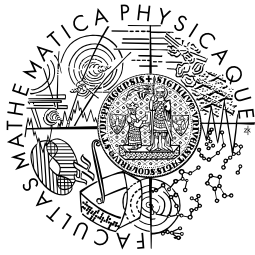


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**FACULTY
OF MATHEMATICS
AND PHYSICS**
Charles University

BACHELOR THESIS

Miroslav Burýšek

**The connection between continuum
mechanics and Riemannian geometry**

Mathematical Institute, Charles University, Prague

Supervisor of the bachelor thesis: RNDr. Michal Pavelka, Ph.D.

Study programme: Physics

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Dedication.

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Author: Miroslav Burýšek

Institute: Mathematical Institute, Charles University, Prague

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1. Mathematical Preliminaries

vymyslet lepší název kapitoly

In this chapter, we shall revise some basic definitions and propositions of the functional calculus which we will need for the proper formulation of the hydrodynamic geometry.

Vylepšit kritiku.

Most of the existing literature on this topic often lacks clarity due to the omission of certain details. This can include the domains on which functionals are defined or regularity conditions.

Over the last century, modern mathematics has worked to develop the theory of generalized functions - distributions, including Dirac's delta distributions and operations with them, such as in [CITATION]. The same applies to functional variations and the conditions on which they exist. Among physicists, however, it is conventional to operate with delta distributions as they would with "ordinary" functions, using dualities as integrals and functional variations as ordinary derivatives.

After introducing the concept of Gateaux and Frechet derivatives, which are suitable for arbitrary normed space, we shall take a closer look on the space of smooth functions and functionals on it.

1.1 Functional derivatives

přidat nějaký kontext

Definition 1 (Gateaux and Frechet derivative). *Let X be a normed space, $A : X \rightarrow \mathbb{R}$ is a functional and $x \in \text{Dom}(A)$.*

1. *Let $h \in D(A)$. Gateaux derivative $\delta A(x, h)$ in the direction h is defined as*

$$\delta A(x, h) := \frac{d}{dt} A(x + th)|_{t=0} = \lim_{t \rightarrow 0} \frac{A(x + th) - A(x)}{t} . \quad (1.1)$$

2. *Fréchet differential $D_x A$ is defined as the unique linear bounded functional satisfying the relation*

$$\lim_{\|h\| \rightarrow 0} \frac{A(x + h) - A(x) - D_x A(h)}{\|h\|} = 0 . \quad (1.2)$$

We are interested in such functionals, where X is the space of smooth functions with The smoothness is required because the functionals would operate with derivatives, and the values being zero on the boundary is required because we shall use integration by parts.

One of possible choices is to take

$$X^k := \{ f \in C^k(\Omega) \mid D^\alpha f = 0 \text{ on } \partial\Omega \text{ for } |\alpha| = 1, \dots, k-1 \} . \quad (1.3)$$

This space can be equipped with the usual supremum norm

$$\|f\|_k := \sum_{|\alpha| \leq k} \sup_{x \in \Omega} |D^\alpha f(x)|. \quad (1.4)$$

There are two main types of functionals on these spaces - the integral functionals and Dirac's delta distributions.

1.1.1 Integral functionals

The first type is the integral functional

$$A[u] = \int_{\Omega} a(x, u(x), u'(x), \dots, u^{(k)}(x)) \, d^n x, \quad \text{eq: integral-functional} \quad (1.5)$$

where $a : \mathbb{R} \times \mathbb{R}^k \rightarrow \mathbb{R}$ is an analytic function. For simplicity we often write this using a physicist's notation

$$A[u] = \int a(u) \, d^n x, \quad (1.6)$$

meaning that $a(u)$ is in fact function $a(x, u(x), u'(x), \dots, u^{(k)}(x))$ and also the domain Ω is specified by the context. We call $a(u)$ the density of functional $A[u]$.

Gateaux derivative of A is computed as follows:

$$\delta A[u, h] = \int_{\Omega} \sum_{i=1}^k \frac{\partial a}{\partial u^{(i)}}(x) h^{(i)}(x) \, d^n x. \quad \text{eq: Gateaux} \quad (1.7)$$

The functional $\delta A[u, h]$ is linear and bounded in h , so the Fréchet derivative of A exists and is given by

$$D_u A[h] = \delta A[u, h]. \quad (1.8)$$

To get rid of the derivatives of $h(x)$ in (1.7), we use integration by parts in each term of the sum (exploiting the fact that boundary terms are zero), obtaining

$$D_u A[h] = \int_{\Omega} \left[\frac{\partial a}{\partial u}(x) - \frac{d}{dx} \frac{\partial a}{\partial u'}(x) + \dots + (-1)^k \frac{d^k}{dx^k} \frac{\partial a}{\partial u^{(k)}}(x) \right] h(x) \, d^n x. \quad \text{eq: per partes} \quad (1.9)$$

It is natural to give a special name to the integrand in (1.9).

Definition 2 (Functional derivative of the integral functional). *Let A be the integral functional given by (1.5). The functional derivative of A is the function*

$$\frac{\delta A}{\delta u(x)} = \frac{\partial a}{\partial u}(x) - \frac{d}{dx} \frac{\partial a}{\partial u'}(x) + \frac{d^2}{dx^2} \frac{\partial a}{\partial u''} - \dots + (-1)^k \frac{d^k}{dx^k} \frac{\partial a}{\partial u^{(k)}}(x). \quad (1.10)$$

The functional derivative can be seen as the density of the integral functional $D_u A$ applied on h :

$$D_u A[h] = \int_{\Omega} \frac{\delta A}{\delta u(x)} h(x) \, dx \quad (1.11)$$

Since the Fréchet derivative is always linear bounded functional, another way to look at this relation is the following: $D_u A$ is a regular distribution represented by some smooth function $\frac{\delta A}{\delta u(x)}$, i.e.

$$D_u A[h] = \left\langle T_{\frac{\delta A}{\delta u(x)}}, h(x) \right\rangle . \quad (1.12)$$

Hodit regulární distribuce před tuhle část?

It follows from theory (CITATION) that the functional derivative beeing zero is a necessary condition for the function beeing the extremal point of the functional (given that the corresponding function is smooth enough).

Theorem 3 (Euler-Lagrange equations). *Let $v \in C^\infty(\Omega)$ such that $v(x)$ is stationary point of the functional A in (1.5), i.e*

$$\delta A[v, h] = 0 . \quad (1.13)$$

Then

$$\frac{\delta A}{\delta v(x)} = 0 . \quad (1.14)$$

The computation above gives sense to the

$$A[u + h] = A[u] + \left\langle \frac{\delta A}{\delta u}, h \right\rangle + o(h) , \quad (1.15)$$

where

$$\lim_{\|h\| \rightarrow 0} \frac{o(h)}{\|h\|} = 0 . \quad (1.16)$$

In physicist's notation, the function h is often denoted as δu .

1.1.2 Distributions

Definition 4 (Distributions). *Let $\mathcal{D}(\Omega)$ be the space of smooth functions with compact support in $\Omega \subseteq \mathbb{R}^n$. The space of distributions $\mathcal{D}'(\Omega)$ is the dual space, i.e. the space of bounded linear functionals on $\mathcal{D}(\Omega)$. An element $T \in \mathcal{D}'(\Omega)$ is called a distribution.*

Definition 5 (Regular distribution). *Let $a \in L^1_{loc}(\Omega)$. A regular distribution is a distribution $T_a \in \mathcal{D}'(\Omega)$ defined as*

$$\langle T_a, \phi(x) \rangle = \int_{\Omega} a(x) \phi(x) d^n x . \quad (1.17)$$

Note that if two regular distributions satisfy $T_a = T_b$, then $a = b$ almost everywhere and vice versa.

