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Connection between continuum mechanics and Riemannian geometry

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Dedication.

Title: Connection between continuum mechanics and Riemannian geometry

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1. Mathematical framework

1.1 Functional derivatives

In hydrodynamics

Definition 1 (Gateuax and Frechet derivative)**.** *Let X be a normed space, $A : X \to \mathbb{R}$ is a functional and $x \in D(A)$.*

1. *Gateaux derivative $\delta A(x,h)$ in the direction h is defined as*

$$\delta A(x,h) = \frac{\mathrm{d}}{\mathrm{d}t} A(x+th)|_{t=0} = \lim_{t\to 0} \frac{A(x+th)-A(x)}{t} \,. \tag{1.1}$$

2. *Fréchet derivative $DA(x)$ is defined as the unique linear bounded functional satisfying the relation*

$$\lim_{h\to 0} \frac{A(x+h)-A(x)-DA(h)}{\|h\|} = 0 \,. \tag{1.2}$$

We are intersted in such functionals, where X is the subspace of functions $f \in C^k(\Omega)$, Ω is a subset of $\mathbb{R}^n$, satisfying $f, f', \cdots, f^{(k-1)} = 0$ on $\partial\Omega$, equipped with the norm

$$\|u\|_X = \sum_{j=1}^{n} \sup_{x\in\Omega} |u^{(j)}(x)| \,. \tag{1.3}$$

There are two main types of functionals on that space.

1.1.1 Integral functionals

The first type is the integral functional

$$A[u] = \int_\Omega a(x, u(x), u'(x), \cdots, u^{(k)}(x)) \,\mathrm{d}^n x \,, \tag{1.4}$$

where $a : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$ is analytic function. For simplicity we often write this using a physical notation

$$A[u] = \int a(u) \,\mathrm{d}^n x \,, \tag{1.5}$$

meaning that $a(u)$ is in fact function $a(x, u(x), u'(x), \cdots, u^{(k)}(x))$ and also the domain Ω is specified by the context. We call $a(u)$ the density of functional $A[u]$.

Gateaux derivative of A is computed as follows:

$$\delta A[u,h] = \int_\Omega \sum_{i=1}^{k} \frac{\partial a}{\partial u^{(i)}}(x) h^{(i)}(x) \,\mathrm{d}^n x \,. \tag{1.6}$$

The functional $\delta A[u,h]$ is linear and bounded in h, so the Fréchet derivative of A exists and is given by

$$DA[u] \cdot h = \delta A[u,h] \,. \tag{1.7}$$

To get rid of the derivatives of $h(x)$ in (1.6), we use integration by parts in each term of the sum, obtaining

$$DA[u] \cdot h = \int_\Omega \left[\frac{\partial a}{\partial u}(x) - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial a}{\partial u'}(x) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial a}{\partial u''}(x) - \cdots + (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}x^k}\frac{\partial a}{\partial u^{(k)}}(x) \right] h(x)\,\mathrm{d}^n x \,. \tag{1.8}$$

It is natural to give a special name to the integrand in (1.8).

Definition 2 (Functional derivative)**.** *Let A be the integral functional given by* (1.4)*. The functional derivative o A is the function*

$$\frac{\delta A}{\delta u(x)} = \frac{\partial a}{\partial u}(x) - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial a}{\partial u'}(x) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial a}{\partial u''} - \cdots + (-1)^n \frac{\mathrm{d}^k}{\mathrm{d}x^k}\frac{\partial a}{\partial u^{(k)}}(x) \,. \tag{1.9}$$

The expression (1.8) is very similar to the scalar product on $L^2(\Omega)$ given by

$$\langle f, g \rangle_{L^2(\Omega)} = \int_\Omega f(x) g(x)\,\mathrm{d}x \,. \tag{1.10}$$

It makes sense then to write

$$DA[u] \cdot h = \int_\Omega \frac{\delta A}{\delta u(x)} h(x)\,\mathrm{d}x = \left\langle \frac{\delta A}{\delta u(x)}, h(x) \right\rangle \,. \tag{1.11}$$

It follows from theory (CITATION) that the functional derivative beeing zero is necessery condition for beeing the extremal point of the functional if the corresponding function is smooth enough.

Theorem 3 (Euler-Lagrange equations)**.** *Let $v \in C^\infty(\Omega)$ such that $v(x)$ is stationary point of the functional A in* (1.4)*, i.e*

$$\delta A[v, h] = 0 \,. \tag{1.12}$$

Then

$$\frac{\delta A}{\delta v(x)} = 0 \,. \tag{1.13}$$

The computation above gives sense to the

$$A[u + h] = A[u] + \left\langle \frac{\delta A}{\delta u}, h \right\rangle + o(h) \,, \tag{1.14}$$

where

$$\lim_{\|h\| \to 0} \frac{o(h)}{\|h\|} = 0 \,. \tag{1.15}$$

In physical notation, the function h is often denoted as δu.

1.1.2 Distributions

Definition 4 (Distributions)**.** *Let $\mathcal{D}(\Omega)$ be the space of functions with compact support in $\Omega \subseteq \mathbb{R}^n$. The space of distributions $\mathcal{D}'(\Omega)$ is the dual space, i.e. the space of bounded linear functionals on $\mathcal{D}(\Omega)$. An element $T \in \mathcal{D}'(\Omega)$ is called a distribution.*

Definition 5 (Regular distribution)**.** *Let $a \in L^1_{loc}(\Omega)$. A regular distribution is a distribution $T_a \in \mathcal{D}'(\Omega)$ defined as*

$$\langle T_a, \phi(x) \rangle = \int_\Omega a(x)\phi(x) \mathrm{d}^n x \, . \tag{1.16}$$

Note that if two regular distributions satisfy $T_a = T_b$, then $a = b$ almost everywhere and vice versa.

