Null frame vielbeins aligned with null radial geodesics

Overview, M.B.Kocic, Version 2, 2017-02-16

A null geodesic is a geodesic, whose tangent vector is a light-like vector everywhere on the geodesic. That is $x(\lambda)$ is a geodesic and

$$2\,\mathcal{L} = g_{\mu\nu}\,\frac{dx^{\mu}}{d\lambda}\,\frac{dx^{\nu}}{d\lambda} = 0$$

for all λ, where λ is an affine parameter along the curve.

In Minkowski spacetime, the nonholonomically constructed null vectors $\{\, \ell^a \,, n^a \,\}$ respectively match the outgoing and ingoing null radial rays.

As an extension of this idea in generic curved spacetimes, $\{\, \ell^a \,, n^a \,\}$ can still be aligned with the tangent vector field of null radial congruence [see Chandrasekhar, The math theory of blackholes]. However, this type of adaption only works for $\{\, t\,, r\,, \theta\,, \phi \,\}$, $\{\, u\,, r\,, \theta\,, \phi \,\}$ or $\{\, v\,, r\,, \theta\,, \phi \,\}$ coordinates where the radial behaviors can be well described, with u and v denote the outgoing (retarded) and ingoing (advanced) null coordinate, respectively.

Comments on NP formalism for the case of family of shear-free, null geodesic light rays spaning the null tetrad.

We are working with a null congruence which is basically a family of null curves; $\ell^a\,\ell_a = 0$ means that ℓ^a is a null vector field, that is, at every point of the manifold, ℓ^a is tangent to a generator of the light cone at this point. There is an infine number of possible null vector fields defining curves on the manifold, but not everyone is a geodesic. In the Newman-Penrose formalism, it is possible, if you already have a null vector field ℓ^a to adapt to it a null tetrad, $\{\ell^a,\, n^a,\, m^a,\, \overline{m}^a\}$. Then, one of the directional NP derivatives of the tetrad would be: $\ell^b\ \nabla_b\,\ell^a = (\epsilon + \overline{\epsilon})\,\ell^a - \overline{\kappa}\,m^a - \kappa\,\overline{m}^a$, where ϵ and κ are spin coefficients (which are part of the full set of the components of the Ricci rotation coefficients). The Goldberg-Sachs theorem states that if you have a congruence of null geodesics which are shear-free (which this is basically a way of saying that the family of null rays, when evolve in time, they don't "distort"), then $\kappa = 0$ (and other spin coefficient also is zero, but it is not here). So, the null geodesic equation can be put as $\left(\ell^b\ \nabla_b\,\ell^a = (\epsilon + \overline{\epsilon})\,\ell^a\right.$, If besides of this, $\epsilon = 0$, you have a congruence of null geodesics that have an affine parameter.

References: Section 2.1.3 (pp 10-12) in J.B. Griffiths, J. Podolský, *Exact Space-Times in Einstein's General Relativity*, Cambridge University Press, 2009, or Null geodesic congruences in Ezra (Ted) Newman and Roger Penrose (2009) *Spin-coefficient formalism*. Scholarpedia, 4(6):7445.

Definitions

We define the most general form of a null frame vielbein with the associated metric. Here we ignore the spherically symmetric radial part $\{\theta, \phi\}$ for $\dot\theta = 0$, $\dot r = 0$.

```
e = (ℓu ℓr
     nu nr);   η = ( 0 -1
                    -1  0);
```

```
g = eᵀ.η.e;
```

The inverse vielbein (not transposed, thus we have $\vec{n}$ and $\vec{\ell}$ vectors as columns):

```
ie = Inverse@e;
ie // MatrixForm
```

$$\begin{pmatrix} \frac{n_r}{\ell_u n_r - \ell_r n_u} & -\frac{\ell_r}{\ell_u n_r - \ell_r n_u} \\ -\frac{n_u}{\ell_u n_r - \ell_r n_u} & \frac{\ell_u}{\ell_u n_r - \ell_r n_u} \end{pmatrix}$$

Utility function to display the uninstantiated geometry ("uninstantiated" = the components are without particular values):

```
fullInfo[rules_: {}, simplify_: Simplify] := {
  Row@
   {gName /. rules, " = ", (conformalFactor (du dr)ᵀ.g.(du
                                                        dr) /. rules // simplify //
      Collect[#, {du, dr}] &)[[1, 1]]},
  Row@{"det ", gName /. rules, " = ", Det[
     conformalFactor g /. rules // simplify]},
  Row@{gName /. rules, " = ", MatrixForm[
     conformalFactor g /. rules // simplify]},
  Row@{"e = ", MatrixForm[e /. rules // simplify]},
  Row@{"det e = ", Det[e /. rules // simplify]},
  Row@{"ℓ̃ = ", MatrixForm[{e[[1]] /. rules // simplify}]},
  Row@{"ñ = ", MatrixForm[{e[[2]] /. rules // simplify}]},
  Row@{"e⁻¹ = ", MatrixForm[ie /. rules // simplify]},
  Row@{"ℓ⃗ = ", MatrixForm[ie[[All, 2]] /. rules // simplify]},
  Row@{"n⃗ = ", MatrixForm[ie[[All, 1]] /. rules // simplify]}
 } /. {gName → "g", conformalFactor → 1}
```

```
Row@Riffle[fullInfo[], ", "]
```

$g = -2\,dr^2 \ell_r n_r - 2\,du^2 \ell_u n_u - 2\,dr\,du\,(\ell_u n_r + \ell_r n_u)$, $\det g = -\ell_u^2 n_r^2 + 2\,\ell_r \ell_u n_r n_u - \ell_r^2 n_u^2$, $g = \begin{pmatrix} -2\,\ell_u n_u & -\ell_u n_r - \ell_r n_u \\ -\ell_u n_r - \ell_r n_u & -2\,\ell_r n_r \end{pmatrix}$, $e = \begin{pmatrix} \ell_u & \ell_r \\ n_u & n_r \end{pmatrix}$, $\det e = \ell_u n_r - \ell_r n_u$, $\tilde{\ell} = \begin{pmatrix} \ell_u & \ell_r \end{pmatrix}$, $\tilde{n} = \begin{pmatrix} n_u & n_r \end{pmatrix}$, $e^{-1} = \begin{pmatrix} \frac{n_r}{\ell_u n_r - \ell_r n_u} & \frac{\ell_r}{-\ell_u n_r + \ell_r n_u} \\ \frac{n_u}{-\ell_u n_r + \ell_r n_u} & \frac{\ell_u}{\ell_u n_r - \ell_r n_u} \end{pmatrix}$, $\vec{\ell} = \begin{pmatrix} \frac{\ell_r}{-\ell_u n_r + \ell_r n_u} \\ \frac{\ell_u}{\ell_u n_r - \ell_r n_u} \end{pmatrix}$, $\vec{n} = \begin{pmatrix} \frac{n_r}{\ell_u n_r - \ell_r n_u} \\ \frac{n_u}{-\ell_u n_r + \ell_r n_u} \end{pmatrix}$

Geodesic equation

The geodesics are obtained by the variational method from the following Lagrangian, [see section 7.6 (and also sections 16.4, 18.3 for examples) in d'Inverno, Introducing Einstein's Relativity, Clarendon Press, 1992]:

$$2\,\mathcal{L} = \dot{x}^{\mu}\, g_{\mu\nu}\, \dot{x}^{\mu}$$

```
ℒ = 1/2 (({{u'[λ]}, {r'[λ]}})ᵀ.g.({{u'[λ]}, {r'[λ]}}))[[1, 1]] // Simplify;
```

For the radial null gedoesics we have $\mathcal{L} = 0$, $\dot{\theta} = 0$, $\dot{\phi} = 0$. In the following we also assume that the frame fields are independent on the u coordinate. Then,

```
Row@{"ℒ = ", ℒ, " = 0"}
D[ℒ, u'[λ]] == const  (* from D[D[ℒ,u'],λ] - D[ℒ,u] == 0 *)
```

$\mathcal{L} = -(\ell_r\, r'[\lambda] + \ell_u\, u'[\lambda])\,(n_r\, r'[\lambda] + n_u\, u'[\lambda]) = 0$

$-n_u\,(\ell_r\, r'[\lambda] + \ell_u\, u'[\lambda]) - \ell_u\,(n_r\, r'[\lambda] + n_u\, u'[\lambda]) == \text{const}$

This gives the "solution" [actually, the vector fields which will yield the solution]:

```
vec$ℓ =
 {u'[λ], r'[λ]} /. Solve[{D[ℒ, u'[λ]] == -1, ℒ == 0}, {u'[λ], r'[λ]}][[1]] // Simplify
```

$$\left\{-\frac{\ell_r}{\ell_u^2\, n_r - \ell_r\, \ell_u\, n_u},\ \frac{1}{\ell_u\, n_r - \ell_r\, n_u}\right\}$$

```
vec$n =
 {u'[λ], r'[λ]} //. Solve[{D[ℒ, u'[λ]] == -1, ℒ == 0}, {u'[λ], r'[λ]}][[2]] // Simplify
```

$$\left\{\frac{n_r}{\ell_u\, n_r\, n_u - \ell_r\, n_u^2},\ \frac{1}{-\ell_u\, n_r + \ell_r\, n_u}\right\}$$