Definition 6 (Dirac delta distribution). *Let $y \in \Omega$. The Dirac delta distribution $\delta(x - y)$ is defined as*

$$\langle \delta(x - y), \phi(x) \rangle = \phi(y) . \quad (1.18)$$

Definition 7 (Derivatives of distributions). *Let $T \in \mathcal{D}'(\Omega)$ be a distribution. We define its derivative by*

$$\langle D^\alpha T, \phi \rangle = (-1)^{|\alpha|} \langle T, D^\alpha \phi \rangle , \quad (1.19)$$

where α is a multiindex and

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_N^{\alpha_N}} . \quad (1.20)$$

The reason, why distributions have such a "nice" behaviour and are often used by mathematicians, is that every distribution is diferentiable infinitely many times. However, if we abandon this property and require only the existence of derivatives of the k -th order, we can extend regular and Dirac distributions on previously defined space X^k . Indeed,

$$| \langle \partial_x^n \delta(x - y), f(x) \rangle | \leq \|f\|_n \quad \forall n = 0, 1, \dots, k , \quad (1.21)$$

so Dirac distributions and their derivatives are also bounded linear functionals on X^k up to the order k .

Rozmysli si, Míro, rozmysli.

Let $a \in C^\infty(\Omega)$. Then it makes sense to define the product of a and arbitrary distribution T by

$$\langle aT, \phi \rangle = \langle T, a\phi \rangle . \quad (1.22)$$

In the next chapter we shall frequently use the following identities of the delta distributions, which are written in the next lemma and which we will refer to as the Dirac identities.

lemma:delta

Lemma 8 (Dirac identities).

$$f(y)\delta(x - y) = f(x)\delta(x - y) , \quad (1.23)$$

$$f(y)\partial_x \delta(x - y) = f(x)\partial_x \delta(x - y) + f'(x)\delta(x - y) . \quad (1.24)$$

Proof. The first relation is trivial. Direct calculation gives

$$\begin{aligned} \langle f(x)\partial_x \delta(x - y), \psi(x) \rangle &= \langle \partial_x \delta(x - y), f(x)\psi(x) \rangle \\ &= - \langle \delta(x - y), f'(x)\psi(x) \rangle - \langle \delta(x - y), f(x)\psi'(x) \rangle \\ &= - f'(y)\psi(y) - f(y) \langle \delta(x - y), \psi'(x) \rangle \\ &= - \langle f'(x)\delta(x - y), \psi(x) \rangle + f(y) \langle \partial_x \delta(x - y), \psi(x) \rangle \\ &= \langle f(y)\delta(x - y) - f'(x)\partial_x \delta(x - y), \psi(x) \rangle . \end{aligned}$$

Therefore

$$\langle f(y)\partial_x \delta(x - y), \psi(x) \rangle = \langle f'(x)\delta(x - y), \psi(x) \rangle + \langle f(x)\partial_x \delta(x - y), \psi(x) \rangle . \quad (1.25)$$

□

The convolution of two distributions is not so simple to construct in rigorous way. The procedure is the following. First the tensor product of distributions is defined

$$\langle A(x) \otimes B(y), \phi(x, y) \rangle = \langle A(x), \langle B(y), \phi(x, y) \rangle \rangle . \quad (1.26)$$

The product defined in this way satisfies comutativity, associativity and it is indeed a distribution (although the proof of these properties is non-trivial). Then the convolution of distribution is defined by

$$\langle A \star B, \phi(x) \rangle = \langle A(x) \otimes B(y), \phi(x + y) \rangle , \quad (1.27)$$

with additional requirements on $A \otimes B$ and ϕ . Convolution defined in this way satisfies associativity and if it exists, then

$$D^\alpha(A \star B) = D^\alpha A \star B = A \star D^\alpha B . \quad (1.28)$$

The convolution with δ and arbitrary distribution A always exists and is equal to

$$A(y) \star \delta(x - y) = A(x) . \quad (1.29)$$

This means, in particular, that given an integral functional $T[u]$ one can write

$$T[u(x)] = T[u(y)] \star \delta(x - y) , \quad (1.30)$$

or in physicist's notation

$$T[u(x)] = \int dy T[u(y)] \delta(x - y) . \quad \text{eq:konvoluce-s-deltou} \quad (1.31)$$

The equation (1.31) is very important in calculations of functional derivatives.

Example 9. Let $u(x)$ be a field variable and $F[u(x)]$ an integral functional. To compute its functional derivative w.r.t. $u(y)$ in different position y , we use the integral expression of F

$$F[u(x)] = \int dy F[u(y)] \delta(x - y) . \quad (1.32)$$

Its Gateaux differential is

$$\delta F[u, h] = \frac{d}{dt} F[u + th] |_{t=0} = \int dy \frac{\partial F}{\partial u}(y) \delta(x - y) h(y) , \quad (1.33)$$

therefore

$$\frac{\delta F(u(x))}{\delta u(y)} = \frac{\partial F}{\partial u}(x) \delta(x - y) . \quad (1.34)$$

Example 10. In the special case of performing functional derivative w.r.t. the same field but in the different position,

$$\frac{\delta u^i(x)}{\delta u^j(y)} = \delta_j^i \delta(x - y) , \quad (1.35)$$

because

$$\frac{\delta u^i(x)}{\delta u^j(y)} = \frac{\delta}{\delta u^j(y)} \int dy u^i(y) \delta(x-y) = \frac{\partial u^i}{\partial u^j}(y) \delta(x-y) = \delta_j^i \delta(x-y) . \quad (1.36)$$

The same applies to

$$\frac{\delta}{\delta u^j(y)} \partial_x u^i(x) = -\delta_j^i \partial_x \delta(x-y) , \quad (1.37)$$

because

$$\frac{\delta}{\delta u^j(y)} \partial_x u^i(x) = \frac{\delta}{\delta u^j(y)} \int dy \partial_y u^i(y) \delta(x-y) \quad (1.38)$$

$$= -\frac{\delta}{\delta u^j(y)} \int dy u^i(y) \partial_y \delta(x-y) = -\delta_j^i \partial_x \delta(x-y) . \quad (1.39)$$

1.2 Geometrical aspects

1.2.1 Riemannian manifolds and covariant derivative

1.2.2 Lie algebras and Poisson brackets

2. The connection between Riemannian geometry and Hamiltonian mechanics

In this chapter we shall prove the theorem of the connection between the hydrodynamic Poisson bracket and the metrics found in the homogeneous system of hydrodynamic type. The original result was given by Dubrovin and Novikov and can be found in ?. The proof itself and the preceding lemmas are based on a considerable amount of technical computations, which we shall clarify in detail in order to offer the reader a comprehensive understanding of the chapter in question.

2.1 Systems of hydrodynamic type

Definition 11 (Homogeneous system of hydrodynamic type). *A homogeneous system of hydrodynamic type is an equation of the form*

$$\frac{\partial u^i}{\partial t} = f_j^{i\alpha}(u) \frac{\partial u^j}{\partial x^\alpha} \quad \text{eq:h-system} \quad (2.1)$$

where $u^i(t, x^\alpha)$ are unknown functions, $i = 1, \dots, N$, $\alpha = 1, \dots, d$.

Sometimes we refer to the number of equations N as the dimension of the hydrodynamical phase space and to the number of spatial variables d as the dimension of the configuration space.

Příklad asi přesunout do jiné kapitoly.

Before we go any further, let us present a few examples.

