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BACHELOR THESIS

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The connection between continuum mechanics and Riemannian geometry

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Dedication.

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1. "Mathematical framework"

vymyslet lepší název kapitoly

In this chapter we shall revise some basic definitions and propositions of the functional calculus, which we will need for the proper formulation of the hydrodynamic geometry. Most of the existing literature on this topic often lacks clarity due to the omittance of certain details. This can include the domains on which functionals are defined or regularity conditions.

Over the last century, modern mathematics has worked to develop the theory of generalized functions - distributions, including Dirac's delta distributions and operations with them, such as in [CITATION]. The same applies to functional variations and the conditions on which they exist. Among physicists, however, it is convential to operate with delta distributions as they would with "ordinary" functions, using dualities as integrals and functional variations as ordinary derivatives.

After introducing the concept of Gateaux and Frechet derivatives, which are suitable for arbitrary normed space, we shall take a closer look on the space of smooth functions and functionals on it.

1.1 Functional derivatives

přidat nějaký kontext

Definition 1 (Gateaux and Frechet derivative). *Let X be a normed space, $A : X \to \mathbb{R}$ is a functional and $x \in \mathrm{Dom}(A)$.*

1. *Let $h \in D(A)$. Gateaux derivative $\delta A(x,h)$ in the direction h is defined as*

$$\delta A(x,h) := \frac{\mathrm{d}}{\mathrm{d}t} A(x+th)|_{t=0} = \lim_{t\to 0} \frac{A(x+th)-A(x)}{t} \,. \tag{1.1}$$

2. *Fréchet diferential $\mathrm{D}_x A$ is defined as the unique linear bounded functional satisfying the relation*

$$\lim_{\|h\|\to 0} \frac{A(x+h)-A(x)-\mathrm{D}_x A(h)}{\|h\|} = 0 \,. \tag{1.2}$$

We are intersted in such functionals, where X is the space of smooth functions with The smoothness is required because the functionals would operate with derivatives, and the values beeing zero on the boundary is required because we shall use integration by parts.

One of possible choices is to take

$$X^k := \left\{ f \in C^k(\Omega) \,|\, \mathrm{D}^\alpha f = 0 \text{ on } \partial\Omega \text{ for } |\alpha| = 1, \cdots, k-1 \right\} \,. \tag{1.3}$$

This space can be equipped with the usual supremum norm

$$\|f\|_k := \sum_{|\alpha|\le k} \sup_{x\in\Omega} |\mathrm{D}^\alpha f(x)| \,. \tag{1.4}$$

There are two main types of functionals on these spaces - the integral functionals and Dirac's delta distributions.

1.1.1 Integral functionals

The first type is the integral functional

$$A[u] = \int_\Omega a(x, u(x), u'(x), \cdots, u^{(k)}(x))\,\mathrm{d}^n x\,, \tag{1.5}$$

eq:integral-functional

where $a : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$ is an analytic function. For simplicity we often write this using a physicist's notation

$$A[u] = \int a(u)\,\mathrm{d}^n x\,, \tag{1.6}$$

meaning that $a(u)$ is in fact function $a(x, u(x), u'(x), \cdots, u^{(k)}(x))$ and also the domain Ω is specified by the context. We call $a(u)$ the density of functional $A[u]$.

Gateaux derivative of A is computed as follows:

$$\delta A[u, h] = \int_\Omega \sum_{i=1}^{k} \frac{\partial a}{\partial u^{(i)}}(x) h^{(i)}(x)\,\mathrm{d}^n x\,. \tag{1.7}$$

eq:Gateaux

The functional $\delta A[u, h]$ is linear and bounded in h, so the Fréchet derivative of A exists and is given by

$$\mathrm{D}_u A[h] = \delta A[u, h]\,. \tag{1.8}$$

To get rid of the derivatives of $h(x)$ in (1.7), we use integration by parts in each term of the sum (exploiting the fact that boundary terms are zero), obtaining

$$\mathrm{D}_u A[h] = \int_\Omega \left[\frac{\partial a}{\partial u}(x) - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial a}{\partial u'}(x) + \cdots + (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}x^k}\frac{\partial a}{\partial u^{(k)}}(x)\right] h(x)\,\mathrm{d}^n x\,. \tag{1.9}$$

eq:perpartes

It is natural to give a special name to the integrand in (1.9).

Definition 2 (Functional derivative of the integral functional)**.** *Let A be the integral functional given by* (1.5)*. The functional derivative of A is the function*

$$\frac{\delta A}{\delta u(x)} = \frac{\partial a}{\partial u}(x) - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial a}{\partial u'}(x) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial a}{\partial u''} - \cdots + (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}x^k}\frac{\partial a}{\partial u^{(k)}}(x)\,. \tag{1.10}$$

The functional derivative can be seen as the density of the integral functional $\mathrm{D}_u A$ applicated on h:

$$\mathrm{D}_u A[h] = \int_\Omega \frac{\delta A}{\delta u(x)} h(x)\,\mathrm{d}x \tag{1.11}$$

Since the Fréchet derivative is always linear bounded functional, another way to look at this relation is the following: $\mathrm{D}_u A$ is a regular distribution represented by some smooth function $\frac{\delta A}{\delta u(x)}$, i.e.

$$\mathrm{D}_u A[h] = \left\langle T_{\frac{\delta A}{\delta u(x)}}, h(x)\right\rangle\,. \tag{1.12}$$

Hodit regulární distribuce před tuhle část?

It follows from theory (CITATION) that the functional derivative beeing zero is a necessery condition for the function beeing the extremal point of the functional (given that the corresponding function is smooth enough).

Theorem 3 (Euler-Lagrange equations). *Let $v \in C^\infty(\Omega)$ such that $v(x)$ is stationary point of the functional A in* (1.5)*, i.e*

$$\delta A[v, h] = 0 \, . \tag{1.13}$$

Then

$$\frac{\delta A}{\delta v(x)} = 0 \, . \tag{1.14}$$

The computation above gives sense to the

$$A[u + h] = A[u] + \left\langle \frac{\delta A}{\delta u}, h \right\rangle + o(h) \, , \tag{1.15}$$

where

$$\lim_{\|h\| \to 0} \frac{o(h)}{\|h\|} = 0 \, . \tag{1.16}$$

In physicist's notation, the function h is often denoted as δu.

1.1.2 Distributions

Definition 4 (Distributions). *Let $\mathcal{D}(\Omega)$ be the space of smooth functions with compact support in $\Omega \subseteq \mathbb{R}^n$. The space of distributions $\mathcal{D}'(\Omega)$ is the dual space, i.e. the space of bounded linear functionals on $\mathcal{D}(\Omega)$. An element $T \in \mathcal{D}'(\Omega)$ is called a distribution.*

Definition 5 (Regular distribution). *Let $a \in L^1_{loc}(\Omega)$. A regular distribution is a distribution $T_a \in \mathcal{D}'(\Omega)$ defined as*

$$\langle T_a, \phi(x) \rangle = \int_\Omega a(x)\phi(x) \mathrm{d}^n x \, . \tag{1.17}$$

Note that if two regular distributions satisfy $T_a = T_b$, then $a = b$ almost everywhere and vice versa.