Now, scaling of the vector fields by n_u (or ℓ_u)does not influence the stream flow. [That is since the vector fields are autonomous, i.e., they do not depend on λ (just parameterized by λ) so the scaling just reparameterizes λ.] Morever, the scaling makes the stream flow regular (in the case if n_u or ℓ_u are zero). As a consequence, we can take the vector fields $\vec{\ell}$ and $\vec{n}$ as generators of the radial null geodesics. Of course, this is the expected fact that the null cone field will generate the null geodesics.

```
{ℓu, ℓu} vec$ℓ // Simplify
Row@{"Equal to ℓ⃗? >>> ", (% == ie[[All, 2]] // Simplify)}
```

$$\left\{\frac{\ell_r}{-\ell_u\, n_r + \ell_r\, n_u},\ \frac{\ell_u}{\ell_u\, n_r - \ell_r\, n_u}\right\}$$

Equal to $\vec{\ell}$? >>> True

```
{nu, nu} vec$n // Simplify
Row@{"Equal to n⃗? >>> ", (% == ie[[All, 1]] // Simplify)}
```

$$\left\{\frac{n_r}{\ell_u\, n_r - \ell_r\, n_u},\ \frac{n_u}{-\ell_u\, n_r + \ell_r\, n_u}\right\}$$

Equal to $\vec{n}$? >>> True

Examples

Null tetrad for Schwarzschild metric in Eddington-Finkestein coordinates

Ingoing chart (v, r)

The ingoing Eddington–Finkelstein coordinates are obtained by replacing the coordinate t with the new coordinate $v = t + r^*$. The metric in these coordinates can be written:

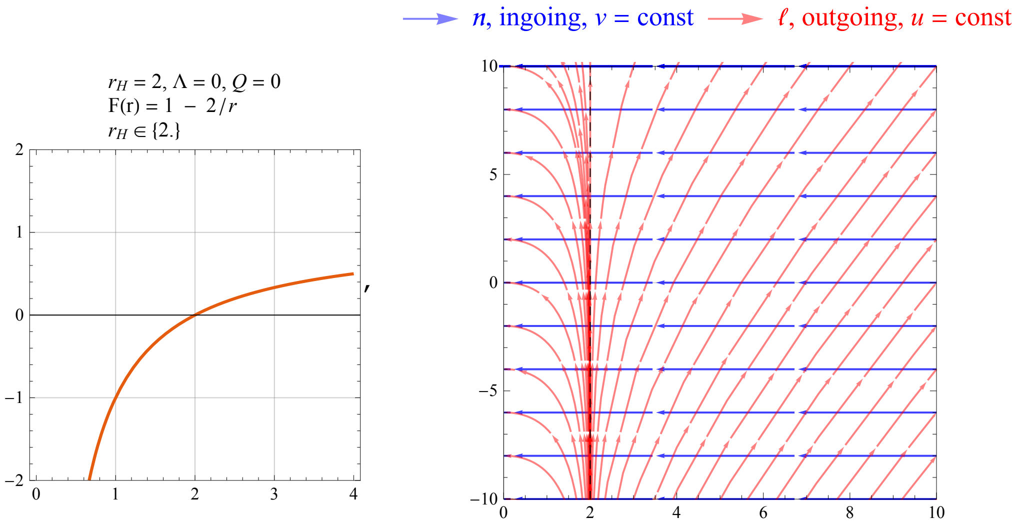
$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dv^2 + 2\, dv\, dr + r^2\, d\Omega^2.$$

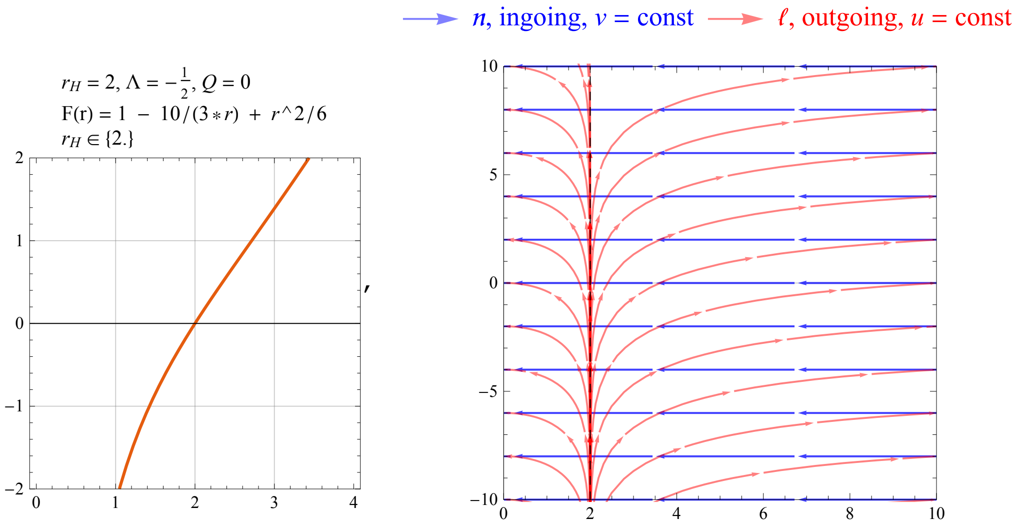
One needs to remember, though, that v (usually called an **advanced time**) is a lightlike coordinate (neither spatial nor temporal). For a second coordinate, we can choose the familiar coordinate r. The congruence of ingoing radial null geodesics is given by v = constant, i.e., coordinate lines of constant v, representing ingoing radial null rays, are plotted on a 45-degree slant, just as they would be in flat spacetime.

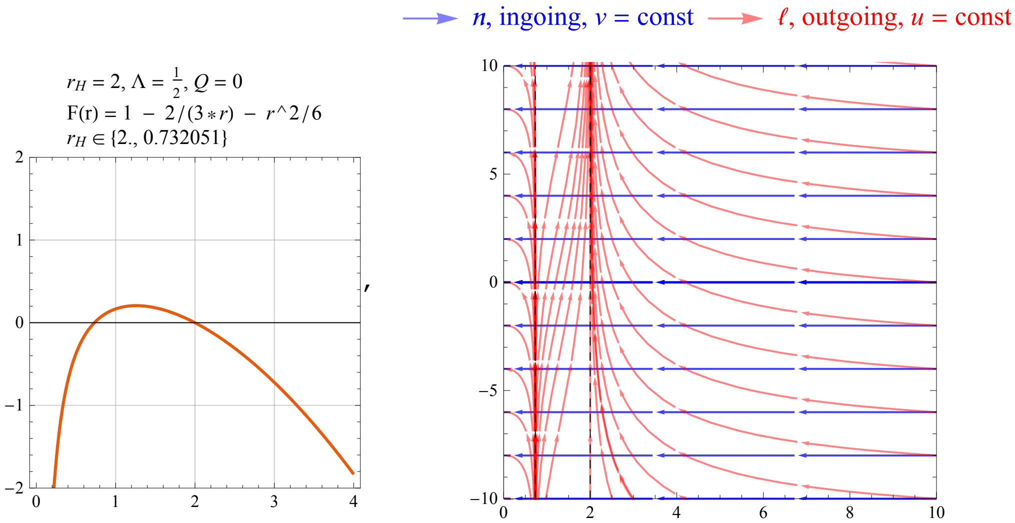
```
CellPrint[Cell["", "PageBreak"]]
Row@Riffle[
  fullInfo[{nr → 0, nu → 1, ℓr → -1, ℓu → F/2}, # /. {dr → dr, du → dv} &], ", "]
Row@Riffle[ingoingNullCongruences[(*rH*) 2, (*Λ*) 0, (*Q*) 0], ", "]
Row@Riffle[ingoingNullCongruences[(*rH*) 2, (*Λ*) -1/2, (*Q*) 0], ", "]
Row@Riffle[ingoingNullCongruences[(*rH*) 2, (*Λ*) 1/2, (*Q*) 0], ", "]
Row@Riffle[ingoingNullCongruences[(*rH*) 0, (*Λ*) 0, (*Q*) 0], ", "]
Row@Riffle[ingoingNullCongruences[(*rH*) 0, (*Λ*) -1/2, (*Q*) 0], ", "]
Row@Riffle[ingoingNullCongruences[(*rH*) 0, (*Λ*) 1/2, (*Q*) 0], ", "]
```

$$g = 2\,dr\,dv - dv^2 F, \quad \det g = -1, \quad g = \begin{pmatrix} -F & 1 \\ 1 & 0 \end{pmatrix}, \quad e = \begin{pmatrix} \frac{F}{2} & -1 \\ 1 & 0 \end{pmatrix}, \quad \det e = 1$$