Example 12. *The energy is given by the functional*

$$E = \int e(\rho, \mathbf{m}, s, \mathbf{F}) d\mathbf{x} \quad (2.2)$$

The evolution equations in the Eulerian frame are

$$\partial_t \rho = -\partial_i(\rho E_{m_i}) \quad (2.3)$$

$$\partial_t m_i = -\partial_j(m_i E_{m_j}) - \rho \partial_i E_\rho - m_j \partial_i E_{m_j} - m_j \partial_i E_{m_j} \quad (2.4)$$

$$-s \partial_i E_s - F_j^j \partial_i E_{F_j^j} + \partial_j(F_I^j E_{F_I^i} + F_O^i E_{F_I^i}) \quad (2.5)$$

$$\partial_t s = -\partial_i(s E_{m_i}) \quad (2.6)$$

$$\partial_t F_I^i = -E_{m_k} \partial_k F_I^i + F_I^j \partial_j E_{m_i} . \quad (2.7)$$

Total energy density e can be prescribed as

$$e = \frac{\mathbf{m}^2}{2\rho} + \epsilon(\rho, s, \mathbf{F}) , \quad (2.8)$$

where ϵ is the elastic and internal energy. In particular, $E_{\mathbf{m}} = \frac{\mathbf{m}}{\rho} = \mathbf{v}$ becomes the velocity. The evolution equation for the deformation gradient then gets the explicit form

$$\partial_t \mathbf{F} = -(\mathbf{v} \cdot \nabla) \mathbf{F} + \nabla \mathbf{V} \cdot \mathbf{F}, \quad (2.9)$$

which is the usual evolution equation for \mathbf{F} in the Eulerian frame.

It was Riemann who noticed that the functions $f_j^{i\alpha}$ in (2.1) are in fact tensors ?.

prop:transformace-A

Proposition 13 (Transformation of $f_j^{i\alpha}$). *Under smooth change of variables $u^i \mapsto v^a$ of the form*

$$u^i = u^i(v^1, \dots, v^N) \quad (2.10)$$

the functions $f_j^{i\alpha}$ transform for each α according to the tensor law

$$f_b^{a\alpha}(v) = \frac{\partial v^a}{\partial u^i} \frac{\partial u^j}{\partial v^b} f_j^{i\alpha}(u). \quad (2.11)$$

Proof. Using a direct calculation we obtain

$$\frac{\partial u^i}{\partial t}(v) = \frac{\partial u^i}{\partial v^a} \frac{\partial v^a}{\partial t}, \quad (2.12)$$

$$\frac{\partial u^j}{\partial x^\alpha}(v) = \frac{\partial u^j}{\partial v^b} \frac{\partial v^b}{\partial x^\alpha}, \quad (2.13)$$

so

$$\frac{\partial u^i}{\partial v^a} \frac{\partial v^a}{\partial t} = \frac{\partial u^j}{\partial v^b} f_j^{i\alpha}(v(u)) \frac{\partial v^b}{\partial x^\alpha} \quad (2.14)$$

and therefore

$$\frac{\partial v^a}{\partial t} = \underbrace{\frac{\partial v^a}{\partial u^i} \frac{\partial u^j}{\partial v^b} f_j^{i\alpha}(v(u))}_{f_b^{a\alpha}(v)} \frac{\partial v^b}{\partial x^\alpha}. \quad (2.15)$$

□

According to the previous lemma, if we denote the space of points $(t, x^\alpha) \subset \mathbb{R} \times \mathbb{R}^N$ as some manifold M , then the unknown variables $u^i(t, x^\alpha)$ can be seen as the local coordinates of that manifold. The functions $f_j^{i\alpha}$ are transformed as tensors of the type (1,1) in the i and j indices.

Our next purpose is to gain a richer geometry of that space M . If the functions $f_j^{i\alpha}$ have a special form, we can introduce a geometry of the Poisson brackets.

Definition 14. *Theorem*

1. A Dubrovin-Novikov bracket of two functionals I_1 a I_2 is defined as

$$\{I_1, I_2\} = \sum_{\alpha=1}^d \int dx \left[\frac{\delta I_1}{\delta u^p(x)} \left(g^{pq\alpha}(u) \frac{d}{dx} + b_s^{pq\alpha}(u) \frac{\partial u^s}{\partial x^\alpha} \right) \frac{\delta I_2}{\delta u^q(x)} \right], \quad (2.16)$$

where $g^{ij\alpha}$ and $b_k^{ij\alpha}$ are certain functions, $i, j, k = 1, \dots, N$ and $\frac{d}{dx^\alpha}$ is the total derivative with respect to the independent variable x^α ,

$$\frac{d}{dx^\alpha} = \frac{\partial}{\partial x^\alpha} + u_\alpha^i \frac{\partial}{\partial u^i} + u_{\alpha\beta}^i \frac{\partial}{\partial u_\beta^i} + \dots + u_{\beta_1 \dots \beta_s \alpha}^i \frac{\partial}{\partial u_{\beta_1 \dots \beta_s}^i} + \dots, \quad (2.17)$$

$$u_{\beta_1 \dots \beta_s}^i = \frac{\partial^s u^i}{\partial x^{\beta_1} \dots \partial x^{\beta_s}}. \quad (2.18)$$

The summation over repeating upper and lower indices is assumed, moreover, in this case the summation over β_1, \dots, β_s is taken over distinct (up to arbitrary permutation) sets of these indices. (One can consider that these sets of indices are ordered: $\beta_1 \leq \dots \leq \beta_s$.)

2. The functional of hydrodynamic type is a functional of the form

$$H[u] = \int_{\mathbb{R}^d} h(u(x)) d^d x, \quad (2.19)$$

where h is independent of u_α , $u_{\alpha\beta}$. The function h is called a hamiltonian density.

3. The hamiltonian system of hydrodynamic type is a system of the form

$$u_t^i(x) = \left[g^{ij\alpha}[u(x)] \frac{\partial^2 h(u)}{\partial u^j \partial u^k} + b_k^{ij\alpha}[u(x)] \frac{\partial h(u)}{\partial u^j} \right] u_\alpha^k(x), \quad (2.20)$$

where $H[u]$ is the functional of the hydrodynamic type and h the hamiltonian density.

...Jednoduché cvičení a zároveň důležité...

Proposition 15 (Dubrovin-Novikov bracket of field variables).

$$\{u^i(x), u^j(y)\} = g^{ij\alpha}[u(x)] \delta_\alpha(x - y) + b_k^{ij\alpha}[u(x)] u_\alpha^k(x) \delta(x - y). \quad (2.21)$$

Proof.

$$\{u^i(x), u^j(y)\} = \int d\xi \frac{\delta u^i(x)}{\delta u^p(\xi)} \left(g^{pq\alpha}[u(\xi)] \frac{d}{d\xi^\alpha} + b_s^{pq\alpha}[u(\xi)] \frac{\partial u^s}{\partial \xi^\alpha} \right) \frac{\delta u^j(y)}{\delta u^q(\xi)} \quad (2.22)$$

$$= \int d\xi \delta(x - \xi) \left[g^{ij\alpha}[u(\xi)] \partial_\alpha \delta(y - \xi) + b_s^{ij\alpha}[u(\xi)] u_\alpha^s \delta(y - \xi) \right] \quad (2.23)$$

$$= g^{ij\alpha}[u(x)] \delta_\alpha(x - y) + b_k^{ij\alpha}[u(x)] u_\alpha^k(x) \delta(x - y). \quad (2.24)$$

□

Proposition 16 (Time derivative of the field variables). *The Hamiltonian system of hydrodynamic type reads as*

$$u_t^i(x) = \{u^i, H\}. \quad (2.25)$$

Proof.

$$\begin{aligned}
\{u^i(x), H\} &= \int d\xi \frac{\delta u^i(x)}{\delta u^p(\xi)} \left[g^{pq\alpha}[u(\xi)] \frac{d}{d\xi} + b_m^{pq\alpha}[u(\xi)] \frac{\partial u^m}{\partial \xi^\alpha} \right] \frac{\delta H}{\delta u^q(\xi)} \\
&= \int d\xi \int dy \delta(x - \xi) \left[g^{ij\alpha}[u(\xi)] \frac{d}{d\xi} \frac{\partial h}{\partial u^j}(\xi) \delta(y - \xi) + b_k^{ij\alpha}[u(\xi)] u_\alpha^k \frac{\partial h}{\partial u^j}(\xi) \delta(y - \xi) \right] \\
&= \left[g^{ij\alpha}[u(x)] \frac{\partial^2 h}{\partial u^j \partial u^k}(x) + b_k^{ij\alpha}(u(x)) \frac{\partial h}{\partial u^j}(x) \right] \frac{\partial u^k}{\partial x^\alpha} .
\end{aligned}$$

□

Dokončit.