Definition 6 (Dirac delta distribution). *Let $y \in \Omega$. The Dirac delta distribution $\delta(x - y)$ is defined as*

$$\langle \delta(x - y), \phi(x) \rangle = \phi(y) \, . \tag{1.18}$$

Definition 7 (Derivatives of distributions). *Let $T \in \mathcal{D}'(\Omega)$ be a distribution. We define its derivative by*

$$\langle D^\alpha T, \phi \rangle = (-1)^{|\alpha|} \langle T, D^\alpha \phi \rangle \, , \tag{1.19}$$

where α is a multiindex and

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_N^{\alpha_N}} \, . \tag{1.20}$$

The reason, why distributions have such a "nice" behaviour and are often used by mathematicians, is that every distribution is diferentiable infinitely many times. However, if we abandon this property and require only the existence of derivatives of the k-th order, we can extend regular and Dirac distributions on previously defined space X^k. Indeed,

$$|\langle \partial_x^n \delta(x-y), f(x) \rangle| \le \|f\|_n \quad \forall n = 0, 1, \cdots, k\,, \tag{1.21}$$

so Dirac distributions and their derivatives are also bounded linear functionals on X^k up to the order k.

Rozmysli si, Míro, rozmysli.

Let $a \in C^\infty(\Omega)$. Then it makes sense to define the product of a and arbitrary distribution T by

$$\langle aT, \phi \rangle = \langle T, a\phi \rangle \ . \tag{1.22}$$

In the next chapter we shall frequently use the following identities of the delta distributions, which are written in the next lemma and which we will refer to as the Dirac identities.

`lemma:delta`

Lemma 8 (Dirac identities)**.**

$$f(y)\delta(x-y) = f(x)\delta(x-y)\,, \tag{1.23}$$

$$f(y)\partial_x\delta(x-y) = f(x)\partial_x\delta(x-y) + f'(x)\delta(x-y)\,. \tag{1.24}$$

Proof. The first relation is trivial. Direct calculation gives

$$\begin{aligned}
\langle f(x)\partial_x\delta(x-y), \psi(x)\rangle &= \langle \partial_x\delta(x-y), f(x)\psi(x)\rangle \\
&= -\langle \delta(x-y), f'(x)\psi(x)\rangle - \langle \delta(x-y), f(x)\psi'(x)\rangle \\
&= -f'(y)\psi(y) - f(y)\langle \delta(x-y), \psi'(x)\rangle \\
&= -\langle f'(x)\delta(x-y), \psi(x)\rangle + f(y)\langle \partial_x\delta(x-y), \psi(x)\rangle \\
&= \langle f(y)\delta(x-y) - f'(x)\partial_x\delta(x-y), \psi(x)\rangle \ .
\end{aligned}$$

Therefore

$$\langle f(y)\partial_x\delta(x-y), \psi(x)\rangle = \langle f'(x)\delta(x-y), \psi(x)\rangle + \langle f(x)\partial_x\delta(x-y), \psi(x)\rangle \ . \tag{1.25}$$

□

The convolution of two distributions is not so simple to construct in rigorous way. The procedure is the following. First the tensor product of distributions is defined

$$\langle A(x) \otimes B(y), \phi(x,y)\rangle = \langle A(x), \langle B(y), \phi(x,y)\rangle\rangle \ . \tag{1.26}$$

The product defined in this way satisfies comutativity, associativity and it is indeed a distribution (altough the proof of these properties is non-trivial). Then the convolution of distribution is defined by

$$\langle A \star B, \phi(x)\rangle = \langle A(x) \otimes B(y), \phi(x+y)\rangle \ , \tag{1.27}$$

with additional requirements on $A \otimes B$ and ϕ. Convolution defined in this way satisfies associativity and if it exists, then

$$D^\alpha(A \star B) = D^\alpha A \star B = A \star D^\alpha B \,. \tag{1.28}$$

The convolution with δ and arbitrary distribution A always exists and is equal to

$$A(y) \star \delta(x-y) = A(x) \,. \tag{1.29}$$

This means, in particular, that given an integral functional $T[u]$ one can write

$$T[u(x)] = T[u(y)] \star \delta(x-y) \,, \tag{1.30}$$

or in physicist's notation

eq:konvoluce-s-deltou

$$T[u(x)] = \int \mathrm{d}y\, T[u(y)]\, \delta(x-y) \,. \tag{1.31}$$

The equation (1.31) is very important in calculations of functional derivatives.

Example 9. *Let $u(x)$ be a field variable and $F[u(x)]$ an integral functional. To compute its functional derivative w.r.t. $u(y)$ in different position y, we use the integral expression of F*

$$F[u(x)] = \int \mathrm{d}y F[u(y)]\delta(x-y) \,. \tag{1.32}$$

Its Gateaux diferential is

$$\delta F[u,h] = \frac{\mathrm{d}}{\mathrm{d}t} F[u+th]|_{t=0} = \int \mathrm{d}y \frac{\partial F}{\partial u}(y)\delta(x-y)h(y) \,, \tag{1.33}$$

therefore

$$\frac{\delta F(u(x))}{\delta u(y)} = \frac{\partial F}{\partial u}(x)\delta(x-y) \,. \tag{1.34}$$

Example 10. *In the special case of performing functional derivative w.r.t. the same field but in the different position,*

$$\frac{\delta u^i(x)}{\delta u^j(y)} = \delta^i_j \delta(x-y) \,, \tag{1.35}$$

because

$$\frac{\delta u^i(x)}{\delta u^j(y)} = \frac{\delta}{\delta u^j(y)} \int \mathrm{d}y u^i(y)\delta(x-y) = \frac{\partial u^i}{\partial u^j}(y)\delta(x-y) = \delta^i_j\delta(x-y) \,. \tag{1.36}$$

The same applies to

$$\frac{\delta}{\delta u^j(y)} \partial_x u^i(x) = -\delta^i_j \partial_x \delta(x-y) \,, \tag{1.37}$$

because

$$\frac{\delta}{\delta u^j(y)} \partial_x u^i(x) = \frac{\delta}{\delta u^j(y)} \int \mathrm{d}y \partial_y u^i(y)\delta(x-y) \tag{1.38}$$

$$= -\frac{\delta}{\delta u^j(y)} \int \mathrm{d}y u^i(y)\partial_y\delta(x-y) = -\delta^i_j \partial_x \delta(x-y) \,. \tag{1.39}$$

1.2 Geometrical aspects

1.2.1 Riemannian manifolds and covariant derivative

1.2.2 Lie algebras and Poisson brackets

2. The connection between Riemannian geometry and Hamiltonian mechanics

In this chapter we shall prove the theorem of the connection between the hydrodynamic Poisson bracket and the metrics found in the homogeneous system of hydrodynamic type. The original result was given by Dubrovin and Novikov and can be found in **?**. The proof itself and the preceding lemmas are based on a considerable amount of technical computations, which we shall clarify in detail in order to offer the reader a comprehensive understanding of the chapter in question.

