Todo list

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The connection between continuum mechanics and Riemannian geometry

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Dedication.

Title: The connection between continuum mechanics and Riemannian geometry

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Contents

1. Mathematical Preliminarities

In this chapter, we shall revise some basic definitions and propositions of the functional calculus which we will need for the proper formulation of the hydrodynamic geometry.

Over the last century, modern mathematics has worked to develop the theory of generalized functions - distributions, including Dirac's delta distributions and operations with them. For the reader who seeks for a nice introduction to this theory accompanied with several physical applications, we would recommend the book by R.P.Kanwal [1]. The same applies to functional variations and the conditions on which they exist. A book written with a deep pedagogical attention on this topic is e.g. the university text by von Brunt [2]. Among physicists, however, it is convential to operate with delta distributions as they would with "ordinary" functions, using dualities as integrals and functional variations as ordinary derivatives.

After introducing the concept of Gateaux and Frechet derivatives, which are suitable for arbitrary normed space, we shall take a closer look on the space of smooth functions and functionals on it.

1.1 Functional derivatives

přidat nějaký kontext

Definition 1 (Gateaux and Frechet derivative). *Let X be a normed space, $A : X \to \mathbb{R}$ a functional and $x \in \mathrm{Dom}(A)$.*

1. *Let $h \in D(A)$. Gateaux derivative $\delta A(x, h)$ in the direction h is defined as*

$$\delta A(x,h) := \frac{\mathrm{d}}{\mathrm{d}t} A(x+th)|_{t=0} = \lim_{t\to 0} \frac{A(x+th) - A(x)}{t} \,. \tag{1.1}$$

2. *Fréchet diferential $\mathrm{D}_x A$ is defined as the unique linear bounded functional satisfying the relation*

$$\lim_{\|h\|\to 0} \frac{A(x+h) - A(x) - \mathrm{D}_x A(h)}{\|h\|} = 0 \,. \tag{1.2}$$

We are intersted in such functionals, where X is the space of smooth functions with The smoothness is required because the functionals would operate with derivatives, and the values beeing zero on the boundary is required because we shall use integration by parts.

One of possible choices is to take

$$X^k := \left\{ f \in C^k(\Omega) \,|\, \mathrm{D}^\alpha f = 0 \text{ on } \partial\Omega \text{ for } |\alpha| = 1, \cdots, k-1 \right\} \,. \tag{1.3}$$

This space can be equipped with the usual supremum norm

$$\|f\|_k := \sum_{|\alpha|\le k} \sup_{x\in\Omega} |\mathrm{D}^\alpha f(x)| \,. \tag{1.4}$$

There are two main types of functionals on these spaces - the integral functionals and Dirac's delta distributions.

1.1.1 Integral functionals

The first type is the integral functional

eq:integral-functional

$$A[u] = \int_\Omega a(x, u(x), u'(x), \cdots, u^{(k)}(x))\, \mathrm{d}^n x\,, \tag{1.5}$$

where $a : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$ is an analytic function. For simplicity we often write this using a physicist's notation

$$A[u] = \int a(u)\, \mathrm{d}^n x\,, \tag{1.6}$$

meaning that $a(u)$ is in fact function $a(x, u(x), u'(x), \cdots, u^{(k)}(x))$ and also the domain Ω is specified by the context. We call $a(u)$ the density of functional $A[u]$.

Gateaux derivative of A is computed as follows:

eq:Gateaux

$$\delta A[u,h] = \int_\Omega \sum_{i=1}^{k} \frac{\partial a}{\partial u^{(i)}}(x) h^{(i)}(x)\, \mathrm{d}^n x\,. \tag{1.7}$$

The functional $\delta A[u,h]$ is linear and bounded in h, so the Fréchet derivative of A exists and is given by

$$\mathrm{D}_u A[h] = \delta A[u,h]\,. \tag{1.8}$$

To get rid of the derivatives of $h(x)$ in (1.7), we use integration by parts in each term of the sum (exploiting the fact that boundary terms are zero), obtaining

eq:perpartes

$$\mathrm{D}_u A[h] = \int_\Omega \left[\frac{\partial a}{\partial u}(x) - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial a}{\partial u'}(x) + \cdots + (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}x^k} \frac{\partial a}{\partial u^{(k)}}(x) \right] h(x)\, \mathrm{d}^n x\,. \tag{1.9}$$

It is natural to give a special name to the integrand in (1.9).

Definition 2 (Functional derivative of the integral functional)**.** *Let A be the integral functional given by* (1.5)*. The functional derivative of A is the function*

$$\frac{\delta A}{\delta u(x)} = \frac{\partial a}{\partial u}(x) - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial a}{\partial u'}(x) + \frac{\mathrm{d}^2}{\mathrm{d}x^2} \frac{\partial a}{\partial u''} - \cdots + (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}x^k} \frac{\partial a}{\partial u^{(k)}}(x)\,. \tag{1.10}$$

The functional derivative can be seen as the density of the integral functional $\mathrm{D}_u A$ applicated on h:

$$\mathrm{D}_u A[h] = \int_\Omega \frac{\delta A}{\delta u(x)} h(x)\, \mathrm{d}x \tag{1.11}$$

Since the Fréchet derivative is always linear bounded functional, another way to look at this relation is the following: $\mathrm{D}_u A$ is a regular distribution represented by some smooth function $\frac{\delta A}{\delta u(x)}$, i.e.

$$\mathrm{D}_u A[h] = \left\langle T_{\frac{\delta A}{\delta u(x)}}, h(x) \right\rangle\,. \tag{1.12}$$

Hodit regulární distribuce před tuhle část?

It follows from theory (CITATION) that the functional derivative beeing zero is a necessery condition for the function beeing the extremal point of the functional (given that the corresponding function is smooth enough).

Theorem 3 (Euler-Lagrange equations)**.** *Let $v \in C^\infty(\Omega)$ such that $v(x)$ is stationary point of the functional A in* (1.5)*, i.e*

$$\delta A[v, h] = 0 \, . \tag{1.13}$$

Then

$$\frac{\delta A}{\delta v(x)} = 0 \, . \tag{1.14}$$

The computation above gives sense to the

$$A[u + h] = A[u] + \left\langle \frac{\delta A}{\delta u}, h \right\rangle + o(h) \, , \tag{1.15}$$

where

$$\lim_{\|h\| \to 0} \frac{o(h)}{\|h\|} = 0 \, . \tag{1.16}$$

In physicist's notation, the function h is often denoted as δu.

1.1.2 Distributions

Definition 4 (Distributions)**.** *Let $\mathcal{D}(\Omega)$ be the space of smooth functions with compact support in $\Omega \subseteq \mathbb{R}^n$. The space of distributions $\mathcal{D}'(\Omega)$ is the dual space, i.e. the space of bounded linear functionals on $\mathcal{D}(\Omega)$. An element $T \in \mathcal{D}'(\Omega)$ is called a distribution.*

Definition 5 (Regular distribution)**.** *Let $a \in L^1_{loc}(\Omega)$. A regular distribution is a distribution $T_a \in \mathcal{D}'(\Omega)$ defined as*

$$\langle T_a, \phi(x) \rangle = \int_\Omega a(x)\phi(x) \mathrm{d}^n x \, . \tag{1.17}$$

Note that if two regular distributions satisfy $T_a = T_b$, then $a = b$ almost everywhere and vice versa.

Definition 6 (Dirac delta distribution)**.** *Let $y \in \Omega$. The Dirac delta distribution $\delta(x - y)$ is defined as*

$$\langle \delta(x - y), \phi(x) \rangle = \phi(y) \, . \tag{1.18}$$

Definition 7 (Derivatives of distributions)**.** *Let $T \in \mathcal{D}'(\Omega)$ be a distribution. We define its derivative by*

$$\langle D^\alpha T, \phi \rangle = (-1)^{|\alpha|} \langle T, D^\alpha \phi \rangle \, , \tag{1.19}$$

where α is a multiindex and

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_N^{\alpha_N}} \, . \tag{1.20}$$

The reason, why distributions have such a "nice" behaviour and are often used by mathematicians, is that every distribution is diferentiable infinitely many times. However, if we abandon this property and require only the existence of derivatives of the k-th order, we can extend regular and Dirac distributions on previously defined space X^k. Indeed,

$$|\left\langle \partial_x^n \delta(x-y), f(x) \right\rangle| \leq \|f\|_n \quad \forall n = 0, 1, \cdots, k \,, \tag{1.21}$$

so Dirac distributions and their derivatives are also bounded linear functionals on X^k up to the order k.

Let $a \in C^\infty(\Omega)$. Then it makes sense to define the product of a and arbitrary distribution T by

$$\langle aT, \phi \rangle = \langle T, a\phi \rangle \ . \tag{1.22}$$

In the next chapter we shall frequently use the following identities of the delta distributions, which are written in the next lemma and which we will refer to as the Dirac identities.

lemma:delta

Lemma 8 (Dirac identities)**.**

$$\begin{aligned} f(y)\delta(x-y) =& f(x)\delta(x-y) \,, & (1.23) \\ f(y)\partial_x\delta(x-y) =& f(x)\partial_x\delta(x-y) + f'(x)\delta(x-y) \,. & (1.24) \end{aligned}$$

Proof. The first relation is trivial. Direct calculation gives

$$\begin{aligned} \langle f(x)\partial_x\delta(x-y), \psi(x) \rangle =& \langle \partial_x\delta(x-y), f(x)\psi(x) \rangle \\ =& - \langle \delta(x-y), f'(x)\psi(x) \rangle - \langle \delta(x-y), f(x)\psi'(x) \rangle \\ =& - f'(y)\psi(y) - f(y) \langle \delta(x-y), \psi'(x) \rangle \\ =& - \langle f'(x)\delta(x-y), \psi(x) \rangle + f(y) \langle \partial_x\delta(x-y), \psi(x) \rangle \\ =& \langle f(y)\delta(x-y) - f'(x)\partial_x\delta(x-y), \psi(x) \rangle \ . \end{aligned}$$

Therefore

$$\langle f(y)\partial_x\delta(x-y), \psi(x) \rangle = \langle f'(x)\delta(x-y), \psi(x) \rangle + \langle f(x)\partial_x\delta(x-y), \psi(x) \rangle \ . \tag{1.25}$$

□

The convolution of two distributions is not so simple to construct in rigorous way. The procedure is the following. First the tensor product of distributions is defined

$$\langle A(x) \otimes B(y), \phi(x,y) \rangle = \langle A(x), \langle B(y), \phi(x,y) \rangle \rangle \ . \tag{1.26}$$

The product defined in this way satisfies comutativity, associativity and it is indeed a distribution (altough the proof of these properties is non-trivial). Then the convolution of distribution is defined by

$$\langle A \star B, \phi(x) \rangle = \langle A(x) \otimes B(y), \phi(x+y) \rangle \ , \tag{1.27}$$

with additional requirements on $A \otimes B$ and ϕ. Convolution defined in this way satisfies associativity and if it exists, then

$$D^\alpha(A \star B) = D^\alpha A \star B = A \star D^\alpha B\,. \tag{1.28}$$

The convolution with δ and arbitrary distribution A always exists and is equal to

$$A(y) \star \delta(x-y) = A(x)\,. \tag{1.29}$$

This means, in particular, that given an integral functional $T[u]$ one can write

$$T[u(x)] = T[u(y)] \star \delta(x-y)\,, \tag{1.30}$$

or in physicist's notation

eq:konvoluce-s-deltou

$$T[u(x)] = \int \mathrm{d}y\, T[u(y)]\, \delta(x-y)\,. \tag{1.31}$$

The equation (1.31) is very important in calculations of functional derivatives.

