SIMON-REED

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1 DOMAINS, GRAPHS, ADJOINTS AND SPECTRUM

It is a fact of life that many of the most important operators which occur in mathematical physics are not bounded. In this chapter we will introduce some of the basic definitions and theorems necessary for dealing with unbounded operators on Hilbert spaces. The Hellinger-Toeplitz theorem (see Section III.5) says that an everywhere-defined operator A which satisfies $(\mathsf{A}\phi, \psi) = (\phi, \mathsf{A}\psi)$ is necessarily a bounded operator suggesting that a general unbounded operator T will only be defined on a dense linear subset of the Hilbert space. Thus an **operator** on a Hilbert space $\mathcal{H}$ is a linear map from its domain, a linear subspace of $\mathcal{H}$, into $\mathcal{H}$. Unless we specify otherwise, we will always suppose that the domain is dense. This subspace, which we denote by $D(\mathsf{T})$, is called the **domain** of the operator T. So, to identify an unbounded operator on a Hilbert space one must first give the domain on which it acts and then specify how it acts on that subspace.

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Example 1 (The position operator). Let $\mathcal{H} = L^2(\mathbb{R})$ and let $D(\mathsf{T})$ be the set of functions ϕ in $L^2(\mathbb{R})$ which satisfy $\int_{-\infty}^{\infty} x^2 |\phi(x)|^2 \, dx < \infty$. For $\phi \in D(\mathsf{T})$ define $(\mathsf{T}\phi)(x) = x\phi(x)$. It is clear that T is unbounded since if we choose ϕ to have support near plus or minus infinity, we can make $\|\mathsf{T}\phi\|$ as large as we like while keeping $\|\phi\| = 1$. Of course, even if $\phi \notin D(\mathsf{T})$, $x\phi(x)$ has a well- defined meaning as a function, but it is not in $L^2(\mathbb{R})$. Thus, if we want to deal only with the Hilbert space $L^2(\mathbb{R})$ we must restrict the domain of T. The domain we have chosen is the largest one for which the range is in $L^2(\mathbb{R})$.

Example 2. Let $\mathcal{H} = L^2(\mathbb{R})$ and $D(\mathsf{T}) = \mathcal{S}(\mathbb{R})$. On $D(\mathsf{T})$ define $\mathsf{T}\psi = -\psi''(x) + x^2\psi(x)$. If $\phi_n(x)$ is the n-th Hermite function (see the appendix to Section V.3), then $\phi_n \in D(\mathsf{T})$ and $\mathsf{T}\phi_n(x) = (2n+1)\phi_n(x)$. Thus T must be unbounded since it has arbitrarily large eigenvalues.

The notion of the graph of a linear transformation, introduced by von Neumann, is very useful for studying unbounded operators.

Definition 1. The **graph** of the linear transformation T is the set of pairs

$$\{[\phi, \mathsf{T}\phi] | \phi \in D(\mathsf{T})\} \ . \tag{1}$$

The graph of T, denoted by $\Gamma(\mathsf{T})$, is thus a subset of $\mathcal{H} \times \mathcal{H}$ which is a Hilbert space with inner product

$$([\phi_1, \psi_1], [\phi_2, \psi_2]) = (\phi_1, \phi_2) + (\psi_1, \psi_2) \, . \tag{2}$$

T is called a closed operator if $\Gamma(\mathsf{T})$ is a closed subset of $\mathcal{H} \times \mathcal{H}$.

Definition 2. Let T_1 and T be operators on $\mathcal{H}$. If $\Gamma(\mathsf{T}_1) \supset \Gamma(\mathsf{T})$, then T_1 is said to be an **extension** of T and we write $\mathsf{T}_1 \supset \mathsf{T}$. Equivalently, $\mathsf{T}_1 \supset \mathsf{T}$ if and only if $D(\mathsf{T}_1) \supset D(\mathsf{T})$ and $\mathsf{T}_1\phi = \mathsf{T}\phi$ for all $\phi \in D(\mathsf{T})$.