2.1 Systems of hydrodynamic type

Definition 11 (Homogeneous system of hydrodynamic type)**.** *A homogeneous system of hydrodynamic type is an equation of the form*

$$\frac{\partial u^i}{\partial t} = f_j^{i\alpha}(u)\frac{\partial u^j}{\partial x^\alpha} \tag{2.1}$$

eq:h-system

where $u^i(t, x^\alpha)$ are unknown functions, $i = 1, \ldots, N$, $\alpha = 1, \ldots, d$.

Sometimes we refer to the number of equations N as the dimension of the hydrodynamical phase space and to the number of spatial variables d as the dimension of the configuration space.

Příklad asi přesunout do jiné kapitoly.

Before we go any further, let us present a few examples.

Example 12. *The energy is given by the functional*

$$E = \int e(\rho, \boldsymbol{m}, s, \boldsymbol{F})\,\mathrm{d}\boldsymbol{x} \tag{2.2}$$

The evolution equations in the Eulerian frame are

$$\begin{aligned}
\partial_t \rho &= -\,\partial_i(\rho E_{m_i}) && (2.3)\\
\partial_t m_i &= -\,\partial_j(m_i E_{m_j}) - \rho\partial_i E_\rho - m_j\partial_i E_{m_j} - m_j\partial_i E_{m_j} && (2.4)\\
&\quad -s\partial_i E_s - F_J^j \partial_i E_{F_J^j} + \partial_j(F_I^j E_{F_I^i} + F_O^i E_{F_I^i}) && (2.5)\\
\partial_t s &= -\,\partial_i(s E_{m_i}) && (2.6)\\
\partial_t F_I^i &= -\,E_{m_k}\partial_k F_I^i + F_I^j \partial_j E_{m_i}\,. && (2.7)
\end{aligned}$$

Total energy density e can be prescribed as

$$e = \frac{\boldsymbol{m}^2}{2\rho} + \epsilon(\rho, s, \boldsymbol{F})\,, \tag{2.8}$$

where ϵ is the elastic and internal energy. In particular, $E_{\boldsymbol{m}} = \frac{\boldsymbol{m}}{\rho} = \boldsymbol{v}$ becomes the velocity. The evolution equation for the deformation gradient then gets the explicit form

$$\partial_t \boldsymbol{F} = -(\boldsymbol{v} \cdot \nabla)\boldsymbol{F} + \nabla \boldsymbol{V} \cdot \boldsymbol{F} \, , \tag{2.9}$$

which is the usual evolution equation for $\boldsymbol{F}$ in the Eulerian frame.

It was Riemann who noticed that the functions $f^{i\alpha}_j$ in (2.1) are in fact tensors [CITACE]**?**.

`prop:transformace-A`

Proposition 13 (Transformation of $f^{i\alpha}_j$)**.** *Under smooth change of variables $u^i \mapsto v^a$ of the form*

$$u^i = u^i(v^1, \cdots, v^N) \tag{2.10}$$

the functions $f^{i\alpha}_j$ transform for each α according to the tensor law

$$f^{a\alpha}_b(v) = \frac{\partial v^a}{\partial u^i}\frac{\partial u^j}{\partial v^b} f^{i\alpha}_j(u) \, . \tag{2.11}$$

Proof. Using a direct calculation we obtain

$$\frac{\partial u^i}{\partial t}(v) = \frac{\partial u^i}{\partial v^a}\frac{\partial v^a}{\partial t} \, , \tag{2.12}$$

$$\frac{\partial u^j}{\partial x^\alpha}(v) = \frac{\partial u^j}{\partial v^b}\frac{\partial v^b}{\partial x^\alpha} \, , \tag{2.13}$$

so

$$\frac{\partial u^i}{\partial v^a}\frac{\partial v^a}{\partial t} = \frac{\partial u^j}{\partial v^b} f^{i\alpha}_j(v(u)) \frac{\partial v^b}{\partial x^\alpha} \tag{2.14}$$

and therefore

$$\frac{\partial v^a}{\partial t} = \underbrace{\frac{\partial v^a}{\partial u^i}\frac{\partial u^j}{\partial v^b} f^{i\alpha}_j(v(u))}_{f^{a\alpha}_b(v)} \frac{\partial v^b}{\partial x^\alpha} \, . \tag{2.15}$$

□

According to the previous lemma, if we denote the space of points $(t, x^\alpha) \subset \mathbb{R} \times \mathbb{R}^N$ as some manifold M, then the unknown variables $u^i(t, x^\alpha)$ can be seen as the local coordinates of that manifold. The functions $f^{i\alpha}_j$ are transformed as tensors of the type (1,1) in the i and j indices.

Our next purpose is to gain a richer geometry of that space M. If the functions $f^{i\alpha}_j$ have a special form, we can introduce a geometry of the Poisson brackets.

Definition 14. *1. A Poisson bracket of the hydrodynamic type of two functionals I_1 a I_2 is defined as*

$$\{I_1, I_2\} = \int \mathrm{d}x \left[\frac{\delta I_1}{\delta u^q(x)} A^{qp} \frac{\delta I_2}{\delta u^p(x)}\right] \tag{2.16}$$

where

$$A = (A^{qp}) = \left(g^{qp\alpha}(u) \frac{\mathrm{d}}{\mathrm{d}x^\alpha} + b^{qp\alpha}_s(u) \frac{\partial u^s}{\partial x^\alpha} \right) . \tag{2.17}$$

where $g^{ij\alpha}$ *and* $b^{ij\alpha}_k$ *are certain functions,* $i, j, k = 1, \ldots, N$ *and* $\alpha = 1, \ldots, d$. *In particular*

$$\left\{ u^i(x), u^j(y) \right\} = g^{ij\alpha}[u(x)]\delta_\alpha(x-y) + b^{ij\alpha}_k[u(x)]u^k_\alpha(x)\delta(x-y) . \tag{2.18}$$

2. *The functional of hydrodynamic type is a functional of the form*

$$H[u] = \int_{\mathbb{R}^d} h(u(x))\mathrm{d}^d x , \tag{2.19}$$

where h *is independent of* u_α, $u_{\alpha\beta}$. *The function* h *is called a hamiltonian density.*

3. *The hamiltonian system of hydrodynamic type is a system of the form*

$$u^i_t(x) = \left[g^{ij\alpha}[u(x)] \frac{\partial^2 h(u)}{\partial u^j \partial u^k} + b^{ij\alpha}_k[u(x)] \frac{\partial h(u)}{\partial u^j} \right] u^k_\alpha(x) , \tag{2.20}$$

where $H[u]$ *is the functional of the hydrodynamic type and* h *the hamiltonian density.*

Proposition 15.

$$u^i_t(x) = \left\{ u^i, H \right\} . \tag{2.21}$$

Proof.