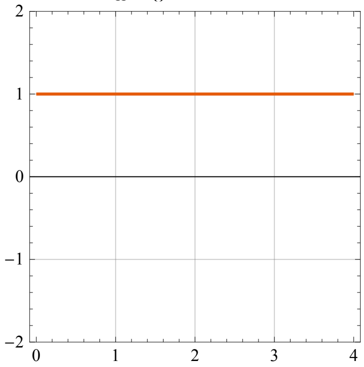
$$, \quad \tilde{\ell} = \begin{pmatrix} \frac{F}{2} & -1 \end{pmatrix}, \quad \tilde{n} = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad e^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & \frac{F}{2} \end{pmatrix}, \quad \vec{\ell} = \begin{pmatrix} 1 \\ \frac{F}{2} \end{pmatrix}, \quad \vec{n} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$







$r_H = 0, \Lambda = 0, Q = 0$
F(r) = 1
$r_H \in \{\}$



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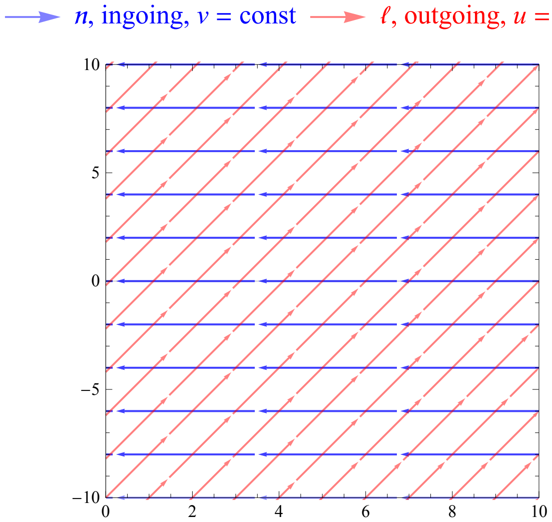
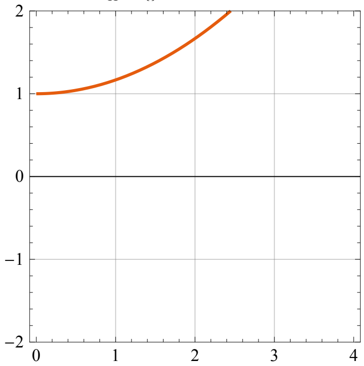


$r_H = 0, \Lambda = -\frac{1}{2}, Q = 0$
F(r) = 1 + r^2/6
$r_H \in \{\}$



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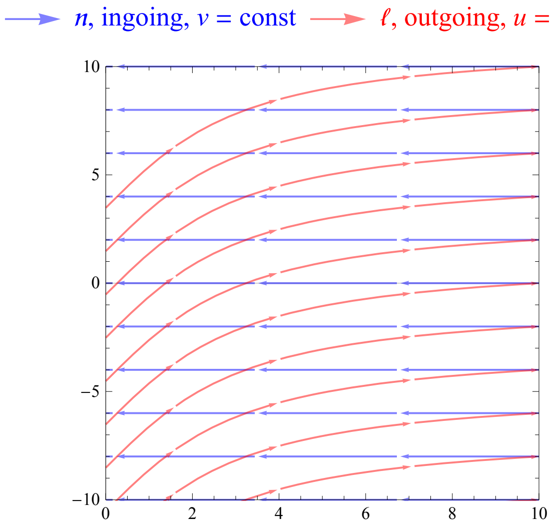
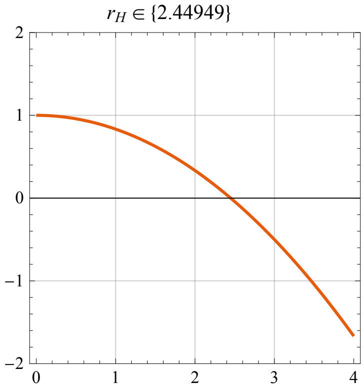


$r_H = 0, \Lambda = \frac{1}{2}, Q = 0$
F(r) = 1 − r^2/6
$r_H \in \{2.44949\}$



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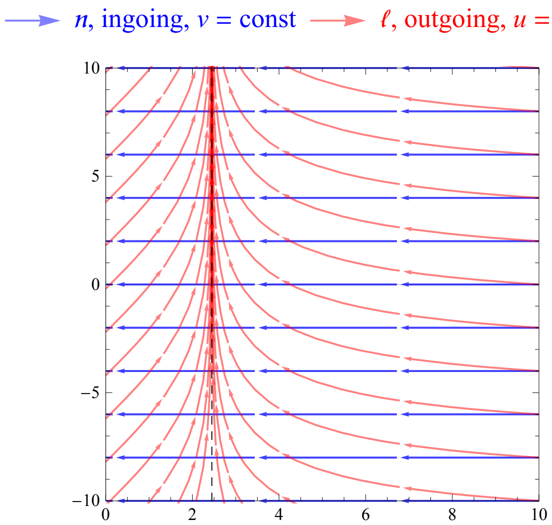


Outgoing chart (u, r)

Another type of the Eddington-Finkelstein coordinates are (u, r), where $u = t - r^*$ is the **retarded time**. The congruence of outgoing radial null geodesies is given by u = constant.

The outgoing Eddington–Finkelstein coordinates are obtained by replacing t with the null coordinate $u = t - r^*$. The metric is then given by,

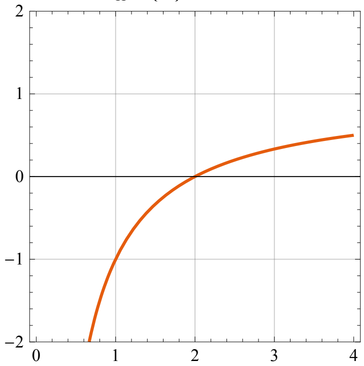
$$\mathrm{d}s^2 = -\left(1 - \tfrac{r_s}{r}\right) \mathrm{d}u^2 - 2\,\mathrm{d}u\,\mathrm{d}r + \; + r^2\,\mathrm{d}\Omega^2.$$

```
CellPrint[Cell["", "PageBreak"]]
Row@Riffle[fullInfo[{nr → 0, nu → 1, ℓr → 1, ℓu → F/2}] /. {dr → dr, du → du}, ", "]
Row@Riffle[outgoingNullCongruences[(*rH*) 2, (*Λ*) 0, (*Q*) 0 ], ", "]
Row@Riffle[outgoingNullCongruences[(*rH*) 2, (*Λ*) -1/2, (*Q*) 0 ], ", "]
Row@Riffle[outgoingNullCongruences[(*rH*) 2, (*Λ*) 1/2, (*Q*) 0 ], ", "]
Row@Riffle[outgoingNullCongruences[(*rH*) 0, (*Λ*) 0, (*Q*) 0 ], ", "]
Row@Riffle[outgoingNullCongruences[(*rH*) 0, (*Λ*) -1/2, (*Q*) 0 ], ", "]
Row@Riffle[outgoingNullCongruences[(*rH*) 0, (*Λ*) 1/2, (*Q*) 0 ], ", "]
```

$$g = -2\,\mathrm{dr}\,\mathrm{du} - \mathrm{du}^2\,F,\ \det g = -1,\ g = \begin{pmatrix} -F & -1 \\ -1 & 0 \end{pmatrix},\ e = \begin{pmatrix} \frac{F}{2} & 1 \\ 1 & 0 \end{pmatrix},\ \det e = -1$$

$$,\ \tilde{\ell} = \begin{pmatrix} \frac{F}{2} & 1 \end{pmatrix},\ \tilde{n} = \begin{pmatrix} 1 & 0 \end{pmatrix},\ e^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -\frac{F}{2} \end{pmatrix},\ \vec{\ell} = \begin{pmatrix} 1 \\ -\frac{F}{2} \end{pmatrix},\ \vec{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$r_H = 2, \Lambda = 0, Q = 0$
F(r) = 1 − 2/r
$r_H \in \{2.\}$



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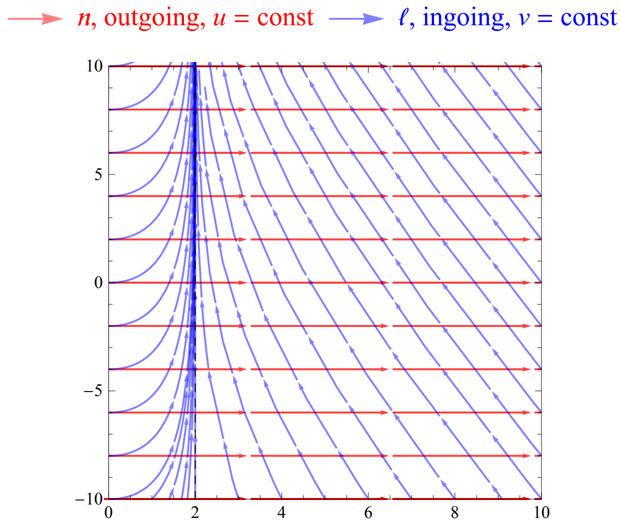
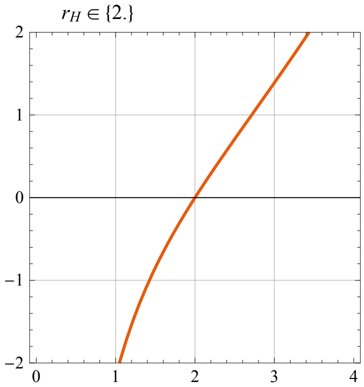