Later we shall show that the Dubrovin-Novikov bracket defined above can be seen as the Poisson bracket (of hydrodynamic type), i.e. satisfies skew-symmetry, Leibniz rule and Jacobi identity. In fact, this will be the key theorem of this chapter.

2.2 One-dimensional case

In this section we first consider the one-dimensional case in which the unknown functions u^i depend on only two variables (t, x) . The hamiltonian system is then

$$\frac{\partial u^i}{\partial t}(t, x) = \left[g^{ij}[u(t, x)] \frac{\partial^2 h(u)}{\partial u^j \partial u^k} + b_k^{ij}[u(t, x)] \frac{\partial h(u)}{\partial u^j} \right] \frac{\partial u^k}{\partial x}(t, x) , \quad (2.26)$$

and the Dubrovin-Novikov bracket is of the form

$$\{I_1, I_2\} = \int dx \left[\frac{\delta I_1}{\delta u^p(x)} \left(g^{pq}[u(x)] \frac{d}{dx} + b_s^{pq}[u(x)] \frac{\partial u^s}{\partial x} \right) \frac{\delta I_2}{\delta u^q(x)} \right] . \quad (2.27)$$

In particular

$$\{u^i(x), u^j(y)\} = g^{ij}[u(x)] \partial_x \delta(x - y) + b_k^{ij}[u(x)] u_x^k \delta(x - y) . \quad \text{eq:Poisson1D} \quad (2.28)$$

Our goal is to specify the conditions for the Dubrovin-Novikov bracket to be Poisson bracket. The simplest property is the Leibniz rule, which is always satisfied.

Na Leibnize se podívat ještě jednou.

Proposition 17 (Leibniz rule for Dubrovin-Novikov bracket).

$$\{v^p(u^i(x)), v^q(u^j(y))\} = \frac{\partial v^p}{\partial u^i}(x) \frac{\partial v^q}{\partial u^j}(y) \{u^i(x), u^j(y)\} . \quad \text{eq:transformace-zavorka} \quad (2.29)$$

Proof. The coordinates are written as functionals in the form

$$v^p(u^i(x)) = \int dx' v^p(u^i(x')) \delta(x - x') . \quad (2.30)$$

Applying this to the definition of the Dubrovin-Novikov bracket (2.28), we have

$$\left\{v^p(u^i(x)), v^q(u^j(y))\right\} = \int dx' \int dy' \underbrace{\frac{\delta v^p}{\delta u^i(x')}}_{\frac{\partial v^p}{\partial u^i}(x')\delta(x-x')} \left\{u^i(x'), u^j(y')\right\} \underbrace{\frac{\delta v^q}{\delta u^j(y')}}_{\frac{\partial v^q}{\partial u^j}(y')\delta(y-y')} = \quad (2.31)$$

$$= \frac{\partial v^p}{\partial u^i}(x) \frac{\partial v^q}{\partial u^j}(y) \left\{u^i(x), u^j(y)\right\}. \quad (2.32)$$

□

Skew-symmetry and Jacobi identity for the bracket are not so easy to investigate. In order to do that, we introduce one more class of functions.

We say that the Poisson bracket is non-degenerate if $\det(g^{ij}) \neq 0$. We shall show later that the non-degeneracy is invariant under the transformation of the coordinates. In that case, we can naturally define its inverse $g_{ij} := (g^{mn})^{-1}$.

Let $g^{ij}(u)$ be non-degenerate. Then we can define functions $\Gamma_{jk}^i(u)$ as

$$\Gamma_{jk}^i(u) := -g_{js}(u)b_k^{si}(u). \quad (2.33)$$

Note that this relation allows to decompose

$$b_k^{ij}(u) = -g^{is}(u)\Gamma_{sk}^j(u). \quad \text{eq: b-g*Gamma} \quad (2.34)$$

The choice of denotation $g^{ij}(u)$ and $\Gamma_{jk}^i(u)$ has its purpose. We shall show soon that these are in fact the metric and affine coefficients, respectively, on M .

Proposition 18 (Transformation of g^{ij} and Γ_{jk}^i). *Let $u^i \mapsto v^a$ be the smooth change of variables. Then*

1. The g^{ij} coefficients transform as the $(2,0)$ tensors, i.e.

$$g^{pq}(v) = \frac{\partial v^p}{\partial u^i} \frac{\partial v^q}{\partial u^j} g^{ij}[u(v)]. \quad \text{eq: transforme-metrika} \quad (2.35)$$

2. The Γ_{jk}^i coefficients transform like Christoffel symbols, i.e.

$$\Gamma_{qr}^p(v) = \frac{\partial v^p}{\partial u^i} \frac{\partial u^j}{\partial v^q} \frac{\partial u^k}{\partial v^r} \Gamma_{jk}^i(u) + \frac{\partial v^p}{\partial u^i} \frac{\partial^2 u^i}{\partial v^q \partial v^r}. \quad \text{eq: transforme-konexe} \quad (2.36)$$

Proof. 1. Let us apply the transformation equation (2.29) to the definition of the Dubrovin-Novikov bracket (2.28). We get

$$\begin{aligned} & g^{pq}[v(u(x))] \partial_x \delta(x-y) + b_s^{pq}[v(u(x))] \frac{\partial v^s}{\partial u^k}(x) \frac{\partial u^k}{\partial x} \delta(x-y) \\ &= \frac{\partial v^p}{\partial u^i}(x) \frac{\partial v^q}{\partial u^j}(y) \left[g^{ij}[u(x)] \partial_x \delta(x-y) + b_k^{ij}[u(x)] \frac{\partial u^k}{\partial x} \delta(x-y) \right]. \quad \text{eq: dosad} \quad (2.37) \end{aligned}$$

From now on (for readability) we denote

$$T_k^a(x) := \frac{\partial v^a}{\partial u^k}(x). \quad (2.38)$$

We apply Dirac identity (Lemma 8 on page 5) to $T_j^q(y)\partial_x\delta(x-y)$ on the right-hand side of (2.37). Hence all functions are then of variable x , from now on we shall not write it explicitly. We have

$$\begin{aligned} & g^{pq}(v)\partial_x\delta(x-y) + b_n^{pq}(v)T_j^n\frac{\partial u^j}{\partial x}\delta(x-y) \\ &= T_i^pT_j^qg^{ij}(u)\partial_x\delta(x-y) + \left(\frac{\partial^2 v^q}{\partial u^j\partial u^k} + T_i^pT_j^qb_k^{ij}\right)\frac{\partial u^k}{\partial x}\delta(x-y). \end{aligned} \quad \text{eq: potovpami} \quad (2.39)$$

The terms with $\partial_x\delta$ must be equal to each other, as must δ terms. From $\partial_x\delta$ terms we get

$$g^{pq}(v) = T_i^pT_j^qg^{ij}(u), \quad (2.40)$$

which is the transformation relation for g .