Definition 3. An operator T is **closable** if it has a closed extension. Every closable operator has a smallest closed extension, called its **closure**, which we denote by $\overline{\mathsf{T}}$.

A natural way to try to obtain a closed extension of an operator T, is to take the closure of its graph in $\mathcal{H} \times \mathcal{H}$. The trouble with this is that $\Gamma(\mathsf{T})$ may not be the graph of an operator (for example, see Problem 1). However, most operators which we deal with will be symmetric operators (introduced Section VIII.2) and we will see that they always have closed extensions.

Preposition 1. *If* T *is closable, then* $\Gamma(\overline{\mathsf{T}}) = \overline{\Gamma(\mathsf{T})}$.

Proof. Suppose that S is a closed extension of T. Then $\overline{\Gamma(\mathsf{T})} \subset \Gamma(\mathsf{S})$ so if $[0, \psi] \in \overline{\Gamma(\mathsf{T})}$ then $\psi = 0$.

Define R with $D(\mathsf{R}) = \left\{ \psi | [\psi, \phi] \in \overline{\Gamma(\mathsf{T})} \text{ pro nějaké } \phi \right\}$ by $\mathsf{R}\psi = \phi$ where $\phi \in \mathcal{H}$ is the unique vector so that $[\psi, \phi] \in \overline{\Gamma(\mathsf{T})}$. Then $\Gamma(\mathsf{R}) = \overline{\Gamma(\mathsf{T})}$ so R is a closed extension of T. But $R \subset \mathsf{S}$ which is an arbitrary closed extension, so $\mathsf{R} = \mathsf{T}$. $\square$

The following example illustrates the concepts we have just introduced.

Example 3. Let $\mathcal{H} = L^2(\mathbb{R})$, $D(\mathsf{T}) = C_0^\infty(\mathbb{R})$ and $D(\mathsf{T}_1) = C_0^1(\mathbb{R})$, the once continuously differentiable functions with compact support. Let $\mathsf{T}f = if'(x)$ if $f \in D(\mathsf{T})$ and $\mathsf{T}_1 f = if'(x)$ if $f \in D(\mathsf{T}_1)$. T_1 is an extension of T.

We will show that $\overline{\Gamma(\mathsf{T})} \supset \Gamma(\mathsf{T}_1)$. When we prove that T is symmetric and therefore closable, it will follow that T extends T_1.

First we introduce the approximate identity, $\{j_\varepsilon(x)\}$. Let $j(x)$ be any positive, infinitely differentiable function with support in $(-1, 1)$ so that $\int_{-\infty}^{\infty} j(x)\mathrm{d}x = 1$. Define $j_\varepsilon(x) = \frac{1}{\varepsilon} j(\frac{x}{\varepsilon})$. If $\phi \in D(\mathsf{T}_1)$, set

$$\phi_\varepsilon(x) = \int_{-\infty}^{\infty} j_\varepsilon(x - t)\phi(t)\,\mathrm{d}t\,. \tag{3}$$

Then

$$|\phi_\varepsilon(x) - \phi(x)| \le \int_{-\infty}^{\infty} j_\varepsilon(x-t)|\phi(t) - \phi(x)|\,\mathrm{d}t \le \tag{4}$$

$$\le \left(\sup_{t:|x-t|<\varepsilon} |\phi(t) - \phi(x)| \right) \int_{-\infty}^{\infty} j_\varepsilon(x-t)\,\mathrm{d}t = \sup_{t:|x-t|<\varepsilon} |\phi(t) - \phi(x)|\,.$$

Since ϕ has compact support, it is uniformly continuous which implies that $\phi_\varepsilon \to \phi$ uniformly. Since the ϕ_ε have support in a fixed compact set, $\phi_\varepsilon \to \phi$ in $L^2(\mathbb{R})$.