$r_H = 2, \Lambda = -\frac{1}{2}, Q = 0$
F(r) = 1 − 10/(3∗r) + r^2/6
$r_H \in \{2.\}$



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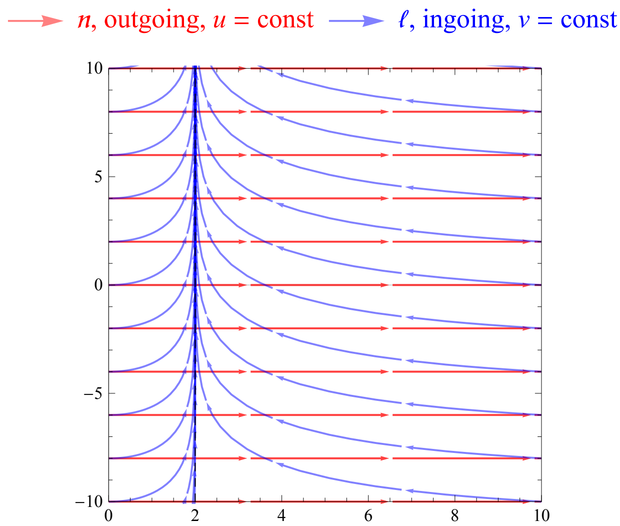
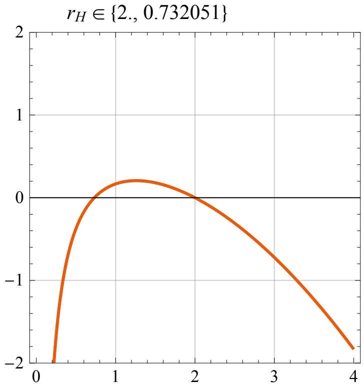


$r_H = 2, \Lambda = \frac{1}{2}, Q = 0$
F(r) = 1 − 2/(3∗r) − r^2/6
$r_H \in \{2., 0.732051\}$



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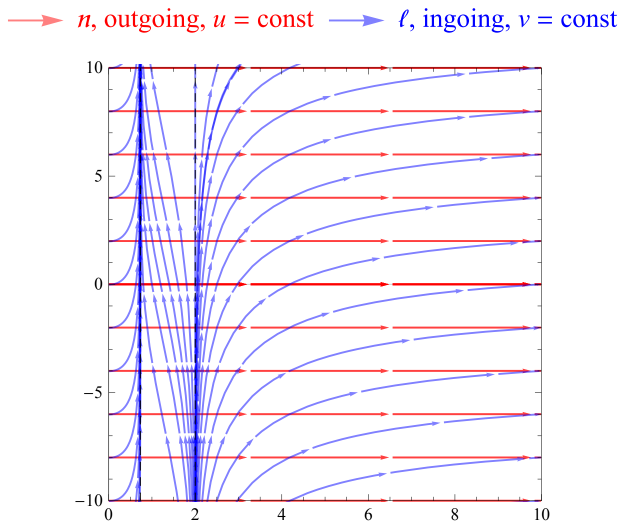
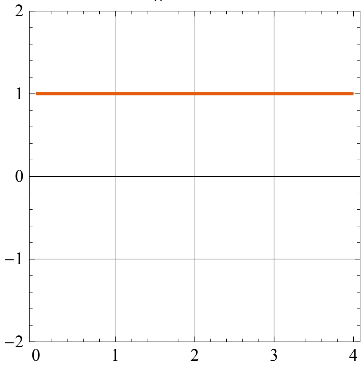


$\longrightarrow$ n, outgoing, u = const $\longrightarrow$ ℓ, ingoing, v = const

$r_H = 0$, $\Lambda = 0$, $Q = 0$
F(r) = 1
$r_H \in \{\}$



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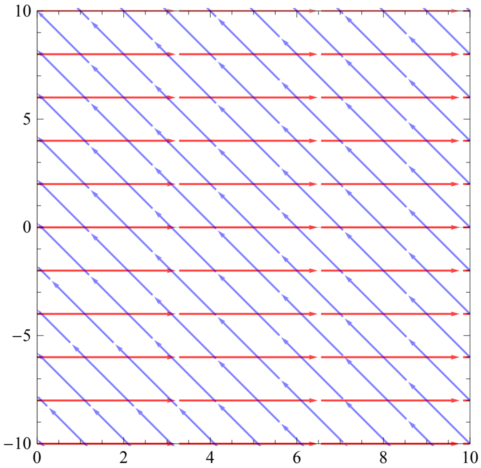
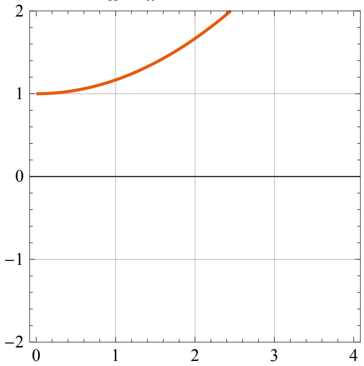


$\longrightarrow$ n, outgoing, u = const $\longrightarrow$ ℓ, ingoing, v = const

$r_H = 0$, $\Lambda = -\frac{1}{2}$, $Q = 0$
F(r) = 1 + r^2/6
$r_H \in \{\}$



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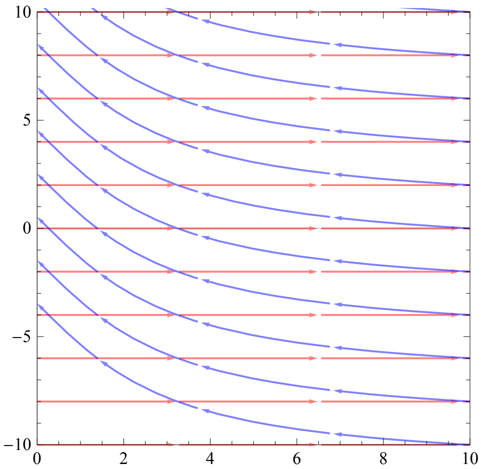
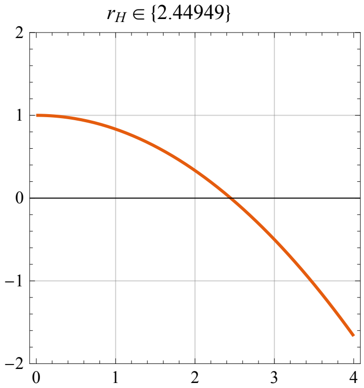


$\longrightarrow$ n, outgoing, u = const $\longrightarrow$ ℓ, ingoing, v = const

$r_H = 0$, $\Lambda = \frac{1}{2}$, $Q = 0$
F(r) = 1 − r^2/6
$r_H \in \{2.44949\}$



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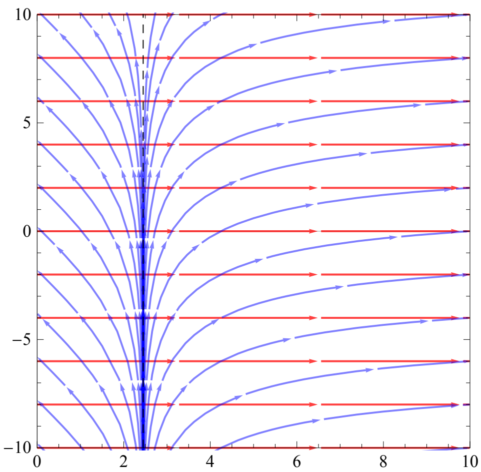


Bim BH metric g in the ingoing chart (v, r)

$$\text{fullInfo@}\left\{n_r \to 0,\ n_u \to -\left(-e^{q/2}\right),\ \ell_r \to -(1),\ \ell_u \to \frac{1}{2} e^{q/2} F\right\} \text{ /. } \boxed{\{dr \to dr,\ du \to dv\}} \text{ // Simplify}$$

$$\left\{g = 2\,dr\,dv\,e^{q/2} - dv^2 e^q F,\ \det g = -e^q,\ g = \begin{pmatrix} -e^q F & e^{q/2} \\ e^{q/2} & 0 \end{pmatrix},\ e = \begin{pmatrix} \frac{1}{2} e^{q/2} F & -1 \\ e^{q/2} & 0 \end{pmatrix},\ \det e = e^{q/2},\ \tilde{\ell} = \begin{pmatrix} \frac{1}{2} e^{q/2} F & -1 \end{pmatrix},\ \tilde{n} = \begin{pmatrix} e^{q/2} & 0 \end{pmatrix},\ e^{-1} = \begin{pmatrix} 0 & e^{-q/2} \\ -1 & \frac{F}{2} \end{pmatrix},\ \vec{\ell} = \begin{pmatrix} e^{-q/2} \\ \frac{F}{2} \end{pmatrix},\ \vec{n} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right\}$$

$$\text{\% /. \{q} \to 0\text{\} // Simplify}$$

$$\left\{g = dv\,(2\,dr - dv\,F),\ \det g = -1,\ g = \begin{pmatrix} -F & 1 \\ 1 & 0 \end{pmatrix},\ e = \begin{pmatrix} \frac{F}{2} & -1 \\ 1 & 0 \end{pmatrix},\ \det e = 1,\ \tilde{\ell} = \begin{pmatrix} \frac{F}{2} & -1 \end{pmatrix},\ \tilde{n} = \begin{pmatrix} 1 & 0 \end{pmatrix},\ e^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & \frac{F}{2} \end{pmatrix},\ \vec{\ell} = \begin{pmatrix} 1 \\ \frac{F}{2} \end{pmatrix},\ \vec{n} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right\}$$