$$\left\{ u^i(x), H \right\} = \int \mathrm{d}x'\mathrm{d}y' \frac{\delta u^i(x)}{\delta u^p(x')} \left[g^{pq}(u(x')) \frac{\mathrm{d}}{\mathrm{d}x'} + b^{pq}_m \frac{\partial u^m}{\partial x} \right] \frac{\delta H}{\delta u^q(y')} . \tag{2.22}$$

$$\frac{\delta u^i(x)}{\delta u^p(x')} = \delta^i_p \delta(x - x') \tag{2.23}$$

$$H[u] = \int h(u(y'))\mathrm{d}y' , \quad \frac{\delta H}{\delta u^q(y')} = \int \frac{\partial h}{\partial u^q}(y)\delta(y-y') \tag{2.24}$$

$$\left\{ u^i(x), H \right\} = \int \mathrm{d}y' \left[g^{iq}(u(x)) \frac{\mathrm{d}}{\mathrm{d}x} + b^{iq}_m(u(x)) \frac{\partial u^m}{\partial x} \right] \frac{\partial h}{\partial u^q}(y)\delta(y-y') \tag{2.25}$$

$$= g^{iq}(u(x)) \frac{\partial^2 h}{\partial u^k \partial u^q} \frac{\partial u^k}{\partial x} + b^{iq}_m(u(x)) \frac{\partial u^m}{\partial x} \frac{\partial h}{\partial u^q} \tag{2.26}$$

$$= \left[g^{ij}(u(x)) \frac{\partial^2 h}{\partial u^j \partial u^k} + b^{ij}_k(u(x)) \frac{\partial h}{\partial u^j} \right] \frac{\partial u^k}{\partial x} . \tag{2.27}$$

□

Dokončit a přidělat α. kde se předělá $\frac{\mathrm{d}}{\mathrm{d}x}$ na $\frac{\mathrm{d}}{\mathrm{d}y}$?

Note that we have yet to show that the Poisson bracket of the hydrodynamic type defined above is indeed a Poisson bracket in the classical sense. We must verify that it satisfies the three properties: skew-symmetry, Leibniz rule and Jacobi identity. In fact, this will be the key theorem of this chapter.

2.2 One-dimensional case

In this section we first consider the one-dimensional case in which the unknown functions u^i depend on only two variables (t, x). The hamiltonian system is then

$$\frac{\partial u^i}{\partial t}(t,x) = \left[g^{ij}[u(t,x)] \frac{\partial^2 h(u)}{\partial u^j \partial u^k} + b^{ij}_k [u(t,x)] \frac{\partial h(u)}{\partial u^j} \right] \frac{\partial u^k}{\partial x}(t,x) \tag{2.28}$$

and the Poisson bracket of hydrodynamic type is of the form

$$\{I_1, I_2\} = \int \mathrm{d}x \left[\frac{\delta I_1}{\delta u^q(x)} A^{qp} \frac{\delta I_2}{\delta u^p(x)} \right] , \quad \text{where } A^{qp} = \left(g^{qp}(u) \frac{\mathrm{d}}{\mathrm{d}x} + b^{qp}_s(u) \frac{\partial u^s}{\partial x} \right) . \tag{2.29}$$

In particular

$$\left\{ u^i(x), u^j(y) \right\} = g^{ij}[u(t,x)] \partial_x \delta(x-y) + b^{ij}_k [u(t,x)] \frac{\partial u^k}{\partial x} \delta(x-y) . \tag{2.30}$$

eq:Poisson1D

We say that the Poisson bracket is non-degenerate if $\det(g^{ij}) \neq 0$. We will show later that the non-degeneracy is invariant under the transformation of the coordinates.

Definition 16. *Let $(g^{ij}(u))$ be a non-degenerate matrix. We define functions $\Gamma^i_{jk}(u)$ by*

$$b^{ij}_k(u) = -g^{is}(u) \Gamma^j_{sk}(u) , \quad i, j, k, s = 1, \ldots, N . \tag{2.31}$$

eq:definice-Gamma

The choice of denotation g^{ij} and Γ^i_{jk} has its purpose. We will show soon that these are in fact the metric and afinne coefficients, respectively, on M.

Proposition 17 (Leibniz rule for Poisson bracket)**.** *The Poisson bracket of the hydrodynamic type satisfies the Leibniz rule*

$$\left\{ v^p(u^i(x)), v^q(u^j(y)) \right\} = \frac{\partial v^p}{\partial u^i}(x) \frac{\partial v^q}{\partial u^j}(y) \left\{ u^i(x), u^j(y) \right\} . \tag{2.32}$$

eq:transformace-zavorka

Proof. The coordinates are written as functionals in the form

$$v^p(u^i(x)) = \int \mathrm{d}x' v^p(u^i(x')) \delta(x - x') . \tag{2.33}$$

Applying this to the definition of the Poisson bracket (2.30), we have

$$\left\{ v^p(u^i(x)), v^q(u^j(y)) \right\} = \int \mathrm{d}x' \int \mathrm{d}y' \underbrace{\frac{\delta v^p}{\delta u^i(x')}}_{\frac{\partial v^p}{\partial u^i}(x')\delta(x-x')} \left\{ u^i(x'), u^j(y') \right\} \underbrace{\frac{\delta v^q}{\delta u^j(y')}}_{\frac{\partial v^q}{\partial u^j}(y')\delta(y-y')} = \tag{2.34}$$

$$= \frac{\partial v^p}{\partial u^i}(x) \frac{\partial v^q}{\partial u^j}(y) \left\{ u^i(x), u^j(y) \right\} . \tag{2.35}$$

□

prop:transoformace

Proposition 18 (Transformation of g and Γ)**.** *Let $u^i \mapsto v^a$ be the smooth change of variables. Then*

1. *The g^{ij} coefficients transform as the (2,0) tensors, i.e.*

eq:transformace-metrika

$$g^{pq}(v) = \frac{\partial v^p}{\partial u^i}\frac{\partial v^q}{\partial u^j} g^{ij}[u(v)] \,, \quad p,q = 1,\dots,N \,. \tag{2.36}$$

2. *The Γ^i_{jk} coefficients transform like Christoffel symbols, i.e.*

eq:transformace-konexe

$$\Gamma^p_{qr}(v) = \frac{\partial v^p}{\partial u^i}\frac{\partial u^j}{\partial v^q}\frac{\partial u^k}{\partial v^r}\Gamma^i_{jk}(u) + \frac{\partial v^p}{\partial u^i}\frac{\partial^2 u^i}{\partial v^q \partial v^r} \,. \tag{2.37}$$

Proof. 1. Let us apply the transformation equation (2.32) to the definition of the Poisson bracket (2.30). We get

eq:dosad

$$\begin{aligned} & g^{pq}[v(u(x))]\partial_x\delta(x-y) + b^{pq}_s[v(u(x))]\frac{\partial v^s}{\partial u^k}(x)\frac{\partial u^k}{\partial x}\delta(x-y) \\ &= \frac{\partial v^p}{\partial u^i}(x)\frac{\partial v^q}{\partial u^j}(y)\left[g^{ij}[u(x)]\partial_x\delta(x-y) + b^{ij}_k[u(x)]\frac{\partial u^k}{\partial x}\delta(x-y)\right] \,. \end{aligned} \tag{2.38}$$

