Tangent Spaces, Overview

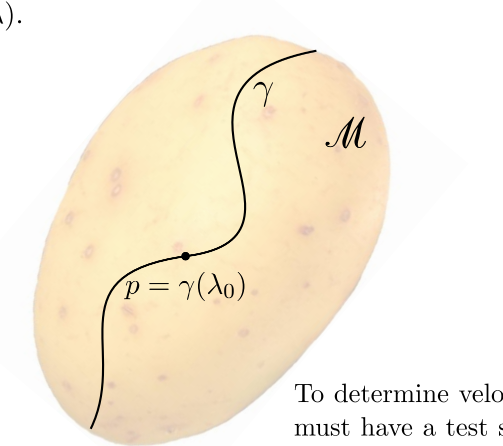
by M.B.Kocic – Version 1.02 (2017-06-03) – Advanced GR – (partly based on F.P. Schuller's lectures, WE-Heraeus, 2015)

1 The velocity of a curve γ at a point $p \in \mathscr{M}$.

Let $(\mathscr{M}, \mathscr{O}, \mathscr{A})$ be a smooth manifold with a topology $\mathscr{O}$ and a smooth atlas $\mathscr{A}$. Let further,

$$\gamma : \mathbb{R} \to \mathscr{M},$$

be at least C^1 smooth curve on $\mathscr{M}$ parameterized by $\lambda \mapsto \gamma(\lambda)$.



To determine velocity, we must have a test scalar field.

The *velocity of a curve* γ *at a point* $p \in \mathscr{M}$ is the **linear map**,

$$v_{\gamma,p} : C^\infty(\mathscr{M}) \to \mathbb{R},$$

defined by,

$$f \mapsto v_{\gamma,p}(f) := (f \circ \gamma)'(\lambda_0),$$

where $p = \gamma(\lambda_0)$.

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\gamma"] \arrow[rr, bend right, "f \circ \gamma"'] & \mathscr{M} \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

Here, $C^\infty(\mathscr{M})$ is the vector space of smooth functions on the manifold $\mathscr{M}$,

$$C^\infty(\mathscr{M}) := \{f : \mathscr{M} \to \mathbb{R} \mid f \text{ is smooth}\}.$$

Note: $(f+g)(p) := v(p) + g(p)$, $(\alpha \cdot f)(p) := \alpha \cdot f(p)$.

2 Tangent space, intrinsic definition

The *tangent space to* $\mathscr{M}$ *at* p, denoted by $T_p\mathscr{M}$, is the **set** of the velocities of all smooth curves in $\mathscr{M}$ passing through p,

$$T_p\mathscr{M} := \{v_{\gamma,p} \mid \gamma \text{ is smooth}\}.$$

Define two binary operations on $T_p\mathscr{M}$,

$$(v_{\gamma,p} + v_{\delta,p})(f) := v_{\gamma,p}(f) + v_{\delta,p}(f),$$

$$(\alpha \cdot v_{\gamma,p})(f) := \alpha \cdot v_{\gamma,p}(f),$$

where $v_{\gamma,p}, v_{\delta,p} \in T_p\mathscr{M}$ and $\alpha \in \mathbb{R}$.

Proposition. $T_p\mathscr{M}$ equipped with $+$ and $\cdot$ from above is a **vector space**.

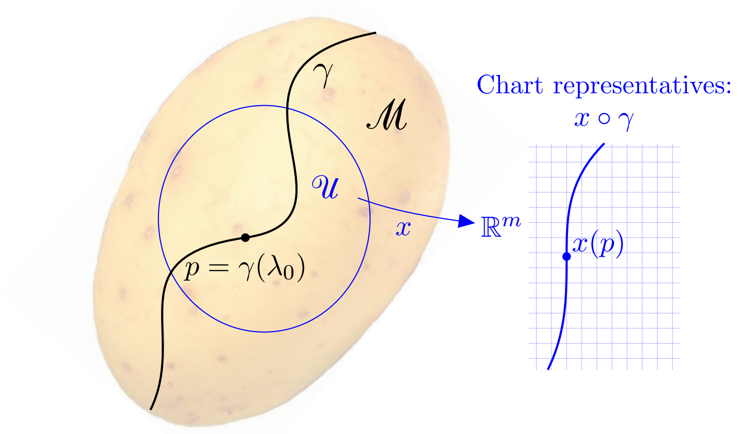
Proof by construction (in an arbitrary chart). E.g., $(x \circ \sigma)(\lambda) = (x \circ \gamma)(\lambda_0 + \lambda) + (x \circ \delta)(\lambda_1 + \lambda) - (x \circ \gamma)(\lambda_0)$ where $p = \gamma(\lambda_0) = \delta(\lambda_1)$.