2. From δ terms in (2.39), we get

$$b_n^{pq}(v)T_k^n = b_k^{ij}(u)T_i^pT_j^q + g^{ij}(u)T_i^p\frac{\partial^2 v^q}{\partial u^i\partial u^k} \quad (2.41)$$

and after (2.34)

$$-g^{ps}(v)\Gamma_{sn}^q(v)T_k^n = -g^{il}(u)\Gamma_{lk}^j(u)T_i^pT_j^q + g^{ij}(u)T_i^p\frac{\partial^2 v^q}{\partial u^i\partial u^k}. \quad (2.42)$$

On the left-hand side we use the transformation of g (2.35)

$$-g^{ij}(u)\Gamma_{sm}^q(v)T_i^pT_j^sT_k^m = -g^{ij}(u)\Gamma_{jk}^lT_i^pT_l^q + g^{ij}(u)\frac{\partial^2 v^q}{\partial u^i\partial u^k}T_i^p \quad (2.43)$$

and after vectoring out $g^{ij}(u)T_i^p$

$$\Gamma_{jk}^l(u)T_l^q = \Gamma_{sm}^q(v)T_j^sT_k^m + \frac{\partial^2 v^q}{\partial u^i\partial u^k}. \quad (2.44)$$

All that remains is to multiply this equation with $(T^{-1})_q^a$ and we get the final transformation relation for Γ

$$\Gamma_{jk}^a(u) = \Gamma_{sm}^q(v)\frac{\partial v^s}{\partial u^j}\frac{\partial v^m}{\partial u^k}\frac{\partial u^a}{\partial v^q} + \frac{\partial^2 v^q}{\partial u^i\partial u^k}\frac{\partial u^a}{\partial v^q}. \quad (2.45)$$

□

Let us summarize the results. We have defined Dubrovin-Novikov bracket in the way that it satisfies the Leibniz rule. It remains to investigate the skew-symmetry and Jacobi identity. For that, we defined g^{ij} and Γ_{jk}^i , which transform like tensors and Christoffel symbols, respectively. It remains to show that g^{ij} is indeed symmetric and Γ_{jk}^i is the Levi-Civita connection, i.e. the corresponding covariant derivative ∇_j has zero torsion and is compatible with the metric, i.e.

$$\nabla_k g^{ij} := \frac{\partial g^{ij}}{\partial u^k} + \Gamma_{ks}^i g^{sj} + \Gamma_{ks}^j g^{is} = 0. \quad (2.46)$$

We are now ready to prove the key theorem of this chapter which shows that these properties are indeed equivalent. Moreover, the corresponding covariant derivative has zero curvature, which indicates that the phase space M is in fact Euclidean space.

Theorem 19 (Dubrovin, Novikov). *Let $\det(g^{ij})$ be non-degenerate. Then the Dubrovin-Novikov bracket (2.28) is a Poisson bracket with skew-symmetry, Leibniz rule and Jacobi identity if and only if these three conditions are satisfied:*

1. $g^{ij} = g^{ji}$, that is, g^{ij} is a metric on the phase space M .
2. $\Gamma_{jk}^i = \Gamma_{kj}^i$, that is, Γ_{jk}^i are affine coefficients, such that the corresponding covariant derivative has zero torsion and annihilate the metric g^{ij} .
3. The covariant derivative corresponding to Γ_{jk}^i has zero curvature.

Corollary: existence of flat coordinates

Corollary 20. *There are local coordinates $w^i = w^i(u^1, \dots, u^N)$, $i = 1, \dots, N$ such that*

$$g^{ij}(w) = \tilde{g}^{ij} = \text{const}, \quad \Gamma_{jk}^i(w) = 0. \quad (2.47)$$

In these coordinates is Poisson bracket reduced to the constant one

$$\{w^i(x), w^j(y)\} = \tilde{g}^{ij} \partial_x \delta(x - y). \quad (2.48)$$

The only local invariant of the Poisson bracket is the signature of the metric $g^{ij}(u)$.

Proof of theorem 19. The proof is divided into three steps.

Step 1: Skew-symmetry of Poisson bracket implies the symmetry of g^{ij} and compatibility with Γ_{jk}^i .

Let the Poisson bracket be skew-symmetric, i.e.

$$\{u^i(x), u^j(y)\} + \{u^j(y), u^i(x)\} = 0. \quad (2.49)$$

The second bracket reads as

$$\{u^j(y), u^i(x)\} = g^{ji}[u(y)] \partial_x \delta(y - x) + b_k^{ji}[u(y)] \frac{\partial u^k}{\partial y} \delta(y - x). \quad (2.50)$$

Let us apply Dirac identities (Lemma 8 on page 5) on $g^{ji}[u(y)] \partial_x \delta(y - x)$. We get

$$\begin{aligned} & \{u^j(y), u^i(x)\} = \\ & = -g^{ji}[u(x)] \partial_x \delta(x - y) - \frac{\partial g^{ji}}{\partial u^k}[u(x)] \frac{\partial u^k}{\partial x} \delta(x - y) + b_k^{ji}[u(x)] \frac{\partial u^k}{\partial x} \delta(x - y). \end{aligned} \quad (2.51)$$

The skew-symmetry is then

$$\begin{aligned} 0 & = \{u^i(x), u^j(y)\} + \{u^j(y), u^i(x)\} = \\ & = [g^{ij}[u(x)] - g^{ji}[u(x)]] \partial_x \delta(x - y) + \left[b_k^{ij} - \frac{\partial g^{ji}}{\partial u^k} + b_k^{ji} \right] \frac{\partial u^k}{\partial x} \delta(x - y). \end{aligned} \quad (2.52)$$

The right-hand side is equal to zero if and only if

$$\begin{aligned} g^{ij}(u) & = g^{ji}(u), & \text{eq: Th. Novikov-symetrie} \\ \frac{\partial g^{ij}}{\partial u^k}(u) & = b_k^{ij}(u) + b_k^{ji}(u). & \text{eq: Th. Novikov-koneks} \end{aligned} \quad \begin{matrix} (2.53) \\ (2.54) \end{matrix}$$

Equation (2.53) says that g^{ij} is a symmetric tensor. By assumption $\det g^{ij}$ is non-degenerate, so it defines a metric structure on M . Equation (2.54) is equivalent to

$$\frac{\partial g^{ij}}{\partial u^k} + g^{is} \Gamma_{sk}^j + g^{sj} \Gamma_{sk}^i = 0. \quad (2.55)$$

If ∇ is the covariant derivative corresponding to Γ_{jk}^i , then $\nabla_k g^{ij} = 0$. So Γ_{ij}^k are affine connection coefficients and are compatible with g^{ij} .

Step 2: Jacobi identity implies zero torsion and curvature.

Let us assume that Jacobi identity holds. We denote

$$\mathcal{J}^{ijk}(x, y, z) = J^{ijk}(x, y, z) + J^{jki}(y, z, x) + J^{kij}(z, x, y) = 0 \quad (2.56)$$

where

$$J^{ijk}(x, y, z) = \left\{ \left\{ u^i(x), u^j(y) \right\}, u^k(z) \right\}. \quad (2.57)$$

From now on we shall use the abbreviated notation

$$g_{,m}^{ij} = \frac{\partial g^{ij}}{\partial u^m}, \quad b_{k,m}^{ij} = \frac{\partial b_k^{ij}}{\partial u^m}, \quad u_x^k = \frac{\partial u^k}{\partial x}. \quad (2.58)$$

One of the Jacobiators reads as

$$J^{ijk}(x, y, z) = \int d\xi \frac{\delta \{u^i(x), u^j(y)\}}{\delta u^m(\xi)} \left[g^{mn}(\xi) \partial_\xi + b_p^{mn} \partial_\xi u^p \right] \frac{\delta u^k(z)}{\delta u^n(\xi)}.$$

The functional derivatives are

$$\frac{\delta u^k(z)}{\delta u^n(\xi)} = \delta_n^k \delta(z - \xi) \quad (2.59)$$

and

$$\begin{aligned} \frac{\delta \{u^i(x), u^j(y)\}}{\delta u^m(\xi)} &= \frac{\delta}{\delta u^m(\xi)} \left[g^{ij}(x) \partial_x \delta(x - y) + b_k^{ij}(x) \partial_x u^k \delta(x - y) \right] = \\ &= \left[g_{,m}^{ij}(x) \partial_x \delta(x - y) + b_{k,m}^{ij}(x) u_x^k \delta(x - y) \right] \delta(x - \xi) - b_m^{ij}(x) \delta(x - y) \partial_\xi \delta(x - \xi). \end{aligned} \quad (2.60)$$