From now on (for readibility) we denote

$$T^a_k(x) := \frac{\partial v^a}{\partial u^k}(x) \,. \tag{2.39}$$

We apply Dirac identity (Lemma 8 on page 5) to $T^q_j(y)\partial_x\delta(x-y)$ on the right-hand side of (2.38). Hence all functions are then of variable x, from now on we shall not write it explicitely. We have

eq:porovnani

$$\begin{aligned} & g^{pq}(v)\partial_x\delta(x-y) + b^{pq}_n(v)T^n_j\frac{\partial u^j}{\partial x}\delta(x-y) \\ &= T^p_i T^q_j g^{ij}(u)\partial_x\delta(x-y) + \left(\frac{\partial^2 v^q}{\partial u^j \partial u^k} + T^p_i T^q_j b^{ij}_k\right)\frac{\partial u^k}{\partial x}\delta(x-y) \,. \end{aligned} \tag{2.40}$$

The terms with $\partial_x\delta$ must be equal to each other, as must δ terms. From $\partial_x\delta$ terms we get

$$g^{pq}(v) = T^p_i T^q_j g^{ij}(u) \,, \tag{2.41}$$

which is the transformation relation for g.

2. From δ terms in (2.40), we get

$$b^{pq}_n(v)T^n_k = b^{ij}_k(u)T^p_i T^q_j + g^{ij}(u)T^p_i\frac{\partial^2 v^q}{\partial u^i \partial u^k} \tag{2.42}$$

and after (2.31)

$$-g^{ps}(v)\Gamma^q_{sn}(v)T^n_k = -g^{il}(u)\Gamma^j_{lk}(u)T^p_i T^q_j + g^{ij}(u)T^p_i\frac{\partial^2 v^q}{\partial u^i \partial u^k} \,. \tag{2.43}$$

On the left-hand side we use the transformation of g (2.36)

$$-g^{ij}(u)\Gamma^q_{sm}(v)T^p_iT^s_jT^m_k = -g^{ij}(u)\Gamma^l_{jk}T^p_iT^q_l + g^{ij}(u)\frac{\partial^2 v^q}{\partial u^i \partial u^k}T^p_i \qquad (2.44)$$

and after vectoring out $g^{ij}(u)T^p_i$

$$\Gamma^l_{jk}(u)T^q_l = \Gamma^q_{sm}(v)T^s_jT^m_k + \frac{\partial^2 v^q}{\partial u^i \partial u^k} \,. \qquad (2.45)$$

All that remains is to multiply this equation with $(T^{-1})^a_q$ and we get the final transformation relation for Γ

$$\Gamma^a_{jk}(u) = \Gamma^q_{sm}(v)\frac{\partial v^s}{\partial u^j}\frac{\partial v^m}{\partial u^k}\frac{\partial u^a}{\partial v^q} + \frac{\partial^2 v^q}{\partial u^i \partial u^k}\frac{\partial u^a}{\partial v^q} \,. \qquad (2.46)$$

$\square$

Let us summarize the results. We have defined Poisson bracket in the way that it satisfies the Leibniz rule. It remains to investigate the skew-symmetry and Jacobi identity. We have also defined g^{pq} and Γ^p_{qr}, which transform like tensors and Christoffel symbols, respectively. It remains to show that g^{pq} is indeed symmetric and Γ is the Levi-Civita connection, i.e. the corresponding covariant derivative annuls the metric. We are now ready to prove the key theorem of this chapter which shows that these properties are indeed equivalent. Moreover, the corresponding covariant derivative has zero curvature, which indicates that the phase space M is in fact Euclidean space.

`theorem1:ekvivalence`

Theorem 19 (Dubrovin, Novikov)**.** *Let* $\det(g^{ij}) \neq 0$. *Then* (2.30) *defines a Poisson bracket with skew-symmetry, Leibniz rule and Jacobi identity if and only if these three conditions are satisfied:*

1. $g^{ij} = g^{ji}$, *that is,* g^{ij} *is a metric on the phase space* M.
2. $\Gamma^i_{jk} = \Gamma^i_{kj}$, *that is,* Γ^i_{jk} *are affine coefficients, such that the corresponding covariant derivative has zero torsion and anihilate the metric* g^{ij}.
3. *The covariant derivative corresponding to* Γ^i_{jk} *has zero curvature.*

Corollary 20. *There are local coordinates* $w^i = w^i(u^1, \dots, u^N)$, $i = 1, \dots, N$ *such that*

$$g^{ij}(w) = \tilde{g}^{ij} = \text{const}\,, \quad \Gamma^i_{jk}(w) = 0\,. \qquad (2.47)$$

In these coordinates is Poisson bracket reduced to the constant one

$$\left\{w^i(x), w^j(y)\right\} = \tilde{g}^{ij}\partial_x\delta(x-y)\,. \qquad (2.48)$$

The only local invariant of the Poisson bracket is the signature of the metric $g^{ij}(u)$.

Proof of theorem 19. The proof is divided into three steps.
Step 1: Skew-symmetry of Poisson bracket implies the symmetry of g^{ij} and compatibility with Γ^i_{jk}.

Let the Poisson bracket be skew-symmetric, i.e.

$$\left\{u^i(x), u^j(y)\right\} + \left\{u^j(y), u^i(x)\right\} = 0 \, . \tag{2.49}$$

The second bracket reads as

$$\left\{u^j(y), u^i(x)\right\} = g^{ji}[u(y)]\partial_x\delta(y-x) + b^{ji}_k[u(y)]\frac{\partial u^k}{\partial y}\delta(y-x) \, . \tag{2.50}$$

Let us apply Dirac identites (Lemma 8 on page 5) on $g^{ji}[u(y)]\partial_x\delta(y-x)$. We get

$$\begin{aligned} &\left\{u^j(y), u^i(x)\right\} = \\ &= -g^{ji}[u(x)]\partial_x\delta(x-y) - \frac{\partial g^{ji}}{\partial u^k}[u(x)]\frac{\partial u^k}{\partial x}\delta(x-y) + b^{ji}_k[u(x)]\frac{\partial u^k}{\partial x}\delta(x-y) \, . \end{aligned} \tag{2.51}$$

The skew-symmetry is then

$$\begin{aligned} &0 = \left\{u^i(x), u^j(y)\right\} + \left\{u^j(y), u^i(x)\right\} = \\ &= \left[g^{ij}[u(x)] - g^{ji}[u(x)]\right]\partial_x\delta(x-y) + \left[b^{ij}_k - \frac{\partial g^{ji}}{\partial u^k} + b^{ji}_k\right]\frac{\partial u^k}{\partial x}\delta(x-y) \, . \end{aligned} \tag{2.52}$$

The right-hand side is equal to zero if and only if

eq:Th.Novikov-symetrie

$$g^{ij}(u) = g^{ji}(u) \, , \tag{2.53}$$

eq:Th.Novikov-konexe

$$\frac{\partial g^{ij}}{\partial u^k}(u) = b^{ij}_k(u) + b^{ji}_k(u) \, . \tag{2.54}$$

Equation (2.53) says that g^{ij} is a symmetric tensor. By assumption $\det g^{ij}$ is non-degenerate, so it defines a metric structure on M. Equation (2.54) is equivalent to

$$\frac{\partial g^{ij}}{\partial u^k} + g^{is}\Gamma^j_{sk} + g^{sj}\Gamma^i_{sk} = 0 \, . \tag{2.55}$$

If ∇ is the covariant derivative corresponding to Γ^i_{jk}, then $\nabla_k g^{ij} = 0$. So Γ^k_{ij} are affine connection coefficients and are compatible with g^{ij}.