Example 9. *Let $u(x)$ be a field variable and $F[u(x)]$ an integral functional. To compute its functional derivative w.r.t. $u(y)$ in different position y, we use the integral expression of F*

$$F[u(x)] = \int \mathrm{d}y F[u(y)]\delta(x-y)\,. \tag{1.32}$$

Its Gateaux diferential is

$$\delta F[u,h] = \frac{\mathrm{d}}{\mathrm{d}t} F[u+th]|_{t=0} = \int \mathrm{d}y \frac{\partial F}{\partial u}(y)\delta(x-y)h(y)\,, \tag{1.33}$$

therefore

$$\frac{\delta F(u(x))}{\delta u(y)} = \frac{\partial F}{\partial u}(x)\delta(x-y)\,. \tag{1.34}$$

Example 10. *In the special case of performing functional derivative w.r.t. the same field but in the different position,*

$$\frac{\delta u^i(x)}{\delta u^j(y)} = \delta^i_j \delta(x-y)\,, \tag{1.35}$$

because

$$\frac{\delta u^i(x)}{\delta u^j(y)} = \frac{\delta}{\delta u^j(y)} \int \mathrm{d}y u^i(y)\delta(x-y) = \frac{\partial u^i}{\partial u^j}(y)\delta(x-y) = \delta^i_j\delta(x-y)\,. \tag{1.36}$$

The same applies to

$$\frac{\delta}{\delta u^j(y)}\partial_x u^i(x) = -\delta^i_j \partial_x \delta(x-y)\,, \tag{1.37}$$

because

$$\frac{\delta}{\delta u^j(y)}\partial_x u^i(x) = \frac{\delta}{\delta u^j(y)} \int \mathrm{d}y \partial_y u^i(y)\delta(x-y) \tag{1.38}$$

$$= -\frac{\delta}{\delta u^j(y)} \int \mathrm{d}y u^i(y)\partial_y\delta(x-y) = -\delta^i_j\partial_x\delta(x-y)\,. \tag{1.39}$$

2. Geometrical preliminarities

The goal of this chapter is to briefly revise some basic definitions and well-known facts about the pseudo-Riemannian manifolds and the Poisson structure. This important part of geometry and topology is covered by hundreds of great books. Since for the rest of the thesis we shall study mostly the work of B. A. Dubrovin and S. P. Novikov, we will recommend three monographs on modern geometry written by our heroes and A. T. Fomenko [3]. Another exquisite book, which has become almost cult among Czechs and Slovaks and which is worth all of the reader's attention, is the book of M.Fecko [4].

Throughout this chapter M represents a finite-dimensional smooth manifold, $\mathcal{F}M$ the space of smooth functions $M \to \mathbb{R}$, T_xM the tangent space in $x \in M$, TM the tangent bundle on M, T^p_qM the bundle of tensors of rank (p,q) on M. We also automatically assume the so called Einstein summation over contravariant and covariant indices. Vectors, covectors and tensors of arbitrary rank are denoted (only in this chapter) by bold symbols, for the purpose of higliting the difference between the abstract indices (denoted by normal letters $i, j, \cdots$) and vector indices (denoted by bold letters $\boldsymbol{A}$, $\boldsymbol{b}$, $\cdots$).

2.1 Pseudo-Riemannian manifolds

Definition 11 (Metric tensor)**.** *A metric tensor $\boldsymbol{g}_{ij} \in T^0_2M$ is symmetric and non-degenerate, i.e.*

$$\boldsymbol{g}_{ij} = \boldsymbol{g}_{ji} \quad \textbf{\textit{(symmetry)}}\,, \tag{2.1}$$

$$\det \boldsymbol{g}_{ij} \neq 0 \quad \textbf{\textit{(non-degeneracy)}}\,. \tag{2.2}$$

If the metric $\boldsymbol{g}_{ij}$ is positive quadratic form, the manifold M is said to be Riemannian. If it is indefinite, the manifold is said to be pseudo-Riemannian.

Non-degeneracy of the metric $\boldsymbol{g}_{ij}$ allows us to define the inverse metric $\boldsymbol{g}^{ij}$ defined by the property

$$\boldsymbol{g}^{ij}\boldsymbol{g}_{jk} = \boldsymbol{\delta}^i_k\,. \tag{2.3}$$

A metric structure is important mainly for three things: 1. it allows to measure lenghts of vectors and angles between them, 2. together with its inverse allows to upper and lower indices of tensors, 3. it enables to build the parallel transport of tensors from one point to another and the concept of covariant derivative.

Definition 12 (Covariant derivative)**.** *Let $\boldsymbol{a} \in TM$. A covariant derivative $\nabla_{\boldsymbol{a}}$ in the direction $\boldsymbol{a}$ is an operation on T^p_qM satisfying these properties:*

$$\nabla_{\boldsymbol{a}}(\boldsymbol{A} + r\boldsymbol{B}) = \nabla_{\boldsymbol{a}}\boldsymbol{A} + r\nabla_{\boldsymbol{a}}\boldsymbol{B} \quad \forall \boldsymbol{A}, \boldsymbol{B} \in T^k_lM\,,\ r \in \mathbb{R}\,, \quad \textbf{\textit{(linearity)}} \tag{2.4}$$

$$\nabla_{\boldsymbol{a}}(\boldsymbol{A} \otimes \boldsymbol{B}) = (\nabla_{\boldsymbol{a}}\boldsymbol{A}) \otimes \boldsymbol{B} + \boldsymbol{A} \otimes (\nabla_{\boldsymbol{a}}\boldsymbol{B}) \quad \forall \boldsymbol{A}, \boldsymbol{B} \in T^k_lM\,, \quad \textbf{\textit{(Leibniz rule)}} \tag{2.5}$$

$$\nabla_{\boldsymbol{a}}(\mathsf{C}\boldsymbol{A}) = \mathsf{C}(\nabla_{\boldsymbol{a}}\boldsymbol{A}) \quad \forall \boldsymbol{A} \in T^k_lM\,, \quad \textbf{\textit{(commutation with contraction)}} \tag{2.6}$$

$$\nabla_{\boldsymbol{a}}f = \boldsymbol{a}[f] = \boldsymbol{a} \cdot \mathrm{d}f \quad \forall f \in \mathcal{F}M\,, \quad \textbf{\textit{(gradient on scalars)}} \tag{2.7}$$

where C *is the contraction operation.*

(Note that the covariant derivative $\nabla_{\boldsymbol{a}}$ has the lower index bold, as it refers to the vector $\boldsymbol{a}$.)

If we multiply the vector field $\boldsymbol{a} \in TM$ by some smooth function $f \in \mathcal{F}M$, then

$$\nabla_{f\boldsymbol{a}}\boldsymbol{A} = f\nabla_{\boldsymbol{a}}\boldsymbol{A} \quad \forall \boldsymbol{A} \in T^p_q M \quad \textbf{(ultralocality)}\,. \tag{2.8}$$

Therefore we can introduce the covariant differential ∇_j (with the normal lower index) as the operation from $\nabla : T^p_q \to T^p_{q+1}$ defined by the property

$$\nabla_{\boldsymbol{a}}\boldsymbol{A} = \boldsymbol{a} \cdot \nabla \boldsymbol{A}\,, \tag{2.9}$$

or equivalently in the index notation

$$\nabla_{\boldsymbol{a}}\boldsymbol{A}^{ij\cdots}_{kl\cdots} = \boldsymbol{a}^k \nabla_k \boldsymbol{A}^{ij\cdots}_{kl\cdots}\,. \tag{2.10}$$

Any covariant diferential ∇_i can be expressed via the partial derivative with respect to the coordinates (x^j) as

$$\nabla_i \boldsymbol{A}^{ab\ldots}_{cd\ldots} = \boldsymbol{A}^{ab\ldots}_{cd\ldots,i} + \boldsymbol{\Gamma}^a_{im}\boldsymbol{A}^{mb\ldots}_{cd\ldots} + \boldsymbol{\Gamma}^b_{im}\boldsymbol{A}^{am\ldots}_{cd\ldots} + \cdots - \boldsymbol{\Gamma}^m_{ic}\boldsymbol{A}^{ab\ldots}_{md\ldots} - \boldsymbol{\Gamma}^m_{id}\boldsymbol{A}^{ab\ldots}_{cm\ldots} - \cdots\,, \tag{2.11}$$

where $\boldsymbol{\Gamma}^a_{bc}$ are the affine connection coefficients (Christoffel symbols). Given two coordinate systems (x^i) and $(\tilde{x}^a)$, the Christoffel symbols transform as

$$\tilde{\boldsymbol{\Gamma}}^a_{bc} = \boldsymbol{\Gamma}^i_{jk}\frac{\partial x^j}{\partial \tilde{x}^b}\frac{\partial x^k}{\partial \tilde{x}^c}\frac{\partial \tilde{x}^a}{\partial x^i} + \frac{\partial \tilde{x}^a}{\partial x^m}\frac{\partial^2 x^m}{\partial \tilde{x}^b \partial \tilde{x}^c}\,. \tag{2.12}$$

Compared to the partial derivative in flat space, the covariant derivative brings several complicacies with it. The information on how much it differs from ordinary derivative is encoded in tensors of torsion and curvature.

Definition 13 (Torsion and curvature tensors)**.** *Let* $[\cdot,\cdot]$ *be the Lie bracket of two vector fields.*

1. *The torsion tensor* $\boldsymbol{T}^k_{mn}$ *is defined by*

$$\boldsymbol{T}^k_{mn}\boldsymbol{a}^m\boldsymbol{b}^n = \boldsymbol{a}^m\nabla_m\boldsymbol{b}^k - \boldsymbol{b}^m\nabla_m\boldsymbol{a}^k - [\boldsymbol{a},\boldsymbol{b}]^k \quad \forall \boldsymbol{a},\boldsymbol{b} \in TM\,. \tag{2.13}$$

2. *The Riemann curvature tensor* $\boldsymbol{R}_{ij}{}^k{}_l$ *is defined by*

$$\boldsymbol{R}_{ij}{}^k{}_l\boldsymbol{a}^i\boldsymbol{b}^j\boldsymbol{c}^l = [\nabla_{\boldsymbol{a}}\nabla_{\boldsymbol{b}} - \nabla_{\boldsymbol{b}}\nabla_{\boldsymbol{a}} - \nabla_{[\boldsymbol{a},\boldsymbol{b}]}]\boldsymbol{c}^k \quad \forall \boldsymbol{a},\boldsymbol{b},\boldsymbol{c} \in TM\,, \tag{2.14}$$

The torsion tensor represents the commutator of the second covariant diferentials applied on functions

$$[\nabla_i\nabla_j - \nabla_j\nabla_i]f = -\boldsymbol{T}^m_{ij}\mathrm{d}_m f \quad \forall f \in \mathcal{F}M\,. \tag{2.15}$$

and Riemann tensor represents the commutator on vectors

$$[\nabla_i\nabla_j - \nabla_j\nabla_i]\boldsymbol{a}^k = -\boldsymbol{T}^m_{ij}\nabla_m\boldsymbol{a}^k + \boldsymbol{R}_{ij}{}^k{}_m\boldsymbol{a}^m\,. \tag{2.16}$$

Proposition 14 (Zero torsion and symmetry of the Christoffel symbols)**.** *The torsion tensor* $\boldsymbol{T}^i_{jk}$ *of the covariant derivative* ∇_i *is zero if and only if the corresponding Christoffel symbols are symmetric in the lower indices, i.e.* $\boldsymbol{\Gamma}^i_{jk} = \boldsymbol{\Gamma}^i_{kj}$.

Among a whole class of covariant derivatives, there is one type that is used most often: the Levi-Civita derivative.

