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**The connection between continuum
mechanics and Riemannian geometry**

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Dedication.

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1. Mathematical framework

In this chapter we shall revise some basic definitions and propositions of the functional calculus, which we will need for the proper formulation of the hydrodynamic geometry. Most of the existing literature on this topic often lacks clarity due to the omittance of certain details. This can include the domains on which functionals are defined or regularity conditions.

Over the last century, modern mathematics has worked to develop the theory of generalized functions - distributions, including Dirac's delta distributions and operations with them, such as in [CITATION]. The same applies to functional variations and the conditions on which they exist. Among physists, however, it is convential to operate with delta distributions as they would with "ordinary" functions, using dualities as integrals and functional variations as ordinary derivatives.

After introducing the concept of Gateaux and Frechet derivatives, which are suitable for arbitrary normed space, we shall take a closer look on the space of smooth functions and functionals on it.

1.1 Functional derivatives

Definition 1 (Gateaux and Frechet derivative). *Let X be a normed space, $A : X \rightarrow \mathbb{R}$ is a functional and $x \in D(A)$.*

1. Gateaux derivative $\delta A(x, h)$ in the direction h is defined as

$$\delta A(x, h) = \frac{d}{dt} A(x + th)|_{t=0} = \lim_{t \rightarrow 0} \frac{A(x + th) - A(x)}{t} . \quad (1.1)$$

2. Fréchet derivative $DA(x)$ is defined as the unique linear bounded functional satisfying the relation

$$\lim_{h \rightarrow 0} \frac{A(x + h) - A(x) - DA(h)}{\|h\|} = 0 . \quad (1.2)$$

We are intersted in such functionals, where X is the subspace of functions $f \in C^k(\Omega)$, Ω is a subset of \mathbb{R}^n , satisfying $f, f', \dots, f^{(k-1)} = 0$ on $\partial\Omega$, equipped with the norm

$$\|u\|_X = \sum_{j=1}^n \sup_{x \in \Omega} |u^{(j)}(x)| . \quad (1.3)$$

There are two main types of functionals on that space.

1.1.1 Integral functionals

The first type is the integral functional

$$A[u] = \int_{\Omega} a(x, u(x), u'(x), \dots, u^{(k)}(x)) \, d^n x , \quad (1.4)$$

where $a : \mathbb{R} \times \mathbb{R}^k \rightarrow \mathbb{R}$ is an analytic function. For simplicity we often write this using a physical notation

$$A[u] = \int a(u) \, d^n x , \quad (1.5)$$

meaning that $a(u)$ is in fact function $a(x, u(x), u'(x), \dots, u^{(k)}(x))$ and also the domain Ω is specified by the context. We call $a(u)$ the density of functional $A[u]$.

Gateaux derivative of A is computed as follows:

$$\delta A[u, h] = \int_{\Omega} \sum_{i=1}^k \frac{\partial a}{\partial u^{(i)}}(x) h^{(i)}(x) \, d^n x . \quad (1.6)$$

The functional $\delta A[u, h]$ is linear and bounded in h , so the Fréchet derivative of A exists and is given by

$$DA[u] \cdot h = \delta A[u, h] . \quad (1.7)$$

To get rid of the derivatives of $h(x)$ in (1.6), we use integration by parts in each term of the sum, obtaining

$$DA[u] \cdot h = \int_{\Omega} \left[\frac{\partial a}{\partial u}(x) - \frac{d}{dx} \frac{\partial a}{\partial u'}(x) + \frac{d^2}{dx^2} \frac{\partial a}{\partial u''}(x) - \dots + (-1)^k \frac{d^k}{dx^k} \frac{\partial a}{\partial u^{(k)}}(x) \right] h(x) \, d^n x . \quad (1.8)$$

It is natural to give a special name to the integrand in (1.8).

Definition 2 (Functional derivative). *Let A be the integral functional given by (1.4). The functional derivative of A is the function*

$$\frac{\delta A}{\delta u(x)} = \frac{\partial a}{\partial u}(x) - \frac{d}{dx} \frac{\partial a}{\partial u'}(x) + \frac{d^2}{dx^2} \frac{\partial a}{\partial u''}(x) - \dots + (-1)^n \frac{d^k}{dx^k} \frac{\partial a}{\partial u^{(k)}}(x) . \quad (1.9)$$

The expression (1.8) is very similar to the scalar product on $L^2(\Omega)$ given by

$$\langle f, g \rangle_{L^2(\Omega)} = \int_{\Omega} f(x) g(x) \, dx . \quad (1.10)$$

Thus, it makes sense to write

$$DA[u] \cdot h = \int_{\Omega} \frac{\delta A}{\delta u(x)} h(x) \, dx = \left\langle \frac{\delta A}{\delta u(x)}, h(x) \right\rangle . \quad (1.11)$$

It follows from theory (CITATION) that the functional derivative being zero is a necessary condition for the function being the extremal point of the functional (given that the corresponding function is smooth enough).

Theorem 3 (Euler-Lagrange equations). *Let $v \in C^\infty(\Omega)$ such that $v(x)$ is stationary point of the functional A in (1.4), i.e*

$$\delta A[v, h] = 0 . \quad (1.12)$$

Then

$$\frac{\delta A}{\delta v(x)} = 0 . \quad (1.13)$$

The computation above gives sense to the

$$A[u + h] = A[u] + \left\langle \frac{\delta A}{\delta u}, h \right\rangle + o(h) , \quad (1.14)$$

where

$$\lim_{\|h\| \rightarrow 0} \frac{o(h)}{\|h\|} = 0 . \quad (1.15)$$

In physical notation, the function h is often denoted as δu .