Definition 6 (Dirac delta distribution)**.** *Let $y \in \Omega$. The Dirac delta distribution $\delta(x-y)$ is defined as*

$$\langle \delta(x-y), \phi(x) \rangle = \phi(y) \, . \tag{1.17}$$

Zmínit se o derivaci distribucí a o tom, že se diraci dají testovat klidně C^1 funkcemi, takže souvisejí s teorií v minulé kapitole a můžeme aplikovat teorii funkcionálních derivací i na distribuce.

Definition 7 (Derivatives of distributions)**.** *Let $T \in \mathcal{D}'(\Omega)$ be a distribution. We define its derivative by*

$$\langle D^\alpha T, \phi \rangle = (-1)^{|\alpha|} \langle T, D^\alpha \phi \rangle q \, , \tag{1.18}$$

where α is a multiindex and

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_N^{\alpha_N}} \, . \tag{1.19}$$

Let $a \in C^\infty(\Omega)$. Then it makes sense to define the product of a and arbitrary distribution T by

$$\langle aT, \phi \rangle = \langle T, a\phi \rangle \, . \tag{1.20}$$

In the next chapter we will often use some identities of the delta distributions, which are written in the next lemma and which we will refer to as the Dirac identities.

Lemma 8 (Dirac identities)**.**

$$f(y)\delta(x-y) = f(x)\delta(x-y) \, , \tag{1.21}$$

$$f(y)\delta'(x-y) = f(x)\delta'(x-y) + f'(x)\delta(x-y) \, . \tag{1.22}$$

Proof. The first relation is trivial. Direct calculation gives

$$\begin{aligned}
\langle f(x)\delta'(x-y), \psi(x) \rangle &= \langle \delta'(x-y), f(x)\psi(x) \rangle \\
&= - \langle \delta(x-y), f'(x)\psi(x) \rangle - \langle \delta(x-y), f(x)\psi'(x) \rangle \\
&= - f'(y)\psi(y) - f(y) \langle \delta(x-y), \psi'(x) \rangle \\
&= - \langle f'(x)\delta(x-y), \psi(x) \rangle + f(y) \langle \delta'(x-y), \psi(x) \rangle \\
&= \langle f(y)\delta(x-y) - f'(x)\delta'(x-y), \psi(x) \rangle \, .
\end{aligned}$$

Therefore

$$\langle f(y)\delta'(x-y),\psi(x)\rangle = \langle f'(x)\delta(x-y),\psi(x)\rangle + \langle f(x)\delta'(x-y),\psi(x)\rangle \ . \tag{1.23}$$

□

Vysvětlit, jak by se dal matematicky přesněji odvodit vztah $A(u(x)) = \int A(u(\xi))\delta(x-\xi)\,\mathrm{d}\xi$.

The function $a(u)$ can be thus identified with a functional $A[u]$ defined by the relation

$$\langle A[u(x)],\phi(x)\rangle = \langle a(u(\xi))\delta(x-\xi),\phi(\xi)\rangle \tag{1.24}$$

In the physical notation

$$A[u(x)] = \int a(u(\xi))\delta(x-\xi)\,\mathrm{d}^n\xi\ . \tag{1.25}$$

Therefore it is also common erase the difference between the function itself and its functional meaning and denote them by the same letter A:

$$A[u(x)] = \int A(u(\xi))\delta(x-\xi)\,\mathrm{d}^n\xi\ . \tag{1.26}$$

Example 9. *Let $u(x)$ be a field variable and $F[u(x)]$ a function(al). To compute its functional derivative w.r.t. $u(y)$ in different position y, we use the integral expression of F*

$$F[u(x)] = \int \mathrm{d}y F(u(y))\delta(x-y)\ . \tag{1.27}$$

Its Gateaux diferential is

$$\delta F[u,h] = \frac{\mathrm{d}}{\mathrm{d}t}F[u+th]|_{t=0} = \int \mathrm{d}y\frac{\partial F}{\partial u}(y)\delta(x-y)h(y)\ , \tag{1.28}$$

therefore

$$\frac{\delta F(u(x))}{\delta u(y)} = \frac{\partial F}{\partial u}(x)\delta(x-y)\ . \tag{1.29}$$

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Example 10. *In the special case of performing functional derivative w.r.t. the same field but in the different position,*

$$\frac{\delta u^i(x)}{\delta u^j(y)} = \delta^i_j\delta(x-y)\ . \tag{1.30}$$

Example 11.

$$\{I,J\} = \int_\Omega \mathrm{d}x' \int_\Omega \mathrm{d}y' \frac{\delta I}{\delta u^i(x)} P^{ij}(x',y')\frac{\delta J}{\delta u^j(y)}\ , \tag{1.31}$$

where

$$I = I[u],\ J = J[u]\ , \quad P^{ij}(x',y') = \alpha^{ij}(x',y')\frac{\mathrm{d}}{\mathrm{d}x'}\ . \tag{1.32}$$

Then

$$\left\{u^a(x), u^b y)\right\} = \int_\Omega \mathrm{d}x' \int_\Omega \mathrm{d}y' \delta^a_i\delta(x-x')\alpha^{ij}(x',y')\delta^b_j\delta(y-y') = \alpha^{ab}(x,y)\ . \tag{1.33}$$

1.2 Geometrical aspects

1.2.1 Riemannian manifolds

1.2.2 Lie algebras and Poisson brackets

2. Connection between Riemannian geometry and Hamiltonian mechanics

In this chapter we will prove the theorem of the connection between the hydrodynamic Poisson bracket and the metrics found in homogeneous system of hydrodynamic type. The original result was given by Dubrovin and Novikov and can be found in Dubrovin and NOVIKOV [1989]. The proof itself and the preceding lemmas are based on quite a few technical computations, which we will try not to omit, in order to be as much readable as possible.

2.1 Systems of hydrodynamic type

Definition 12 (Homogeneous system of hydrodynamic type)**.** *A homogeneous system of hydrodynamic type is an equation of the form*

$$\frac{\partial u^i}{\partial t} = f_j^{i\alpha}(u)\frac{\partial u^j}{\partial x^\alpha} \tag{2.1}$$

where $u^i(t, x^\alpha)$ are unknown functions, $i = 1, \dots, N$, $\alpha = 1, \dots, d$.