Definition 15 (Levi-Civita covariant derivative)**.** *Let* M *be a pseudo-Riemannian manifold with the metric* $\boldsymbol{g}_{ij}$. *Levi-Civita covariant derivative* $\boldsymbol{\nabla}_k$ *is defined by the following properties:*

$$\begin{aligned} \boldsymbol{\nabla}_k \boldsymbol{g}_{ij} = \boldsymbol{g}_{ij,k} - \boldsymbol{\Gamma}^m_{ki}\boldsymbol{g}_{mj} - \boldsymbol{\Gamma}^m_{kj}\boldsymbol{g}_{im} &= 0 \quad \textbf{\textit{(compatibility with metric)}}\,, && (2.17)\\ \boldsymbol{T}^i_{jk} &= 0 \quad \textbf{\textit{(zero torsion)}}\,, && (2.18)\end{aligned}$$

where $\boldsymbol{T}^i_{jk}$ *is the torsion corresponding to* $\boldsymbol{\nabla}_j$.

If $\boldsymbol{g}_{ij}$ is non-degenerate, i.e. $\det \boldsymbol{g}_{ij} \neq 0$, then the Levi-Civita covariant derivative is always defined and its Christoffel symbols are given by

$$\boldsymbol{\Gamma}^m_{ij} = \frac{1}{2}\boldsymbol{g}^{mn}\left(\boldsymbol{g}_{jn,i} + \boldsymbol{g}_{ni,j} - \boldsymbol{g}_{ij,n}\right)\,. \tag{2.19}$$

The covariant derivative compatible with the metric commutes with the operation of lowering and uppering indices of a tensor. For example:

$$\nabla_i \boldsymbol{T}^{ab} = \nabla_i(\boldsymbol{g}^{am}\boldsymbol{T}^b_m) = \boldsymbol{g}^{am}\nabla_i \boldsymbol{T}^b_m\,. \tag{2.20}$$

The Riemann curvature tensor for the Levi-Civita covariant derivative is

$$\boldsymbol{R}_{ij}{}^k{}_l = \boldsymbol{\Gamma}^k_{jl,i} - \boldsymbol{\Gamma}^k_{il,j} + \boldsymbol{\Gamma}^k_{im}\boldsymbol{\Gamma}^m_{jl} - \boldsymbol{\Gamma}^k_{jm}\boldsymbol{\Gamma}^m_{il}\,. \tag{2.21}$$

2.2 Poisson brackets

Definition 16 (Poisson bracket)**.** *A Poisson bracket is a bilinear operation* $\{\cdot,\cdot\} : \mathcal{F}M \times \mathcal{F}M \to \mathcal{F}M$ *which satisfies*

$$\begin{aligned} &\{f,g\} = -\{g,f\} \quad \textbf{\textit{(skew-symmetry)}}\,, && (2.22)\\ &\{fg,h\} = f\{g,h\} + g\{f,h\} \quad \textbf{\textit{(Leibniz identity)}}\,, && (2.23)\\ &\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0 \quad \textbf{\textit{(Jacobi identity)}}\,. && (2.24)\end{aligned}$$

Definition 17 (Symplectic 2-form)**.** *A skew-symmetric 2-form* $\boldsymbol{\omega} \in T^0_2 M$ *is said to be symplectic if it is closed and nondegenerate, i.e.*

$$\begin{aligned} \mathrm{d}\boldsymbol{\omega} &= 0 \quad \textbf{\textit{(closedness)}}\,, && (2.25)\\ \det \boldsymbol{\omega} &\neq 0 \quad \textbf{\textit{(nondegeneracy)}}\,. && (2.26)\end{aligned}$$

A pair $(M, \boldsymbol{\omega})$ *is called sympletic manifold.*

Since in odd dimension every skew-symmetric 2-form is degenerate, the dimension m of the symplectic manifold M must be even number. The following statement holds.

Theorem 18 (Darboux theorem - symplectic forms formulation)**.** *Let* $(M, \boldsymbol{\omega})$ *be a* $2n$*-dimensional symplectic manifold and* $x \in M$*. Then there is a neighbourhood* $U(x)$ *and a coordinate system* $(p_1, \cdots, p_n, q^1, \cdots, q^n)$ *such that* $\boldsymbol{\omega}$ *has the form*

eq:standart form

$$\boldsymbol{\omega} = \mathrm{d}p_a \wedge \mathrm{d}q^a \quad \textit{on } U(x)\,. \tag{2.27}$$

We refer to this coordinate system as to the standard coordinates $(x^1, \cdots, x^{2n}) = (p_1, \cdots, p_n, q^1, \cdots, q^n)$ and to the form (**??**) as to the standard form of $\boldsymbol{\omega}$.

Definition 19 (Poisson bivector)**.** *Let* $(M, \boldsymbol{\omega})$ *be a symplectic manifold. The Poisson bivector* $\boldsymbol{P} \in T^2_0 M$ *is the inverse of* $\boldsymbol{\omega}$*, i.e.*

$$\boldsymbol{P}^{ij}\boldsymbol{\omega}_{jk} = \boldsymbol{\delta}^i_k\,. \tag{2.28}$$

Thus a symplectic manifold $(M, \boldsymbol{\omega})$ has its associated Poisson bracket given by

$$\{f, g\} := \boldsymbol{P}(\mathrm{d}f, \mathrm{d}g) = \boldsymbol{P}^{ab}\mathrm{d}_a f \mathrm{d}_b g \quad \forall f, g \in \mathcal{F}M\,. \tag{2.29}$$

In the standard coordinates

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} \right)\,. \tag{2.30}$$

In the standard coordinates we have

$$\left\{x^i, x^j\right\} = \boldsymbol{P}^{ij}\,, \tag{2.31}$$

$$\left\{\left\{x^i, x^j\right\}, x^k\right\} = \frac{\partial \boldsymbol{P}^{ij}}{\partial x^m}\frac{\partial x^k}{\partial x^n}\boldsymbol{P}^{mn} = \frac{\partial \boldsymbol{P}^{ij}}{\partial x^m}\boldsymbol{P}^{mk}\,, \tag{2.32}$$

so the Jacobi identity is given by

$$\frac{\partial \boldsymbol{P}^{ij}}{\partial x^m}\boldsymbol{P}^{mk} + \frac{\partial \boldsymbol{P}^{jk}}{\partial x^m}\boldsymbol{P}^{mi} + \frac{\partial \boldsymbol{P}^{ki}}{\partial x^m}\boldsymbol{P}^{mj} = 0\,. \tag{2.33}$$

Definition 20 (The Casimir of the Poisson bracket)**.** *The function* $g \in \mathcal{F}M$ *is called a Casimir for the given Poisson bracket* $\{\cdot, \cdot\}$*, if*

$$\{f, g\} = 0 \quad \forall g \in \mathcal{F}M\,. \tag{2.34}$$

3. Systems of hydrodynamic type: one-dimensional case

In this chapter we shall prove the theorem of the connection between the hydrodynamic Poisson bracket and the metrics found in the homogeneous system of hydrodynamic type. The original result was given in 1983 by B.A.Dubrovin and S.P.Novikov and can be found in [5]. Since then, many authors have contributed to the further development of this theory, e.g. Grinberg in [6], Ferapontov in [7] and Mochov in [8]. The proof itself and the preceding lemmas are based on a considerable amount of technical computations, which we shall clarify in detail in order to offer the reader a comprehensive understanding of the chapter in question.

3.1 Systems of hydrodynamic type

Definition 21 (Homogeneous system of hydrodynamic type)**.** *A homogeneous system of hydrodynamic type is an equation of the form*

$$\frac{\partial u^i}{\partial t} = f_j^{i\alpha}(u)\frac{\partial u^j}{\partial x^\alpha}\,, \tag{3.1}$$

eq:h-system

where $u^i(t,x^\alpha)$ are unknown functions, $i = 1,\dots,N$, $\alpha = 1,\dots,d$.

Sometimes we refer to the number of equations N as the dimension of the hydrodynamical phase space and to the number of spatial variables d as the dimension of the configuration space.

Příklad asi přesunout do jiné kapitoly.

Before we go any further, let us present a few examples.

Example 22. *The energy is given by the functional*

$$E = \int e(\rho, \boldsymbol{m}, s, \boldsymbol{F})\,\mathrm{d}\boldsymbol{x}\,. \tag{3.2}$$

The evolution equations in the Eulerian frame are

$$\begin{aligned}
\partial_t\rho &= -\,\partial_i(\rho E_{m_i})\,, && (3.3)\\
\partial_t m_i &= -\,\partial_j(m_i E_{m_j}) - \rho\partial_i E_\rho - m_j\partial_i E_{m_j} - m_j\partial_i E_{m_j}\,, && (3.4)\\
&\quad -s\partial_i E_s - F_J^j\partial_i E_{F_J^j} + \partial_j(F_I^j E_{F_I^i} + F_O^i E_{F_I^i})\,, && (3.5)\\
\partial_t s &= -\,\partial_i(sE_{m_i})\,, && (3.6)\\
\partial_t F_I^i &= -\,E_{m_k}\partial_k F_I^i + F_I^j\partial_j E_{m_i}\,. && (3.7)
\end{aligned}$$

Total energy density e can be prescribed as

$$e = \frac{\boldsymbol{m}^2}{2\rho} + \epsilon(\rho, s, \boldsymbol{F})\,, \tag{3.8}$$

where ϵ is the elastic and internal energy. In particular, $E_{\boldsymbol{m}} = \frac{\boldsymbol{m}}{\rho} = \boldsymbol{v}$ becomes the velocity. The evolution equation for the deformation gradient then gets the explicit form

$$\partial_t \boldsymbol{F} = -(\boldsymbol{v} \cdot \nabla)\boldsymbol{F} + \nabla \boldsymbol{V} \cdot \boldsymbol{F} \,, \tag{3.9}$$

which is the usual evolution equation for $\boldsymbol{F}$ in the Eulerian frame.

It was Riemann who noticed that the functions $f_j^{i\alpha}$ in (3.1) are in fact tensors [5].

prop:transformace-A

Proposition 23 (Transformation of $f_j^{i\alpha}$)**.** *Under smooth change of variables $u^i \mapsto v^a$ of the form*

$$u^i = u^i(v^1, \cdots, v^N) \,, \tag{3.10}$$

the functions $f_j^{i\alpha}$ transform for each α according to the tensor law

$$f_b^{a\alpha}(v) = \frac{\partial v^a}{\partial u^i} \frac{\partial u^j}{\partial v^b} f_j^{i\alpha}(u) \,. \tag{3.11}$$

Proof. Using a direct calculation, we obtain

$$\frac{\partial u^i}{\partial t}(v) = \frac{\partial u^i}{\partial v^a} \frac{\partial v^a}{\partial t} \,, \tag{3.12}$$

$$\frac{\partial u^j}{\partial x^\alpha}(v) = \frac{\partial u^j}{\partial v^b} \frac{\partial v^b}{\partial x^\alpha} \,. \tag{3.13}$$

Hence

$$\frac{\partial u^i}{\partial v^a} \frac{\partial v^a}{\partial t} = \frac{\partial u^j}{\partial v^b} f_j^{i\alpha}(v(u)) \frac{\partial v^b}{\partial x^\alpha} \,, \tag{3.14}$$

therefore

$$\frac{\partial v^a}{\partial t} = \underbrace{\frac{\partial v^a}{\partial u^i} \frac{\partial u^j}{\partial v^b} f_j^{i\alpha}(v(u))}_{f_b^{a\alpha}(v)} \frac{\partial v^b}{\partial x^\alpha} \,. \tag{3.15}$$

□

According to the previous lemma, if we denote the space of points $(t, x^\alpha) \subset \mathbb{R} \times \mathbb{R}^N$ as some manifold M, then the unknown variables $u^i(t, x^\alpha)$ can be seen as the local coordinates of that manifold. The functions $f_j^{i\alpha}$ are transformed as tensors of the type $(1, 1)$ in the i and j indices.

Our next purpose is to gain a richer geometry of that space M. If the functions $f_j^{i\alpha}$ have a special form, we can introduce a geometry of the Poisson brackets.

