelope)$	Universal instantiation
4.	A(Penelope)	Hypothesis
5.	$\neg(\neg A(Penelope))$	Double Negation law, 4
6.	$\neg(M(Penelope) \vee D(Penelope))$	Modus tollens, 3,5
7.	$\neg M(Penelope) \wedge \neg D(Penelope)$	De Morgan's Law, 6
8.	$\neg D(Penelope)$	Simplification, 7·

1. $\forall((M(x) \vee D(x)) \rightarrow \neg A(x))$ is a hypothesis.
2. Penelope, a student in the class is a hypothesis.
3. $M(Penelope) \vee D(Penelope) \rightarrow \neg A(Penelope)$ is true by universal instantiation applied to line 1 and 2.
4. A(Penelope) is a hypothesis.
5. $\neg(\neg A(Penelope))$ is true by Double Negation Law applied to line 4.
6. $\neg(M(Penelope) \vee D(Penelope))$ is true by Modus tollens applied to line 3 and 5,
7. $\neg M(Penelope) \wedge \neg D(Penelope)$ is true by De Morgan's Law applied to line 6.
8. $\neg D(Penelope)$ is true by simplification applied to line 7.

Therefore, the argument is valid.

2 Question Six

1. Exercise 2.4.1, section d

   Theorem: The product of two odd integers is an odd integer.

   Proof:

   Assume that x and y are two odd integers.

   Since x is odd, there is an integer a such that x = 2a + 1. And since y is odd, there is an integer b such that y = 2b + 1.

   Plug x = 2a + 1 and y = 2b + 1 into $x \times y$ to get:

   $x \times y = (2a + 1) \times (2b + 1)$

   $= 4ab + 2a + 2b + 1$

   $= 2(2ab + a + b) + 1$

   Since a and b are two integers, 2ab + a + b is also an integer.

   Since $x \times y = 2m + 1$, where m = 2ab + a + b is an integer, the product of two odd integer x and y is odd.

   ■

2. Exercise 2.4.3, section b
   Theorem: If x is a real number and $x \leq 3$, then $12 - 7x + x^2 \geq 0$.

   Proof:

   Assume x is a real number and $x \leq 3$. We shall prove that $12-7x+x^2 \geq 0$.

   Subtract 3 to both sides of the inequality $x \leq 3$, we get $x - 3 \leq 0$.

   Subtract 4 to both sides of the inequality $x \leq 3$, we get $x - 4 \leq -1$.

   Apparently x - 3 and x - 4 are both negative integers, their product $(x - 3) \times (x - 4)$ must be a positive integer.

   $(x - 3) \times (x - 4) \geq 0$

   Multiplying out the left hand side of the inequality gives the conclusion of the theorem:

   $12 - 7x + x^2 \geq 0$

   ■

3 Question Seven

1. Exercise 2.5.1, section d
   Prove each statement by contrapositive:

   Theorem: For every integer n, if $n^2 - 2n + 7$ is even, then n is odd.

   Prove:

   Let n be a integer. We will assume that it is not true that n is odd and will show that $n^2 - 2n + 7$ is odd.

   Since n is not odd, then n is even.

   Since n is even, there is an integer a such that n = 2a.

   Therefore $n^2 - 2n + 7 = (2a)^2 - 2(2a) + 7$

   $= 4a^2 - 4a + 7$

   $= 2(2a^2 - 2a + 3) + 1$

   since a is an integer, $2a^2 - 2a + 3$ is also an integer.

   since $n^2 - 2n + 7$ is equal to an integer plus 1, $n^2 - 2n + 7$ is odd.

   ■

2. Exercise 2.5.4, section a,b
   (a.)

   Theorem: For every pair of real numbers x and y, if $x^3 + xy^2 \leq x^2y + y^3$, then $x \leq y$.

   Prove:

   Let x and y be two real numbers, we will assume that $x > y$, and will show that $x^3 + xy^2 > x^2y + y^3$.

   Since x and y are two real numbers and $x > y$, then x and y can not be zero at the same time.

   Therefore, $x^2 + y^2 > 0$

   Since$x^2 + y^2$ is positive, multiplying $x^2 + y^2$ to both sides of $x > y$ will not change the direction of inequality $x > y$:

   $x(x^2 + y^2) > y(x^2 + y^2)$

   $x^3 + xy^2 > x^2y + y^3$

   Therefore, the argument is valid.

   ■

   (b.)

   Theorem: For every pair of real numbers x and y, if $x + y > 20$, then $x > 10$ or $y > 10$.

   Prove:

Let x and y be two real numbers, we will assume that $x \leq 10$ and $y \leq 10$, and will show that $x + y \leq 20$

add y to both sides of inequality $x \leq 10$, we get $x + y \leq 10 + y$,

add 10 to both sides of the inequality $y \leq 10$, we get $10 + y \leq 10 + 10$

$10 + y \leq 20$

Since we know $x + y \leq 10 + y$ and $10 + y \leq 20$, we can say that $x + y \leq 10 + y \leq 20$

Therefore $x + y \leq 20$. Overall, the argument is valid.

■

3. Exercise 2.5.5, section c

   Theorem: For every non-zero real number x, if x is irrational, then $\frac{1}{x}$ is also irrational.

   Prove:

   Let x be a non-zero real number, we will assume that $\frac{1}{x}$ is rational, and will show that x is also rational.

   Since $\frac{1}{x}$ is rational and $x \neq 0$, $\frac{1}{x} = \frac{a}{b}$, where a and b are integers and $a \neq 0, b \neq 0$.

   Then $x = \frac{b}{a}$,

   Since x is equal to the ratio of two integers where the denominator $\neq 0$, then x is rational.

   ■

4 Question Eight

Exercise 2.6.6, sections c,d

(c.)

Theorem:

The average of three real numbers is greater than or equal to at least one of the numbers.

Prove:

Assume there exists three real numbers x, y, and z. And their average number k is smaller than all of them.

Since k is the average of x, y and z, k is equal to $\frac{(x+y+z)}{3}$;

Since $k < x$,$k < y$, and $k < z$, then $3k < x + y + z$.

Let both sides of the inequality divided by 3, we get $k < \frac{(x+y+z)}{3}$.

Therefore, k cannot be equal to $\frac{(x+y+z)}{3}$.

This contradicts the assumption that k is the average number of x, y, and z.

Therefore, the average of three real numbers must be greater than or equal to at least one of the numbers.

■

(d.)

Theorem: There is no smallest integer.

Prove:

Assume that there is a smallest integer called r.

Since r is an integer, r - 1 is also an integer. Since they are both integers, $r - 1 < r$.

r - 1 is an integer that is smaller than r.

This contradicts the assumption that r is the smallest integer. Therefore, there is no smallest integer.

■

5 Question Nine

Theorem:

If integers x and y have the same parity, then x + y is even.

Prove:

If integers x and y have the same parity, we consider two cases:

Case 1: x and y are both even. Since x is even, there is an integer a such that x = 2a. Since y is even, there is an integer b such that y = 2b.

Plug in the expression x + y.

x + y = 2a + 2b = 2(a + b)

Since a and b are two integers, a + b is also an integer. We can conclude that if x and y are even, then x + y is also even.

Case 2: x and y are both odd. Since x is odd, there is an integer k such that x = 2k+1. Since y is odd, there is an integer i such that y = 2i + 1.

Plug in the expression x + y.

x + y = 2k + 1 + 2i + 1 = 2(k + i + 1)

Since k, i are both integers, k + i + 1 is also an integer. We can conclude that if x and y are odd, then x + y is even.

■