Sometimes we refer to number of equations N as the dimensinon of the hydrodynamical phase space and to the number of as the configuration space.

Before we go any further, let us mention a few examples.

Example 13. *The energy is given by the functional*

$$E = \int e(\rho, \boldsymbol{m}, s, \boldsymbol{F})\,\mathrm{d}\boldsymbol{x} \tag{2.2}$$

The evolution equations in the Eulerian frame are

$$\begin{aligned}
\partial_t \rho &= -\,\partial_i(\rho E_{m_i}) && (2.3)\\
\partial_t m_i &= -\,\partial_j(m_i E_{m_j}) - \rho\partial_i E_\rho - m_j \partial_i E_{m_j} - m_j \partial_i E_{m_j} && (2.4)\\
&\quad -s\partial_i E_s - F_J^j \partial_i E_{F_J^j} + \partial_j(F_I^j E_{F_I^i} + F_O^i E_{F_I^i}) && (2.5)\\
\partial_t s &= -\,\partial_i(s E_{m_i}) && (2.6)\\
\partial_t F_I^i &= -\,E_{m_k}\partial_k F_I^i + F_I^j \partial_j E_{m_i}\,. && (2.7)
\end{aligned}$$

Total energy density e can be prescribed as

$$e = \frac{\boldsymbol{m}^2}{2\rho} + \epsilon(\rho, s, \boldsymbol{F})\,, \tag{2.8}$$

where ϵ is the elastic and internal energy. In particular, $E_{\boldsymbol{m}} = \frac{\boldsymbol{m}}{\rho} = \boldsymbol{v}$ becomes the velocity. The evolution equation for the deformation gradient then gets the explicit form

$$\partial_t \boldsymbol{F} = -(\boldsymbol{v}\cdot\nabla)\boldsymbol{F} + \nabla\boldsymbol{V}\cdot\boldsymbol{F}\,, \tag{2.9}$$

which is the usual evolution equation for $\boldsymbol{F}$ in the Eulerian frame.

It was Riemann who noticed that the theory of systems of the form (2.1) is the theory of tensors.

Proposition 14 (Transformation of $f_j^{i\alpha}$)**.** *Under smooth change of variables $u^i \mapsto v^a$ of the form*

$$u^i = u^i(v^1, \cdots, v^N) \tag{2.10}$$

the functions $f_j^{i\alpha}$ transform for each a according to the tensor law

$$f_b^{a\alpha}(v) = \frac{\partial v^a}{\partial u^i}\frac{\partial u^j}{\partial v^b} f_j^{i\alpha}(u)\,. \tag{2.11}$$

Proof. Using a direct calculation we obtain

$$\frac{\partial u^i}{\partial t}(v) = \frac{\partial u^i}{\partial v^a}\frac{\partial v^a}{\partial t}\,, \tag{2.12}$$

$$\frac{\partial u^j}{\partial x^\alpha}(v) = \frac{\partial u^j}{\partial v^b}\frac{\partial v^b}{\partial x^\alpha}\,, \tag{2.13}$$

so

$$\frac{\partial u^i}{\partial v^a}\frac{\partial v^a}{\partial t} = \frac{\partial u^j}{\partial v^b} f_j^{i\alpha}(v(u))\frac{\partial v^b}{\partial x^\alpha} \tag{2.14}$$

and therefore

$$\frac{\partial v^a}{\partial t} = \underbrace{\frac{\partial v^a}{\partial u^i}\frac{\partial u^j}{\partial v^b} f_j^{i\alpha}(v(u))}_{f_b^{a\alpha}(v)}\frac{\partial v^b}{\partial x^\alpha}\,. \tag{2.15}$$

□

According to the previous lemma, if we denote the space of points $(t, x^\alpha) \subset \mathbb{R} \times \mathbb{R}^N$ as some manifold M, then the unknown variables $u^i(t, x^\alpha)$ can be seen as the local coordinates on that manifold. The functions $f_j^{i\alpha}$ are transformed as tensors of the type (1,1) in the i and j indices.

Our next purpose is to gain a richer geometry on that space M. If the functions $f_j^{i\alpha}$ have a special form, we can introduce a geometry of the Poisson brackets.

Definition 15. *1. A Poisson bracket of the hydrodynamic type of two functionals I_1 a I_2 is defined as*

$$\{I_1, I_2\} = \int \mathrm{d}x \left[\frac{\delta I_1}{\delta u^q(x)} A^{qp} \frac{\delta I_2}{\delta u^p(x)}\right] \tag{2.16}$$

where

$$A = (A^{qp}) = \left(g^{qp\alpha}(u)\frac{\mathrm{d}}{\mathrm{d}x^\alpha} + b_s^{qp\alpha}(u)\frac{\partial u^s}{\partial x^\alpha}\right)\,. \tag{2.17}$$

where $g^{ij\alpha}$ and $b_k^{ij\alpha}$ are certain functions, $i, j, k = 1, \ldots, N$ and $\alpha = 1, \ldots, d$. In particular

$$\left\{u^i(x), u^j(y)\right\} = g^{ij\alpha}[u(x)]\delta_\alpha(x-y) + b_k^{ij\alpha}[u(x)]u_\alpha^k(x)\delta(x-y)\,. \tag{2.18}$$

2. *Functional of the hydrodynamic type is a functional of the form*

$$H[u] = \int_{\mathbb{R}^d} h(u(x))\mathrm{d}^d x\,, \tag{2.19}$$

where h is independent u_α, $u_{\alpha\beta}$. The function h is called a hamiltonian density.

3. *The hamiltonian system of hydrodynamic type is a system of the form*

$$u^i_t(x) = \left[g^{ij\alpha}[u(x)]\frac{\partial^2 h(u)}{\partial u^j \partial u^k} + b^{ij\alpha}_k[u(x)]\frac{\partial h(u)}{\partial u^j}\right] u^k_\alpha(x)\,, \tag{2.20}$$

where $H[u]$ is the functional of the hydrodynamic type and h the hamiltonian density.