Similarly,

$$i\frac{\mathrm{d}}{\mathrm{d}x}\phi_\varepsilon(x) = \int_{-\infty}^{\infty} i\frac{\mathrm{d}}{\mathrm{d}x} j_\varepsilon(x-t)\phi(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} -i\left(\frac{\mathrm{d}}{\mathrm{d}t} j_\varepsilon(x-t)\right)\phi(t)\,\mathrm{d}t = \tag{5}$$

$$= \int_{-\infty}^{\infty} j_\varepsilon(x-t) i\frac{\mathrm{d}}{\mathrm{d}t}\phi(t)\,\mathrm{d}t \xrightarrow{L^2(\mathbb{R})} i\frac{\mathrm{d}}{\mathrm{d}x}\phi(x)\,.$$

Since $j_\varepsilon(x)$ has compact support and is infinitely differentiable, $\phi_\varepsilon \in C_0^\infty(\mathbb{R})$. Thus, $\phi_\varepsilon \in D(\mathsf{T})$ for each $\varepsilon > 0$. What we have shown above is that

$$\phi_\varepsilon \xrightarrow{L^2(\mathbb{R})} \phi\,, \quad \mathsf{T}\phi_\varepsilon \xrightarrow{L^2(\mathbb{R})} \mathsf{T}\phi\,. \tag{6}$$

Thus, the closure of the graph of T contains the graph of T_1. The notion of adjoint operator can be extended to the unbounded case.

Definition 4. Let T be a densely defined linear operator on a Hilbert space $\mathcal{H}$. Let $D(\mathsf{T}^\star)$ be the set of $\phi \in \mathcal{H}$ for which there is an $\eta \in \mathcal{H}$ with

$$(\mathsf{T}\psi, \phi) = (\psi, \eta) \quad \text{for all } \psi \in D(\mathsf{T})\,. \tag{7}$$

For each such $\phi \in D(\mathsf{T}^\star)$ we define $\mathsf{T}^\star\phi = \eta$. $\mathsf{T}^\star$ is called the **adjoint** of T. By the Riesz lemma, $\phi \in D(\mathsf{T}^\star)$ if and only if $|(\mathsf{T}\psi, \phi)| \le C\,\|\psi\|$ for all $\psi \in D(\mathsf{T})$.

We note that $\mathsf{S} \subset \mathsf{T}$ implies $\mathsf{T}^\star \subset \mathsf{S}^\star$. Notice that for η to be uniquely determined by (VIII.1) we need the fact that $D(\mathsf{T})$ is dense. Unlike the case of bounded operators, the domain of $\mathsf{T}^\star$ may not be dense as the following example shows. As a matter of fact it is possible to have $D(\mathsf{T}^\star) = \{0\}$.

Example 4. Suppose that f is a bounded measurable function, but that $f \notin L^2(\mathbb{R})$. Let $D(\mathsf{T}) = \left\{\psi \in L^2(\mathbb{R}) | \int_{-\infty}^{\infty} f(x)\psi(x)\,\mathrm{d}x < \infty\right\}$. $D(\mathsf{T})$ certainly contains all the L^2 functions with compact support so $D(\mathsf{T})$ is dense in $L^2(\mathbb{R})$. Let ψ_0 be some fixed vector in $L^2(\mathbb{R})$ and define $\mathsf{T}\psi = (f,\psi)\psi_0$ for $\psi \in D(\mathsf{T})$. Suppose that $\phi \in D(\mathsf{T}^\star)$, then for all $\psi \in D(\mathsf{T})$

$$(\psi, \mathsf{T}^\star\phi) = (\mathsf{T}\psi, \phi) = ((f,\psi)\psi_0, \phi) = \overline{(f,\psi)}(\psi_0,\phi) = (\psi, (\psi_0,\phi)f)\,. \tag{8}$$

Thus $\mathsf{T}^\star\phi = (\psi_0,\phi)f$. Since $f \notin L^2(\mathbb{R})$, $(\psi_0,\phi) = 0$. Thus any $\phi \in D(\mathsf{T}^\star)$ is orthogonal to ψ_0, so $D(\mathsf{T}^\star)$ is not dense. In fact, $D(\mathsf{T}^\star)$ is just the vectors perpendicular to ψ_0, and on that domain $\mathsf{T}^\star$ is the zero operator.