3 The components of a vector $\in T_p\mathscr{M}$ wrt a chart.

Let $(\mathscr{U}, x)$ be a chart, $(\mathscr{U}, x) \in \mathscr{A}$, covering the curve γ at p,

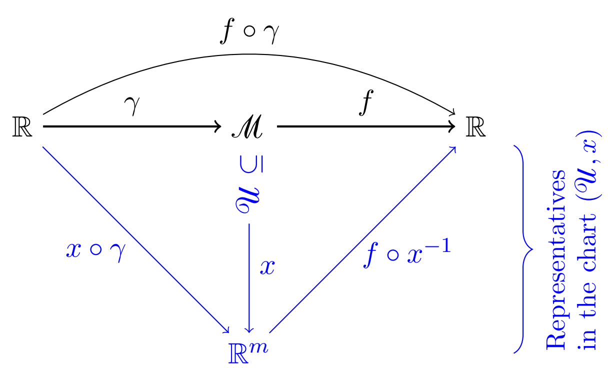
$$p = \gamma(\lambda_0) \in \mathscr{U} \subseteq \mathscr{M}.$$

The coordinates of p are given by scalar functions $x^\mu(p)$, where $\mu = 1, ..., m$.



Expand $v_{\gamma,p}(f)$ in terms of chart representatives,

$$\begin{aligned} v_{\gamma,p}(f) &= (f \circ \gamma)'(\lambda_0) \\ &= \left((f \circ x^{-1}) \circ (x \circ \gamma)\right)(\lambda_0) \\ &= (x^\mu \circ \gamma)'(\lambda_0) \cdot \partial_\mu (f \circ x^{-1})\,(x(p)) \end{aligned}$$



Define,

$$\dot\gamma^\mu_{(x),p} := (x^\mu \circ \gamma)'(\lambda_0) \qquad \leftarrow \text{components}$$

$$\left(\frac{\partial f}{\partial x^\mu}\right)_p := \partial_\mu (f \circ x^{-1})\,(x(p)) \quad \leftarrow \text{basis}$$

Then: $v_{\gamma,p}(f) = \dot\gamma^\mu_{(x),p} \left(\frac{\partial f}{\partial x^\mu}\right)_p \implies v_{\gamma,p} = \dot\gamma^\mu_{(x),p} \left(\frac{\partial}{\partial x^\mu}\right)_p.$

4 Chart induced basis:

Choice of a chart in $\mathscr{M}$ gives the choice of a basis in $T_p\mathscr{M}$.

Proposition. $\left(\frac{\partial}{\partial x^1}\right)_p, \ldots \left(\frac{\partial}{\partial x^m}\right)_p \in T_p\mathscr{U}$ constitutes a basis of $T_p\mathscr{U} \subseteq T_p\mathscr{M}$.

Proof of the linear independence. Apply $\left(\frac{\partial}{\partial x^\mu}\right)_p$ on x^ν,

$$0 = \lambda^\mu \left(\frac{\partial x^\nu}{\partial x^\mu}\right)_p = \lambda^\mu \partial_\mu (x^\nu \circ x^{-1})\,(x(p)) = \lambda^\mu \delta^\nu_\mu = \lambda^\nu.$$

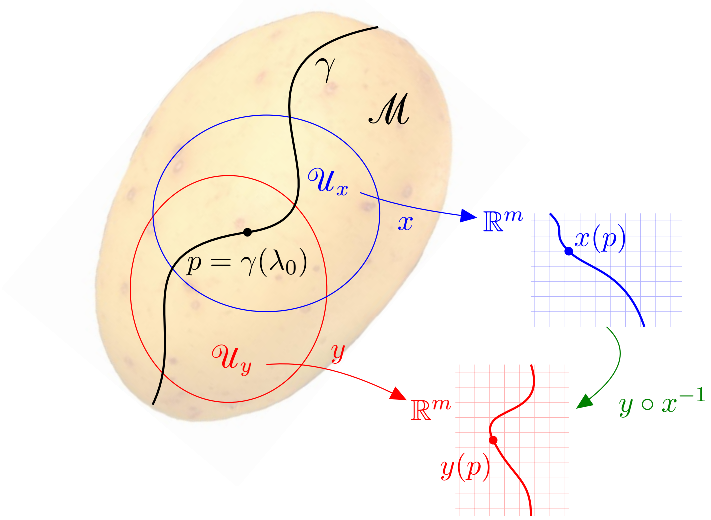
Corollary: $\dim T_p\mathscr{M} = \dim T_p\mathscr{U} = \dim \mathscr{U} = \dim \mathscr{M}$.