We have

$$J^{ijk}(x, y, z) = J_1^{ijk}(x, y, z) + J_2^{ijk}(x, y, z) \quad (2.61)$$

where

$$\begin{aligned} J_1^{ijk}(x, y, z) &= \int d\xi \left[g_{,m}^{ij}(x) \partial_x \delta(x - y) \delta(x - \xi) + b_{k,m}^{ij}(x) u_x^k \delta(x - y) \delta(x - \xi) \right] \\ &\quad \left[g^{mn}(\xi) \partial_\xi + b_p^{mn} \partial_\xi u^p \right] \delta_n^k \delta(z - \xi) \end{aligned}$$

and

$$J_2^{ijk}(x, y, z) = - \int d\xi b_m^{ij}(x) \delta(x - y) \partial_\xi \delta(x - \xi) \left[g^{mn}(\xi) \partial_\xi + b_p^{mn} \partial_\xi u^p \right] \delta_n^k \delta(z - \xi).$$

In J_1 we use integration with $\delta(x - \xi)$ and get rid of the integral:

$$\begin{aligned} J_1^{ijk}(x, y, z) &= \left[g_{,m}^{ij}(x) \partial_x \delta(x - y) + b_{k,m}^{ij}(x) u_x^k \delta(x - y) \right] \left[g^{mk}(\xi) \partial_x + b_p^{mk} \partial_x u^p \right] \delta(z - x) \\ &= \left[g_{,m}^{ij} g^{mk} \partial_x \delta(x - y) + b_{k,m}^{ij} g^{mk} u_x^k \delta(x - y) \right] \partial_x \delta(z - x) + \\ &\quad + \left[g_{,m}^{ij} b_p^{mk} u_x^p \partial_x \delta(x - y) + b_{k,m}^{ij} b_p^{mk} u_x^p u_x^k \delta(x - y) \right] \delta(z - x) . \end{aligned}$$

In J_2 we first apply integration by parts to get rid of $\partial_\xi \delta(x - \xi)$ and then integrate:

$$\begin{aligned} J_2^{ijk}(x, y, z) &= \int d\xi b_m^{ij}(x) \delta(x - y) \delta(x - \xi) \partial_\xi \left[g^{mk}(\xi) \partial_\xi \delta(z - \xi) + b_p^{mk} \partial_\xi u^p \delta(z - \xi) \right] \\ &= b_m^{ij}(x) \delta(x - y) \partial_x \left[g^{mn}(x) \partial_x \delta(z - x) + b_p^{mn} \partial_x u^p \delta(z - x) \right] \\ &= b_m^{ij}(x) \delta(x - y) \left[g_{,s}^{mk} u_x^s \partial_x \delta(z - x) + g^{mk} \partial_x \partial_x \delta(z - x) \right. \\ &\quad \left. + b_{p,s}^{mk}(x) u_x^s u_x^p \delta(z - x) + b_p^{mn}(x) u_{xx}^p \delta(z - x) + b_p^{mk}(x) u_x^p \partial_x \delta(z - x) \right] \end{aligned}$$

$$\begin{aligned} J_2^{ijk}(x, y, z) &= b_m^{ij} g_{,s}^{mk} u_x^s \delta(x - y) \partial_x \delta(z - x) + b_m^{ij} g^{mk} \delta(x - y) \partial_x \partial_x \delta(z - x) \\ &\quad + b_m^{ij} b_{p,s}^{mk} u_x^s u_x^p \delta(x - y) \delta(z - x) + b_m^{ij} b_p^{mn} u_{xx}^p \delta(x - y) \delta(z - x) \\ &\quad + b_m^{ij} b_p^{mk} u_x^p \delta(x - y) \partial_x \delta(z - x) . \end{aligned}$$

In total we end with terms

$$J^{ijk}(x, y, z) = E^{ijk}(x) \delta(x - y) \delta(z - x) \quad (2.62)$$

$$+ F^{ijk}(x) \delta(x - y) \partial_x \delta(z - x) \quad (2.63)$$

$$+ G^{ijk}(x) \partial_x \delta(x - y) \delta(z - x) \quad (2.64)$$

$$+ H^{ijk}(x) \partial_x \delta(x - y) \partial_x \delta(z - x) \quad (2.65)$$

$$+ I^{ijk}(x) \delta(x - y) \partial_x \partial_x \delta(z - x) \quad (2.66)$$

where

$$E^{ijk} = (b_m^{ij} b_{p,s}^{mk} + b_{s,m}^{ij} b^{mk}) u_x^p u_x^s + b_m^{ij} b_p^{mk} u_{xx}^p \quad (2.67)$$

$$F^{ijk} = (b_{s,m}^{ij} g^{mk} + g_{,s}^{mk} b_m^{ij} + b_m^{ij} b_s^{mk}) u_x^s \quad (2.68)$$

$$G^{ijk} = g_{,m}^{ij} b_s^{mk} u_x^s \quad (2.69)$$

$$H^{ijk} = g_{,m}^{ij} g^{mk} \quad (2.70)$$

$$I^{ijk} = b_m^{ij} g^{mk} . \quad (2.71)$$

Now the procedure is the following: firstly we have to add the cyclic permutations $J^{jki}(y, z, x)$ and $J^{kij}(z, x, y)$ to obtain $\mathcal{J}^{ijk}(x, y, z)$. Secondly, the resulting generalized function $\mathcal{J}^{ijk}(x, y, z)$ is equal to zero, i.e.

$$\langle \mathcal{J}^{ijk}(x, y, z), p_i(x) q_j(y) r_k(z) \rangle = 0 \quad (2.72)$$

or in the integral notation

$$\iiint dx dy dz J^{ijk}(x, y, z) p_i(x) q_j(y) r_k(z) = 0 . \quad (2.73)$$

We want to reduce this integral into a one-dimensional integral

$$\int dx \sum_{a,b=0}^2 A_{ab}^{ijk} p_i(x) q_j^{(a)}(x) r_k^{(b)}(x) , \quad (2.74)$$

where $f^{(a)}$ means the a -th derivative. This integral is equal to zero if and only if

$$A_{ab}^{ijk} = 0 \quad \forall a, b = 0, 1, 2 . \quad (2.75)$$

From this condition we shall obtain the zero torsion and curvature. We illustrate the procedure of reduction to the one-dimensional integral on one particular expression. For instance, we have

$$\int \left[F^{ijk}(x) \delta(x-y) \partial_x \delta(z-x) + F^{jki}(y) \delta(y-z) \partial_y \delta(x-y) + F^{kij}(z) \delta(z-x) \partial_z \delta(y-z) \right] \times \\ \times p_i(x) q_j(y) r_k(z) dx dy dz . \quad \blacksquare$$

We use the antisymmetry of δ -derivatives

$$\partial_x \delta(z-x) = -\partial_z \delta(z-x) , \quad \partial_y \delta(x-y) = -\partial_x \delta(x-y) , \quad \partial_z \delta(y-z) = -\partial_y \delta(y-z) , \quad \blacksquare$$

and obtain

$$\int dx F^{ijk}(x) p_i q_j r'_k + \int dx F^{jki}(x) p'_i q_j r_k + \int dx F^{kij}(x) p_i q'_j r_k .$$

In the term containing p'_i we perform integration by parts. The result is

$$\int dx (F^{ijk} - F^{jki}) p_i q_j r'_k + \int dx (F^{kij} - F^{jki}) p_i q'_j r_k - \int dx \partial_x F^{jki} p_i q_j r_k .$$

In this particular case, we end up with terms, which contribute to A_{01}^{ijk} , A_{10}^{ijk} and A_{00}^{ijk} , respectively. Other terms in $\mathcal{J}^{ijk}(x, y, z)$ are handled in the similar way.