Definition 24. *1. The Dubrovin-Novikov bracket of two functionals I_1 a I_2 is defined as*

eq:Dubrovin-Novikov bracket

$$\{I_1, I_2\} = \sum_{\alpha=1}^{d} \int \mathrm{d}x \left[\frac{\delta I_1}{\delta u^p(x)} \left(g^{pq\alpha}(u) \frac{\mathrm{d}}{\mathrm{d}x^\alpha} + b_s^{pq\alpha}(u) \frac{\partial u^s}{\partial x^\alpha} \right) \frac{\delta I_2}{\delta u^q(x)} \right] , \tag{3.16}$$

where $g^{ij\alpha}$ and $b_k^{ij\alpha}$ are certain functions, $i, j, k = 1, \ldots, N$ and $\frac{\mathrm{d}}{\mathrm{d}x^\alpha}$ is the total derivative with respect to the independent variable x^α,

$$\frac{\mathrm{d}}{\mathrm{d}x^\alpha} = \frac{\partial}{\partial x^\alpha} + u^i_\alpha \frac{\partial}{\partial u^i} + u^i_{\alpha\beta} \frac{\partial}{\partial u^i_\beta} + \cdots + u^i_{\beta_1 \cdots \beta_s \alpha} \frac{\partial}{\partial u^i_{\beta_1 \cdots \beta_s}} + \cdots \,, \tag{3.17}$$

$$u^i_{\beta_1 \cdots \beta_s} = \frac{\partial^s u^i}{\partial x^{\beta_1} \cdots \partial x^{\beta_s}} \,. \tag{3.18}$$

The summation over repeating upper and lower indices is assumed, moreover, in this case the summation over $\beta_1, \cdots, \beta_s$ is taken over distinct (up to arbitrary permutation) sets of these indices. (One can consider that these sets of indices are ordered: $\beta_1 \leq \cdots \leq \beta_s$.)

2. *The functional of hydrodynamic type is a functional of the form*

$$H[u] = \int_{\mathbb{R}^d} h(u(x)) \mathrm{d}^d x \,, \tag{3.19}$$

where h is independent of u_α, $u_{\alpha\beta}$. The function h is called a hamiltonian density.

3. *The hamiltonian system of hydrodynamic type is a system of the form*

$$u^i_t(x) = \left[g^{ij\alpha}[u(x)] \frac{\partial^2 h(u)}{\partial u^j \partial u^k} + b_k^{ij\alpha}[u(x)] \frac{\partial h(u)}{\partial u^j} \right] u^k_\alpha(x) \,, \tag{3.20}$$

where $H[u]$ is the functional of the hydrodynamic type and h the hamiltonian density.

...Jednoduché cvičení a zároveň důležité...

Proposition 25 (Dubrovin-Novikov bracket of field variables)**.**

$$\left\{ u^i(x), u^j(y) \right\} = g^{ij\alpha}[u(x)] \delta_\alpha(x-y) + b_k^{ij\alpha}[u(x)] u^k_\alpha(x) \delta(x-y) \,. \tag{3.21}$$

Proof.

$$\left\{ u^i(x), u^j(y) \right\} = \int \mathrm{d}\xi \frac{\delta u^i(x)}{\delta u^p(\xi)} \left(g^{pq\alpha}[u\xi] \frac{\mathrm{d}}{\mathrm{d}\xi^\alpha} + b_s^{pq\alpha}[u(\xi)] \frac{\partial u^s}{\partial \xi^\alpha} \right) \frac{\delta u^j(y)}{\delta u^q(\xi)} \tag{3.22}$$

$$= \int \mathrm{d}\xi \delta(x-\xi) \left[g^{ij\alpha}[u(\xi)] \partial_\alpha \delta(y-\xi) + b_s^{ij\alpha}[u(\xi)] u^s_\alpha \delta(y-\xi) \right] \tag{3.23}$$

$$= g^{ij\alpha}[u(x)] \delta_\alpha(x-y) + b_k^{ij\alpha}[u(x)] u^k_\alpha(x) \delta(x-y) \,. \tag{3.24}$$

□

Proposition 26 (Time derivative of the field variables)**.** *The Hamiltonian system of hydrodynamic type reads as*

$$u^i_t(x) = \left\{ u^i, H \right\} \,. \tag{3.25}$$

Proof.

$$\begin{aligned}
\left\{u^i(x), H\right\} &= \int \mathrm{d}\xi \frac{\delta u^i(x)}{\delta u^p(\xi)} \left[g^{pq\alpha}[u(\xi)] \frac{\mathrm{d}}{\mathrm{d}\xi} + b^{pq\alpha}_m[u(\xi)] \frac{\partial u^m}{\partial \xi^\alpha} \right] \frac{\delta H}{\delta u^q(\xi)} \\
&= \int \mathrm{d}\xi \int \mathrm{d}y \delta(x-\xi) \left[g^{ij\alpha}[u(\xi)] \frac{\mathrm{d}}{\mathrm{d}\xi} \frac{\partial h}{\partial u^j}(\xi)\delta(y-\xi) + b^{ij\alpha}_k[u(\xi)] u^k_\alpha \frac{\partial h}{\partial u^j}(\xi)\delta(y-\xi) \right] \\
&= \left[g^{ij\alpha}[u(x)] \frac{\partial^2 h}{\partial u^j \partial u^k}(x) + b^{ij\alpha}_k(u(x)) \frac{\partial h}{\partial u^j}(x) \right] \frac{\partial u^k}{\partial x^\alpha} \,.
\end{aligned}$$

□

Dokončit.

Later we shall show that the Dubrovin-Novikov bracket defined above can be seen as the Poisson bracket (of hydrodynamic type), i.e. satisfies skew-symmetry, Leibniz rule and Jacobi identity. In fact, this will be the key theorem of this chapter.

3.2 One-dimensional case

In this section we first consider the one-dimensional case in which the unknown functions u^i depend on only two variables (t, x). The hamiltonian system is then

$$\frac{\partial u^i}{\partial t}(t,x) = \left[g^{ij}[u(t,x)] \frac{\partial^2 h(u)}{\partial u^j \partial u^k} + b^{ij}_k[u(t,x)] \frac{\partial h(u)}{\partial u^j} \right] \frac{\partial u^k}{\partial x}(t,x) \,, \tag{3.26}$$

and the Dubrovin-Novikov bracket is of the form

eq:Dubrovin-Novikov bracket 1D

$$\{I_1, I_2\} = \int \mathrm{d}x \left[\frac{\delta I_1}{\delta u^p(x)} \left(g^{pq}[u(x)] \frac{\mathrm{d}}{\mathrm{d}x} + b^{pq}_s[u(x)] \frac{\partial u^s}{\partial x} \right) \frac{\delta I_2}{\delta u^q(x)} \right] . \tag{3.27}$$

In particular

eq:Poisson1D

$$\left\{u^i(x), u^j(y)\right\} = g^{ij}[u(x)] \partial_x \delta(x-y) + b^{ij}_k[u(x)] u^k_x \delta(x-y) \,. \tag{3.28}$$

Our goal is to specify the conditions for the Dubrovin-Novikov bracket to be Poisson bracket. The simplest property is the Leibniz rule, which is always satisfied.

Jakou to má souvislost s Leibnizem $\{fg, h\} = f\{g,h\} + \{f,h\} g$? Není toto spíš vlastnost "transformace Poissonova bivektoru jakožto tensoru"?

Proposition 27 (Leibniz rule for Dubrovin-Novikov bracket)**.**

eq:transformace-zavorka

$$\left\{v^p(u^i(x)), v^q(u^j(y))\right\} = \frac{\partial v^p}{\partial u^i}(x) \frac{\partial v^q}{\partial u^j}(y) \left\{u^i(x), u^j(y)\right\} . \tag{3.29}$$

Proof. The coordinates are written as functionals in the form

$$v^p(u^i(x)) = \int \mathrm{d}x' v^p(u^i(x')) \delta(x - x') \,. \tag{3.30}$$

Applying this to the definition of the Dubrovin-Novikov bracket (3.28), we have

$$\left\{v^p(u^i(x)), v^q(u^j(y))\right\} = \int \mathrm{d}x' \int \mathrm{d}y' \underbrace{\frac{\delta v^p}{\delta u^i(x')}}_{\frac{\partial v^p}{\partial u^i}(x')\delta(x-x')} \left\{u^i(x'), u^j(y')\right\} \underbrace{\frac{\delta v^q}{\delta u^j(y')}}_{\frac{\partial v^q}{\partial u^j}(y')\delta(y-y')} = \tag{3.31}$$

$$= \frac{\partial v^p}{\partial u^i}(x) \frac{\partial v^q}{\partial u^j}(y) \left\{u^i(x), u^j(y)\right\} . \tag{3.32}$$

□

Skew-symmetry and Jacobi identity for the bracket are not so easy to investigate. In order to do that, we introduce one more class of functions.

We say that the Poisson bracket is non-degenerate if $\det(g^{ij}) \neq 0$. We shall show later that the non-degeneracy is invariant under the transformation of the coordinates. In that case, we can naturally define its inverse $g_{ij} := (g^{mn})^{-1}$.

Let $g^{ij}(u)$ be non-degenerate. Then we can define functions $\Gamma^i_{jk}(u)$ as

eq:Gamma=g*b

$$\Gamma^i_{jk}(u) := -g_{js}(u) b^{si}_k(u) . \tag{3.33}$$

Note that this relation allows to decompose

eq:b=g*Gamma

$$b^{ij}_k(u) = -g^{is}(u)\Gamma^j_{sk}(u) . \tag{3.34}$$

The choice of denotation $g^{ij}(u)$ and $\Gamma^i_{jk}(u)$ has its purpose. We shall show soon that these are in fact the metric and afinne coefficients, respectively, on M.

prop:transoformace

Proposition 28 (Transformation of g^{ij} and Γ^i_{jk})**.** *Let $u^i \mapsto v^a$ be the smooth change of variables. Then*

1. *The g^{ij} coefficients transform as the (2,0) tensors, i.e.*

eq:transformace-metrika

$$g^{pq}(v) = \frac{\partial v^p}{\partial u^i}\frac{\partial v^q}{\partial u^j} g^{ij}[u(v)] . \tag{3.35}$$

2. *The Γ^i_{jk} coefficients transform like Christoffel symbols, i.e.*

eq:transformace-konexe

$$\Gamma^p_{qr}(v) = \frac{\partial v^p}{\partial u^i}\frac{\partial u^j}{\partial v^q}\frac{\partial u^k}{\partial v^r}\Gamma^i_{jk}(u) + \frac{\partial v^p}{\partial u^i}\frac{\partial^2 u^i}{\partial v^q \partial v^r} . \tag{3.36}$$

Proof. 1. Let us apply the transformation equation (3.29) to the definition of the Dubrovin-Novikov bracket (3.28). We get

eq:dosad

$$\begin{aligned} &g^{pq}[v(u(x))]\partial_x\delta(x-y) + b^{pq}_s[v(u(x))]\frac{\partial v^s}{\partial u^k}(x)\frac{\partial u^k}{\partial x}\delta(x-y) \\ &= \frac{\partial v^p}{\partial u^i}(x)\frac{\partial v^q}{\partial u^j}(y)\left[g^{ij}[u(x)]\partial_x\delta(x-y) + b^{ij}_k[u(x)]\frac{\partial u^k}{\partial x}\delta(x-y)\right] . \end{aligned} \tag{3.37}$$

From now on (for readibility) we denote

$$T^a_k(x) := \frac{\partial v^a}{\partial u^k}(x) . \tag{3.38}$$

We apply Dirac identity (Lemma 8 on page 5) to $T^q_j(y)\partial_x\delta(x-y)$ on the right-hand side of (3.37). Hence all functions are then of variable x, from now on we shall not write it explicitely. We have

$$g^{pq}(v)\partial_x\delta(x-y) + b^{pq}_n(v)T^n_j\frac{\partial u^j}{\partial x}\delta(x-y)$$
$$= T^p_iT^q_jg^{ij}(u)\partial_x\delta(x-y) + \left(\frac{\partial^2 v^q}{\partial u^j\partial u^k} + T^p_iT^q_jb^{ij}_k\right)\frac{\partial u^k}{\partial x}\delta(x-y)\,. \tag{3.39}$$

eq:porovnani

The terms with $\partial_x\delta$ must be equal to each other, as must δ terms. From $\partial_x\delta$ terms we get

$$g^{pq}(v) = T^p_iT^q_jg^{ij}(u)\,, \tag{3.40}$$

which is the transformation relation for g.