Proposition 16.

$$u^i_t(x) = \left\{u^i, H\right\}\,. \tag{2.21}$$

Proof.

$$\left\{u^i(x), H\right\} = \int \mathrm{d}x'\mathrm{d}y' \frac{\delta u^i(x)}{\delta u^p(x')}\left[g^{pq}(u(x'))\frac{\mathrm{d}}{\mathrm{d}x'} + b^{pq}_m \frac{\partial u^m}{\partial x}\right]\frac{\delta H}{\delta u^q(y')}\,. \tag{2.22}$$

$$\frac{\delta u^i(x)}{\delta u^p(x')} = \delta^i_p \delta(x - x') \tag{2.23}$$

$$H[u] = \int h(u(y'))\mathrm{d}y'\,, \quad \frac{\delta H}{\delta u^q(y')} = \int \frac{\partial h}{\partial u^q}(y)\delta(y - y') \tag{2.24}$$

$$\left\{u^i(x), H\right\} = \int \mathrm{d}y' \left[g^{iq}(u(x))\frac{\mathrm{d}}{\mathrm{d}x} + b^{iq}_m(u(x))\frac{\partial u^m}{\partial x}\right]\frac{\partial h}{\partial u^q}(y)\delta(y - y') \tag{2.25}$$

$$= g^{iq}(u(x))\frac{\partial^2 h}{\partial u^k \partial u^q}\frac{\partial u^k}{\partial x} + b^{iq}_m(u(x))\frac{\partial u^m}{\partial x}\frac{\partial h}{\partial u^q} \tag{2.26}$$

$$= \left[g^{ij}(u(x))\frac{\partial^2 h}{\partial u^j \partial u^k} + b^{ij}_k(u(x))\frac{\partial h}{\partial u^j}\right]\frac{\partial u^k}{\partial x}\,. \tag{2.27}$$

□

Dokončit a přidělat α. kde se předělá $\frac{\mathrm{d}}{\mathrm{d}x}$ na $\frac{\mathrm{d}}{\mathrm{d}y}$?

We have not shown yet that the Poisson bracket of the hydrodynamic type defined above is indeed a Poisson bracket in the sense of a classical meaning, that means it satisfies the three properties, skew-symmetry, Leibniz rule and Jacobi identity. In fact, this will be the key theorem of this chapter.

2.2 One-dimensional case

Consider first the one-dimensional case, the unknown functions u^i are depending only on two variables (t, x). The hamiltonian system is then

$$\frac{\partial u^i}{\partial t}(t,x) = \left[g^{ij}[u(t,x)] \frac{\partial^2 h(u)}{\partial u^j \partial u^k} + b^{ij}_k [u(t,x)] \frac{\partial h(u)}{\partial u^j} \right] \frac{\partial u^k}{\partial x}(t,x) \tag{2.28}$$

and the Poisson bracket of thy hydrodynamic type is of the form

$$\{I_1, I_2\} = \int \mathrm{d}x \left[\frac{\delta I_1}{\delta u^q(x)} A^{qp} \frac{\delta I_2}{\delta u^p(x)} \right] , \quad A^{qp} = \left(g^{qp}(u) \frac{\mathrm{d}}{\mathrm{d}x} + b^{qp}_s(u) \frac{\partial u^s}{\partial x} \right) . \tag{2.29}$$

In particular

$$\left\{ u^i(x), u^j(y) \right\} = g^{ij}[u(t,x)] \delta'(x-y) + b^{ij}_k [u(t,x)] \frac{\partial u^k}{\partial x} \delta(x-y) . \tag{2.30}$$

We begin to gain a Riemannian geometry on the space M.

Definition 17. *We say that the Poisson bracket is non-degenerate if* $\det(g^{ij}) \neq 0$.

We will show later that the non-degeneracy is invariant under the transformation of the coordinates.

Definition 18. *Let $(g^{ij}(u))$ be a non-degenerate matrix. We define functions $\Gamma^i_{jk}(u)$ by*

$$b^{ij}_k(u) = -g^{is}(u) \Gamma^j_{sk}(u) , \quad i,j,k,s = 1, \dots, N . \tag{2.31}$$

The choice of the denoting g^{ij} and Γ^i_{jk} has its purpose. We will show soon that these are in fact the metrics and afinne coeficients on M.

Proposition 19 (Leibniz rule for Poisson bracket)**.** *The Poisson brackets of the hydrodynamic type satisfy the Leibniz rule*

$$\left\{ v^p(u^i(x)), v^q(u^j(y)) \right\} = \frac{\partial v^p}{\partial u^i}(x) \frac{\partial v^q}{\partial u^j}(y) \left\{ u^i(x), u^j(y) \right\} . \tag{2.32}$$

Proof. The coordinates are written as functionals in the form

$$v^p(u^i(x)) = \int \mathrm{d}x' v^p(u^i(x')) \delta(x - x') . \tag{2.33}$$

Applying this to the definition of the Poisson bracket, we have

$$\left\{ v^p(u^i(x)), v^q(u^j(y)) \right\} = \int \mathrm{d}x' \int \mathrm{d}y' \underbrace{\frac{\delta v^p}{\delta u^i(x')}}_{\frac{\partial v^p}{\partial u^i}(x')\delta(x-x')} \left\{ u^i(x'), u^j(y') \right\} \underbrace{\frac{\delta v^q}{\delta u^j(y')}}_{\frac{\partial v^q}{\partial u^j}(y')\delta(y-y')} = \tag{2.34}$$

$$= \frac{\partial v^p}{\partial u^i}(x) \frac{\partial v^q}{\partial u^j}(y) \left\{ u^i(x), u^j(y) \right\} . \tag{2.35}$$

□

Proposition 20 (O transformaci proměnných)**.** *Uvažujme transformaci $u^i \mapsto v^i$ danou vztahem*

$$v^i = v^i(u^1, \ldots, u^N)\,, \quad i = 1, \ldots, N\,, \tag{2.36}$$

která je diffeomorfismem třídy C^2. Pak platí následující tvrzení:

1. *Poissonovy závorky se transformují jako tensory typu (2,0), tj.*

2. *Koeficienty $g^{ij}(u)$ se transformují jako tensory typu (0,2), tj.*

$$g^{pq}(v) = \frac{\partial v^p}{\partial u^i}\frac{\partial v^q}{\partial u^j} g^{ij}[u(v)]\,, \quad p, q = 1, \ldots, N\,. \tag{2.37}$$

3. *Koeficienty Γ^i_{jk} se transformují jako Christoffelovy symboly (složky afinní konexe), tj.*

$$\Gamma^p_{qr}(v) = \frac{\partial v^p}{\partial u^i}\frac{\partial u^j}{\partial v^q}\frac{\partial u^k}{\partial v^r}\Gamma^i_{jk}(u) + \frac{\partial v^p}{\partial u^i}\frac{\partial^2 u^i}{\partial v^q \partial v^r}\,. \tag{2.38}$$

Proof. 1. Přímým dosazením do definice

2. Rozepišme transformaci závorky (2.32) pomocí (2.30)

$$g^{pq}[v(u(x))]\delta'(x-y) + b^{pq}_s[v(u(x))]\frac{\partial v^s}{\partial u^k}(x)\frac{\partial u^k}{\partial x}\delta(x-y) = \frac{\partial v^p}{\partial u^i}(x)\frac{\partial v^q}{\partial u^j}(y)\Bigg[g^{ij}[u(x)]\delta'(x-y \tag{2.39}$$

V dalším pro přehlednost označme

$$T^a_k(x) := \frac{\partial v^a}{\partial u^k}(x)\,. \tag{2.40}$$

Na pravé straně rovnice (2.39) vyjádříme $T^q_j(y)\delta'(x-y)$ pomocí lemmatu 8 on page 4. Tím převedeme všechny funkce do proměnné x, což už dále nebudeme explicitně vypisovat. Dostaneme

$$g^{pq}(v)\delta'(x-y) + b^{pq}_n(v)T^n_j\frac{\partial u^j}{\partial x}\delta(x-y) = T^p_i T^q_j g^{ij}(u)\delta'(x-y) + \left(\frac{\partial^2 v^q}{\partial u^j \partial u^k} + T^p_i T^q_j b^{ij}_k\right)\frac{\partial u}{\partial x} \tag{2.41}$$

Porovnáním členů s δ' dostaneme vztah pro transformaci metriky (2.37).

3. Porovnáním členů s δ dostaneme

$$b^{pq}_n(v)T^n_k = b^{ij}_k(u)T^p_i T^q_j + g^{ij}(u)T^p_i\frac{\partial^2 v^q}{\partial u^i \partial u^k} \tag{2.42}$$

a po rozpisu pomocí (2.31)

$$-g^{ps}(v)\Gamma^q_{sn}(v)T^n_k = -g^{il}(u)\Gamma^j_{lk}(u)T^p_i T^q_j + g^{ij}(u)T^p_i\frac{\partial^2 v^q}{\partial u^i \partial u^k}\,. \tag{2.43}$$

Na levé straně využijeme transformaci metriky (2.37) a dostaneme

$$-g^{ij}(u)\Gamma^q_{sm}(v)T^p_i T^s_j T^m_k = -g^{ij}(u)\Gamma^l_{jk}T^p_i T^q_l + g^{ij}(u)\frac{\partial^2 v^q}{\partial u^i \partial u^k}T^p_i \tag{2.44}$$

a po zkrácení $g^{ij}(u)T^p_i$

$$\Gamma^l_{jk}(u)T^q_l = \Gamma^q_{sm}(v)T^s_j T^m_k + \frac{\partial^2 v^q}{\partial u^i \partial u^k}\,. \tag{2.45}$$

V posledním kroku přenásobíme inverzní maticí $(T^{-1})^a_q$ a dostaneme

$$\Gamma^a_{jk}(u) = \Gamma^q_{sm}(v)\frac{\partial v^s}{\partial u^j}\frac{\partial v^m}{\partial u^k}\frac{\partial u^a}{\partial v^q} + \frac{\partial^2 v^q}{\partial u^i \partial u^k}\frac{\partial u^a}{\partial v^q}\,, \tag{2.46}$$

což je hledaný transformační vztah (2.38).

□

2.2.1 Ekvivalence riemannovské a poissonovské struktury

Theorem 21 (Ekvivalence pseudoriemannovské a hamiltonovské struktury)**.** *Nechť* $\det(g^{ij}) \neq 0$. *Pak je vztahem* (2.30) *definována Poissonova závorka splňující antisymetrii, Leibnizovo pravidlo a Jacobiho identitu právě tehdy, když jsou splněny následující podmínky:*

1. g^{ij} *je symetrický tensor, tj. definuje pseudo-riemannovskou metriku na prostoru* M^N.
2. *Koeficienty* Γ^i_{jk} *jsou složky Levi-Civitovy konexe příslušející metrice* g^{ij}.
3. *Odpovídající konexe má nulovou torzi a křivost.*

Corollary. Existují lokální souřadnice $w^i = w^i(u^1, \ldots, u^N)$, $i = 1, \ldots, N$ takové, že

$$\tilde{g}^{ij}(w) = \text{konst}\,, \quad b^{ij}_k(w) = 0\,. \tag{2.47}$$

V těchto souřadnicích je Poissonova závorka konstantní:

$$\left\{w^i(x), w^j(y)\right\} = \tilde{g}^{ij}\delta'(x-y)\,. \tag{2.48}$$

Úplný lokální invariant Poissonových závorek je signatura pseudo-eukleidovské metriky $\tilde{g}^{ij}$.

Co se tímhle myslí?