1.1.2 Distributions

Definition 4 (Distributions). *Let $\mathcal{D}(\Omega)$ be the space of functions with compact support in $\Omega \subseteq \mathbb{R}^n$. The space of distributions $\mathcal{D}'(\Omega)$ is the dual space, i.e. the space of bounded linear functionals on $\mathcal{D}(\Omega)$. An element $T \in \mathcal{D}'(\Omega)$ is called a distribution.*

Definition 5 (Regular distribution). *Let $a \in L^1_{loc}(\Omega)$. A regular distribution is a distribution $T_a \in \mathcal{D}'(\Omega)$ defined as*

$$\langle T_a, \phi(x) \rangle = \int_{\Omega} a(x) \phi(x) d^n x . \quad (1.16)$$

Note that if two regular distributions satisfy $T_a = T_b$, then $a = b$ almost everywhere and vice versa.

Definition 6 (Dirac delta distribution). *Let $y \in \Omega$. The Dirac delta distribution $\delta(x - y)$ is defined as*

$$\langle \delta(x - y), \phi(x) \rangle = \phi(y) . \quad (1.17)$$

Zmínit se o derivaci distribucí a o tom, že se diraci dají testovat klidně C^1 funkcemi, takže souvisejí s teorií v minulé kapitole a můžeme aplikovat teorii funkcionálních derivací i na distribuce.

Definition 7 (Derivatives of distributions). *Let $T \in \mathcal{D}'(\Omega)$ be a distribution. We define its derivative by*

$$\langle D^\alpha T, \phi \rangle = (-1)^{|\alpha|} \langle T, D^\alpha \phi \rangle , \quad (1.18)$$

where α is a multiindex and

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_N^{\alpha_N}} . \quad (1.19)$$

Let $a \in C^\infty(\Omega)$. Then it makes sense to define the product of a and arbitrary distribution T by

$$\langle aT, \phi \rangle = \langle T, a\phi \rangle . \quad (1.20)$$

In the next chapter we shall frequently use the following identities of the delta distributions, which are written in the next lemma and which we will refer to as the Dirac identities.

Lemma 8 (Dirac identities).

$$f(y)\delta(x-y) = f(x)\delta(x-y) , \quad (1.21)$$

$$f(y)\delta'(x-y) = f(x)\delta'(x-y) + f'(x)\delta(x-y) . \quad (1.22)$$

Proof. The first relation is trivial. Direct calculation gives

$$\begin{aligned} \langle f(x)\delta'(x-y), \psi(x) \rangle &= \langle \delta'(x-y), f(x)\psi(x) \rangle \\ &= - \langle \delta(x-y), f'(x)\psi(x) \rangle - \langle \delta(x-y), f(x)\psi'(x) \rangle \\ &= - f'(y)\psi(y) - f(y) \langle \delta(x-y), \psi'(x) \rangle \\ &= - \langle f'(x)\delta(x-y), \psi(x) \rangle + f(y) \langle \delta'(x-y), \psi(x) \rangle \\ &= \langle f(y)\delta(x-y) - f'(x)\delta'(x-y), \psi(x) \rangle . \end{aligned}$$

Therefore

$$\langle f(y)\delta'(x-y), \psi(x) \rangle = \langle f'(x)\delta(x-y), \psi(x) \rangle + \langle f(x)\delta'(x-y), \psi(x) \rangle . \quad (1.23)$$

□

Vysvětlit, jak by se dal matematicky přesněji odvodit vztah $A(u(x)) = \int A(u(\xi))\delta(x-\xi) d\xi$.

The function $a(u)$ can be thus identified with a functional $A[u]$ defined by the relation

$$\langle A[u(x)], \phi(x) \rangle = \langle a(u(\xi))\delta(x-\xi), \phi(\xi) \rangle \quad (1.24)$$

In the physical notation

$$A[u(x)] = \int a(u(\xi))\delta(x-\xi) d^n\xi . \quad (1.25)$$

Therefore it is also common not to differentiate between the function itself and its corresponding functional form and denote them by the same letter A :

$$A[u(x)] = \int A(u(\xi))\delta(x-\xi) d^n\xi . \quad (1.26)$$

Example 9. Let $u(x)$ be a field variable and $F[u(x)]$ a function(al). To compute its functional derivative w.r.t. $u(y)$ in different position y , we use the integral expression of F

$$F[u(x)] = \int dy F(u(y))\delta(x-y) . \quad (1.27)$$

Its Gateaux differential is

$$\delta F[u, h] = \frac{d}{dt} F[u + th]|_{t=0} = \int dy \frac{\partial F}{\partial u}(y)\delta(x-y)h(y) , \quad (1.28)$$

therefore

$$\frac{\delta F(u(x))}{\delta u(y)} = \frac{\partial F}{\partial u}(x)\delta(x-y) . \quad (1.29)$$

Example 10. *In the special case of performing functional derivative w.r.t. the same field but in the different position,*

$$\frac{\delta u^i(x)}{\delta u^j(y)} = \delta_j^i \delta(x - y) . \quad (1.30)$$

Example 11.

$$\{I, J\} = \int_{\Omega} dx' \int_{\Omega} dy' \frac{\delta I}{\delta u^i(x)} P^{ij}(x', y') \frac{\delta J}{\delta u^j(y)} , \quad (1.31)$$

where

$$I = I[u], \quad J = J[u] , \quad P^{ij}(x', y') = \alpha^{ij}(x', y') \frac{d}{dx'} . \quad (1.32)$$

Then

$$\{u^a(x), u^b(y)\} = \int_{\Omega} dx' \int_{\Omega} dy' \delta_i^a \delta(x - x') \alpha^{ij}(x', y') \delta_j^b \delta(y - y') = \alpha^{ab}(x, y) . \quad (1.33)$$

1.2 Geometrical aspects

1.2.1 Riemannian manifolds

1.2.2 Lie algebras and Poisson brackets

2. The connection between Riemannian geometry and Hamiltonian mechanics

In this chapter we shall prove the theorem of the connection between the hydrodynamic Poisson bracket and the metrics found in the homogeneous system of hydrodynamic type. The original result was given by Dubrovin and Novikov and can be found in ?. The proof itself and the preceding lemmas are based on a considerable amount of technical computations, which we shall clarify in detail in order to offer the reader a comprehensive understanding of the chapter in question.