2. From δ terms in (3.39), we get

$$b^{pq}_n(v)T^n_k = b^{ij}_k(u)T^p_iT^q_j + g^{ij}(u)T^p_i\frac{\partial^2 v^q}{\partial u^i\partial u^k} \tag{3.41}$$

and after (3.34)

$$-g^{ps}(v)\Gamma^q_{sn}(v)T^n_k = -g^{il}(u)\Gamma^j_{lk}(u)T^p_iT^q_j + g^{ij}(u)T^p_i\frac{\partial^2 v^q}{\partial u^i\partial u^k}\,. \tag{3.42}$$

On the left-hand side we use the transformation of g (3.35)

$$-g^{ij}(u)\Gamma^q_{sm}(v)T^p_iT^s_jT^m_k = -g^{ij}(u)\Gamma^l_{jk}T^p_iT^q_l + g^{ij}(u)\frac{\partial^2 v^q}{\partial u^i\partial u^k}T^p_i \tag{3.43}$$

and after vectoring out $g^{ij}(u)T^p_i$

$$\Gamma^l_{jk}(u)T^q_l = \Gamma^q_{sm}(v)T^s_jT^m_k + \frac{\partial^2 v^q}{\partial u^i\partial u^k}\,. \tag{3.44}$$

All that remains is to multiply this equation with $(T^{-1})^a_q$ and we get the final transformation relation for Γ

$$\Gamma^a_{jk}(u) = \Gamma^q_{sm}(v)\frac{\partial v^s}{\partial u^j}\frac{\partial v^m}{\partial u^k}\frac{\partial u^a}{\partial v^q} + \frac{\partial^2 v^q}{\partial u^i\partial u^k}\frac{\partial u^a}{\partial v^q}\,. \tag{3.45}$$

□

3.3 Dubrovin-Novikov theorem

Let us summarize, what we have investigated so far. We have defined Dubrovin-Novikov bracket in the way that it satisfies the Leibniz rule. It remains to investigate the skew-symmetry and Jacobi identity. For that, we have defined g^{ij} and Γ^i_{jk}, which transform like tensors and Christoffel symbols, respectively. It remains to show that g^{ij} is indeed symmetric and Γ^i_{jk} is the Levi-Civita connection, i.e. the corresponding covariant derivative ∇_j has zero torsion and is compatible with the metric.

We are now ready to prove the key theorem of this chapter which shows that these properties are indeed equivalent. Moreover, the corresponding covariant derivative has zero curvature, which indicates that the phase space M is in fact Euclidean space.

Theorem-Dubrovin-Novikov-1D

Theorem 29 (Dubrovin, Novikov;[5]). *Let g^{ij} be non-degenerate. Then the Dubrovin-Novikov bracket* (3.28) *is a Poisson bracket with skew-symmetry, Leibniz rule and Jacobi identity if and only if these three conditions are satisfied:*

1. $g^{ij} = g^{ji}$*, that is, g^{ij} is a metric on the phase space M.*
2. $\Gamma^i_{jk} = \Gamma^i_{kj}$*, that is, Γ^i_{jk} are Christoffel symbols corresponding to the Levi-Civita covariant derivative compatible with g^{ij}.*
3. *The covariant derivative corresponding to Γ^i_{jk} has zero curvature.*

Corollary: existence of flat coordinates

Corollary 30. *There exist local coordinates $w^i = w^i(u^1, \dots, u^N)$, $i = 1, \dots, N$ such that the metric is flat:*

$$g^{ij}(w) = \eta^{ij} = \text{const}\,, \quad \Gamma^i_{jk}(w) = 0\,. \tag{3.46}$$

In these flat coordinates is the Dubrovin-Novikov bracket reduced to the constant form

$$\Big\{w^i(x), w^j(y)\Big\} = \eta^{ij}\partial_x\delta(x-y)\,. \tag{3.47}$$

The only local invariant of the Poisson bracket is the signature of the metric $g^{ij}(u)$.

Due to the relatively large length of the proof we divide it into four steps.

- **Step 1:** We show that the skew-symmetry of the bracket implies the symmetry of the metric g^{ij} and compatibility with the Christoffel symbols Γ^i_{jk}.
- **Step 2:** We formulate the conditions under which the Jacobi identity for the bracket holds.
- **Step 3:** We show that two of these conditions implies the zero torsion and zero curvature of the covariant derivative associated to Γ^i_{jk}. That finishes the implication in one direction.
- **Step 4:** We prove the converse implication via the existence of the flat coordinates.

lemma:skew-symmetry

Lemma 31 (Step 1: skew-symmetry). *If the skew-symmetry of the Dubrovin-Novikov bracket holds, then*

eq:skew-1

$$g^{ij} = g^{ji}\,, \tag{3.48}$$

eq:skew-2

$$g^{ij}_{,m} = b^{ij}_m + b^{ji}_m\,. \tag{3.49}$$

Proof. The skew-symmetry reads as

$$\Big\{u^i(x), u^j(y)\Big\} + \Big\{u^j(y), u^i(x)\Big\} = 0\,. \tag{3.50}$$

According to the definition, the exact form of the second bracket is

$$\Big\{u^j(y), u^i(x)\Big\} = g^{ji}[u(y)]\partial_x\delta(y-x) + b^{ji}_k[u(y)]\frac{\partial u^k}{\partial y}\delta(y-x)\,. \tag{3.51}$$

Let us apply Dirac identites (Lemma 8 on page 5) on $g^{ji}[u(y)]\partial_x\delta(y-x)$. We get

$$\begin{aligned}&\left\{u^j(y),u^i(x)\right\}=\\ =&-g^{ji}[u(x)]\partial_x\delta(x-y)-\frac{\partial g^{ji}}{\partial u^k}[u(x)]\frac{\partial u^k}{\partial x}\delta(x-y)+b^{ji}_k[u(x)]\frac{\partial u^k}{\partial x}\delta(x-y)\,.\end{aligned}\tag{3.52}$$

The skew-symmetry is then

$$\begin{aligned}0&=\left\{u^i(x),u^j(y)\right\}+\left\{u^j(y),u^i(x)\right\}=\\ &=\left[g^{ij}[u(x)]-g^{ji}[u(x)]\right]\partial_x\delta(x-y)+\left[b^{ij}_k-\frac{\partial g^{ji}}{\partial u^k}+b^{ji}_k\right]\frac{\partial u^k}{\partial x}\delta(x-y)\,.\end{aligned}\tag{3.53}$$

The right-hand side is equal to zero if and only if

eq:Th.Novikov-symetrie

$$g^{ij}(u)=g^{ji}(u)\,,\tag{3.54}$$

eq:Th.Novikov-konexe

$$\frac{\partial g^{ij}}{\partial u^k}(u)=b^{ij}_k(u)+b^{ji}_k(u)\,.\tag{3.55}$$

□

lemma:Jacobi

Lemma 32 (Step 2: Jacobi identity)**.** *The Jacobi identity for the bracket is equivalent to*

eq:Jacobi-1

$$b^{ij}_m g^{mk}=b^{ik}_m g^{mj}\,,\tag{3.56}$$

eq:Jacobi-2

$$b^{ij}_m b^{mk}_s-b^{ik}_m b^{mj}_s=g^{mi}\left(b^{jk}_{s,m}-b^{jk}_{m,s}\right)\,,\tag{3.57}$$

eq:Jacobi-3

$$\sum_{(i,j,k)}\left[(b^{ij}_{t,p}-b^{ij}_{p,t})b^{tk}_s+(b^{ij}_{t,s}-b^{ij}_{s,t})b^{tk}_p\right]=0\,,\tag{3.58}$$

where the summation $\sum_{(i,j,k)}$ means summation over cyclic permutations of indices i,j,k.

Proof. We denote

$$\mathcal{J}^{ijk}(x,y,z)=J^{ijk}(x,y,z)+J^{jki}(y,z,x)+J^{kij}(z,x,y)=0\tag{3.59}$$

where

$$J^{ijk}(x,y,z)=\left\{\left\{u^i(x),u^j(y)\right\},u^k(z)\right\}\,.\tag{3.60}$$

From now on we shall use the abbreviated notation

$$g^{ij}_{,m}=\frac{\partial g^{ij}}{\partial u^m}\,,\quad b^{ij}_{k,m}=\frac{\partial b^{ij}_k}{\partial u^m}\,,\quad u^k_x=\frac{\partial u^k}{\partial x}\,.\tag{3.61}$$

One of the Jacobiators reads as

$$J^{ijk}(x,y,z)=\int \mathrm{d}\xi\frac{\delta\left\{u^i(x),u^j(y)\right\}}{\delta u^m(\xi)}\left[g^{mn}(\xi)\partial_\xi+b^{mn}_p\partial_\xi u^p\cdot\right]\frac{\delta u^k(z)}{\delta u^n(\xi)}\,.$$

The functional derivatives are

$$\frac{\delta u^k(z)}{\delta u^n(\xi)}=\delta^k_n\delta(z-\xi)\tag{3.62}$$

and

$$\frac{\delta\left\{u^i(x),u^j(y)\right\}}{\delta u^m(\xi)}=\frac{\delta}{\delta u^m(\xi)}\left[g^{ij}(x)\partial_x\delta(x-y)+b^{ij}_k(x)\partial_x u^k\delta(x-y)\right]=$$
$$=\left[g^{ij}_{,m}(x)\partial_x\delta(x-y)+b^{ij}_{k,m}(x)u^k_x\delta(x-y)\right]\delta(x-\xi)-b^{ij}_m(x)\delta(x-y)\partial_\xi\delta(x-\xi)\ . \tag{3.63}$$

We have

$$J^{ijk}(x,y,z)=J^{ijk}_1(x,y,z)+J^{ijk}_2(x,y,z) \tag{3.64}$$

where

$$J^{ijk}_1(x,y,z)=\int \mathrm{d}\xi\left[g^{ij}_{,m}(x)\partial_x\delta(x-y)\delta(x-\xi)+b^{ij}_{k,m}(x)u^k_x\delta(x-y)\delta(x-\xi)\right]$$
$$\left[g^{mn}(\xi)\partial_\xi+b^{mn}_p\partial_\xi u^p\cdot\right]\delta^k_n\delta(z-\xi)$$

and

$$J^{ijk}_2(x,y,z)=-\int \mathrm{d}\xi b^{ij}_m(x)\delta(x-y)\partial_\xi\delta(x-\xi)\left[g^{mn}(\xi)\partial_\xi+b^{mn}_p\partial_\xi u^p\cdot\right]\delta^k_n\delta(z-\xi)\ .$$

In J_1 we use integration with $\delta(x-\xi)$ and get rid of the integral:

$$\begin{aligned}J^{ijk}_1(x,y,z)&=\left[g^{ij}_{,m}(x)\partial_x\delta(x-y)+b^{ij}_{k,m}(x)u^k_x\delta(x-y)\right]\left[g^{mk}(\xi)\partial_x+b^{mk}_p\partial_x u^p\cdot\right]\delta(z-x)\\&=\left[g^{ij}_{,m}g^{mk}\partial_x\delta(x-y)+b^{ij}_{k,m}g^{mk}u^k_x\delta(x-y)\right]\partial_x\delta(z-x)+\\&+\left[g^{ij}_{,m}b^{mk}_p u^p_x\partial_x\delta(x-y)+b^{ij}_{k,m}b^{mk}_p u^p_x u^k_x\delta(x-y)\right]\delta(z-x)\ .\end{aligned}$$