Důkaz věty 21. **Krok 1: Antisymetrie Poissonovy závorky implikuje symetrii metriky a kompatibilitu s Γ^i_{jk}**

Poissonova závorka je antisymetrická, tj.

$$\left\{u^i(x), u^j(y)\right\} + \left\{u^j(y), u^i(x)\right\} = 0\,. \tag{2.49}$$

Rozepišme druhou závorku

$$\left\{u^j(y), u^i(x)\right\} = g^{ji}[u(y)]\delta'(y-x) + b^{ji}_k[u(y)]\frac{\partial u^k}{\partial y}\delta(y-x)\,. \tag{2.50}$$

Využijeme vztahů

$$\delta(y-x) = \delta(x-y)\,, \quad \delta'(y-x) = -\delta'(x-y) \tag{2.51}$$

a lemmatu 8 on page 4 aplikovaného na $g^{ji}[u(y)]\delta'(y-x)$. Dostaneme

$$\left\{u^j(y), u^i(x)\right\} = -g^{ji}[u(x)]\delta'(x-y) - \frac{\partial g^{ji}}{\partial u^k}[u(x)]\frac{\partial u^k}{\partial x}\delta(x-y) + b^{ji}_k[u(x)]\frac{\partial u^k}{\partial x}\delta(x-y)\,. \tag{2.52}$$

Celkově

$$0 = \left\{u^i(x), u^j(y)\right\} + \left\{u^j(y), u^i(x)\right\} = \left[g^{ij}[u(x)] - g^{ji}[u(x)]\right]\delta'(x-y) + \left[b^{ij}_k - \frac{\partial g^{ji}}{\partial u^k} + b^{ji}_k\right]\frac{\partial u^k}{\partial x}\delta(\tag{2.53}$$

Výraz na pravé straně bude nulový právě tehdy, jestliže

$$g^{ij} = g^{ji}\,, \tag{2.54}$$

$$\frac{\partial g^{ij}}{\partial u^k} = b^{ij}_k + b^{ji}_k\,. \tag{2.55}$$

Rovnice (2.54) říká, že g^{ij} je symetrický tenzor, dle předpokladu $\det g^{ij} \neq 0$ je nedegenerovaný, tj. definuje metriku na varietě M^N. Rovnice (2.55) dává

$$\frac{\partial g^{ij}}{\partial u^k} + g^{is}\Gamma^j_{sk} + g^{sj}\Gamma^i_{sk} = 0\,, \tag{2.56}$$

takže Γ^k_{ij} dává konexi kompatibilní s metrikou g^{ij}.

Krok 2: Jacobiho identita je ekvivalentní nulové torzi a křivosti

Abychom dokázali, že je nulová křivost a torze, využijeme Jacobiho identity. Položme

$$J^{ijk}(x,y,z) = \left\{\left\{u^i(x), u^j(y)\right\}, u^k(z)\right\} + \left\{\left\{u^j(y), u^k(z)\right\}, u^i(x)\right\} + \left\{\left\{u^k(z), u^i(x)\right\}, u^j(y)\right\}\,. \tag{2.57}$$

Musíme ukázat, že je $J^{ijk}(x,y,z) = 0$ ve smyslu distribucí, tedy

$$\left\langle J^{ijk}(x,y,z), p_i(x)q_j(y)r_k(z)\right\rangle = 0 \quad \text{pro } p_i, q_j, r_k \in \mathcal{D}'R \tag{2.58}$$

a protože se jedná o regulární distribuci, ověřujeme

$$\int_{\mathbb{R}^3} J^{ijk}(x,y,z)p_i(x)q_j(y)r_k(z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = 0\,. \tag{2.59}$$

Takový integrál lze převést na jednodimenzionální integrál

$$\int_{\mathbb{R}} \sum_{\sigma,\tau=0}^{2} A^{ijk}_{\sigma\tau} p_i q_j^{(\sigma)} r_k^{(\tau)}\,\mathrm{d}x = 0\,, \tag{2.60}$$

kde koeficienty $A^{ijk}_{\sigma\tau}$ jsou nezávislé na p, q, r. Obdržíme tedy systém rovnic

$$A^{ijk}_{\sigma\tau} = 0 \quad \forall i, j, k = 1, \dots, N \; ; \quad 0 \le \sigma, \tau \le 2 \, . \tag{2.61}$$

Rád bych si funkce napsal explicitně, určitě se v nich objeví derivace druhého řádu někde? Ale nedaří se mi to sestavit.

Přepišme tyto rovnice explicitně.

$$\left\{ \left\{ u^i(x), u^j(y) \right\}, u^k(z) \right\} =$$
$$= \int \mathrm{d}\xi \frac{\delta}{\delta u^m(\xi)} \left[g^{ij}(x)\partial_x \delta(x-y) + b^{ij}_k(x)\partial_x u^k \delta(x-y) \right] \left[g^{mn}(\xi)\partial_\xi + b^{mn}_p \partial_\xi u^p \right] \frac{\delta u^k(z)}{\delta u^n(\xi)} \, .$$

Jednotlivé členy jsou

$$\frac{\delta u^k(z)}{\delta u^n(\xi)} = \delta^k_n \delta(z-\xi) \, , \tag{2.62}$$

$$\frac{\delta}{\delta u^m(\xi)} \left[g^{ij}(x)\partial_x \delta(x-y) + b^{ij}_k(x)\partial_x u^k \delta(x-y) \right] = \tag{2.63}$$

$$= \left[g^{ij}_{,m} \partial_x \delta(x-y) + b^{ij}_{,m}(x) u^k_x \delta(x-y) \right] \delta(x-\xi) \, , \tag{2.64}$$

kde jsme označili

$$g^{ij}_{,m} = \frac{\partial g^{ij}}{\partial u^m} \, , \quad b^{ij}_{k,m} = \frac{\partial b^{ij}_k}{\partial u^m} \, , \quad u^k_x = \frac{\partial u^k}{\partial x} \, , \quad \partial_x \delta(x-y) = \frac{\partial}{\partial x} \delta(x-y) \, . \tag{2.65}$$