Step 2: Jacobi identity implies zero torsion and curvature.

Let us assume that Jacobi identity holds. We denote

$$\mathcal{J}^{ijk}(x,y,z) = J^{ijk}(x,y,z) + J^{jki}(y,z,x) + J^{kij}(z,x,y) = 0 \tag{2.56}$$

where

$$J^{ijk}(x,y,z) = \left\{\left\{u^i(x), u^j(y)\right\}, u^k(z)\right\} \, . \tag{2.57}$$

From now on we shall use the abbreviated notation

$$g^{ij}_{,m} = \frac{\partial g^{ij}}{\partial u^m} \, , \quad b^{ij}_{k,m} = \frac{\partial b^{ij}_k}{\partial u^m} \, , \quad u^k_x = \frac{\partial u^k}{\partial x} \, . \tag{2.58}$$

One of the Jacobiators reads as

$$J^{ijk}(x,y,z)=\int \mathrm{d}\xi \frac{\delta\left\{u^i(x),u^j(y)\right\}}{\delta u^m(\xi)}\left[g^{mn}(\xi)\partial_\xi+b^{mn}_p\partial_\xi u^p\cdot\right]\frac{\delta u^k(z)}{\delta u^n(\xi)}\;.$$

The functional derivatives are

$$\frac{\delta u^k(z)}{\delta u^n(\xi)}=\delta^k_n\delta(z-\xi) \tag{2.59}$$

and

$$\begin{aligned}\frac{\delta\left\{u^i(x),u^j(y)\right\}}{\delta u^m(\xi)}&=\frac{\delta}{\delta u^m(\xi)}\left[g^{ij}(x)\partial_x\delta(x-y)+b^{ij}_k(x)\partial_x u^k\delta(x-y)\right]=\\ =\left[g^{ij}_{,m}(x)\partial_x\delta(x-y)+b^{ij}_{k,m}(x)u^k_x\delta(x-y)\right]&\delta(x-\xi)-b^{ij}_m(x)\delta(x-y)\partial_\xi\delta(x-\xi)\;.\end{aligned} \tag{2.60}$$

We have

$$J^{ijk}(x,y,z)=J^{ijk}_1(x,y,z)+J^{ijk}_2(x,y,z) \tag{2.61}$$

where

$$\begin{aligned}J^{ijk}_1(x,y,z)=\int \mathrm{d}\xi&\left[g^{ij}_{,m}(x)\partial_x\delta(x-y)\delta(x-\xi)+b^{ij}_{k,m}(x)u^k_x\delta(x-y)\delta(x-\xi)\right]\\ &\left[g^{mn}(\xi)\partial_\xi+b^{mn}_p\partial_\xi u^p\cdot\right]\delta^k_n\delta(z-\xi)\end{aligned}$$

and

$$J^{ijk}_2(x,y,z)=-\int \mathrm{d}\xi b^{ij}_m(x)\delta(x-y)\partial_\xi\delta(x-\xi)\left[g^{mn}(\xi)\partial_\xi+b^{mn}_p\partial_\xi u^p\cdot\right]\delta^k_n\delta(z-\xi)\;.$$

In J_1 we use integration with $\delta(x-\xi)$ and get rid of the integral:

$$\begin{aligned}J^{ijk}_1(x,y,z)&=\left[g^{ij}_{,m}(x)\partial_x\delta(x-y)+b^{ij}_{k,m}(x)u^k_x\delta(x-y)\right]\left[g^{mk}(\xi)\partial_x+b^{mk}_p\partial_x u^p\cdot\right]\delta(z-x)\\ &=\left[g^{ij}_{,m}g^{mk}\partial_x\delta(x-y)+b^{ij}_{k,m}g^{mk}u^k_x\delta(x-y)\right]\partial_x\delta(z-x)+\\ &+\left[g^{ij}_{,m}b^{mk}_p u^p_x\partial_x\delta(x-y)+b^{ij}_{k,m}b^{mk}_p u^p_x u^k_x\delta(x-y)\right]\delta(z-x)\;.\end{aligned}$$

In J_2 we first apply integration by parts to get rid of $\partial_\xi\delta(x-\xi)$ and then integrate:

$$\begin{aligned}J^{ijk}_2(x,y,z)=\int \mathrm{d}\xi b^{ij}_m(x)\delta(x-y)\delta(x-\xi)\partial_\xi&\left[g^{mk}(\xi)\partial_\xi\delta(z-\xi)+b^{mk}_p\partial_\xi u^p\delta(z-\xi)\right]\\ =b^{ij}_m(x)\delta(x-y)\partial_x&\left[g^{mn}(x)\partial_x\delta(z-x)+b^{mn}_p\partial_x u^p\delta(z-x)\right]\\ =b^{ij}_m(x)\delta(x-y)&\Big[g^{mk}_{,s}u^s_x\partial_x\delta(z-x)+g^{mk}\partial_x\partial_x\delta(z-x)\\ +b^{mk}_{p,s}(x)u^s_x u^p_x\delta(z-x)+b^{mn}_p(x)u^p_{xx}\delta(z-x)&+b^{mk}_p(x)u^p_x\partial_x\delta(z-x)\Big]\end{aligned}$$

$$\begin{aligned}J^{ijk}_2(x,y,z)=&b^{ij}_m g^{mk}_{,s}u^s_x\delta(x-y)\partial_x\delta(z-x)+b^{ij}_m g^{mk}\delta(x-y)\partial_x\partial_x\delta(z-x)\\ &+b^{ij}_m b^{mk}_{p,s}u^s_x u^p_x\delta(x-y)\delta(z-x)+b^{ij}_m b^{mn}_p u^p_{xx}\delta(x-y)\delta(z-x)\\ &+b^{ij}_m b^{mk}_p u^p_x\delta(x-y)\partial_x\delta(z-x)\;.\end{aligned}$$