The list of the resulting functions is the following:

$$A_{00}^{ijk} = E^{ijk} + E^{kij} + E^{jki} + \partial_x \partial_x I^{jki} - \partial_x F^{jki} - \partial_x G^{kij} , \quad (2.76)$$

$$A_{01}^{ijk} = F^{ijk} - F^{jki} + G^{jki} - G^{kij} - \partial_x H^{jki} + 2\delta_x I^{jki} , \quad (2.77)$$

$$A_{10}^{ijk} = F^{kij} - F^{jki} + G^{ijk} - G^{kij} - \partial_x H^{kij} + 2\partial_x I^{jki} , \quad (2.78)$$

$$A_{11}^{ijk} = H^{ijk} - H^{jki} - H^{kij} + 2\partial_x I^{jki} , \quad (2.79)$$

$$A_{02}^{ijk} = I^{ijk} + I^{jki} - H^{jki} , \quad (2.80)$$

$$A_{20}^{ijk} = I^{jki} + I^{kij} - H^{kij} , \quad (2.81)$$

$$A_{12}^{ijk} = A_{21}^{ijk} = 0 . \quad (2.82)$$

We are interested only in functions A_{00}^{ijk} and A_{02}^{ijk} , since they correspond to the zero curvature and torsion, respectively. We shall also use the compatibility condition (2.54), which is

$$g_{,m}^{ij} = b_m^{ij} + b_m^{ji} . \quad (2.83)$$

The A_{02}^{ijk} function reads as

$$A_{02}^{ijk} = b_m^{ij} g^{mk} - b_m^{kj} g^{mi} = -g^{ip} \Gamma_{pm}^j g^{mk} + g^{ks} \Gamma_{sm}^j g^{mi} = 0 .$$

After multiplying this relation with $g_{ai}g_{bk}$, we get

$$\Gamma_{ab}^j - \Gamma_{ba}^j = 0, \quad (2.84)$$

therefore Christoffel symbols are symmetric in the lower indices, which implies that the covariant derivative corresponding to them has zero torsion.

The A_{00}^{ijk} function reads as

$$A_{00}^{ijk} = B_p^{ijk}(u)u_{xx}^p + C_{ps}^{ijk}(u)u_x^p u_x^s = 0, \quad (2.85)$$

where

$$B_p^{ijk} = (b_{s,p}^{jk} - b_{p,s}^{jk})g^{si} + b_s^{ij}b_p^{sk} - b_s^{ik}b_p^{sj}. \quad (2.86)$$

The function B_p^{ijk} being zero implies the zero curvature, which follows from

$$\begin{aligned} B_p^{ijk} &= -g^{is} \left(g_{,p}^{jm} \Gamma_{ms}^k + g^{jm} \Gamma_{ms,p}^k - g_{,s}^{jm} \Gamma_{mp}^k - g^{jm} \Gamma_{mp,s}^k \right) \\ &\quad + g^{im} \Gamma_{ms}^j g^{sn} \Gamma_{np}^k - g^{im} \Gamma_{ms}^k g^{sn} \Gamma_{np}^j \\ &= g^{is} g^{jn} \left(\Gamma_{np}^m \Gamma_{ms}^k - \Gamma_{ns}^m \Gamma_{mp}^k + \Gamma_{mp,s}^k - \Gamma_{ms,p}^k \right) \\ &\quad + g^{is} g^{nm} \left(\Gamma_{np}^j \Gamma_{ms}^k - \Gamma_{ns}^j \Gamma_{mp}^k \right) + g^{im} g^{sn} \left(\Gamma_{ms}^j \Gamma_{np}^k - \Gamma_{ms}^k \Gamma_{np}^j \right) \\ &= -g^{is} g^{jn} \left(\Gamma_{ns}^m \Gamma_{mp}^k - \Gamma_{np}^m \Gamma_{ms}^k + \Gamma_{ms,p}^k - \Gamma_{mp,s}^k \right) \\ &= -g^{is} g^{jn} R_{nps}^k. \end{aligned}$$

So far we proved that if Dubrovin-Novikov bracket satisfies skew-symmetry and Jacobi identity, the conditions 1-3 are satisfied.

Step 3: Zero curvature and torsion imply properties of the Poisson bracket.

If ∇ has zero torsion and curvature, there are coordinates $w^i = w^i(u^1, \dots, u^N)$ for $i = 1, \dots, N$ such that $g^{ij} = \text{const}$ a $b_k^{ij} = 0$. In these coordinates

$$\{w^i(x), w^j(y)\} = \tilde{g}^{ij} \partial_x \delta(x - y). \quad (2.87)$$

Jacobi identity, skew-symmetry and Leibniz rule are for that bracket satisfied trivially. Therefore the three conditions are also sufficient. That finishes the proof. \square

It follows from the theorem 19 on page 14 that we can write the original functions f_j^i in (2.1) by the Laplace-Beltrami operator.

Proposition 21 (Hamiltonian system by the Laplace operator). *Given Hamiltonian system of hydrodynamic type*

$$\frac{\partial u^i}{\partial t} = f_k^i(u) \frac{\partial u^k}{\partial x}, \quad f_k^i(u) = g^{ij}(u) \frac{\partial^2 h}{\partial u^j \partial u^k} - g^{is}(u) \Gamma_{sk}^j(u) \frac{\partial h_{\text{eq:hamiltonovsky-system}}}{\partial u^j}, \quad (2.88)$$

one can write

$$f_k^i(u) = \nabla^i \nabla_k h(u), \quad (2.89)$$

where ∇_j is Levi-Civita covariant derivative compatible with g_{ij} and $\nabla^i = g^{is} \nabla_s$.

Proof. Since h is ordinary function, application of ∇_k is just the partial derivative w.r.t. u^k :

$$\nabla_k h(u) = \frac{\partial h}{\partial u^k} . \quad (2.90)$$

Now this term $\frac{\partial h}{\partial u^k}$ is a covector, so ∇_i is given by the partial derivative and the Γ_{ij}^k part by

$$\nabla^i \nabla_k h(u) = g^{is} \nabla_s \frac{\partial h}{\partial u^k} = g^{is} \left(\frac{\partial^2 h}{\partial u^s \partial u^k} - \Gamma_{sk}^j \frac{\partial h}{\partial u^j} \right) = f_k^i(u) . \quad (2.91)$$

□

Another direct application of the theorem 19 on page 14 is the sufficient condition for the determining, whether is a given system Hamiltonian or not.

Theorem 22 (The sufficient condition the a system to be Hamiltonian). *A system of hydrodynamic type*

$$u_t^i = f_j^i(u) u_x^j , \quad (2.92)$$

is Hamiltonian if and only if there is a non-degenerate metric $g^{ij}(u)$ and corresponding covariant derivative ∇_j , such that

V tomto vztahu nesedí indexy.

$$g^{ij} f_j^k = g_{jk} f_i^k , \quad (2.93) \quad \text{eq:eff}$$

$$\nabla_i f_j^k = \nabla_j f_i^k . \quad (2.94) \quad \text{eq:nabla f}$$

In particular, (2.94) is equivalent to the zero torsion of ∇_i .

Proof. If the system is Hamiltonian, then $f_j^k = \nabla^k \nabla_j h$ and ∇_j has zero torsion, that means

$$\nabla_i f_j^k = g^{km} \nabla_i \nabla_m \nabla_j h = g^{km} \nabla_j \nabla_m \nabla_i h = \nabla_j f_i^k , \quad (2.95)$$

and

$$g^{ij} f_j^k = g^{ij} g^{mk} \nabla_m \nabla_j h = \nabla^i \nabla^k h = \nabla^k \nabla_i h \quad (2.96)$$

□

2.3 Multi-dimensional case

In the multi-dimensional case, we have a set of metrics $g^{ij\alpha}$ and connections $b_k^{ij\alpha}$, where $\alpha = 1, \dots, d$. We can again introduce Christoffel symbols defined by

$$b_k^{ij\alpha} = -g^{is\alpha}\Gamma_{sk}^{j\alpha}, \quad (2.97)$$

(there is no summation over α) and the corresponding set of covariant derivatives ∇_j^α . The first observation is, that these coefficients represent coordinates of a vector under a orthogonal transformation of the space coordinates x^α .

Skutečně se mění jenom závislost u na x^α , nebo i něco jiného? Může to platit i při nějaké obecné transformaci?

