Homework 11

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1 Question 5

1. **a. Prove that for any positive integer n, 3 divide** $n^3 + 2n$

   Proof.

   BY induction on n.

   Base case: n = 1

   When n = 1, $n^3 + 2n = 1 + 2 = 3 = 3 \times 1$

   Therefore, for n = 1, 3 divide $n^3 + 2n$.

   Inductive step: We will show that for any integer k, if 3 divide $k^3 + 2k$, then 3 divide $(k + 1)^3 + 2(k + 1)$

   By the inductive hypothesis, 3 divides k^3+2k, which means that $k^3+2k = 3m$ for some integer m.

   Starting with $(k + 1)^3 + 2(k + 1)$:

   $(k + 1)^3 + 2(k + 1) = k^3 + 3k^2 + 3k + 1 + 2k + 2$ by algebra

   $= 3m + 3k^2 + 3k + 3$ by the inductive hypothesis

   $= 3(m + k^2 + k + 1)$ by algebra

   Since k is an integer, m is an integer, $m + k^2 + k + 1$ is also an integer. Therefore, $(k + 1)^3 + 2(k + 1)$ is equal to 3 times an integer which means $(k + 1)^3 + 2(k + 1)$ is divisible by 3. ■

2. **b. Use strong induction to prove that any positive integer n(**$n \geq 2$**) can be written as a product of primes.**

   Proof.

   By strong induction on n.

   Base case:

   n = 2. Since 2 is a prime number, it already is a product of one prime number: 2.

   Inductive step: Assume that for k ≥ 2, k can be expressed as a product of prime numbers. We will show that k+1 can be expressed as a product of prime numbers.

   If k+1 is prime, then k+1 is a product of one prime number: k+1.

   If k+1 is not prime, k+1 is composite and can be expressed as the product of two integers a/b which are both at least 2. We need to show that both a and b fall in the range of 2 to k to apply in the inductive hypothesis.

   Since k+1 = ab, then a = (k+1)/b. Furthermore, since b ≥ 2, then a = (k+1)/b ¡ k+1. If a is an integer which is strictly less than k+1, then a≤ k.

   Therefore, a and b are both in the range from a to k. Therefore, they can be expressed as a product of primes.

a = $p_1 p_2 p_l$ b = $q_1 q_2 q_m$

Therefore k+1 can be expressed as product of primes:

k+1 = a·b = $(p_1 p_2 p_l) \cdot q_1 q_2 q_m$.

■

2 Question 6

1. **a) Exercise 7.4.1, sections a-g**

   a.

   P(3) = $1^2 + 2^2 + 3^2 = 1 + 4 + 9 = 14 = \dfrac{3 \times (3+1) \times (2 \times 3 + 1)}{6}$

   Therefore, for n = 3, $\sum_{j=1}^{n} j^2 = \dfrac{n(n+1)(2n+1)}{6}$ is true.

   b.

   Take P = k into the equation, $\sum_{j=1}^{k} j^2 = \dfrac{k(k+1)(2k+1)}{6}$

   c.

   Take P = k + 1 into the equation, $\sum_{j=1}^{k+1} j^2 = \dfrac{(k+1) \times (k+1+1) \times (2(k+1)+1)}{6}$

   d.

   Since n can be any positive integer, n $\geq$ 1.

   Therefore, for the base case, n should be 1. Then P(1) must be proven in the base case.

   e.

   For the inductive step, we must be proven that for every k $\geq$ 1, if P(k) is true, then P(k+1) is true.

   f.

   the inductive hypothesis in the inductive step is P(k) is true, which is $\sum_{j=1}^{k} j^2 = \dfrac{k(k+1)(2k+1)}{6}$ is true.

   g.

   Theorem: For n $\geq$ 1, $\sum_{j=1}^{n} j^2 = \dfrac{n(n+1)(2n+1)}{6}$.

   Proof.

   By induction on n.

   Base case: n = 1.
   $\dfrac{1 \times (1+1) \times (2 \times 1 + 1)}{6} = \dfrac{1 \times 2 \times 3}{6} = 1 = 1^2$

   Therefore, for n = 1, $\sum_{j=1}^{n} j^2 = \dfrac{n(n+1)(2n+1)}{6}$.