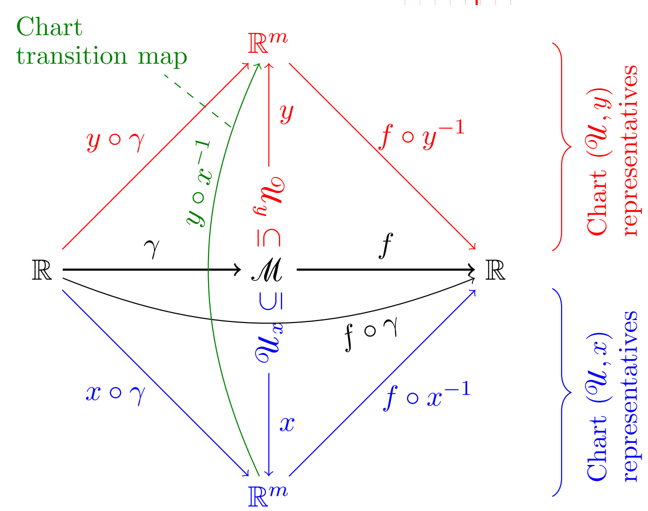
5 Change of vector components under a change of chart.

Let $(\mathscr{U}_x, x)$ and $(\mathscr{U}_y, y)$ be two charts from the atlas $\mathscr{A}$, such that,

$$p = \gamma(\lambda_0) \in \mathscr{U},$$

where $\mathscr{U} = \mathscr{U}_x \cap \mathscr{U}_y \subseteq \mathscr{M}$.





Consider a vector $V_p \in T_p\mathscr{M}$. Then, there are unique components $V^\mu_{(x),p}$ and $V^\mu_{(y),p}$ in the chart induces bases with respect to $(\mathscr{U}_x, x)$ and $(\mathscr{U}_y, y)$, such that,

$$V_p = V^\mu_{(x),p} \left(\frac{\partial}{\partial x^\mu}\right)_p = V^\nu_{(y),p} \left(\frac{\partial}{\partial y^\nu}\right)_p.$$

But, $\left(\frac{\partial f}{\partial x^\mu}\right)_p = \partial_\mu (f \circ x^{-1})\,(x(p))$

$$\begin{aligned} &= \partial_\mu \left((f \circ y^{-1}) \circ (y \circ x^{-1})\right)(x(p)) \\ &= \partial_\mu (y^\nu \circ x^{-1})\,(x(p)) \cdot \partial_\nu (f \circ y^{-1})\,(y(p)) \\ &= \left(\frac{\partial y^\nu}{\partial x^\mu}\right)_p \left(\frac{\partial f}{\partial y^\nu}\right)_p, \end{aligned}$$

so that: $V^\mu_{(x),p} \left(\frac{\partial y^\nu}{\partial x^\mu}\right)_p \left(\frac{\partial f}{\partial y^\nu}\right)_p = V^\nu_{(y),p} \left(\frac{\partial}{\partial y^\nu}\right)_p.$

Therefore,

$$V^\nu_{(y),p} = V^\mu_{(x),p} \left(\frac{\partial y^\nu}{\partial x^\mu}\right)_p$$

- Passive (or alias) transformation
- Active (or alibi) transformation

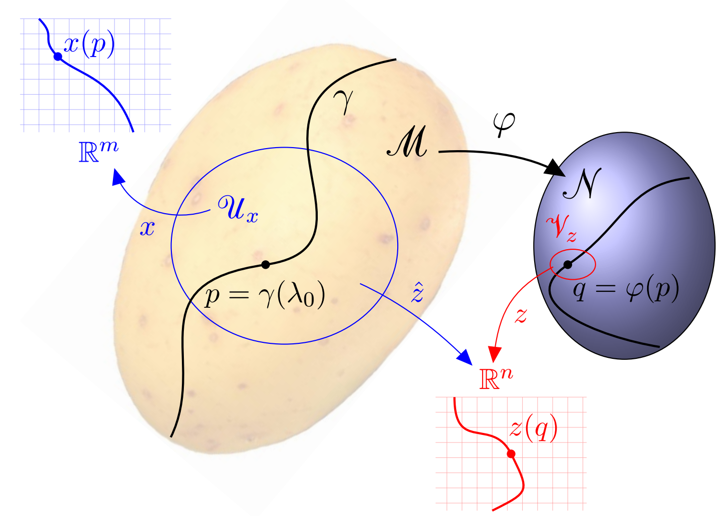
6 Change of vector components under a pushforward.