Proposition 23 (Orthogonal transformation the space coordinates). *For any changes of the spatial variables $\tilde{x}^\alpha := c_\beta^\alpha x^\beta$, where $\alpha = 1, \dots, d$ and $\det(c_\beta^\alpha) = 1$, the metrics $g^{ij\alpha}$ and $b_k^{ij\alpha}$ transforms as components of a vector.*

Proof. The only variable that changes is $\frac{\partial u^i}{\partial x^\alpha}$

$$\frac{\partial u^i}{\partial x^\beta} = \frac{\partial u^i}{\partial \tilde{x}^\alpha} \frac{\partial \tilde{x}^\alpha}{\partial x^\beta} = \frac{\partial u^i}{\partial \tilde{x}^\alpha} c_\beta^\alpha. \quad (2.98)$$

So

$$g^{ij\alpha}[u(\tilde{x})] = c_\beta^\alpha g^{ij\beta}[u(x)], \quad b_k^{ij\alpha}[u(\tilde{x})] = c_\beta^\alpha b_k^{ij\beta}[u(x)]. \quad (2.99)$$

□

Definition 24 (Strongly non-degenerate metrics). *We say that the vector of metrics $g^{ij\alpha}$ is strongly non-degenerate, if there are some constants $c_\alpha \in \mathbb{R}$ such that $c_\alpha g^{ij\alpha}$ is non-degenerate metric.*

(Of course, linear combination of symmetric tensors is symmetric, so the only condition for strong non-degeneracy is to have $\det(c_\alpha g^{ij\alpha}) \neq 0$.)

Theorem 25. *Let the Hamiltonian system be strongly non-degenerate. Then*

1. for $N = 1$ the system can be reduced to constant form

$$g^{ij}(u) = \tilde{g}^{ij}(u), \quad (2.100)$$

2. for $N \geq 2$ can be reduced to linear form

$$g^{ij\alpha}(u) = g_k^{ij\alpha} u^k + \tilde{g}^{ij\alpha}, \quad \alpha = 1, \dots, d, \quad (2.101)$$

$$(2.102)$$

where the coefficients $g_{,k}^{ij\alpha} = b_k^{ij\alpha} + \tilde{g}^{ij\alpha}$ and $b_k^{ij\alpha}$ are constant.

The formulation of the generalized form of the Dubrovin-Novikov theorem (Theorem 19 on page 14) is given by Mokhov in ?.

Theorem 26 (Mokhov). *A Dubrovin-Novikov bracket (2.16) is a Poisson bracket if and only if the following relations for the coefficients are fulfilled:*

$$g^{ij\alpha} = g^{ji\alpha} , \quad \text{eq: Mokhov-A} \quad (2.103)$$

$$g_{,k}^{ij\alpha} = b_k^{ij\alpha} + b_k^{ji\alpha} , \quad \text{eq: Mokhov-B} \quad (2.104)$$

$$\sum_{(\alpha,\beta)} (g^{mi\alpha} b_m^{jk\beta} - g^{mj\beta} b_m^{ik\alpha}) = 0 , \quad \text{eq: Mokhov-C} \quad (2.105)$$

$$\sum_{(i,j,k)} (g^{mi\alpha} b_m^{jk\beta} - g^{mj\beta} b_m^{ik\alpha}) = 0 , \quad \text{eq: Mokhov-D} \quad (2.106)$$

$$\sum_{(\alpha,\beta)} \left[g^{si\alpha} (b_{s,q}^{jk\beta} - b_{q,s}^{jk\beta}) + b_s^{ij\alpha} b_q^{sk\beta} - b_s^{ik\alpha} b_q^{kj\beta} \right] = 0 , \quad \text{eq: Mokhov-E} \quad (2.107)$$

$$g^{si\beta} b_{q,s}^{jk\alpha} - b_s^{ij\beta} b_q^{sk\alpha} - b_s^{ik\beta} b_q^{js\alpha} = g^{sj\alpha} b_{q,s}^{ik\beta} - b_s^{ji\alpha} b_q^{sk\beta} - b_q^{is\beta} b_s^{jk\alpha} , \quad \text{eq: Mokhov-F} \quad (2.108)$$

$$\begin{aligned} & \frac{\partial}{\partial u^r} \left[g^{si\alpha} (b_{s,q}^{jk\beta} - b_{q,s}^{jk\beta}) + b_s^{ij\alpha} b_q^{sk\beta} - b_s^{ik\alpha} b_q^{sj\beta} \right] \\ & + \frac{\partial}{\partial u^q} \left[g^{si\beta} (b_{s,k}^{jk\alpha} - b_{r,s}^{jk\alpha}) + b_s^{ij\beta} b_{s,r}^{sk\alpha} + b_s^{ij\beta} b_r^{sk\alpha} - b_s^{ik\beta} b_s^{kj\alpha} \right] \\ & + \sum_{(i,j,k)} \left[b_q^{si\beta} (b_{r,s}^{jk\alpha} - b_{s,r}^{jk\alpha}) + b_r^{si\alpha} (b_{q,s}^{jk\beta} - b_{s,q}^{jk\beta}) \right] = 0 . \end{aligned} \quad \text{eq: Mokhov-G} \quad (2.109)$$

The summation $\sum_{(\alpha,\beta)}$ and $\sum_{(i,j,k)}$ means summations over all cyclic permutations of the indicated indices.

Remark. In the proof of the Dubrovin-Novikov theorem (19 on page 14) in **Step 2** we end up with six functions A_{ab}^{ijk} , which must be equal to zero if the bracket is Poisson. However, for the rest of the proof, we used only A_{00}^{ijk} and A_{02}^{ijk} , because they corresponded to the zero torsion and curvature, which gave the existence of the flat coordinates. Although their existence was formulated as a corollary (20 on page 14), they played the essential role in the **Step 3**, because in such coordinates Jacobi identity and skew-symmetry holds immediately. Therefore we did not need to assume the rest of functions A_{ab}^{ijk} to be zero, because that was satisfied automatically.

Indeed, in the case of $d = 1$, the relation (2.106) follows from (2.105) and the relation (2.108) follows from (2.104), (2.105) and (2.107). If we assume non-degeneracy, then (2.103) gives the symmetry of metric, (2.104) the compatibility, (2.105) is equivalent with (2.104), (2.107) gives the zero curvature and the last relation (2.108) follows from (2.108) and (2.105).

Asi rozepsat celé ještě jednou pro přehlednost.

Unfortunately, in the multi-dimensional case, the zero torsion and curvature are not sufficient.

Definition 27. *We define an obstruction tensor*

$$T_{jk}^{i\alpha\beta}(u) = \Gamma_{jk}^{i\beta}(u) - \Gamma_{jk}^{i\alpha}(u) \quad (2.110)$$

for each pair of distinct indices α and β .

We shall also use its fully contravariant form

$$T^{ijk\alpha\beta}(u) = g^{ks\beta}(u)g^{ir\alpha}(u)T_{rs}^{j\alpha\beta}(u) . \quad (2.111)$$

Theorem 28 (Mokhov). *Flat non-degenerate metrics $g^{ij\alpha}$ define a multidimensional Poisson bracket if and only if the following relations are fulfilled:*

$$\begin{aligned} T^{ijk\alpha\beta}(u) &= T^{kji\alpha\beta}(u) , & \text{eq:Mokhov-1} \\ \sum_{(i,j,k)} T^{ijk\alpha\beta}(u) &= 0 , & \text{eq:Mokhov-2} \\ T^{ijs\alpha\beta}(u)T_{st}^{k\alpha\beta}(u) &= T^{iks\alpha\beta}(u)T_{st}^{j\alpha\beta}(u) , & \text{eq:Mokhov-3} \\ \nabla_r^\alpha T^{ijk\alpha\beta}(u) &= 0 , & \text{eq:Mokhov-4} \end{aligned}$$

where ∇_k^α is the Levi-Civita covariant derivative generated by the metric $g^{ij\alpha}(u)$ and $\Gamma_{jk}^{i\alpha}(u)$ are the corresponding Christoffel symbols.

Remark. The relations (2.112) and (2.114) were found by Dubrovin and Novikov in ?. In ? it was proved that this set of tensor relations is not complete and the remaining two were obtained.

List of Figures

A. Appendix