In total we end with terms

$$
\begin{aligned}
J^{ijk}(x,y,z) =& E^{ijk}(x)\delta(x-y)\delta(z-x) && (2.62)\\
&+F^{ijk}(x)\delta(x-y)\partial_x\delta(z-x) && (2.63)\\
&+G^{ijk}(x)\partial_x\delta(x-y)\delta(z-x) && (2.64)\\
&+H^{ijk}(x)\partial_x\delta(x-y)\partial_x\delta(z-x) && (2.65)\\
&+I^{ijk}(x)\delta(x-y)\partial_x\partial_x\delta(z-x) && (2.66)
\end{aligned}
$$

where

$$
\begin{aligned}
E^{ijk} =& (b^{ij}_m b^{mk}_{p,s} + b^{ij}_{s,m} b^{mk}) u^p_x u^s_x + b^{ij}_m b^{mk}_p u^p_{xx} && (2.67)\\
F^{ijk} =& (b^{ij}_{s,m} g^{mk} + g^{mk}_{,s} b^{ij}_m + b^{ij}_m b^{mk}_s) u^s_x && (2.68)\\
G^{ijk} =& g^{ij}_{,m} b^{mk}_s u^s_x && (2.69)\\
H^{ijk} =& g^{ij}_{,m} g^{mk} && (2.70)\\
I^{ijk} =& b^{ij}_m g^{mk} \,. && (2.71)
\end{aligned}
$$

Now the procedure is the following: firstly we have to add the cyclic permutations $J^{jki}(y,z,x)$ and $J^{kij}(z,x,y)$ to obtain $\mathcal{J}^{ijk}(x,y,z)$. Secondly, the resulting generalized function $\mathcal{J}^{ijk}(x,y,z)$ is equal to zero, i.e.

$$\left\langle \mathcal{J}^{ijk}(x,y,z), p_i(x)q_j(y)r_k(z)\right\rangle = 0 \tag{2.72}$$

or in the integral notation

$$\iiint \mathrm{d}x\mathrm{d}y\mathrm{d}z\, J^{ijk}(x,y,z)p_i(x)q_j(y)r_k(z) = 0 \,. \tag{2.73}$$

We want to reduce this integral into a one-dimensional integral

$$\int \mathrm{d}x \sum_{a,b=0}^{2} A^{ijk}_{ab} p_i(x) q^{(a)}_j(x) r^{(b)}_k(x) \,, \tag{2.74}$$

where $f^{(a)}$ means the a-th derivative. This integral is equal to zero if and only if

$$A^{ijk}_{ab} = 0 \quad \forall a,b = 0,1,2 \,. \tag{2.75}$$

From this condition we shall obtain the zero torsion and curvature. We illustrate the procedure of reduction to the one-dimensional integral on one particular expression. For instance, we have

$$\int \Big[F^{ijk}(x)\delta(x-y)\partial_x\delta(z-x) + F^{jki}(y)\delta(y-z)\partial_y\delta(x-y) + F^{kij}(z)\delta(z-x)\partial_z\delta(y-z)\Big]\times$$
$$\times p_i(x)q_j(y)r_k(z)\mathrm{d}x\mathrm{d}y\mathrm{d}z \,.$$

We use the antisymmetry of δ-derivatives

$$\partial_x\delta(z-x) = -\partial_z\delta(z-x)\,,\ \partial_y\delta(x-y) = -\partial_x\delta(x-y)\,,\ \partial_z\delta(y-z) = -\partial_y\delta(y-z)\,,$$

and obtain

$$\int \mathrm{d}x F^{ijk}(x) p_i q_j r'_k + \int \mathrm{d}x F^{jki}(x) p'_i q_j r_k + \int \mathrm{d}x F^{kij}(x) p_i q'_j r_k \,.$$

In the term containing p_i' we perform integration by parts. The result is

$$\int \mathrm{d}x(F^{ijk} - F^{jki})p_i q_j r_k' + \int \mathrm{d}x(F^{kij} - F^{jki})p_i q_j' r_k - \int \mathrm{d}x \partial_x F^{jki} p_i q_j r_k \,.$$

In this particular case, we end up with terms, witch contribute to A_{01}^{ijk}, A_{10}^{ijk} and A_{00}^{ijk}, respectively. Other terms in $\mathcal{J}^{ijk}(x,y,z)$ are handled in the similar way.

We are insterested only in functions A_{00}^{ijk} and A_{02}^{ijk}. We shall also use the compatibility condition (2.54), which is

$$g^{ij}_{,m} = b^{ij}_m + b^{ji}_m \,. \tag{2.76}$$

The A_{02}^{ijk} function reads as

$$\begin{aligned} A_{02}^{ijk} =& I^{ijk} + I^{jki} - H^{jki} \\ =& b^{ij}_m g^{mk} + b^{jk}_m g^{mi} - g^{jk}_{,m} g^{mi} \\ =& b^{ij}_m g^{mk} + b^{jk}_m g^{mi} - b^{jk}_m g^{mi} - b^{kj}_m g^{mi} \\ =& b^{ij}_m g^{mk} - b^{kj}_m g^{mi} \\ =& - g^{ip}\Gamma^j_{pm} g^{mk} + g^{ks}\Gamma^j_{sm} g^{mi} \,, \end{aligned}$$

therefore, after multiplying with $g_{ai}g_{bk}$

$$\Gamma^j_{ab} - \Gamma^j_{ba} = 0 \,. \tag{2.77}$$

So the covariant derivative corresponding to Γ^j_{ab} has zero torsion.

The A_{00}^{ijk} function reads as

$$A_{00}^{ijk} = E^{ijk} + E^{jki} + E^{kij} - \partial_x F^{jki} - \partial_x G^{kij} \tag{2.78}$$

which can be modified into

$$A_{00}^{ijk} = B_p^{ijk}(u) u^p_{xx} + C_{ps}^{ijk}(u) u^p_x u^s_x = 0 \,, \tag{2.79}$$

where

$$B_p^{ijk} = (b^{jk}_{s,p} - b^{jk}_{p,s}) g^{si} + b^{ij}_s b^{sk}_p - b^{ik}_s b^{sj}_p \,. \tag{2.80}$$

The function B_p^{ijk} beeing zero implies the zero curvature, which follows from

$$\begin{aligned} B_p^{ijk} =& - g^{is}\left(g^{jm}_{,p}\Gamma^k_{ms} + g^{jm}\Gamma^k_{ms,p} - g^{jm}_{,s}\Gamma^k_{mp} - g^{jm}\Gamma^k_{mp,s} \right) \\ &+ g^{im}\Gamma^j_{ms} g^{sn}\Gamma^k_{np} - g^{im}\Gamma^k_{ms} g^{sn}\Gamma^j_{np} = \\ =& g^{is} g^{jn}\left(\Gamma^m_{np}\Gamma^k_{ms} - \Gamma^m_{ns}\Gamma^k_{mp} + \Gamma^k_{mp,s} - \Gamma^k_{ms,p} \right) \\ &+ g^{is} g^{nm}\left(\Gamma^j_{np}\Gamma^k_{ms} - \Gamma^j_{ns}\Gamma^k_{mp} \right) + g^{im} g^{sn}\left(\Gamma^j_{ms}\Gamma^k_{np} - \Gamma^k_{ms}\Gamma^j_{np} \right) = \\ =& - g^{is} g^{jn}\left(\Gamma^m_{ns}\Gamma^k_{mp} - \Gamma^m_{np}\Gamma^k_{ms} + \Gamma^k_{ms,p} - \Gamma^k_{mp,s} \right) = \\ =& - g^{is} g^{jn} R^k_{nps} \,. \end{aligned}$$

So far we proved that if hydrodynamical Poisson bracket satisfies skew-symmetry and Jacobi identity, the conditions 1-3 are satisfied.