In J_2 we first apply integration by parts to get rid of $\partial_\xi\delta(x-\xi)$ and then integrate:

$$\begin{aligned}J^{ijk}_2(x,y,z)=\int \mathrm{d}\xi b^{ij}_m(x)\delta(x-y)\delta(x-\xi)\partial_\xi&\left[g^{mk}(\xi)\partial_\xi\delta(z-\xi)+b^{mk}_p\partial_\xi u^p\delta(z-\xi)\right]\\=b^{ij}_m(x)\delta(x-y)\partial_x&\left[g^{mn}(x)\partial_x\delta(z-x)+b^{mn}_p\partial_x u^p\delta(z-x)\right]\\=b^{ij}_m(x)\delta(x-y)&\Big[g^{mk}_{,s}u^s_x\partial_x\delta(z-x)+g^{mk}\partial_x\partial_x\delta(z-x)\\+b^{mk}_{p,s}(x)u^s_x u^p_x\delta(z-x)+b^{mn}_p(x)u^p_{xx}&\delta(z-x)+b^{mk}_p(x)u^p_x\partial_x\delta(z-x)\Big]\end{aligned}$$

$$\begin{aligned}J^{ijk}_2(x,y,z)=&b^{ij}_m g^{mk}_{,s}u^s_x\delta(x-y)\partial_x\delta(z-x)+b^{ij}_m g^{mk}\delta(x-y)\partial_x\partial_x\delta(z-x)\\&+b^{ij}_m b^{mk}_{p,s}u^s_x u^p_x\delta(x-y)\delta(z-x)+b^{ij}_m b^{mn}_p u^p_{xx}\delta(x-y)\delta(z-x)\\&+b^{ij}_m b^{mk}_p u^p_x\delta(x-y)\partial_x\delta(z-x)\ .\end{aligned}$$

In total we end with terms

$$J^{ijk}(x,y,z)=E^{ijk}(x)\delta(x-y)\delta(z-x) \tag{3.65}$$
$$+F^{ijk}(x)\delta(x-y)\partial_x\delta(z-x)\ , \tag{3.66}$$
$$+G^{ijk}(x)\partial_x\delta(x-y)\delta(z-x)\ , \tag{3.67}$$
$$+H^{ijk}(x)\partial_x\delta(x-y)\partial_x\delta(z-x)\ , \tag{3.68}$$
$$+I^{ijk}(x)\delta(x-y)\partial_x\partial_x\delta(z-x)\ , \tag{3.69}$$

where

$$
\begin{aligned}
E^{ijk} =& (b^{ij}_m b^{mk}_{p,s} + b^{ij}_{s,m} b^{mk}) u^p_x u^s_x + b^{ij}_m b^{mk}_p u^p_{xx} \,, && (3.70)\\
F^{ijk} =& (b^{ij}_{s,m} g^{mk} + g^{mk}_{,s} b^{ij}_m + b^{ij}_m b^{mk}_s) u^s_x \,, && (3.71)\\
G^{ijk} =& g^{ij}_{,m} b^{mk}_s u^s_x \,, && (3.72)\\
H^{ijk} =& g^{ij}_{,m} g^{mk} \,, && (3.73)\\
I^{ijk} =& b^{ij}_m g^{mk} \,. && (3.74)
\end{aligned}
$$

Now the procedure is the following: firstly we have to add the cyclic permutations $J^{jki}(y,z,x)$ and $J^{kij}(z,x,y)$ to obtain $\mathcal{J}^{ijk}(x,y,z)$. Secondly, the resulting generalized function $\mathcal{J}^{ijk}(x,y,z)$ is equal to zero, i.e.

$$
\Big\langle \mathcal{J}^{ijk}(x,y,z), p_i(x) q_j(y) r_k(z) \Big\rangle = 0 \tag{3.75}
$$

or in the integral notation

$$
\iiint \mathrm{d}x\mathrm{d}y\mathrm{d}z J^{ijk}(x,y,z) p_i(x) q_j(y) r_k(z) = 0 \,. \tag{3.76}
$$

We want to reduce this integral into a one-dimensional integral

$$
\int \mathrm{d}x \sum_{a,b=0}^{2} A^{ijk}_{ab} p_i(x) q^{(a)}_j(x) r^{(b)}_k(x) \,, \tag{3.77}
$$

where $f^{(a)}$ means the a-th derivative. This integral is equal to zero if and only if

$$
A^{ijk}_{ab} = 0 \quad \forall a,b = 0,1,2 \,. \tag{3.78}
$$

From this condition we shall obtain the zero torsion and curvature. We illustrate the procedure of reduction to the one-dimensional integral on one particular expression. For instance, we have

$$
\begin{aligned}
\int \Big[& F^{ijk}(x)\delta(x-y)\partial_x\delta(z-x) + F^{jki}(y)\delta(y-z)\partial_y\delta(x-y) \\
& + F^{kij}(z)\delta(z-x)\partial_z\delta(y-z) \Big] p_i(x) q_j(y) r_k(z) \,\mathrm{d}x\mathrm{d}y\mathrm{d}z \,.
\end{aligned}
$$

We use the antisymmetry $\partial_x\delta(z-x) = -\partial_z\delta(z-x)$ and similarly in each term and then get rid of the integrals with $\mathrm{d}y$ and $\mathrm{d}z$, obtaining

$$
\int \mathrm{d}x F^{ijk}(x) p_i q_j r'_k + \int \mathrm{d}x F^{jki}(x) p'_i q_j r_k + \int \mathrm{d}x F^{kij}(x) p_i q'_j r_k \,.
$$

In the term containing p'_i we perform integration by parts. The result is

$$
\int \mathrm{d}x (F^{ijk} - F^{jki}) p_i q_j r'_k + \int \mathrm{d}x (F^{kij} - F^{jki}) p_i q'_j r_k - \int \mathrm{d}x \partial_x F^{jki} p_i q_j r_k \,.
$$

In this particular case, we end up with terms, witch contribute to A^{ijk}_{01}, A^{ijk}_{10} and A^{ijk}_{00}, respectively. Other terms in $\mathcal{J}^{ijk}(x,y,z)$ are handled in the similar way.

The list of the resulting functions is the following:

$$
\begin{aligned}
A^{ijk}_{00} =& E^{ijk} + E^{kij} + E^{jki} + \partial_x \partial_x I^{jki} - \partial_x F^{jki} - \partial_x G^{kij} \,, & (3.79)\\
A^{ijk}_{01} =& F^{ijk} - F^{jki} + G^{jki} - G^{kij} - \partial_x H^{jki} + 2\partial_x I^{jki} \,, & (3.80)\\
A^{ijk}_{10} =& F^{kij} - F^{jki} + G^{ijk} - G^{kij} - \partial_x H^{kij} + 2\partial_x I^{jki} \,, & (3.81)\\
A^{ijk}_{11} =& H^{ijk} - H^{jki} - H^{kij} + 2\partial_x I^{jki} \,, & (3.82)\\
A^{ijk}_{02} =& I^{ijk} + I^{jki} - H^{jki} \,, & (3.83)\\
A^{ijk}_{20} =& I^{jki} + I^{kij} - H^{kij} \,, & (3.84)\\
A^{ijk}_{12} =& A^{ijk}_{21} = 0 \,. & (3.85)
\end{aligned}
$$

Note that (3.84) is just a cyclic permutation of (3.83). If we also use the symmetry conditon (3.48) and the compatibility condition (3.49), then the equation (3.56) follows from (3.83). Equation (3.79) has the form

$$A^{ijk}_{00} = B^{ijk}_s u^s_{xx} + C^{ijk}_{sp} u^s_x u^p_x \,. \tag{3.86}$$

The expression on the right hand side is zero if and only if both B^{ijk}_s and $C^{ijk}_{sp} = 0$. From the function B^{ijk}_s beeing zero follows (3.57) and from C^{ijk}_{sp} follows (3.58). Finally, equations (3.80), (3.81) and (3.82) are satisfied automatically if (3.79) and (3.83) hold. □

It is useful to write once more all the equations from lemmas 31 on page 17 and 32 on page 18 together. Since they first appeared in [6], we refer to them as to the Grinberg conditions.

`Grinberg conditions`

Lemma 33 (Grinberg conditions; [6])**.** *If the Dubrovin-Novikov bracket satisfies skew-symmetry and Jacobi identity, then the following conditions hold:*

$$
\begin{aligned}
g^{ij} &= g^{ji} \,, & (3.87)\\
g^{ij}_{,m} &= b^{ij}_m + b^{ji}_m \,, & (3.88)\\
b^{ij}_m g^{mk} &= b^{ik}_m g^{mj} \,, & (3.89)\\
b^{ij}_m b^{mk}_s - b^{ik}_m b^{mj}_s &= g^{mi} \left(b^{jk}_{s,m} - b^{jk}_{m,s} \right) \,, & (3.90)\\
\sum_{(i,j,k)} \left[(b^{ij}_{t,p} - b^{ij}_{p,t}) b^{tk}_s + (b^{ij}_{t,s} - b^{ij}_{s,t}) b^{tk}_p \right] &= 0 \,. & (3.91)
\end{aligned}
$$

Up to now, we did not assume non-degeneracy of the bracket. Note that non-degeneracy allows to use $\Gamma^i_{jk} = -g_{js} b^{si}_k$, since in the degenerate case the inverse g_{js} does not exist. If we reformulate the Grinberg conditions in terms of Γ^i_{jk}, we get the compatiblity, zero torsion and curvature.

Lemma 34 (Step 3: Zero curvature and torsion)**.** *If the Grinberg conditions hold and the g^{ij} is non-degenerate, then g^{ij} is a metric, the covariant derivative ∇_i associated to Γ^i_{jk} is compatible with g^{ij} and it has zero torsion and curvature, i.e. ∇_i is a Levi-Civita covariant derivative with zero curvature.*

Proof. Equation (3.87) says that g^{ij} is a symmetric tensor. By assumption $\det g^{ij}$ is non-degenerate, so it defines a metric structure on M. Equation (3.88) is equivalent to

$$g^{ij}_{,k} + g^{is}\Gamma^j_{sk} + g^{sj}\Gamma^i_{sk} = 0 \,. \tag{3.92}$$

If ∇_k is the covariant derivative corresponding to Γ^i_{jk}, then $\nabla_k g^{ij} = 0$. So Γ^k_{ij} are affine connection coefficients, compatible with g^{ij}.

To obtain zero torsion, we use (3.89), which reads as

$$g^{mk} g^{sj} \Gamma^i_{ms} = g^{mk} g^{sj} \Gamma^i_{sm} \,. \tag{3.93}$$

Hence the Christoffel symbols are symmetric in the lower indices, which implies that the covariant derivative corresponding to them has zero torsion.

Zero curvature follows from (3.90), which reads as

$$g^{in} g^{ml} (\Gamma^j_{nm} \Gamma^k_{ls} - \Gamma^k_{nm} \Gamma^j_{ls}) = -g^{im} \frac{\partial}{\partial u^m} \left(g^{jl} \Gamma^k_{ls} \right) + g^{im} \frac{\partial}{\partial u^s} \left(g^{jl} \Gamma^k_{lm} \right) \,, \tag{3.94}$$

and reduces to

$$g^{im} g^{jn} \left(\Gamma^l_{nm} \Gamma^k_{ls} - \Gamma^l_{ns} \Gamma^k_{lm} + \Gamma^k_{nm,s} - \Gamma^k_{ns,m} \right) = 0 \,. \tag{3.95}$$

This means that the Riemann tensor ${R_{nm}}^k{}_s$ is equal to zero. □

The final step is to prove the converse statement and the corollary 30 on page 17, which turns out to be very simple due to the non-degeneracy of the bracket.