If the domain of $\mathsf{T}^\star$ is dense, then we can define $\mathsf{T}^\star = (\mathsf{T}^\star)^\star$. There is a simple relationship between the notions of adjoint and closure.

Theorem 1. *Let* T *be a densely defined operator on a Hilbert space* $\mathcal{H}$*. Then:*

- $\mathsf{T}^\star$ *is closed.*
- T *is closable if and only if* $D(\mathsf{T}^\star)$ *is dense in which case* $\overline{\mathsf{T}} = \mathsf{T}^{\star\star}$*.*
- *If* T *is closable, then* $(\overline{\mathsf{T}})^\star = \mathsf{T}^\star$*.*

Proof. We define a unitary operator V on $\mathcal{H} \times \mathcal{H}$ by

$$\mathsf{V}[\phi,\psi] = [-\psi,\phi]\,. \tag{9}$$

Since V is unitary, $\mathsf{V}[E^\perp] = (\mathsf{V}[E])^\perp$ for any subspace E. Let T be a linear operator on $\mathcal{H}$ and suppose $[\phi,\eta] \subset \mathcal{H} \times \mathcal{H}$. Then $[\phi,\eta] \in \mathsf{V}[\Gamma(\mathsf{T})]^\perp$

if and only if $([\phi,\eta],[-\mathsf{T}\psi,\psi]) = 0$ for all $\psi \in D(\mathsf{T})$ which holds if and only if $(\phi,\mathsf{T}\psi) = (\eta,\psi)$ for all $\psi \in D(\mathsf{T})$, that is, if and only if $[\phi,\eta] \in \Gamma(\mathsf{T}^\star)$. Thus $\Gamma(\mathsf{T}^\star) = \mathsf{V}\,[\Gamma(\mathsf{T})]^\perp$. Since $\mathsf{V}[\Gamma(\mathsf{T})]^\perp$ is always a closed subspace of $\mathcal{H}\times\mathcal{H}$, this proves (1).

To prove (2), observe that $\Gamma(\mathsf{T})$ is a linear subset of $\mathcal{H}\times\mathcal{H}$ so

$$\overline{\Gamma(\mathsf{T})} = \left[\Gamma(\mathsf{T})^\perp\right]^\perp = \left[\mathsf{V}^2\Gamma(\mathsf{T})^\perp\right]^\perp = \left[\mathsf{V}(\mathsf{V}\Gamma(\mathsf{T}))^\perp\right]^\perp . \tag{10}$$

Thus, by the proof of (1), if $\mathsf{T}^\star$ is densely defined, $\Gamma(\mathsf{T})$ is the graph of $\mathsf{T}^{\star\star}$.

Conversely, suppose that $D(\mathsf{T}^\star)$ is not dense and that $\psi \in D(\mathsf{T}^\star)^\star$. A simple computation shows that $[\psi,0] \in [\Gamma(\mathsf{T}^\star)]^\perp$ so $\mathsf{V}\,[\mathsf{T}(\mathsf{T}^\star)]^\perp$ is not the graph of a (single-valued) operator. Since $\overline{\Gamma(\mathsf{T})} = \mathsf{V}\,[\mathsf{T}(\mathsf{T}^\star)]^\perp$, we see that T is not closable.

To prove (3), notice that if T is closable,

$$\mathsf{T}^\star = \overline{(\mathsf{T}^\star)} = \mathsf{T}^{\star\star\star} = (\overline{\mathsf{T}})^\star . \tag{11}$$

□

MB: If M is an subspace of $\mathcal{H}$, then $M^\perp$ is always closed. If $\{y_n\}_n \subset M^\perp$ which $y_n \to y$, then $(y_n, x) = 0$ for all $x \in M$ and so $(y,x) = 0$ for all $x \in M$, therefore $y \in \mathcal{H}$. Also $(M^\perp)^\perp = \overline{M}$.