2.1 Systems of hydrodynamic type

Definition 12 (Homogeneous system of hydrodynamic type). *A homogeneous system of hydrodynamic type is an equation of the form*

$$\frac{\partial u^i}{\partial t} = f_j^{i\alpha}(u) \frac{\partial u^j}{\partial x^\alpha} \quad (2.1)$$

where $u^i(t, x^\alpha)$ are unknown functions, $i = 1, \dots, N$, $\alpha = 1, \dots, d$.

Sometimes we refer to the number of equations N as the dimension of the hydrodynamical phase space and to the number of spatial variables d as the dimension of the configuration space.

Before we go any further, let us present a few examples.

Example 13. *The energy is given by the functional*

$$E = \int e(\rho, \mathbf{m}, s, \mathbf{F}) \, d\mathbf{x} \quad (2.2)$$

The evolution equations in the Eulerian frame are

$$\partial_t \rho = -\partial_i(\rho E_{m_i}) \quad (2.3)$$

$$\partial_t m_i = -\partial_j(m_i E_{m_j}) - \rho \partial_i E_\rho - m_j \partial_i E_{m_j} - m_j \partial_i E_{m_j} \quad (2.4)$$

$$-s \partial_i E_s - F_j^i \partial_i E_{F_j^j} + \partial_j(F_I^j E_{F_I^i} + F_O^i E_{F_I^i}) \quad (2.5)$$

$$\partial_t s = -\partial_i(s E_{m_i}) \quad (2.6)$$

$$\partial_t F_I^i = -E_{m_k} \partial_k F_I^i + F_I^j \partial_j E_{m_i} . \quad (2.7)$$

Total energy density e can be prescribed as

$$e = \frac{\mathbf{m}^2}{2\rho} + \epsilon(\rho, s, \mathbf{F}) , \quad (2.8)$$

where ϵ is the elastic and internal energy. In particular, $E_{\mathbf{m}} = \frac{\mathbf{m}}{\rho} = \mathbf{v}$ becomes the velocity. The evolution equation for the deformation gradient then gets the explicit form

$$\partial_t \mathbf{F} = -(\mathbf{v} \cdot \nabla) \mathbf{F} + \nabla \mathbf{V} \cdot \mathbf{F} , \quad (2.9)$$

which is the usual evolution equation for \mathbf{F} in the Eulerian frame.

It was Riemann who noticed that the functions $f_j^{i\alpha}$ in (2.1) are in fact tensors [CITACE]?

Proposition 14 (Transformation of $f_j^{i\alpha}$). *Under smooth change of variables $u^i \mapsto v^a$ of the form*

$$u^i = u^i(v^1, \dots, v^N) \quad (2.10)$$

the functions $f_j^{i\alpha}$ transform for each α according to the tensor law

$$f_b^{a\alpha}(v) = \frac{\partial v^a}{\partial u^i} \frac{\partial u^j}{\partial v^b} f_j^{i\alpha}(u) . \quad (2.11)$$

Proof. Using a direct calculation we obtain

$$\frac{\partial u^i}{\partial t}(v) = \frac{\partial u^i}{\partial v^a} \frac{\partial v^a}{\partial t} , \quad (2.12)$$

$$\frac{\partial u^j}{\partial x^\alpha}(v) = \frac{\partial u^j}{\partial v^b} \frac{\partial v^b}{\partial x^\alpha} , \quad (2.13)$$

so

$$\frac{\partial u^i}{\partial v^a} \frac{\partial v^a}{\partial t} = \frac{\partial u^j}{\partial v^b} f_j^{i\alpha}(v(u)) \frac{\partial v^b}{\partial x^\alpha} \quad (2.14)$$

and therefore

$$\frac{\partial v^a}{\partial t} = \underbrace{\frac{\partial v^a}{\partial u^i} \frac{\partial u^j}{\partial v^b} f_j^{i\alpha}(v(u))}_{f_b^{a\alpha}(v)} \frac{\partial v^b}{\partial x^\alpha} . \quad (2.15)$$

□

According to the previous lemma, if we denote the space of points $(t, x^\alpha) \subset \mathbb{R} \times \mathbb{R}^N$ as some manifold M , then the unknown variables $u^i(t, x^\alpha)$ can be seen as the local coordinates of that manifold. The functions $f_j^{i\alpha}$ are transformed as tensors of the type (1,1) in the i and j indices.

Our next purpose is to gain a richer geometry of that space M . If the functions $f_j^{i\alpha}$ have a special form, we can introduce a geometry of the Poisson brackets.

Definition 15. 1. A Poisson bracket of the hydrodynamic type of two functionals I_1 a I_2 is defined as

$$\{I_1, I_2\} = \int dx \left[\frac{\delta I_1}{\delta u^q(x)} A^{qp} \frac{\delta I_2}{\delta u^p(x)} \right] \quad (2.16)$$

where

$$A = (A^{qp}) = \left(g^{qp\alpha}(u) \frac{d}{dx^\alpha} + b_s^{qp\alpha}(u) \frac{\partial u^s}{\partial x^\alpha} \right) . \quad (2.17)$$

where $g^{ij\alpha}$ and $b_k^{ij\alpha}$ are certain functions, $i, j, k = 1, \dots, N$ and $\alpha = 1, \dots, d$.

In particular

$$\{u^i(x), u^j(y)\} = g^{ij\alpha}[u(x)] \delta_\alpha(x - y) + b_k^{ij\alpha}[u(x)] u_\alpha^k(x) \delta(x - y) . \quad (2.18)$$

2. The functional of hydrodynamic type is a functional of the form

$$H[u] = \int_{\mathbb{R}^d} h(u(x)) \mathrm{d}^d x, \quad (2.19)$$

where h is independent of u_α , $u_{\alpha\beta}$. The function h is called a hamiltonian density.

3. The hamiltonian system of hydrodynamic type is a system of the form

$$u_t^i(x) = \left[g^{ij\alpha}[u(x)] \frac{\partial^2 h(u)}{\partial u^j \partial u^k} + b_k^{ij\alpha}[u(x)] \frac{\partial h(u)}{\partial u^j} \right] u_\alpha^k(x), \quad (2.20)$$

where $H[u]$ is the functional of the hydrodynamic type and h the hamiltonian density.