Lemma 35 (Step 4: The converse statement). *If g^{ij} is non-degenerate metric and Γ^i_{jk} correspond to the Levi-Civita covariant derivative ∇ with zero curvature, then the Dubrovin-Novikov bracket is a Poisson bracket, i.e. it satisfies skew-symmetry, Leibniz rule and Jacobi identity.*

Proof. If ∇ has zero torsion and curvature, there are coordinates $w^i = w^i(u^1, \ldots, u^N)$ for $i = 1, \ldots, N$ such that $g^{ij} =: \eta^{ij} = \text{const}$ a $b^{ij}_k = 0$. In these coordinates

$$\left\{ w^i(x), w^j(y) \right\} = \eta^{ij} \partial_x \delta(x - y) \,. \tag{3.96}$$

Jacobi identity, skew-symmetry and Leibniz rule are for that bracket satisfied trivially due to the antisymmetry of the delta distribution. □

Thus the proof of the Dubrovin-Novikov theorem 29 on page 17 is finished.

3.4 Some consequences of Dubrovin-Novikov theorem

It follows from the theorem 29 on page 17 that we can write the original functions f^i_j in (3.1) by the Laplace-Beltrami operator.

Proposition 36 (Hamiltonian system by the Laplace operator). *Given Hamiltonian system of hydrodynamic type*

$$\frac{\partial u^i}{\partial t} = f^i_k(u) \frac{\partial u^k}{\partial x} \,, \quad f^i_k(u) = g^{ij}(u) \frac{\partial^2 h}{\partial u^j \partial u^k} - g^{is}(u) \Gamma^j_{sk}(u) \frac{\partial h}{\partial u^j} \,, \quad \text{eq:hamiltonovsky-system} \tag{3.97}$$

one can write

$$f^i_k(u) = \nabla^i \nabla_k h(u) \,, \tag{3.98}$$

where ∇_j is Levi-Civita covariant derivative compatible with g_{ij} and $\nabla^i = g^{is} \nabla_s$.

Proof. Since h is ordinary function, application of ∇_k is just the partial derivative w.r.t. u^k:

$$\nabla_k h(u) = \frac{\partial h}{\partial u^k}\,. \tag{3.99}$$

Now this term $\frac{\partial h}{\partial u^k}$ is a covector, so ∇_i is given by the partial derivative and the Γ^k_{ij} part by

$$\nabla^i \nabla_k h(u) = g^{is} \nabla_s \frac{\partial h}{\partial u^k} = g^{is} \left(\frac{\partial^2 h}{\partial u^s \partial u^k} - \Gamma^j_{sk} \frac{\partial h}{\partial u^j} \right) = f^i_k(u)\,. \tag{3.100}$$

□

Another direct application of the theorem 29 on page 17 is the sufficient condition for the determinating, whether is a given system Hamiltonian or not.

Theorem 37 (The sufficient condition for a system to be Hamiltonian)**.** *A system of hydrodynamic type*

$$u^i_t = f^i_j(u) u^j_x\,, \tag{3.101}$$

is Hamiltonian if and only if there is a non-degenerate metric $g^{ij}(u)$ and corresponding covariant derivative ∇_j, such that

V tomto vztahu nesedí indexy.

$$g^{ij} f^k_j = g_{jk} f^k_i\,, \tag{3.102}$$
eq:gf=gf

$$\nabla_i f^k_j = \nabla_j f^k_i\,. \tag{3.103}$$
eq:nablaf

In particular, (3.103) *is equivalent to the zero torsion of ∇_i.*

Proof. If the system is Hamiltonian, then $f^k_j = \nabla^k \nabla_j h$ and ∇_j has zero torsion, that means

$$\nabla_i f^k_j = g^{km} \nabla_i \nabla_m \nabla_j h = g^{km} \nabla_j \nabla_m \nabla_i h = \nabla_j f^k_i\,, \tag{3.104}$$

and

$$g^{ij} f^k_j = g^{ij} g^{mk} \nabla_m \nabla_j h = \nabla^i \nabla^k h = \nabla^k \nabla_i h \tag{3.105}$$

□

4. Systems of hydrodynamic type: multi-dimensional case

When one moves to more dimensions, the theory becomes much more complicated, opposite to the one dimension, where the non-degeneracy of the bracket together with skew-symmetry and Jacobi identity leads not only to the Levi-Civita covariant derivative with zero curvature, but also to the existence of the flat coordinates. The questions are

1. To find conditions under which the multi-dimensional bracket is Poisson, i.e. satisfies skew-symmetry, Leibniz rule and Jacobi identity.

2. To find such coordinates, where the nondegenerate Poisson bracket is reduced to the constant form.

3. To classify the degenerate brackets.

The first problem was solved in 2006 by O.I.Mochov in [8], where also the complete classification of nonsingular nondegenerate brackets was found. A key role in the second problem is played by the theory of compatible metrics. The classification of two-dimensional systems was given by Ferapontov, Lorenzoni and Savoldi. The main results are summarised in the Doctoral Thesis of the latter author [9].

4.1 Multi-dimensional brackets. Mokhov conditions

In the case with multi-dimensional Dubrovin-Novikov bracket

$$\{I_1, I_2\} = \sum_{\alpha=1}^{d} \int \mathrm{d}x \left[\frac{\delta I_1}{\delta u^p(x)} \left(g^{pq\alpha}(u) \frac{\mathrm{d}}{\mathrm{d}x^\alpha} + b_s^{pq\alpha}(u) \frac{\partial u^s}{\partial x^\alpha} \right) \frac{\delta I_2}{\delta u^q(x)} \right], \tag{4.1}$$

we have a set of metrics $g^{ij\alpha}$ and connections $b_k^{ij\alpha}$, where $\alpha = 1, \cdots, d$. We can again introduce Christoffel symbols defined by

$$b_k^{ij\alpha} := -g^{is\alpha} \Gamma_{sk}^{j\alpha}, \tag{4.2}$$

(there is no summation over α) and the corresponding set of covariant derivatives ∇_j^α. The first observation is that these coefficients represent coordinates of a vector under an transformation of the space coordinates x^α.

Skutečně se mění jenom závislost u na x^α, nebo i něco jiného? Může to platit i při nějaké obecné transformaci?

Proposition 38 (Transformation of the space coordinates)**.** *For any changes of the spatial variables $\tilde{x}^\alpha := c_\beta^\alpha x^\beta$, where $\alpha = 1, \cdots, d$ and $\det(c_\beta^\alpha) = 1$, the metrics $g^{ij\alpha}$ and $b_k^{ij\alpha}$ transform as components of a vector.*

Proof. The only variable that changes is $\frac{\partial u^i}{\partial x^\alpha}$

$$\frac{\partial u^i}{\partial x^\beta} = \frac{\partial u^i}{\partial \tilde{x}^\alpha}\frac{\partial \tilde{x}^\alpha}{\partial x^\beta} = \frac{\partial u^i}{\partial \tilde{x}^\alpha} c^\alpha_\beta \, . \tag{4.3}$$

Hence

$$g^{ij\alpha}[u(\tilde{x})] = c^\alpha_\beta g^{ij\beta}[u(x)] \, , \quad b^{ij\alpha}_k[u(\tilde{x})] = c^\alpha_\beta b^{ij\beta}_k[u(x)] \, . \tag{4.4}$$

□

The formulation of the generalized form of the Dubrovin-Novikov theorem (Theorem 29 on page 17) is given (without proof) by Mokhov in [8].

Theorem 39 (Mokhov conditions)**.** *A Dubrovin-Novikov bracket* (3.16) *is a Poisson bracket if and only if the following relations for the coefficients are fulfilled:*

$$g^{ij\alpha} = g^{ji\alpha} \, , \tag*{eq:Mochov-A (4.5)}$$

$$g^{ij\alpha}_{,k} = b^{ij\alpha}_k + b^{ji\alpha}_k \, , \tag*{eq:Mochov-B (4.6)}$$

$$\sum_{(\alpha,\beta)} (g^{mi\alpha} b^{jk\beta}_m - g^{mj\beta} b^{ik\alpha}_m) = 0 \, , \tag*{eq:Mochov-C (4.7)}$$

$$\sum_{(i,j,k)} (g^{mi\alpha} b^{jk\beta}_m - g^{mj\beta} b^{ik\alpha}_m) = 0 \, , \tag*{eq:Mochov-D (4.8)}$$

$$\sum_{(\alpha,\beta)} \left[g^{si\alpha}(b^{jk\beta}_{s,q} - b^{jk\beta}_{q,s}) + b^{ij\alpha}_s b^{sk\beta}_q - b^{ik\alpha}_s b^{kj\beta}_q \right] = 0 \, , \tag*{eq:Mochov-E (4.9)}$$

$$g^{si\beta} b^{jk\alpha}_{q,s} - b^{ij\beta}_s b^{sk\alpha}_q - b^{ik\beta}_s b^{js\alpha}_q = g^{sj\alpha} b^{ik\beta}_{q,s} - b^{ji\alpha}_s b^{sk\beta}_q - b^{is\beta}_q b^{jk\alpha}_s \, , \tag*{eq:Mochov-F (4.10)}$$

$$\begin{aligned} &\frac{\partial}{\partial u^r} \left[g^{si\alpha}(b^{jk\beta}_{s,q} - b^{jk\beta}_{q,s}) + b^{ij\alpha}_s b^{sk\beta}_q - b^{ik\alpha}_s b^{sj\beta}_q \right] \\ &+ \frac{\partial}{\partial u^q} \left[g^{si\beta}(b^{jk\alpha}_{s,k} - b^{jk\alpha}_{r,s}) + b^{ij\beta}_s b^{sk\alpha}_{s,r} + b^{ij\beta}_s b^{sk\alpha}_r - b^{ik\beta}_s b^{kj\alpha}_s \right] \\ &+ \sum_{(i,j,k)} \left[b^{si\beta}_q (b^{jk\alpha}_{r,s} - b^{jk\alpha}_{s,r}) + b^{si\alpha}_r (b^{jk\beta}_{q,s} - b^{jk\beta}_{s,q}) \right] = 0 \, . \end{aligned} \tag*{eq:Mochov-G (4.11)}$$

The summations $\sum_{(\alpha,\beta)}$ and $\sum_{(i,j,k)}$ mean summations over all cyclic permutations of the indicated indices. Note that in (4.7)-(4.11) the indices α and β do not have to be distinct indices (the cyclic permutation $\sum_{(\alpha,\beta)}$ can be ommited in that case). Thus, in particular, we get an important lemma.

Lemma 40. *Every summand on the right-side of the multi-dimensional bracket* (4.1) *is one-dimensional Dubrovin-Novikov bracket* (3.27)*, i.e. each multidimensional bracket of the form* (4.1) *is always the sum of one-dimensional brackets* (3.27) *with respect to each of the independent variables* x^α.