Bim BH metric f in the ingoing chart (v, r)

$$\text{fullInfo@}\left\{\text{gName} \to \text{"f"},\ n_r \to e^{q/2}\,\Sigma\,\tau\, e^{-q/2}\,\frac{\Sigma-\tau}{F\,\Sigma\,\tau},\ n_u \to -e^{q/2}\,\Sigma\,\tau\left(-\frac{1}{\Sigma}\right),\ \ell_r \to -e^{q/2}\,\Sigma\,\tau\left(e^{-q/2}\,\frac{\Sigma+\tau}{2\,\Sigma\,\tau}\right),\ \ell_u \to e^{q/2}\,\Sigma\,\tau\,\frac{1}{2\,\Sigma}\,F\right\} \text{ /. } \boxed{\{dr \to dr,\ du \to dv\}} \text{ // Simplify}$$

$$\left\{f = 2\,dr\,dv\,e^{q/2}\tau^2 - dv^2 e^q F \tau^2 + \frac{dr^2\left(\Sigma^2-\tau^2\right)}{F},\ \det f = -e^q \Sigma^2 \tau^2,\ f = \begin{pmatrix} -e^q F \tau^2 & e^{q/2}\tau^2 \\ e^{q/2}\tau^2 & \frac{(\Sigma-\tau)(\Sigma+\tau)}{F} \end{pmatrix},\ e = \begin{pmatrix} \frac{1}{2} e^{q/2} F \tau & \frac{1}{2}(-\Sigma-\tau) \\ e^{q/2}\tau & \frac{\Sigma-\tau}{F} \end{pmatrix},\ \det e = e^{q/2}\Sigma\tau,\ \tilde{\ell} = \begin{pmatrix} \frac{1}{2} e^{q/2} F \tau & \frac{1}{2}(-\Sigma-\tau) \end{pmatrix},\ \tilde{n} = \begin{pmatrix} e^{q/2}\tau & \frac{\Sigma-\tau}{F} \end{pmatrix},\ e^{-1} = \begin{pmatrix} \frac{e^{-q/2}(\Sigma-\tau)}{F\,\Sigma\,\tau} & \frac{e^{-q/2}(\Sigma+\tau)}{2\,\Sigma\,\tau} \\ -\frac{1}{\Sigma} & \frac{F}{2\,\Sigma} \end{pmatrix},\ \vec{\ell} = \begin{pmatrix} \frac{e^{-q/2}(\Sigma+\tau)}{2\,\Sigma\,\tau} \\ \frac{F}{2\,\Sigma} \end{pmatrix},\ \vec{n} = \begin{pmatrix} \frac{e^{-q/2}(\Sigma-\tau)}{F\,\Sigma\,\tau} \\ -\frac{1}{\Sigma} \end{pmatrix}\right\}$$

$$\text{\% /. \{q} \to 0,\ \Sigma \to 1,\ \tau \to 1,\ R \to 1\text{\} // Simplify}$$

$$\left\{f = dv\,(2\,dr - dv\,F),\ \det f = -1,\ f = \begin{pmatrix} -F & 1 \\ 1 & 0 \end{pmatrix},\ e = \begin{pmatrix} \frac{F}{2} & -1 \\ 1 & 0 \end{pmatrix},\ \det e = 1,\ \tilde{\ell} = \begin{pmatrix} \frac{F}{2} & -1 \end{pmatrix},\ \tilde{n} = \begin{pmatrix} 1 & 0 \end{pmatrix},\ e^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & \frac{F}{2} \end{pmatrix},\ \vec{\ell} = \begin{pmatrix} 1 \\ \frac{F}{2} \end{pmatrix},\ \vec{n} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right\}$$

Bim BH metric *g* in the outgoing chart (*u*, *r*)

$$\texttt{fullInfo@}\left\{n_r \to 0,\ n_u \to -\left(-e^{q/2}\right),\ \ell_r \to +(1),\ \ell_u \to \frac{1}{2}\, e^{q/2}\, F\right\} \texttt{ /. } \boxed{\{dr \to dr,\ du \to du\}} \texttt{ // Simplify}$$

$$\left\{g = -2\, dr\, du\, e^{q/2} - du^2\, e^q\, F,\ \det g = -e^q,\ g = \begin{pmatrix} -e^q F & -e^{q/2} \\ -e^{q/2} & 0 \end{pmatrix},\right.$$
$$e = \begin{pmatrix} \frac{1}{2} e^{q/2} F & 1 \\ e^{q/2} & 0 \end{pmatrix},\ \det e = -e^{q/2},\ \tilde{\ell} = \begin{pmatrix} \frac{1}{2} e^{q/2} F & 1 \end{pmatrix},$$
$$\left.\tilde{n} = \begin{pmatrix} e^{q/2} & 0 \end{pmatrix},\ e^{-1} = \begin{pmatrix} 0 & e^{-q/2} \\ 1 & -\frac{F}{2} \end{pmatrix},\ \vec{\ell} = \begin{pmatrix} e^{-q/2} \\ -\frac{F}{2} \end{pmatrix},\ \vec{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$$

$$\texttt{\% /. } \{q \to 0\} \texttt{ // Simplify}$$

$$\left\{g = -du\,(2\, dr + du\, F),\ \det g = -1,\ g = \begin{pmatrix} -F & -1 \\ -1 & 0 \end{pmatrix},\ e = \begin{pmatrix} \frac{F}{2} & 1 \\ 1 & 0 \end{pmatrix},\right.$$
$$\left.\det e = -1,\ \tilde{\ell} = \begin{pmatrix} \frac{F}{2} & 1 \end{pmatrix},\ \tilde{n} = \begin{pmatrix} 1 & 0 \end{pmatrix},\ e^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -\frac{F}{2} \end{pmatrix},\ \vec{\ell} = \begin{pmatrix} 1 \\ -\frac{F}{2} \end{pmatrix},\ \vec{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$$

Bim BH metric *f* in the outgoing chart (*u*, *r*)

$$\texttt{fullInfo@}\left\{\texttt{gName} \to \texttt{"f"},\ n_r \to -e^{q/2}\, \Sigma\, \tau\, e^{-q/2}\, \frac{\Sigma - \tau}{F\, \Sigma\, \tau},\ n_u \to -e^{q/2}\, \Sigma\, \tau \left(-\frac{1}{\Sigma}\right),\right.$$
$$\left.\ell_r \to e^{q/2}\, \Sigma\, \tau \left(e^{-q/2}\, \frac{\Sigma + \tau}{2\, \Sigma\, \tau}\right),\ \ell_u \to e^{q/2}\, \Sigma\, \tau\, \frac{1}{2\, \Sigma}\, F\right\} \texttt{ /. } \boxed{\{dr \to dr,\ du \to du\}} \texttt{ // Simplify}$$

$$\left\{f = -2\, dr\, du\, e^{q/2}\, \tau^2 - du^2\, e^q\, F\, \tau^2 + \frac{dr^2\left(\Sigma^2 - \tau^2\right)}{F},\right.$$
$$\det f = -e^q\, \Sigma^2\, \tau^2,\ f = \begin{pmatrix} -e^q F \tau^2 & -e^{q/2} \tau^2 \\ -e^{q/2} \tau^2 & \frac{(\Sigma-\tau)(\Sigma+\tau)}{F} \end{pmatrix},\ e = \begin{pmatrix} \frac{1}{2} e^{q/2} F \tau & \frac{\Sigma+\tau}{2} \\ e^{q/2} \tau & \frac{-\Sigma+\tau}{F} \end{pmatrix},$$
$$\det e = -e^{q/2}\, \Sigma\, \tau,\ \tilde{\ell} = \begin{pmatrix} \frac{1}{2} e^{q/2} F \tau & \frac{\Sigma+\tau}{2} \end{pmatrix},\ \tilde{n} = \begin{pmatrix} e^{q/2} \tau & \frac{-\Sigma+\tau}{F} \end{pmatrix},$$
$$\left.e^{-1} = \begin{pmatrix} \frac{e^{-q/2}(\Sigma-\tau)}{F \Sigma \tau} & \frac{e^{-q/2}(\Sigma+\tau)}{2 \Sigma \tau} \\ \frac{1}{\Sigma} & -\frac{F}{2 \Sigma} \end{pmatrix},\ \vec{\ell} = \begin{pmatrix} \frac{e^{-q/2}(\Sigma+\tau)}{2 \Sigma \tau} \\ -\frac{F}{2 \Sigma} \end{pmatrix},\ \vec{n} = \begin{pmatrix} \frac{e^{-q/2}(\Sigma-\tau)}{F \Sigma \tau} \\ \frac{1}{\Sigma} \end{pmatrix}\right\}$$