Proposition 16.

$$u_t^i(x) = \{u^i, H\}. \quad (2.21)$$

Proof.

$$\{u^i(x), H\} = \int \mathrm{d}x' \mathrm{d}y' \frac{\delta u^i(x)}{\delta u^p(x')} \left[g^{pq}(u(x')) \frac{\mathrm{d}}{\mathrm{d}x'} + b_m^{pq} \frac{\partial u^m}{\partial x} \right] \frac{\delta H}{\delta u^q(y')}. \quad (2.22)$$

$$\frac{\delta u^i(x)}{\delta u^p(x')} = \delta_p^i \delta(x - x') \quad (2.23)$$

$$H[u] = \int h(u(y')) \mathrm{d}y', \quad \frac{\delta H}{\delta u^q(y')} = \int \frac{\partial h}{\partial u^q}(y) \delta(y - y') \quad (2.24)$$

$$\{u^i(x), H\} = \int \mathrm{d}y' \left[g^{iq}(u(x)) \frac{\mathrm{d}}{\mathrm{d}x} + b_m^{iq}(u(x)) \frac{\partial u^m}{\partial x} \right] \frac{\partial h}{\partial u^q}(y) \delta(y - y') \quad (2.25)$$

$$= g^{iq}(u(x)) \frac{\partial^2 h}{\partial u^k \partial u^q} \frac{\partial u^k}{\partial x} + b_m^{iq}(u(x)) \frac{\partial u^m}{\partial x} \frac{\partial h}{\partial u^q} \quad (2.26)$$

$$= \left[g^{ij}(u(x)) \frac{\partial^2 h}{\partial u^j \partial u^k} + b_k^{ij}(u(x)) \frac{\partial h}{\partial u^j} \right] \frac{\partial u^k}{\partial x}. \quad (2.27)$$

□

Dokončit a přidělat α . kde se předělá $\frac{\mathrm{d}}{\mathrm{d}x}$ na $\frac{\mathrm{d}}{\mathrm{d}y}$?

Note that we have yet to show that the Poisson bracket of the hydrodynamic type defined above is indeed a Poisson bracket in the classical sense. We must verify that it satisfies the three properties: skew-symmetry, Leibniz rule and Jacobi identity. In fact, this will be the key theorem of this chapter.

2.2 One-dimensional case

In this section we first consider the one-dimensional case in which the unknown functions u^i depend on only two variables (t, x) . The hamiltonian system is then

$$\frac{\partial u^i}{\partial t}(t, x) = \left[g^{ij}[u(t, x)] \frac{\partial^2 h(u)}{\partial u^j \partial u^k} + b_k^{ij}[u(t, x)] \frac{\partial h(u)}{\partial u^j} \right] \frac{\partial u^k}{\partial x}(t, x) \quad (2.28)$$

and the Poisson bracket of hydrodynamic type is of the form

$$\{I_1, I_2\} = \int dx \left[\frac{\delta I_1}{\delta u^q(x)} A^{qp} \frac{\delta I_2}{\delta u^p(x)} \right], \quad \text{where } A^{qp} = \left(g^{qp}(u) \frac{d}{dx} + b_s^{qp}(u) \frac{\partial u^s}{\partial x} \right). \quad (2.29)$$

In particular

$$\{u^i(x), u^j(y)\} = g^{ij}[u(t, x)] \partial_x \delta(x - y) + b_k^{ij}[u(t, x)] \frac{\partial u^k}{\partial x} \delta(x - y). \quad (2.30)$$

We say that the Poisson bracket is non-degenerate if $\det(g^{ij}) \neq 0$. We will show later that the non-degeneracy is invariant under the transformation of the coordinates.

Definition 17. Let $(g^{ij}(u))$ be a non-degenerate matrix. We define functions $\Gamma_{jk}^i(u)$ by

$$b_k^{ij}(u) = -g^{is}(u) \Gamma_{sk}^j(u), \quad i, j, k, s = 1, \dots, N. \quad (2.31)$$

The choice of denotation g^{ij} and Γ_{jk}^i , respectively, has its purpose. We will show soon that these are in fact the metric and affine coefficients, respectively, on M .

Proposition 18 (Leibniz rule for Poisson bracket). *The Poisson brackets of the hydrodynamic type satisfy the Leibniz rule*

$$\{v^p(u^i(x)), v^q(u^j(y))\} = \frac{\partial v^p}{\partial u^i}(x) \frac{\partial v^q}{\partial u^j}(y) \{u^i(x), u^j(y)\}. \quad (2.32)$$

Proof. The coordinates are written as functionals in the form

$$v^p(u^i(x)) = \int dx' v^p(u^i(x')) \delta(x - x'). \quad (2.33)$$

Applying this to the definition of the Poisson bracket, we have

$$\begin{aligned} \{v^p(u^i(x)), v^q(u^j(y))\} &= \int dx' \int dy' \underbrace{\frac{\delta v^p}{\delta u^i(x')}}_{\frac{\partial v^p}{\partial u^i}(x') \delta(x-x')} \{u^i(x'), u^j(y')\} \underbrace{\frac{\delta v^q}{\delta u^j(y')}}_{\frac{\partial v^q}{\partial u^j}(y') \delta(y-y')} = \\ &= \frac{\partial v^p}{\partial u^i}(x) \frac{\partial v^q}{\partial u^j}(y) \{u^i(x), u^j(y)\}. \end{aligned} \quad (2.34)$$

$$= \frac{\partial v^p}{\partial u^i}(x) \frac{\partial v^q}{\partial u^j}(y) \{u^i(x), u^j(y)\}. \quad (2.35)$$

□

Proposition 19 (Transformation of g and Γ). *Let $u^i \mapsto v^a$ be the smooth change of variables. Then*