Remark. Let us take once again a closer look to the case of $d = 1$ and compare it to the Grinberg conditions (33 on page 21). The Mokhov conditions are following:

eq:Mochov1D-A
$$g^{ij} = g^{ji} \qquad \textbf{(symmetry of metric)}\,, \tag{4.12}$$

eq:Mochov1D-B
$$g^{ij}_{,k} = b^{ij}_k + b^{ji}_k \qquad \textbf{(compatibility condition)}\,, \tag{4.13}$$

eq:Mochov1D-C
$$g^{mi} b^{jk}_m - g^{mj} b^{ik}_m = 0 \qquad \textbf{(zero torsion)}\,, \tag{4.14}$$

eq:Mochov1D-D
$$\sum_{(i,j,k)} g^{mi} b^{jk}_m - g^{mj} b^{ik}_m = 0\,, \tag{4.15}$$

eq:Mochov1D-E
$$g^{si}(b^{jk}_{s,q} - b^{jk}_{q,s}) + b^{ij}_s b^{sk}_q - b^{ik}_s b^{kj}_q = 0 \quad \textbf{(zero curvature)}\,, \tag{4.16}$$

eq:Mochov1D-F
$$g^{si} b^{jk}_{q,s} - b^{ij}_s b^{sk}_q - b^{ik}_s b^{js}_q = g^{sj} b^{ik}_{q,s} - b^{ji}_s b^{sk}_q - b^{is}_q b^{jk}_s\,, \tag{4.17}$$

eq:Mochov1D-G
$$\begin{aligned} &\frac{\partial}{\partial u^r}\left[g^{si}(b^{jk}_{s,q} - b^{jk}_{q,s}) + b^{ij}_s b^{sk}_q - b^{ik}_s b^{sj}_q\right] \\ +&\frac{\partial}{\partial u^q}\left[g^{si}(b^{jk}_{s,k} - b^{jk}_{r,s}) + b^{ij\beta}_s b^{sk}_{s,r} + b^{ij}_s b^{sk}_r - b^{ik}_s b^{kj}_s\right] \\ +&\sum_{(i,j,k)}\left[b^{si}_q(b^{jk}_{r,s} - b^{jk}_{s,r}) + b^{si}_r(b^{jk}_{q,s} - b^{jk}_{s,q})\right] = 0\,. \end{aligned} \tag{4.18}$$

It is evident that (4.15) is just (4.14) with a cyclic permutation hence it holds trivially. It is not so obvious, but also true, that (4.17) follows from the compatibility condition, zero torsion and zero curvature. The last relation (4.18) is the one remaining Grinberg condition (3.91). Only if we assume non-degeneracy, it follows from the zero torsion and curvature, otherwise it must be involved as a separate assumption.

4.2 Tensor obstructions

Definition 41 (The obstruction tensors)**.** *We define the obstruction tensors*

$$T^{i\alpha\beta}_{jk}(u) := \Gamma^{i\beta}_{jk}(u) - \Gamma^{i\alpha}_{jk}(u)\,, \tag{4.19}$$

for each pair of distinct indices α and β.

We shall also use their fully contravariant form

$$T^{ijk\alpha\beta}(u) := g^{ir\alpha}(u) g^{ks\beta}(u) T^{j\alpha\beta}_{rs}(u)\,. \tag{4.20}$$

Note that $T^{i\alpha\beta}_{jk}(u)$ are transformed as tensors only in the upper α, β indices (since in the remaining they transform as Christoffel symbols).

lemma:T=0

Lemma 42. *Non-degenerate Dubrovin-Novikov brackets can be reduced to a constant bracket by a local change of coordinates if and only if all the obstruction tensors $T^{ij\alpha}_{jk}(u)$ are identically equal to zero.*

Proof. For any constant bracket all the connections $\Gamma^{i\alpha}_{jk}(u)$ are equal to zero identically, therefore all obstruction tensors are zero. Conversely, if all the $T^{ij\alpha}_{jk}(u)$ are identically zero, then all the connections $\Gamma^{i\alpha}_{jk}(u)$ are equal to each other. Consequently, they are all zero in any flat coordinates of the metric $g^{ij1}(u)$. All the remaining metrics $g^{ij\beta}(u)$ are necessarily constant in these coordinates by virtue of compatiblity of the metrics with the corresponding connections. ☐

Theorem 43 (Mokhov)**.** *Flat non-degenerate metrics $g^{ij\alpha}$ define a multidimensional Poisson bracket if and only if the following relations are fulfilled:*

$$T^{ijk\alpha\beta}(u) = T^{kji\alpha\beta}(u)\,, \tag*{eq:Mochov-1 (4.21)}$$

$$\sum_{(i,j,k)} T^{ijk\alpha\beta}(u) = 0\,, \tag*{eq:Mochov-2 (4.22)}$$

$$T^{ijs\alpha\beta}(u)T^{k\alpha\beta}_{st}(u) = T^{iks\alpha\beta}(u)T^{j\alpha\beta}_{st}(u)\,, \tag*{eq:Mochov-3 (4.23)}$$

$$\nabla^{\alpha}_{r} T^{ijk\alpha\beta}(u) = 0\,, \tag*{eq:Mochov-4 (4.24)}$$

where ∇^{α}_{k} is the Levi-Civita covariant derivative generated by the metric $g^{ij\alpha}(u)$ and $\Gamma^{i\alpha}_{jk}(u)$ are the corresponding Christoffel symbols.

Remark. The relations (4.21) and (4.23) were found by Dubrovin and Novikov in [5]. In [8] it was proved that this set of tensor relations is not complete and the remaining two were obtained.

Proof. The first equation (4.21) reads as

$$g^{ir\alpha}g^{ks\beta}(\Gamma^{j\beta}_{rs} - \Gamma^{j\alpha}_{rs}) = g^{kr\alpha}g^{is\beta}(\Gamma^{j\beta}_{rs} - \Gamma^{j\alpha}_{rs})\,, \tag{4.25}$$

which is equivalent to

$$\Gamma^{j\beta}_{rs} - \Gamma^{j\alpha}_{rs} = 0\,, \tag{4.26}$$

thus follows from (4.7).

The equation (4.22) is equivalent to (4.8), (4.23) is equivalent to (4.9), (4.24) is equivalent to (4.10). The last condition (4.11) is a direct consequence of the relations (4.5)-(4.10) in the non-degenerate case.

Je potřeba to rozepisovat podrobněji?

□

Example 44 (Classification of one-component brackets)**.** *It is not complicated to obtain a complete classification of all one-component brackets*

$$\frac{\partial u}{\partial t} = \left(g^{\alpha}(u)\frac{\partial^2 h}{\partial u^2} + b^{\alpha}(u)\frac{\partial h}{\partial u}\right)\frac{\partial u}{\partial x^{\alpha}}\,, \qquad \alpha = 1, \cdots, d\,. \tag{4.27}$$

The multi-dimensional bracket is

$$\{u(x), u(y)\} = g^{\alpha}(u)\partial_{\alpha}\delta(x-y) + b^{\alpha}(u)\delta(x-y)\,. \tag{4.28}$$

and the decomposition is

$$b^{\alpha} = -g^{\alpha}\Gamma^{\alpha}\,. \tag{4.29}$$

The Mokhov conditions (4.5), (4.7), (4.9) *and* (4.11) *are automatically fulfilled. The relations* (4.6) *and* (4.8) *are*

$$\frac{\partial g^{\alpha}}{\partial u} = 2b^{\alpha}(u)\,, \tag*{eq:1comp-1 (4.30)}$$

$$g^{\alpha}(u)b^{\beta}(u) = g^{\beta}(u)g^{\alpha}(u)\,. \tag*{eq:1comp-2 (4.31)}$$

The last condition (4.10) *follows from them. If we apply* (4.30) *into* (4.31)*, we get*

$$\frac{1}{g^\alpha}\frac{\partial g^\alpha}{\partial u} = \frac{1}{g^\beta}\frac{\partial g^\beta}{\partial u}\,, \tag{4.32}$$

hence

$$g^\alpha(u) = 2c^\alpha g(u)\,, \quad b^\alpha(u) = c^\alpha \quad \forall \alpha = 1,\cdots d\,, \tag{4.33}$$

where c^α are arbitrary constants. Thus, for $g(u) \neq 0$ all the obstruction tensors are identically zero:

$$T^{\alpha\beta} = \Gamma^\beta - \Gamma^\alpha = -\frac{b^\alpha}{g^\alpha} + \frac{b^\beta}{g^\beta} = -\frac{1}{2g} + \frac{1}{2g} = 0\,. \tag{4.34}$$

By the virtue of lemma 42 on page 26 by local changing of coordinates we can always find a transform in witch the bracket has a constant form. The

$$\frac{\partial w}{\partial t} = \eta^\alpha \frac{\partial w}{\partial x^\alpha}\,. \tag{4.35}$$

The degenerate case is also trivial. If for some α is $g^\alpha(u) = 0$, then from (4.6) *follows $b^\alpha(u) = 0$.*

4.3 Compatible brackets

In this section we introduce the theory of compatible metrics constructed by Mochov in [8] and show that the study of non-degenerate multi-dimensional brackets is connected to the study of compatible one-dimensional brackets.

Definition 45 (Compatible metrics)**.** *Let $g^{ij1}(u)$ and $g^{ij2}(u)$ be non-degenerate psudo-Riemannian metrics with Levi-Civita connections $\Gamma^{ij1}_k(u)$, $\Gamma^{ij2}_k(u)$. We say that g^{ij1} and g^{ij2} are compatible if for any linear combination of these metrics*

$$g^{ij}(u) = \lambda_1 g^{ij1}(u) + \lambda_2 g^{ij2}(u)\,, \tag{4.36}$$

where $\lambda_{1,2}$ are arbitrary constant such that $\det g^{ij} \neq 0$, the coefficients of the corresponding Levi-Civita connections and the components of the corresponding Riemann tensors are related by the same linear formula:

eq:compatible-Gamma

$$\Gamma^{ij}_k = \lambda_1 \Gamma^{ij1}_k + \lambda_2 \Gamma^{ij2}_k\,, \tag{4.37}$$

eq:compatible-R

$$R_{ab}{}^{k}{}_{c} = \lambda_1 R_{ab}{}^{k}{}_{c}{}^{1} + \lambda_2 R_{ab}{}^{k}{}_{c}{}^{2}\,. \tag{4.38}$$

If for any linear combination of metrics only (4.37) *is fulfilled, then the metrics are called almost compatible.*

Definition 46 (Compatible Poisson brackets)**.** *We say that two Poisson brackets are compatible if each their linear combination is a Poisson bracket.*

Definition 47 (L-affinor)**.** *Let $g^{ij1}(u)$ and $g^{ij2}(u)$ be two metrics. We define the L-affinor L^i_j by*

$$L^i_j(u) = g^{ik2}(u) g_{kj1}(u)\,, \tag{4.39}$$

Theorem 48 (Vanishing of the Nijenhuis tensor). *The metrics $g^{ij1}(u)$ and $g^{ij2}(u)$ are almost compatible if and only if the Nijenhuis tensor of the L-affinor is zero, i.e.*

$$N^i_{jk} := L^s_j L^i_{k,s} - L^s_k L^i_{j,s} - L^i_s L^s_{k,j} - L^i_s L^s_{j,k} = 0\,. \tag{4.40}$$

Note that that there are examples of metrics that are only almost compatible, but not compatible, and for which the Nijenhuis tensor is zero [10].

Theorem 49 (Mutual compatibility of the metrics). *All metrics $g^{ij\alpha}(u)$, $1 \leq \alpha \leq N$, defining a multi-dimensional Poisson bracket* (4.1) *are mutually compatible. All one-dimensional Dubrovin-Novikov brackets forming a multi-dimensional bracket are mutually compatible.*

Podívat se na důkaz tohoto.

Proof. The equation (4.21) is equivalent to (4.7) and also to the vanishing of the obstruction tensor, witch means

$$\Gamma^{i\alpha}_{jk} = \Gamma^{i\beta}_{jk}\,. \tag{4.41}$$

Hence their linear combination is also Levi-Civita connection for the same linear combination of metrics. The equation (4.23) is equivalent to (4.9) and thus to

$$R_{ab}{}^{k\,\alpha}_{\;l} = R_{ab}{}^{k\,\beta}_{\;l}\,. \tag{4.42}$$

□

Theorem 50 (Mokhov classification theorem). *If for a nondegenerate multidimensional Poisson bracket one of the metrics $g^{ij\alpha}(u)$ forms nonsingular pairs with all the remaining metrics, i.e.*

$$, \tag{4.43}$$

then this Poisson bracket can be reduced to a constant form by a local change of coordinates.

Proof. [podívat se na důkaz] □ inline

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