Step 3: Zero curvature and torsion imply properties of the Poisson bracket.

If ∇ has zero torsion and curvature, there are coordinates $w^i = w^i(u^1, \ldots, u^N)$ for $i = 1, \ldots, N$ such that $g^{ij} = \text{const}$ a $b^{ij}_k = 0$. In these coordinates

$$\left\{w^i(x), w^j(y)\right\} = \tilde{g}^{ij} \partial_x \delta(x-y)\,. \tag{2.81}$$

Jacobi identity, skew-symmetry and Leibniz rule are for that bracket satisfied trivially. Therefore the three conditions are also sufficient. That finishes the proof. □

It follows from the theorem 19 on page 13 that we can write the original functions f^i_j in (2.1) by the Laplace-Beltrami operator.

Proposition 21 (Hamiltonian system by the Laplace operator)**.** *Given Hamiltonian system of hydrodynamic type*

$$\frac{\partial u^i}{\partial t} = f^i_k(u) \frac{\partial u^k}{\partial x}\,, \qquad f^i_k(u) = g^{ij}(u) \frac{\partial^2 h}{\partial u^j \partial u^k} - g^{is}(u) \Gamma^j_{sk}(u) \frac{\partial h}{\partial u^j}\,, \tag{2.82}$$

eq:hamiltonovsky-system

one can write

$$f^i_k(u) = \nabla^i \nabla_k h(u)\,, \tag{2.83}$$

where ∇_j is Levi-Civita covariant derivative compatible with g_{ij} and $\nabla^i = g^{is}\nabla_s$.

Proof. Since h is ordinary function, application of ∇_k is just the partial derivative w.r.t. u^k:

$$\nabla_k h(u) = \frac{\partial h}{\partial u^k}\,. \tag{2.84}$$

Now this term $\frac{\partial h}{\partial u^k}$ is a covector, so ∇_i is given by the partial derivative and the Γ^k_{ij} part by

$$\nabla^i \nabla_k h(u) = g^{is} \nabla_s \frac{\partial h}{\partial u^k} = g^{is} \left(\frac{\partial^2 h}{\partial u^s \partial u^k} - \Gamma^j_{sk} \frac{\partial h}{\partial u^j} \right) = f^i_k(u)\,. \tag{2.85}$$

□

Another direct application of the theorem 19 on page 13 is the sufficient condition for the determinating, whether is a given system Hamiltonian or not.

Theorem 22 (The sufficient condition the a system to be Hamiltonian)**.** *A system of hydrodynamic type*

$$u^i_t = f^i_j(u) u^j_x\,, \tag{2.86}$$

is Hamiltonian if and only if there is a non-degenerate metric $g^{ij}(u)$ and corresponding covariant derivative ∇_j, such that

V tomto vztahu nesedí indexy.

eq:gf=gf

$$g^{ij} f^k_j = g_{jk} f^k_i\,, \tag{2.87}$$

eq:nablaf

$$\nabla_i f^k_j = \nabla_j f^k_i\,. \tag{2.88}$$

In particular, (2.88) *is equivalent to the zero torsion of ∇_i.*

Proof. If the system is Hamiltonian, then $f^k_j = \nabla^k \nabla_j h$ and ∇_j has zero torsion, that means

$$\nabla_i f^k_j = g^{km} \nabla_i \nabla_m \nabla_j h = g^{km} \nabla_j \nabla_m \nabla_i h = \nabla_j f^k_i \,, \tag{2.89}$$

and

$$g^{ij} f^k_j = g^{ij} g^{mk} \nabla_m \nabla_j h = \nabla^i \nabla^k h = \nabla^k \nabla_i h \tag{2.90}$$

□

2.3 Multi-dimensional case

In the multi-dimensional case, we have a set of metrics $g^{ij\alpha}$ and connections $b^{ij\alpha}_k$ (identified with $\nabla^{(\alpha)}_j$ and $\Gamma^{i(\alpha)}_{jk}$), where $\alpha = 1, \cdots, d$. The first observation is, that they represent coordinates of a vector under a orthogonal transformation of the space coordinates x^α.

Skutečně se mění jenom závislost u na x^α, nebo i něco jiného? Může to platit i při nějaké obecné transformaci?

Proposition 23 (Orthogonal transformation the space coordinates)**.** *For any changes of the spatial variables $\tilde{x}^\alpha := c^\alpha_\beta x^\beta$, where $\alpha = 1, \cdots, d$ and $\det(c^\alpha_\beta) = 1$, the metrics $g^{ij\alpha}$ and $b^{ij\alpha}_k$ transforms as components of a vector.*

Proof. The only variable that changes is $\frac{\partial u^i}{\partial x^\alpha}$

$$\frac{\partial u^i}{\partial x^\beta} = \frac{\partial u^i}{\partial \tilde{x}^\alpha} \frac{\partial \tilde{x}^\alpha}{\partial x^\beta} = \frac{\partial u^i}{\partial \tilde{x}^\alpha} c^\alpha_\beta \,. \tag{2.91}$$

So

$$g^{ij\alpha}[u(\tilde{x})] = c^\alpha_\beta g^{ij\beta}[u(x)] \,, \quad b^{ij\alpha}_k[u(\tilde{x})] = c^\alpha_\beta b^{ij\beta}_k[u(x)] \,. \tag{2.92}$$

□

Definition 24 (Strongly non-degenerate metrics)**.** *We say that the vector of metrics $g^{ij\alpha}$ is strongly non-degenerate, if there are some constants $c_\alpha \in \mathbb{R}$ such that $c_\alpha g^{ij\alpha}$ is non-degenerate metric.*

(Of course, linear combination of symmetric tensors is symmetric, so the only condition for strong non-degeneracy is to have $\det(c_\alpha g^{ij\alpha}) \neq 0$.)

Theorem 25. *Let the Hamiltonian system be strongly non-degenerate. Then*

1. *for $N = 1$ the system can be reduced to constant form*

$$g^{ij}(u) = \tilde{g}^{ij}(u) \,, \tag{2.93}$$

2. *for $N \geq 2$ can be reduced to linear form*

$$g^{ij\alpha}(u) = g^{ij\alpha}_k u^k + \tilde{g}^{ij\alpha} \,, \quad \alpha = 1, \cdots, d \,, \tag{2.94}$$

$$\tag{2.95}$$

where the coefficients $g^{ij\alpha}_{,k} = b^{ij\alpha}_k + b^{ij\alpha}_k$, $\tilde{g}^{ij\alpha}$ and $b^{ij\alpha}_k$ are constant.

List of Figures

A. Appendix