1. *The coefficients g^{ij} transform as the $(2,0)$ tensors, i.e.*

$$g^{pq}(v) = \frac{\partial v^p}{\partial u^i} \frac{\partial v^q}{\partial u^j} g^{ij}[u(v)], \quad p, q = 1, \dots, N. \quad (2.36)$$

2. *The coefficients Γ_{jk}^i transform like the affine connection symbols, i.e.*

$$\Gamma_{qr}^p(v) = \frac{\partial v^p}{\partial u^i} \frac{\partial u^j}{\partial v^q} \frac{\partial u^k}{\partial v^r} \Gamma_{jk}^i(u) + \frac{\partial v^p}{\partial u^i} \frac{\partial^2 u^i}{\partial v^q \partial v^r}. \quad (2.37)$$

Proof. 1. Let us apply the transformation equation (2.32) to the definition of the Poisson bracket (2.2). We get

$$\begin{aligned} & g^{pq}[v(u(x))] \partial_x \delta(x-y) + b_s^{pq}[v(u(x))] \frac{\partial v^s}{\partial u^k}(x) \frac{\partial u^k}{\partial x} \delta(x-y) \\ &= \frac{\partial v^p}{\partial u^i}(x) \frac{\partial v^q}{\partial u^j}(y) \left[g^{ij}[u(x)] \partial_x \delta(x-y) + b_k^{ij}[u(x)] \frac{\partial u^k}{\partial x} \delta(x-y) \right]. \end{aligned} \quad (2.38)$$

From now we denote for readability

$$T_k^a(x) := \frac{\partial v^a}{\partial u^k}(x). \quad (2.39)$$

On the right-hand side of (2.38) we apply Dirac identity (Lemma 8 on page 5) on $T_j^q(y) \partial_x \delta(x-y)$. All functions are then of variable x , so from now we won't write it explicitly. We have

$$\begin{aligned} & g^{pq}(v) \partial_x \delta(x-y) + b_n^{pq}(v) T_j^n \frac{\partial u^j}{\partial x} \delta(x-y) \\ &= T_i^p T_j^q g^{ij}(u) \partial_x \delta(x-y) + \left(\frac{\partial^2 v^q}{\partial u^j \partial u^k} + T_i^p T_j^q b_k^{ij} \right) \frac{\partial u^k}{\partial x} \delta(x-y). \end{aligned} \quad (2.40)$$

The terms with $\partial_x \delta$ must be equal to each other, as must δ terms. From $\partial_x \delta$ terms we get

$$g^{pq}(v) = T_i^p T_j^q g^{ij}(u), \quad (2.41)$$

which is the transformation relation for g .

2. From δ terms in (2.40), we get

$$b_n^{pq}(v) T_k^n = b_k^{ij}(u) T_i^p T_j^q + g^{ij}(u) T_i^p \frac{\partial^2 v^q}{\partial u^i \partial u^k} \quad (2.42)$$

and after (2.31)

$$-g^{ps}(v) \Gamma_{sn}^q(v) T_k^n = -g^{il}(u) \Gamma_{lk}^j(u) T_i^p T_j^q + g^{ij}(u) T_i^p \frac{\partial^2 v^q}{\partial u^i \partial u^k}. \quad (2.43)$$

On the left-hand side we use the transformation of g (2.36)

$$-g^{ij}(u)\Gamma_{sm}^q(v)T_i^pT_j^sT_k^m = -g^{ij}(u)\Gamma_{jk}^lT_i^pT_l^q + g^{ij}(u)\frac{\partial^2 v^q}{\partial u^i \partial u^k}T_i^p \quad (2.44)$$

and after vectoring out $g^{ij}(u)T_i^p$

$$\Gamma_{jk}^l(u)T_l^q = \Gamma_{sm}^q(v)T_j^sT_k^m + \frac{\partial^2 v^q}{\partial u^i \partial u^k} . \quad (2.45)$$

All that remains is to multiply this equation with $(T^{-1})_q^a$ and we get the final transformation relation for Γ

$$\Gamma_{jk}^a(u) = \Gamma_{sm}^q(v)\frac{\partial v^s}{\partial u^j}\frac{\partial v^m}{\partial u^k}\frac{\partial u^a}{\partial v^q} + \frac{\partial^2 v^q}{\partial u^i \partial u^k}\frac{\partial u^a}{\partial v^q} . \quad (2.46)$$

□

Let us summarize the results. We have defined Poisson bracket in the way that it satisfies the Leibniz rule. It remains to investigate the skew-symmetry and Jacobi identity. We have also defined g^{pq} and Γ_{qr}^p , which transform like tensors and Christoffel symbols, respectively. It remains to show that g^{pq} is indeed symmetric and Γ is the Levi-Civita connection, i.e. the corresponding covariant derivative annuls the metric. We are now ready to prove the key theorem of this chapter which shows that these properties are indeed equivalent. Moreover, the corresponding covariant derivative has zero curvature, which indicates that the phase space M is in fact Euclidean space.

Theorem 20 (Dubrovin, Novikov). *Let $\det(g^{ij}) \neq 0$. Then (2.2) defines a Poisson bracket with skew-symmetry, Leibniz rule and Jacobi identity if and only if these three conditions are satisfied:*

1. $g^{ij} = g^{ji}$, that is, g^{ij} is a metric on the phase space M .
2. $\Gamma_{jk}^i = \Gamma_{kj}^i$, that is, Γ_{jk}^i are affine coefficients, such that the corresponding covariant derivative has zero torsion and annihilate the metric g^{ij} .
3. The covariant derivative corresponding to Γ_{jk}^i has zero curvature.

Corollary 21. *There are local coordinates $w^i = w^i(u^1, \dots, u^N)$, $i = 1, \dots, N$ such that*

$$g^{ij}(w) = \tilde{g}^{ij} = \text{const} , \quad \Gamma_{jk}^i(w) = 0 . \quad (2.47)$$

In these coordinates is Poisson bracket reduced to the constant one

$$\{w^i(x), w^j(y)\} = \tilde{g}^{ij} \partial_x \delta(x - y) . \quad (2.48)$$

The only local invariant of the Poisson bracket is the signature of the metric $g^{ij}(u)$.