Let $\varphi : \mathscr{M} \to \mathscr{N}$ be a smooth map between manifolds. Let further $g : \mathscr{N} \to \mathbb{R}$ be a smooth function on $\mathscr{N}$.

Then the *pullback of the function* g *by* φ to $p \in \mathscr{M}$ is the precomposition of g by φ at p,

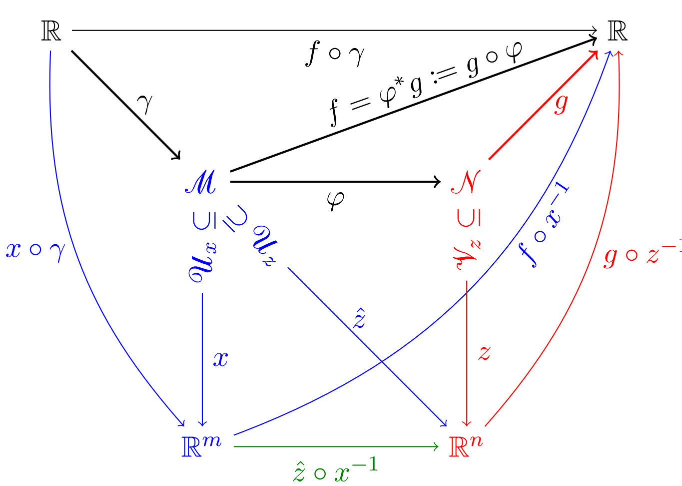
$$(\varphi^* g)(p) := (g \circ \varphi)(p) = g\left(\varphi(p)\right) = g(q),$$

where $q = \varphi(p) \in \mathscr{N}$.



Let V_p be a vector in $T_p\mathscr{M}$. Then the *pushforward of* V_p *by* φ to $T_q\mathscr{N}$ (to act on $g \in C^\infty(\mathscr{N})$) is the composition of V_p with the pullback of g by φ,

$$(\varphi_* V_p)(g) := V_p\,(\varphi^* g) = V_p\,(g \circ \varphi) = V_p(f).$$



Let $(\mathscr{U}_x, x)$ be a chart on $\mathscr{M}$ and $(\mathscr{V}_z, z)$ a chart on $\mathscr{N}$, such that $\mathscr{U}_x = \text{preim}_\varphi \mathscr{V}_z$ and $p \in \mathscr{U}_x$, $q \in \mathscr{V}_z$.

We have, $(\varphi_* V_p)_q\,(g) = (\varphi_* V_p)^\alpha_{(z),q} \left(\frac{\partial g}{\partial z^\alpha}\right)_q$, but also:

$$\begin{aligned} &(\varphi_* V_p)_q\,(g) = V_p\,(\varphi^* g) = V_p\,(f) = v_{\gamma,p}(f) = v_{\gamma,p}(g \circ \varphi) \\ &= (g \circ \varphi \circ \gamma)'\,(\lambda_0) = \left((g \circ z^{-1}) \circ (\hat z \circ \gamma)\right)'(\lambda_0) \\ &= \left((g \circ z^{-1}) \circ (\hat z \circ x^{-1}) \circ (x \circ \gamma)\right)'(\lambda_0) \\ &= (x^\mu \circ \gamma)'\,(\lambda_0) \cdot \partial_\mu \left(\hat z^\nu \circ x^{-1}\right)(x(p)) \cdot \partial_\nu \left(g \circ z^{-1}\right)(z(q)) \\ &= V^\mu_{(x),p} \left(\frac{\partial \hat z^\alpha}{\partial x^\mu}\right)_p \left(\frac{\partial g}{\partial z^\alpha}\right)_q. \end{aligned}$$

Hence, $(\varphi_*)^\alpha{}_\mu = \left(\frac{\partial \hat z^\alpha}{\partial x^\mu}\right)_p$ and,

$$(\varphi_* V_p)^\alpha_{(z),q} = V^\mu_{(x),p} \left(\frac{\partial \hat z^\alpha}{\partial x^\mu}\right)_p$$

If φ is bijective and φ^{-1} is also smooth, then φ is called a *diffeomorphism*.