$$\texttt{\% /. } \{q \to 0,\ \Sigma \to 1,\ \tau \to 1,\ R \to 1\} \texttt{ // Simplify}$$

$$\left\{f = -du\,(2\, dr + du\, F),\ \det f = -1,\ f = \begin{pmatrix} -F & -1 \\ -1 & 0 \end{pmatrix},\ e = \begin{pmatrix} \frac{F}{2} & 1 \\ 1 & 0 \end{pmatrix},\right.$$
$$\left.\det e = -1,\ \tilde{\ell} = \begin{pmatrix} \frac{F}{2} & 1 \end{pmatrix},\ \tilde{n} = \begin{pmatrix} 1 & 0 \end{pmatrix},\ e^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -\frac{F}{2} \end{pmatrix},\ \vec{\ell} = \begin{pmatrix} 1 \\ -\frac{F}{2} \end{pmatrix},\ \vec{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$$

Part 2: pp-Wave

The Schwarzschild metric in the chart $\{u, r, \theta, \phi\}$ with $F = F(r)$ has the similar structure as the pp-wave metric in $\{u, v, x, y\}$ with $F = F(u, x, y)$ (especially if we put x, y on a cylinder).

Let us introduce the light-cone coordinate system $(u, v, \vec{r})$, where $u = t - z$, $v = t + z$ and $\vec{r} = (x, y)$. Here, v is an advanced while u is a retarted coordinate.

$$u = t - z, \; v = t + z \implies t = (u + v)/2, \; z = (v - u)/2,$$

$$u\text{-axis: } v = \text{const} = 0, u = p \in \mathbb{R} \implies t = p/2, \; z = -p/2,$$

$$v\text{-axis: } u = \text{const} = 0, v = p \in \mathbb{R} \implies t = p/2, \; z = p/2.$$

In these coordinates, a generic pp-wave spacetime has the following metric:

$$ds^2 = -du\, dv + F(u, r)\, du^2 + dr^2.$$

This geometry enjoys the null Killing vector ∂_v.

The following profile of a **sandwich wave** solves the MGr EoM (see X.O. Camanho, G.L. Gómez, R. Rahman, *Causality Constraints on Massive Gravity*, https://arxiv.org/abs/1610.02033):

$$F = A(u) K_0(m|\vec{x}|)$$

$$A(u) = \begin{cases} a \exp\left[-\frac{\lambda^2 u^2}{(u^2-\lambda^2)^2}\right] & \text{if } u \in [-\lambda, \lambda], \\ 0 & \text{otherwise,} \end{cases}$$

The sandwich wave moves at the speed of light in the v-direction (i.e., in the z-direction since: $v = t + z$). Its amplitude and width are deffined by a and λ respectively.

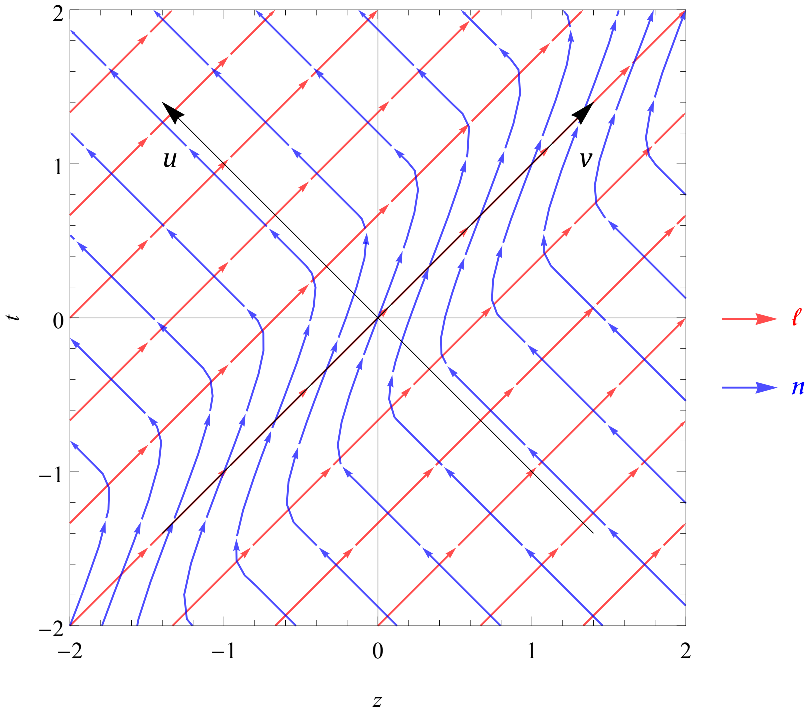
Remark. We choose the amplitude a such that the integral of $\int_{-\lambda}^{\lambda} A(u)\, du$ is λ. This amounts to the choice $a \approx 0.933$. We also choose the width λ to be larger than the resolution length of the effective field theory: $\lambda > 1/\Lambda$. The latter choice is possible for a very large energy of the source: $E \gg \Lambda$. This situation is completely acceptable and does not at all invalidate the effective field theory descripton. (The situation is analogous to having a macroscopic (super-Planckian) black hole in General Relativity.)

```
Fg[u_, r_, a_, λ_, m_] := A[u, a, λ] { BesselK[0, m Abs@r]   m Abs@r ≥ 10^-30
                                       70                     True

A[u_, a_, λ_] := { a Exp[- λ² u² / (u² - λ²)²]   -λ < u < λ
                   0                             True

If[False, Manipulate[
  SliceDensityPlot3D[
   Fg[(*u*) t - z, (*r*) (x² + y²)^(1/2), 0.93, 1, 1],
   Thread[(-z) == Subdivide[-3, 3, 10]],
   {z, -3, 3}, {x, -2, 2}, {y, -2, 2}, AxesLabel → {z, x, y}, BoxRatios → {2, 1, 1},
   BoundaryStyle → None, BaseStyle → {FontFamily → "Cambria", 12},
   ColorFunction →
    (Directive[Blue, Opacity@Rescale[#, {0.1, 1}, {0.15, 0.8}]] &)],
  {{t, 0}, -3, 3}, SaveDefinitions → True
 ]]
```

```
CellPrint[Cell["", "PageBreak"]]
fullInfo@{𝓃r → 2^-1/2, 𝓃u → -F/2^1/2, ℓr → 0, ℓu → 2^-1/2} /. {du → du, dr → dv}
Block[
 {F, v$ℓ, v$𝓃, plot, r = 0.1, range = 1},
 F = Fg[N@u, N@r, 0.93, 1., 1.];
 (* t = ( v - u ) / 2, z = ( u + v ) / 2 *)
 v$ℓ[z_, t_] = {#[[2]] - #[[1]], #[[1]] + #[[2]]} / 2 &@ ({0, 2^1/2} /. {u → t - z, v → t + z});
 v$𝓃[z_, t_] = {#[[2]] - #[[1]], #[[1]] + #[[2]]} / 2 &@ ({2^1/2, 2^1/2 F} /. {u → t - z, v → t + z});
 plot = Show[
   StreamPlot[v$ℓ[z, t],
    {z, -2 range, 2 range}, {t, -2 range, 2 range}, FrameLabel → {z, t},
    PlotTheme → "Scientific",
    PlotRange → {{-2, 2}, {-2, 2}} range,
    StreamStyle → {Opacity[0.7], Red},
    StreamPoints → {Table[{-u, u}, {u, -2 range, 2 range, range / 3}], 5, 5},
    PlotLegends → Placed[{Style[ℓ, 14, Red]}, Right],
    WorkingPrecision → 30, BaseStyle → {FontFamily → "Cambria", 12}
   ],
   StreamPlot[v$𝓃[z, t],
    {z, -2 range, 2 range}, {t, -2 range, 2 range}, FrameLabel → {z, t},
    PlotTheme → "Scientific",
    StreamStyle → {Opacity[0.7], Blue},
    StreamPoints → {Table[{v, v}, {v, -2 range, 2 range, range / 3}], 5, 5},
    PlotLegends → Placed[{Style[𝓃, 14, Blue]}, Right],
    WorkingPrecision → 30, BaseStyle → {FontFamily → "Cambria", 12}
   ],
   Graphics[{Black,
     Arrow[1.4 {{-1, -1}, {1, 1}}], Text[Style[v, 15], 1.5 {0.9, 0.7}],
     Arrow[1.4 {{1, -1}, {-1, 1}}], Text[Style[u, 15], 1.5 {-0.9, 0.7}]
    }],
   ImageSize → Scaled[0.6]
  ]
]
```