   Inductive step: We will show that for any integer k $\geq$ 1, if $\sum_{j=1}^{k} j^2 = \dfrac{k(k+1)(2k+1)}{6}$

   then $\sum_{j=1}^{k+1} j^2 = \dfrac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$ is also true.

Starting with the left side of the equation to be proven:
$\sum_{j=1}^{k+1} j^2 = 1^2 + 2^2 + 3^2 + 4^2 + ... + (k+1)^2$
$= \sum_{j=1}^{k} j^2 + (k+1)^2$ by separating out the last term
$= \dfrac{k(k+1)(2k+1)}{6} + (k+1)^2$ by the inductive hypothesis
$= \dfrac{k(2k+1)(k+1)}{6} + (k+1)^2$
$= \dfrac{(2k^2+k)(k+1)+(6k+6)(k+1)}{6}$
$= \dfrac{(k+1)(2k^2+7k+6)}{6}$
$= \dfrac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$ by algebra
Therefore, $\sum_{j=1}^{k+1} j^2 = \dfrac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$
■

b) Exercise 7.4.3, section c

Proof.

By induction on n.

Base case: n = 1.

When n = 1, the left side $\dfrac{1}{n^2} = \dfrac{1}{1^2} = 1 \leq 2 - \dfrac{1}{n} = 1$

Therefore, for n = 1, $\sum_{j=1}^{n} \dfrac{1}{n^2} \leq 2 - \dfrac{1}{n}$

Inductive step: We will show that for any integer k $\geq$ 1, if $\sum_{j=1}^{k} \dfrac{1}{k^2} \leq 2 - \dfrac{1}{k}$ then $\sum_{j=1}^{k+1} \dfrac{1}{(k+1)^2} \leq 2 - \dfrac{1}{(k+1)^2}$
Starting with the left side:
$\sum_{j=1}^{k+1} \dfrac{1}{(k+1)^2} = \sum_{j=1}^{k} \dfrac{1}{k^2} + \dfrac{1}{(k+1)^2}$ by separating out the last term
$\leq 2 - \dfrac{1}{k} + \dfrac{1}{(k+1)^2}$ by the inductive hypothesis
$\leq 2 - \dfrac{1}{k} + \dfrac{1}{k(k+1)}$ because $\dfrac{1}{(k+1)^2} \leq \dfrac{1}{k(k+1)}$
$= 2 - \dfrac{k+1-1}{k(k+1)}$

= 2 - $\dfrac{k}{k(k+1)}$

= 2 - $\dfrac{1}{k+1}$

$\leq 2 - \dfrac{1}{(k+1)^2}$ by algebra

Therefore, $\sum_{j=1}^{k+1} \dfrac{1}{(k+1)^2} \leq 2 - \dfrac{1}{(k+1)^2}$

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c) Exercise 7.5.1, section a

Prove that for any positive integer n, 4 evenly divides $3^{2n} - 1$

Proof.

By induction on n.

Base case: n = 1

$3^{2n} - 1 = 3^{2\times1} - 1 = 3^2 - 1 = 8$. Since 4 evenly divides 8, the theorem holds for the case n = 1.

Inductive step: Suppose that for the positive integer k, 4 evenly divides $3^{2k} - 1$. Then, we will show that 4 evenly divides $3^{2(k+1)} - 1$.

By the inductive hypothesis, 4 evenly divides $3^{2k} - 1$, which means that $3^{2k} - 1 = 4$m for some integer m. By adding 1 to both side of the equation 3m = $3^{2k} - 1$, we get 3^{2k} = 3m + 1 which is an equivalent statement of the inductive hypothesis.

We must show that $3^{2(k+1)} - 1$ can be expressed as 4 times an integer.

$3^{2(k+1)} - 1 = 3^{2k+2} - 1$

$= 9 \times 3^{2k} - 1$ by algebra

$= 9(4m + 1) - 1$ by the inductive hypothesis

$= 36m + 8$

= 4(9m+2) by algebra

Since m is an integer, (9m+2) is also an integer. Therefore, $3^{2(k+1)} - 1$ is equal to 4 times an integer which means that $3^{2(k+1)} - 1$ is divisible by 4.

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