V integrálu tedy předěláme ξ na x díky členu $\delta(x-\xi)$ a indexy n na k díky δ^n_k. Zůstane

$$\left[g^{ij}_{,m}(x)\partial_x \delta(x-y) + b^{ij}_{k,m}(x) u^k_x \delta(x-y) \right] \left[g^{mk}(x)\partial_x \delta(z-x) + b^{mk}_p(x) u^p_x \delta(z-x) \right] .$$
$$= \left[g^{ij}_{,m} g^{mk} \partial_x u^m \partial_x \delta(x-y) + b^{ij}_{k,m} g^{mk} u^k_x \delta(x-y) \right] \partial_x \delta(z-x) +$$
$$+ \left[g^{ij}_{,m} b^{mk}_p u^p_x \partial_x \delta(x-y) + b^{ij}_{k,m} b^{mk}_p u^p_x u^k_x \delta(x-y) \right] \delta(z-x) \, .$$

Dostaneme

$$\left\{ \left\{ u^i(x), u^j(y) \right\}, u^k(z) \right\} = g^{ij}_{,m} g^{mk} \partial_x \delta(x-y) \partial_x \delta(z-x) + b^{ij}_{k,m} g^{mk} u^k_x \delta(x-y) \partial_x \delta(z-x)$$
$$+ g^{ij}_m b^{mk}_p u^p_x \partial_x \delta(x-y) \delta(z-x) + b^{ij}_{k,m} b^{mk}_p u^p_x u^k_x \delta(x-y) \delta(z-x) \, .$$

Jednotlivé členy označíme

$$\left\{ \left\{ u^i(x), u^j(y) \right\}, u^k(z) \right\} = E(x) \partial_x \delta(x-y) \partial_x \delta(z-x) + F(x) \delta(x-y) \partial_x \delta(z-x)$$
$$+ G(x) \partial_x \delta(x-y) \delta(z-x) + H(x) \delta(x-y) \delta(z-x) \, .$$

Jedna z těchto rovnic dává

$$A^{ijk}_{02} = b^{ij}_s g^{sk} - b^{kj}_s g^{si} = -g^{ip} \Gamma^j_{ps} g^{sk} + g^{km} \Gamma^j_{ms} g^{si} = 0 \, , \tag{2.66}$$

odtud (po přenásobení $g_{ai} g_{bk}$)

$$\Gamma^j_{ab} - \Gamma^j_{ba} = 0 \, . \tag{2.67}$$

Tento vztah říká, že jsou koeficienty afinní konexe symetrické, tj. příslušná konexe má nulovou torzi.

Další z rovnic dává

$$A^{ijk}_{00} = B^{ijk}_p(u)u^p_{xx} + C^{ijk}_{pq}(u)u^p_x u^q_x = 0\,, \tag{2.68}$$

kde

Je v původním článku typo??

$$B^{ijk}_p = (b^{jk}_{s,p} - b^{jk}_{p,s})g^{si} + b^{ij}_s b^{sk}_p - b^{ik}_s b^{sj}_p\,. \tag{2.69}$$

Nulovost B^{ijk}_p dává nulovou křivost, jak plyne z rozpisu

$$B^{ijk}_p = - g^{is}\left(g^{jm}_{,p}\Gamma^k_{ms} + g^{jm}\Gamma^k_{ms,p} - g^{jm}_{,s}\Gamma^k_{mp} - g^{jm}\Gamma^k_{mp,s}\right) + g^{im}\Gamma^j_{ms}g^{sn}\Gamma^k_{np} - g^{im}\Gamma^k_{ms}g^{sn}\Gamma^j_{np} = \tag{2.70}$$

$$= g^{is}g^{jn}\left(\Gamma^m_{np}\Gamma^k_{ms} - \Gamma^m_{ns}\Gamma^k_{mp} + \Gamma^k_{mp,s} - \Gamma^k_{ms,p}\right) + g^{is}g^{nm}\left(\Gamma^j_{np}\Gamma^k_{ms} - \Gamma^j_{ns}\Gamma^k_{mp}\right) + g^{im}g^{sn}\left(\Gamma^j_{ms}\Gamma^k_{n}\right. \tag{2.71}$$

$$= - g^{is}g^{jn}\left(\Gamma^m_{ns}\Gamma^k_{mp} - \Gamma^m_{np}\Gamma^k_{ms} + \Gamma^k_{ms,p} - \Gamma^k_{mp,s}\right) = \tag{2.72}$$

$$= - g^{is}g^{jn}R^k_{nps}\,. \tag{2.73}$$

Takže křivost metriky g^{ij} je nulová. Tím jsme dokázali, že pokud je vztahem (2.30) definována Poissonova závorka, jsou splněny podmínky 1-3.

Krok 3: Souřadnice, ve kterých je závorka triviální, dávají postačitelnost

Jestliže mají koeficienty afinní konexe Γ^k_{ij} nulovou torzi i křivost, existují souřadnice $w^i = w^i(u^1, \ldots, u^N)$ pro $i = 1, \ldots, N$ takové, že $g^{ij} =$ konst a $b^{ij}_k = 0$. V těchto souřadnicích je Poissonova závorka konstantní

$$\left\{w^i(x), w^j(y)\right\} = \tilde{g}^{ij}\delta'(x-y)\,. \tag{2.74}$$

Jacobiho identita, antisymetrie i Leibnizovo pravidlo jsou pro tuto závorku triviálně splněny. Tím jsme ukázali postačitelnost podmínek (1)-(3) pro vlastnosti Poissonovy závorky. □

2.2.2 Kdy je hydrodynamický systém hamiltonovský?

Nyní můžeme explicitně přepsat podmínky, při kterých je obecný systém hydrodynamických rovnic hamiltonovský vůči nějaké nedegenerované Poissonově závorce. Nejprve si povšimneme, že funkce $f^i_j(u)$ lze přepsat pomocí Laplaceova–Beltramiho operátoru.

Proposition 22 (O zápisu pomocí Laplaceova–Beltramiho operátoru)**.** *Mějme hamiltonovský systém rovnic*

$$\frac{\partial u^i}{\partial t} = f^i_k(u)\frac{\partial u^k}{\partial x}\,, \quad f^i_k(u) = g^{ij}(u)\frac{\partial^2 h}{\partial u^j \partial u^k} - g^{is}(u)\Gamma^j_{sk}(u)\frac{\partial h}{\partial u^j}\,. \tag{2.75}$$

Pak lze psát

$$f^i_k(u) = \nabla^i\nabla_k h(u)\,, \tag{2.76}$$

kde ∇_j je Levi-Civitova kovariantní derivace metriky g_{ij} a $\nabla^i = g^{is}\nabla_s$.