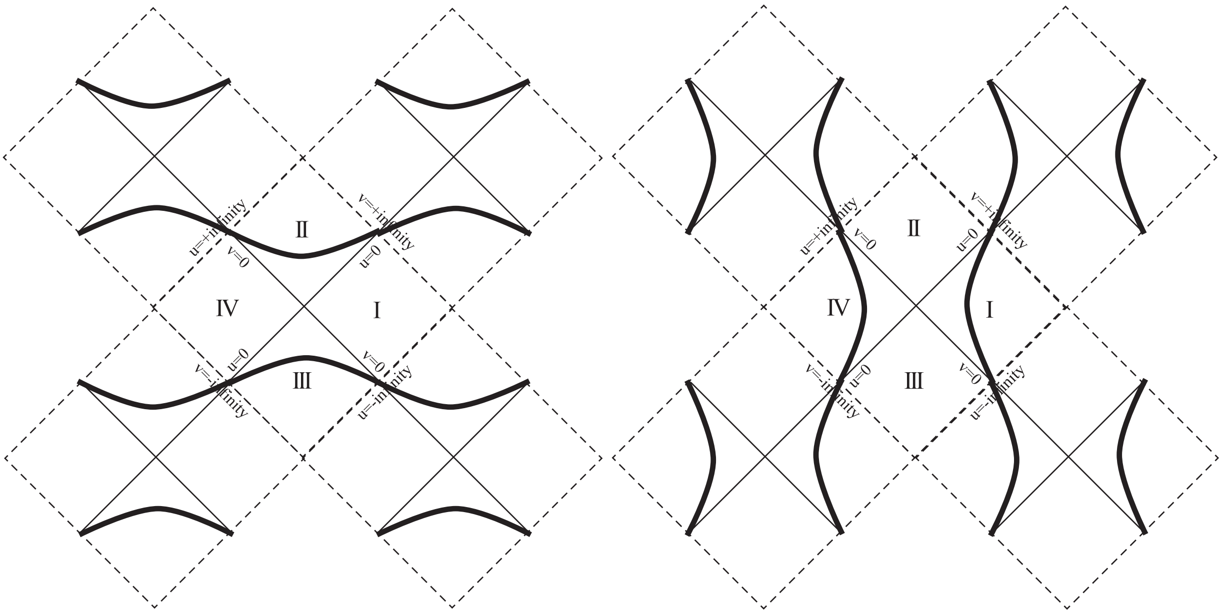
$$\left\{g = -du\,dv + du^2\,F,\ \det g = -\frac{1}{4},\ g = \begin{pmatrix} F & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix},\ e = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ -\frac{F}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix},\ \det e = \frac{1}{2},\right.$$

$$\left.\tilde{\ell} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \end{pmatrix},\ \tilde{n} = \begin{pmatrix} -\frac{F}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix},\ e^{-1} = \begin{pmatrix} \sqrt{2} & 0 \\ \sqrt{2}\,F & \sqrt{2} \end{pmatrix},\ \vec{\ell} = \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix},\ \vec{n} = \begin{pmatrix} \sqrt{2} \\ \sqrt{2}\,F \end{pmatrix}\right\}$$

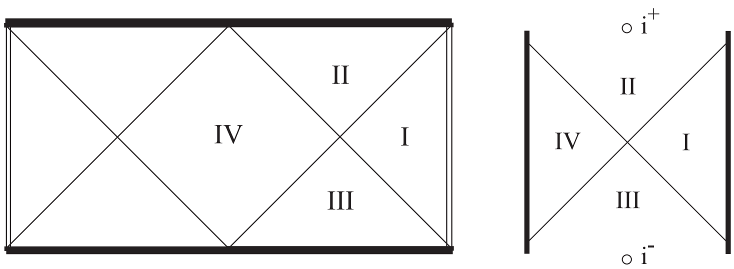


de Sitter & anti de Sitter Penrose-Carter diagrams

The Penrose-Carter diagrams for the de Sitter spacetime (left below) and the anti-de Sitter spacetime (right below). The same copies of the basic portion are attached at the dashed lines. Thick curves correspond to the (A)dS infinity. Observe that the AdS infinity is *timelike*.



Each diagram below is obtained from one maximally extended portion in the corresponding upper diagram. The double lines in the lower left diagram are identified. The isolated points i^{+} and i^{-} in the lower right diagram are future and past timelike infinities, respectively. (See: G.W. Gibbons, S.W. Hawking, *Cosmological event horizons, thermodynamics, and particle creation*, Physical Review D. 15 (1977) 2738–2751.)



References

J.B. Griffiths and J. Podolský, *Exact Space-Times in Einstein's General Relativity* (CUP, 2009).

Schwarzschild

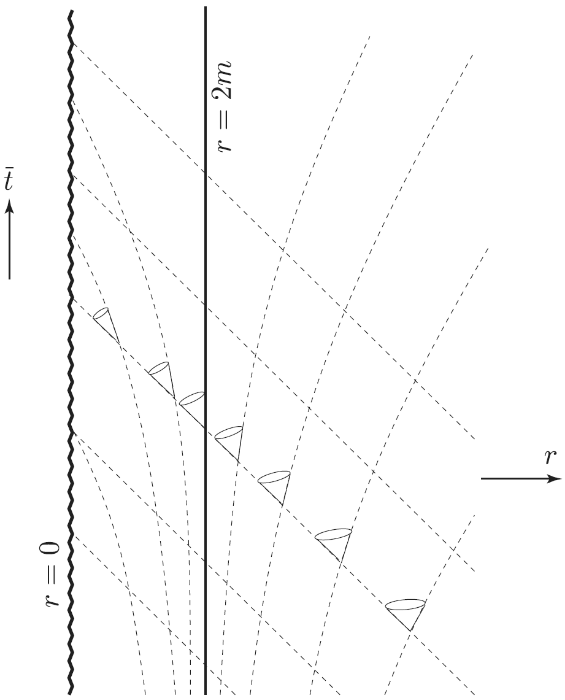


Fig. 8.1 Radial null geodesics in Eddington–Finkelstein coordinates with θ and ϕ constant. Ingoing null geodesics are represented by lines on which $\bar{t} + r =$ const., while null geodesics propagating in the opposite direction have increasing values of r for $r > 2m$, but decreasing values for $r < 2m$. Some future light cones are also indicated.

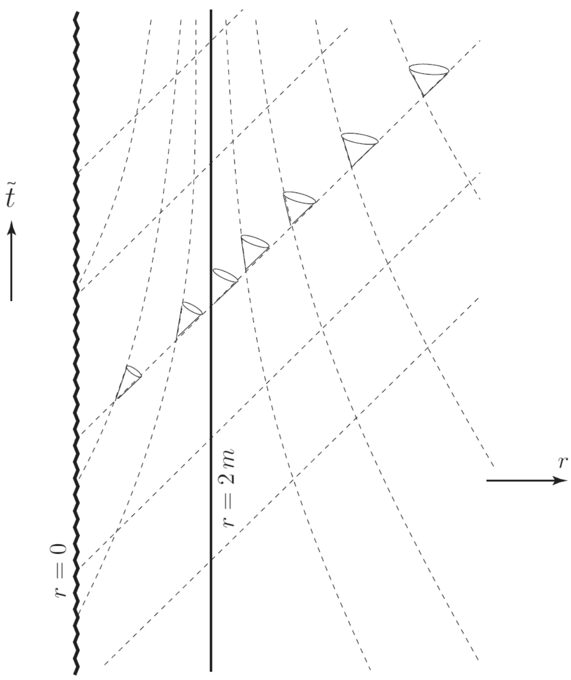


Fig. 8.3 Radial null geodesics in Eddington–Finkelstein-like coordinates with θ and ϕ constant for the Schwarzschild space-time interpreted as a white hole. Some outgoing and ingoing null geodesics and some future light cones are indicated.

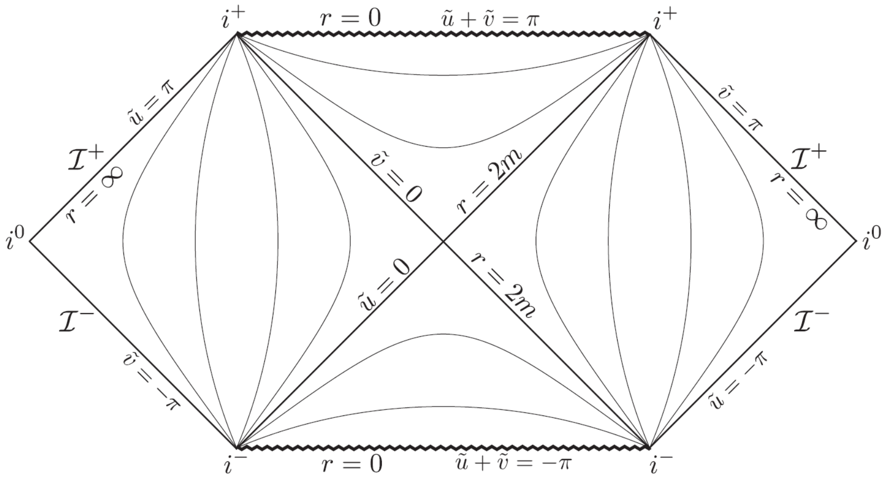


Fig. 8.5 Penrose diagram for the complete Schwarzschild space-time. The θ and ϕ coordinates are suppressed so that each point represents a 2-sphere of radius r. All lines shown are hypersurfaces on which r is a constant.