Co se tímhle myslí? Pořád nemám rozmyšleno.

Proof of theorem 20. The proof is divided into three steps.

Step 1: Skew-symmetry of Poisson bracket implies the symmetry of g^{ij} and compatibility with Γ_{jk}^i .

Let the Poisson bracket be skew-symmetric, i.e.

$$\{u^i(x), u^j(y)\} + \{u^j(y), u^i(x)\} = 0. \quad (2.49)$$

The second bracket reads as

$$\{u^j(y), u^i(x)\} = g^{ji}[u(y)]\partial_x\delta(y-x) + b_k^{ji}[u(y)]\frac{\partial u^k}{\partial y}\delta(y-x). \quad (2.50)$$

Let us apply Dirac identities (Lemma 8 on page 5) on $g^{ji}[u(y)]\partial_x\delta(y-x)$. We get

$$\begin{aligned} & \{u^j(y), u^i(x)\} = \\ & = -g^{ji}[u(x)]\partial_x\delta(x-y) - \frac{\partial g^{ji}}{\partial u^k}[u(x)]\frac{\partial u^k}{\partial x}\delta(x-y) + b_k^{ji}[u(x)]\frac{\partial u^k}{\partial x}\delta(x-y). \end{aligned} \quad (2.51)$$

The skew-symmetry is then

$$\begin{aligned} 0 & = \{u^i(x), u^j(y)\} + \{u^j(y), u^i(x)\} = \\ & = [g^{ij}[u(x)] - g^{ji}[u(x)]]\partial_x\delta(x-y) + \left[b_k^{ij} - \frac{\partial g^{ij}}{\partial u^k} + b_k^{ji}\right]\frac{\partial u^k}{\partial x}\delta(x-y). \end{aligned} \quad (2.52)$$

The right-hand side is equal to zero if and only if

$$g^{ij}(u) = g^{ji}(u), \quad (2.53)$$

$$\frac{\partial g^{ij}}{\partial u^k}(u) = b_k^{ij}(u) + b_k^{ji}(u). \quad (2.54)$$

Equation (2.53) says that g^{ij} is a symmetric tensor. By assumption $\det g^{ij}$ is non-degenerate, so it defines a metric structure on M . Equation (2.54) is equivalent to

$$\frac{\partial g^{ij}}{\partial u^k} + g^{is}\Gamma_{sk}^j + g^{sj}\Gamma_{sk}^i = 0. \quad (2.55)$$

If ∇ is the covariant derivative corresponding to Γ_{jk}^i , then $\nabla_k g^{ij} = 0$. So Γ_{ij}^k are affine connection coefficients and are compatible with g^{ij} .

Step 2: Jacobi identity is equivalent to zero torsion and curvature.

Let us assume that Jacobi identity holds. We denote

$$\mathcal{J}^{ijk}(x, y, z) = J^{ijk}(x, y, z) + J^{jki}(y, z, x) + J^{kij}(z, x, y) = 0 \quad (2.56)$$

where

$$J^{ijk}(x, y, z) = \{u^k(z), \{u^i(x), u^j(y)\}\}. \quad (2.57)$$

[TADY TO JEŠTĚ NENÍ HOTOVÉ.] From now on we shall use the abbreviated notation

$$g_{,m}^{ij} = \frac{\partial g^{ij}}{\partial u^m}, \quad b_{k,m}^{ij} = \frac{\partial b_k^{ij}}{\partial u^m}, \quad u_x^k = \frac{\partial u^k}{\partial x}. \quad (2.58)$$

$$J^{ijk}(x, y, z) = \int d\xi \frac{\delta u^k(z)}{\delta u^n(\xi)} \left[g^{mn}(\xi) \partial_\xi + b_p^{mn} \partial_\xi u^p \right] \frac{\delta \{u^i(x), u^j(y)\}}{\delta u^m(\xi)}.$$

The functional derivatives are

$$\frac{\delta u^k(z)}{\delta u^n(\xi)} = \delta_n^k \delta(z - \xi), \quad (2.59)$$

$$\begin{aligned} \frac{\delta \{u^i(x), u^j(y)\}}{\delta u^m(\xi)} &= \frac{\delta}{\delta u^m(\xi)} \left[g^{ij}(x) \partial_x \delta(x - y) + b_k^{ij}(x) \partial_x u^k \delta(x - y) \right] = \\ &= \left[g_{,m}^{ij} \partial_x \delta(x - y) + b_{,m}^{ij}(x) u_x^k \delta(x - y) \right] \delta(x - \xi). \end{aligned} \quad (2.60)$$

Thus we can eliminate the integration in ξ and get rid of the integral.

$$\begin{aligned} J^{ijk}(x, y, z) &= \int d\xi \delta_n^k \delta(z - \xi) \left[g^{mn}(\xi) \partial_\xi + b_p^{mn} \partial_\xi u^p \right] \\ &\quad \left[g_{,m}^{ij}(x) \partial_x \delta(x - y) \delta(x - \xi) + b_{k,m}^{ij}(x) u_x^k \delta(x - y) \delta(x - \xi) \right] \\ &= \left[g^{mk}(\xi) \partial_z + b_p^{mk} \partial_z u^p \right] \left[g_{,m}^{ij}(x) \partial_x \delta(x - y) \delta(x - \xi) + b_{k,m}^{ij}(x) u_x^k \delta(x - y) \delta(x - \xi) \right] \\ &= \left[g_{,m}^{ij} g^{mk} \partial_x u^m \partial_x \delta(x - y) + b_{k,m}^{ij} g^{mk} u_x^k \delta(x - y) \right] \partial_x \delta(z - x) + \\ &+ \left[g_{,m}^{ij} b_p^{mk} u_x^p \partial_x \delta(x - y) + b_{k,m}^{ij} b_p^{mk} u_x^p u_x^k \delta(x - y) \right] \delta(z - x). \end{aligned}$$