Proof. Přímým dosazením

$$\nabla_k h(u) = \frac{\partial h}{\partial u^k} , \tag{2.77}$$

$$\nabla^i \nabla_k h(u) = g^{is} \nabla_s \frac{\partial h}{\partial u^k} = g^{is} \left(\frac{\partial^2 h}{\partial u^s \partial u^k} - \Gamma^j_{sk} \frac{\partial h}{\partial u^j} \right) = f^i_k(u) . \tag{2.78}$$

□

Pomocí tohoto zápisu snadno můžeme sepsat postačující podmínky hamiltonovskosti. Důkaz plyne ihned z teorému 21 on page 12.

Theorem 23 (Postačující podmínka hamitonovskosti systému)**.** *Systém $u^i_t = f^i_j(u) u^j_x$ je hamiltonovský právě tehdy, když existuje nedegenerovaná metrika $g^{ij}(u)$, jejíž afinní konexe má nulovou křivost a splňuje*

$$g_{ij} f^k_j = g_{jk} f^k_i , \tag{2.79}$$

$$\nabla_i f^k_j = \nabla_j f^k_i . \tag{2.80}$$

Speciálně vztah (2.80) říká, že ∇_i má nulovou torzi.

2.2.3 Rekonstrukce metriky z $f^i_j(u)$?

Máme-li zadaný hamiltonovský systém s maticí $f^i_j(u)$, lze metriku $g^{ij}(u)$ zkonstruovat jednoznačně? Tuto otázku nyní vyřešíme pro $N \geq 3$.

Označme λ_α vlastní čísla matice $f^i_j(u)$. (Mohou být komplexní.) Předpokládejme, že jsou navzájem různá. Označme odpovídající bázi vlastních vektorů $e_\alpha(u)$. Definujme koeficient $c^\gamma_{\alpha\beta}(u)$ vztahem

$$[e_\alpha, e_\beta] = c^\gamma_{\alpha\beta} e_\gamma , \tag{2.81}$$

kde $[\cdot,\cdot]$ značí obyčejný komutátor funkcí. Předpokládejme dále, že pro navzájem různé α, β, γ je $c^\gamma_{\alpha\beta}$ různé od nuly .

Definition 24. *Matici $f^i_j(u)$ splňující podmínky výše nazveme hamiltonovskou maticí.*

Theorem 25 (O rekonstrukci metriky)**.** *Nechť $N \geq 3$. Nechť je dána hamiltonovská matice $f^i_j(u)$. Pak lze zkonstruovat nedegenerovanou metriku $g^{ij}(u)$ s nulovou křivostí jednoznačně až na násobek konstantou.*

Tomuhle nerozumím a myslím si, že v článku mají typo.

Proof. Z rovnice (2.79) je vidět, že v bázi e_α je metrika g^{ij} diagonální.

Opravdu?

V této bázi pak bude mít rovnice (2.80) tvar (zde se nesčítá přes opakované indexy)

$$\partial_\alpha \lambda_\beta \delta^\gamma_\beta - \partial_\beta \lambda_\alpha \delta^\gamma_\alpha + (\Gamma^\gamma_{\alpha\beta} - \Gamma^\gamma_{\beta\alpha}) \lambda_\gamma + \Gamma^\gamma_{\beta\alpha} (\lambda_\beta - \lambda_\alpha) = 0 . \tag{2.82}$$

Zde ∂_α je derivace ve směru e_α a konexe $\Gamma^\gamma_{\alpha\beta}$ jsou definované rovnostmi

$$\nabla_{e_\beta} e_\alpha = \sum \Gamma^\gamma_{\alpha\beta} e_\gamma \,. \tag{2.83}$$

Normalizujme nyní vlastní vektory e_α tak, aby v této bázi byla metrika jednotková matice, tj. $g^{\alpha\beta} = \delta^{\alpha\beta}$.

Výraz $c^\gamma_{\alpha\beta}$ má význam torze (z definice) a platí

$$c^\gamma_{\beta\alpha} = \Gamma^\gamma_{\alpha\beta} - \Gamma^\gamma_{\beta\alpha} \,. \tag{2.84}$$

□

2.3 Vícerozměrný případ

Ve vícerozměrném případě máme lineární bundle metrik a s nimi spojených konexí. Pro každou záměnu prostorových proměnných $x^\alpha \mapsto c^\alpha_\beta x^\beta$ pro $\alpha = 1, \cdots, d$ splňující $\det c^\alpha_\beta = 1$ se metrika $g^{ij\alpha}$ a konexe $b^{ij\alpha}_k$ transformují jako komponenty vektoru.

Definition 26. *Řekneme, že bundle metrik $g^{ij\alpha}$ je silně nedegenerovaný, jestliže pro nějakou sadu c_α je lineární kombinace $c_\alpha g^{ij\alpha}$ nedegenerovaná matice.*

Theorem 27. *Nechť je hamiltonovský systém silně nedegenerovaný.*

1. *Pro $N = 1$ lze Poissonovu závorku redukovat na konstantní formu*

$$g^{ij}(u) = \tilde{g}^{ij}(u) \,. \tag{2.85}$$

2. *Pro $N \geq 2$ lze Poissonovu závorku redukovat na lineární formu*

$$g^{ij\alpha}(u) = g^{ij\alpha}_k u^k + \tilde{g}^{ij\alpha} \,, \quad \alpha = 1, \cdots, d \,, \tag{2.86}$$

(2.87)

kde koeficienty $g^{ij\alpha}_k = b^{ij\alpha}_k + b^{ij\alpha}_k$, $\tilde{g}^{ij\alpha}$ a $b^{ij\alpha}_k$ jsou konstantní.

Bibliography

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List of Figures

A. Appendix