We get four terms which are δ distributions and their derivatives.

$$\begin{aligned} J^{ijk}(x, y, z) &= g_{,m}^{ij} g^{mk} \partial_x \delta(x - y) \partial_x \delta(z - x) + b_{k,m}^{ij} g^{mk} u_x^k \delta(x - y) \partial_x \delta(z - x) \\ &+ g_{,m}^{ij} b_p^{mk} u_x^p \partial_x \delta(x - y) \delta(z - x) + b_{k,m}^{ij} b_p^{mk} u_x^p u_x^k \delta(x - y) \delta(z - x). \end{aligned}$$

Jednotlivé členy označíme

$$\begin{aligned} J^{ijk}(x, y, z) &= E(x) \partial_x \delta(x - y) \partial_x \delta(z - x) + F(x) \delta(x - y) \partial_x \delta(z - x) \\ &+ G(x) \partial_x \delta(x - y) \delta(z - x) + H(x) \delta(x - y) \delta(z - x). \end{aligned}$$

Díky symetrii, resp. antisymetrii δ funkcí

$$\delta(x - y) = \delta(y - x), \quad \partial_x \delta(x - y) = -\partial_y \delta(y - x) \quad (2.61)$$

Lze členy předělat na tvar

$$J^{ijk}(x, y, z) = E(x) \partial_y \delta(y - x) \partial_z \delta(x - z) - F(x) \delta(y - x) \partial_z \delta(x - z) \quad (2.62)$$

$$-G(x) \partial_y \delta(y - x) \delta(x - z) + H(x) \delta(x - y) \delta(x - z) \quad (2.63)$$

[ODTEŽ UŽ JE TO DODĚLANÉ.] One of these equations reads as

$$A_{02}^{ijk} = b_s^{ij} g^{sk} - b_s^{kj} g^{si} = -g^{ip} \Gamma_{ps}^j g^{sk} + g^{km} \Gamma_{ms}^j g^{si} = 0, \quad (2.64)$$

therefore, after multiplying with $g_{ai} g_{bk}$

$$\Gamma_{ab}^j - \Gamma_{ba}^j = 0. \quad (2.65)$$

So the corresponding covariant derivative has zero torsion.

Another equation reads as

$$A_{00}^{ijk} = B_p^{ijk}(u)u_{xx}^p + C_{pq}^{ijk}(u)u_x^p u_x^q = 0, \quad (2.66)$$

where

$$B_p^{ijk} = (b_{s,p}^{jk} - b_{p,s}^{jk})g^{si} + b_s^{ij}b_p^{sk} - b_s^{ik}b_p^{sj}. \quad (2.67)$$

The function B_p^{ijk} being zero implies the zero curvature, which follows from

$$B_p^{ijk} = -g^{is} \left(g_{,p}^{jm} \Gamma_{ms}^k + g^{jm} \Gamma_{ms,p}^k - g_{,s}^{jm} \Gamma_{mp}^k - g^{jm} \Gamma_{mp,s}^k \right) \quad (2.68)$$

$$+ g^{im} \Gamma_{ms}^j g^{sn} \Gamma_{np}^k - g^{im} \Gamma_{ms}^k g^{sn} \Gamma_{np}^j = \quad (2.69)$$

$$= g^{is} g^{jn} \left(\Gamma_{np}^m \Gamma_{ms}^k - \Gamma_{ns}^m \Gamma_{mp}^k + \Gamma_{mp,s}^k - \Gamma_{ms,p}^k \right) \quad (2.70)$$

$$+ g^{is} g^{nm} \left(\Gamma_{np}^j \Gamma_{ms}^k - \Gamma_{ns}^j \Gamma_{mp}^k \right) + g^{im} g^{sn} \left(\Gamma_{ms}^j \Gamma_{np}^k - \Gamma_{ms}^k \Gamma_{np}^j \right) = \quad (2.71)$$

$$= -g^{is} g^{jn} \left(\Gamma_{ns}^m \Gamma_{mp}^k - \Gamma_{np}^m \Gamma_{ms}^k + \Gamma_{ms,p}^k - \Gamma_{mp,s}^k \right) = \quad (2.72)$$

$$= -g^{is} g^{jn} R_{nps}^k. \quad (2.73)$$

So far we proved that if hydrodynamical Poisson bracket satisfies skew-symmetry and Jacobi identity, the conditions 1-3 are satisfied.

Step 3:

If ∇ has zero torsion and curvature, there are coordinates $w^i = w^i(u^1, \dots, u^N)$ for $i = 1, \dots, N$ such that $g^{ij} = \text{const}$ and $b_k^{ij} = 0$. In these coordinates

$$\{w^i(x), w^j(y)\} = \tilde{g}^{ij} \partial_x \delta(x - y). \quad (2.74)$$

Jacobi identity, skew-symmetry and Leibniz rule are for that bracket satisfied trivially. Therefore the three conditions are also sufficient. That finishes the proof. \square

2.2.1 Kdy je hydrodynamický systém hamiltonovský?

Nyní můžeme explicitně přepsat podmínky, při kterých je obecný systém hydrodynamických rovnic hamiltonovský vůči nějaké nede degenerované Poissonově závorce. Nejprve si povšimneme, že funkce $f_j^i(u)$ lze přepsat pomocí Laplaceova–Beltramiho operátoru.

Proposition 22 (O zápisu pomocí Laplaceova–Beltramiho operátoru). *Mějme hamiltonovský systém rovnic*

$$\frac{\partial u^i}{\partial t} = f_k^i(u) \frac{\partial u^k}{\partial x}, \quad f_k^i(u) = g^{ij}(u) \frac{\partial^2 h}{\partial u^j \partial u^k} - g^{is}(u) \Gamma_{sk}^j(u) \frac{\partial h}{\partial u^j}. \quad (2.75)$$

Pak lze psát

$$f_k^i(u) = \nabla^i \nabla_k h(u), \quad (2.76)$$

kde ∇_j je Levi-Civita kovariantní derivace metriky g_{ij} a $\nabla^i = g^{is} \nabla_s$.

Proof. Přímým dosazením

$$\nabla_k h(u) = \frac{\partial h}{\partial u^k}, \quad (2.77)$$

$$\nabla^i \nabla_k h(u) = g^{is} \nabla_s \frac{\partial h}{\partial u^k} = g^{is} \left(\frac{\partial^2 h}{\partial u^s \partial u^k} - \Gamma_{sk}^j \frac{\partial h}{\partial u^j} \right) = f_k^i(u). \quad (2.78)$$

□

Pomocí tohoto zápisu snadno můžeme sepsat postačující podmínky hamiltonovskosti. Důkaz plyne ihned z teorému 20 on page 12.

Theorem 23 (Postačující podmínka hamiltonovskosti systému). *Systém $u_t^i = f_j^i(u)u_x^j$ je hamiltonovský právě tehdy, když existuje nedegenerovaná metrika $g^{ij}(u)$, jejíž afinní konexe má nulovou křivost a splňuje*

$$g_{ij}f_j^k = g_{jk}f_i^k, \quad (2.79)$$

$$\nabla_i f_j^k = \nabla_j f_i^k. \quad (2.80)$$

Speciálně vztah (2.80) říká, že ∇_i má nulovou torzi.

2.2.2 Rekonstrukce metriky z $f_j^i(u)$?

Máme-li zadaný hamiltonovský systém s maticí $f_j^i(u)$, lze metriku $g^{ij}(u)$ zkonstruovat jednoznačně? Tuto otázku nyní vyřešíme pro $N \geq 3$.

Označme λ_α vlastní čísla matice $f_j^i(u)$. (Mohou být komplexní.) Předpokládejme, že jsou navzájem různá. Označme odpovídající bázi vlastních vektorů $e_\alpha(u)$. Definujme koeficient $c_{\alpha\beta}^\gamma(u)$ vztahem

$$[e_\alpha, e_\beta] = c_{\alpha\beta}^\gamma e_\gamma, \quad (2.81)$$

kde $[\cdot, \cdot]$ značí obyčejný komutátor funkcí. Předpokládejme dále, že pro navzájem různé α, β, γ je $c_{\alpha\beta}^\gamma$ různé od nuly.

Definition 24. *Matici $f_j^i(u)$ splňující podmínky výše nazveme hamiltonovskou maticí.*

Theorem 25 (O rekonstrukci metriky). *Nechť $N \geq 3$. Nechť je dána hamiltonovská matice $f_j^i(u)$. Pak lze zkonstruovat nedegenerovanou metriku $g^{ij}(u)$ s nulovou křivostí jednoznačně až na násobek konstantou.*

Tomuhle nerozumím a myslím si, že v článku mají typo.

Proof. Z rovnice (2.79) je vidět, že v bázi e_α je metrika g^{ij} diagonální.

Opravdu?

V této bázi pak bude mít rovnice (2.80) tvar (zde se nesčítá přes opakované indexy)

$$\partial_\alpha \lambda_\beta \delta_\beta^\gamma - \partial_\beta \lambda_\alpha \delta_\alpha^\gamma + (\Gamma_{\alpha\beta}^\gamma - \Gamma_{\beta\alpha}^\gamma) \lambda_\gamma + \Gamma_{\beta\alpha}^\gamma (\lambda_\beta - \lambda_\alpha) = 0. \quad (2.82)$$

Zde ∂_α je derivace ve směru e_α a konexe $\Gamma_{\alpha\beta}^\gamma$ jsou definované rovnostmi

$$\nabla_{e_\beta} e_\alpha = \sum \Gamma_{\alpha\beta}^\gamma e_\gamma. \quad (2.83)$$

Normalizujme nyní vlastní vektory e_α tak, aby v této bázi byla metrika jednotková matice, tj. $g^{\alpha\beta} = \delta^{\alpha\beta}$.

Výraz $c_{\alpha\beta}^\gamma$ má význam torze (z definice) a platí

$$c_{\beta\alpha}^\gamma = \Gamma_{\alpha\beta}^\gamma - \Gamma_{\beta\alpha}^\gamma. \quad (2.84)$$

□

2.3 Multi-dimensional case

Proposition 26 (Changing the space coordinates). *For any changes of the spatial variables $x^\alpha \mapsto c_\beta^\alpha x^\beta$, where $\alpha = 1, \dots, d$ and $\det(c_\beta^\alpha) = 1$, the metrics $g^{ij\alpha}$ and $b_k^{ij\alpha}$ transforms as components of a vector.*

Ve vícerozměrném případě máme lineární bundle metrik a s nimi spojených konexí. Pro každou záměnu prostorových proměnných $x^\alpha \mapsto c_\beta^\alpha x^\beta$ pro $\alpha = 1, \dots, d$ splňující $\det c_\beta^\alpha = 1$ se metrika $g^{ij\alpha}$ a konexe $b_k^{ij\alpha}$ transformují jako komponenty vektoru.

Definition 27. Řekneme, že bundle metrik $g^{ij\alpha}$ je silně nedegenerovaný, jestliže pro nějakou sadu c_α je lineární kombinace $c_\alpha g^{ij\alpha}$ nedegenerovaná matice.

Theorem 28. Nechť je hamiltonovský systém silně nedegenerovaný.

1. Pro $N = 1$ lze Poissonovu závorku redukovat na konstantní formu

$$g^{ij}(u) = \tilde{g}^{ij}(u). \quad (2.85)$$

2. Pro $N \geq 2$ lze Poissonovu závorku redukovat na lineární formu

$$g^{ij\alpha}(u) = g_k^{ij\alpha} u^k + \tilde{g}^{ij\alpha}, \quad \alpha = 1, \dots, d, \quad (2.86)$$

$$(2.87)$$

kde koeficienty $g_k^{ij\alpha} = b_k^{ij\alpha} + b_k^{ij\alpha}$, $\tilde{g}^{ij\alpha}$ a $b_k^{ij\alpha}$ jsou konstantní.

List of Figures

A. Appendix