Schwarzschild when $m < 0$

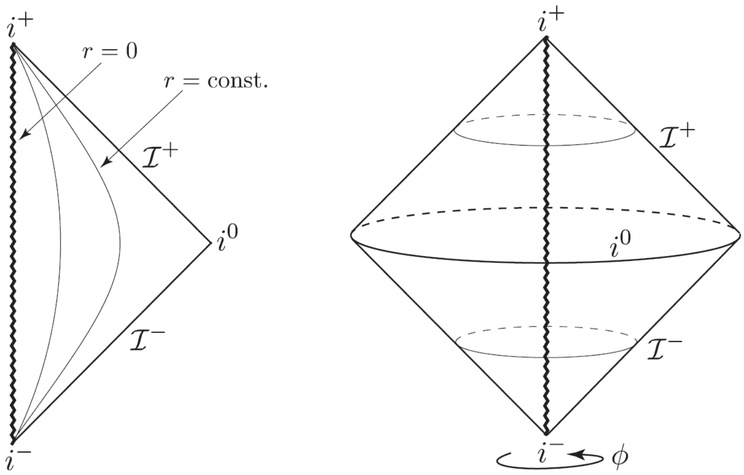


Fig. 8.8 Penrose and conformal diagrams for the Schwarzschild space-time when $m < 0$. In this case, there is a globally naked timelike singularity at $r = 0$.

de Sitter

$$\mathrm{d}s^2 = -\left(1 - \frac{\Lambda}{3}R^2\right)\mathrm{d}T^2 + \left(1 - \frac{\Lambda}{3}R^2\right)^{-1}\mathrm{d}R^2 + R^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2). \quad (4.9)$$

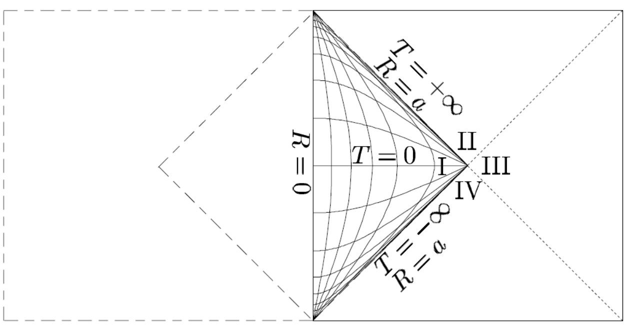


Fig. 4.5 The Penrose diagram of de Sitter space-time with static spherically symmetric coordinates. Each point represents a 2-sphere spanned by θ, ϕ.

Anti de Sitter

$$\mathrm{d}s^2 = -\left(1+\frac{R^2}{a^2}\right)\mathrm{d}T^2 + \left(1+\frac{R^2}{a^2}\right)^{-1}\mathrm{d}R^2 + R^2(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2), \quad (5.6)$$

$$\mathrm{d}s^2 = -\mathrm{d}t_{_{-1}}^2 + a^2\cos^2\frac{t_{_{-1}}}{a}\left(\mathrm{d}\rho^2+\sinh^2\rho\,(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2)\right). \qquad (5.18)$$

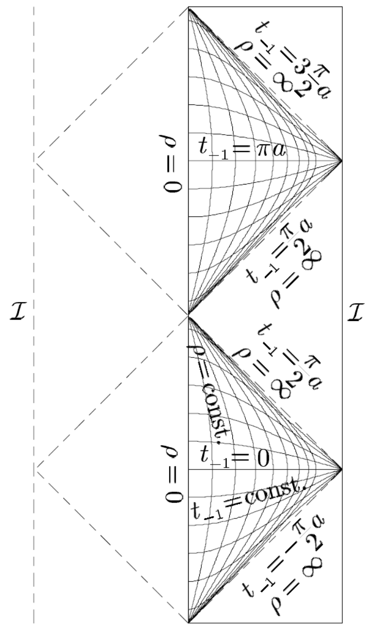


Fig. 5.8 The Penrose diagram of the part of anti-de Sitter space-time covered by the coordinates $(t_{_{-1}}, \rho, \theta, \phi)$. Each point represents a 2-sphere spanned by θ, ϕ.

Schwarzschild - de Sitter

Now consider the generalisation of the Schwarzschild solution which, in addition to mass parameter m, includes an arbitrary cosmological constant Λ. The metric for this case was discovered by Kottler (1918), Weyl (1919b) and Trefftz (1922), and can be written in the form

$$\begin{aligned}\mathrm{d}s^2 = -\left(1-\frac{2m}{r}-\frac{\Lambda}{3}r^2\right)\mathrm{d}t^2 + \left(1-\frac{2m}{r}-\frac{\Lambda}{3}r^2\right)^{-1}\mathrm{d}r^2 \\ +r^2\left(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2\right),\end{aligned} \qquad (9.26)$$

where, $t \in (-\infty,\infty)$, $r \in (0,\infty)$, $\theta \in [0,\pi]$ and $\phi \in [0,2\pi)$. This clearly reduces to the Schwarzschild metric (8.1) when $\Lambda = 0$, and to the de Sitter or anti-de Sitter metrics in their spherically symmetric forms (4.9) or (5.6) when $m = 0$. When $m \neq 0$, it has a curvature singularity at $r = 0$.

$$g_{tt}(r) = \frac{1}{r}\left(\frac{\Lambda}{3}r^3 - r + 2m\right), \qquad (9.27)$$

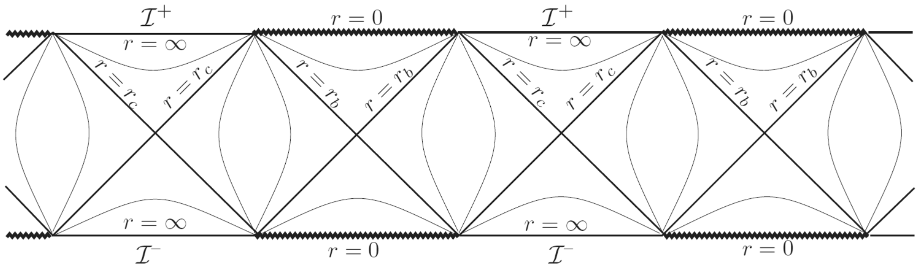


Fig. 9.11 A schematic Penrose diagram for the Schwarzschild–de Sitter space-time when $0 < 9\Lambda m^2 < 1$. This contains an infinite sequence of Schwarzschild-like and de Sitter-like regions. All lines shown have constant values of r.

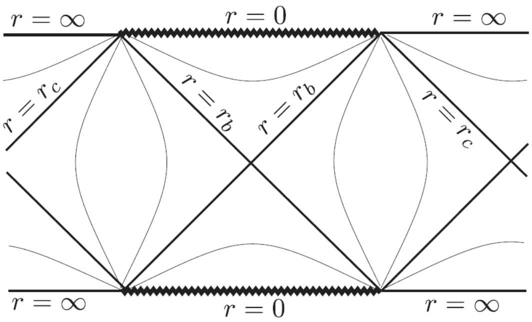


Fig. 9.12 A schematic Penrose diagram for the Schwarzschild–de Sitter space-time in which the spatial edges are identified. This would represent a single Schwarzschild-like black/white hole in a single de Sitter-like universe.

Schwarzschild - Anti de Sitter

For the case when $\Lambda < 0$, it can be seen that the metric function (9.27) must have just a single positive root. This follows from the fact that the sum of the roots must be zero, and that $\frac{\Lambda}{3}r^3 - r + 2m$ is positive when $r = 0$. (We are assuming here that $0 < r < \infty$, and that $m > 0$, so that the space-time has a traditional Schwarzschild limit as $\Lambda \to 0$.) This single root r_b corresponds to a Schwarzschild-like black/white hole horizon.

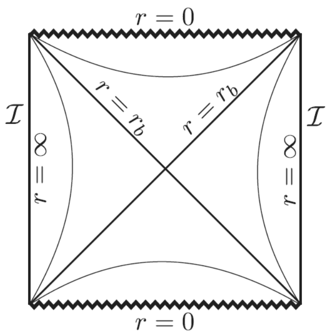


Fig. 9.14 A schematic Penrose diagram for the Schwarzschild–anti-de Sitter space-time. In this case, conformal infinity is timelike.

pp-wave

The family of *pp*-wave space-times was first discussed by Brinkmann (1925), and interpreted in terms of gravitational waves by Peres (1959). Using a coordinate u to label the null wave surfaces such that $k_\mu = -u_{,\mu}$, and an affine parameter r defined such that $\boldsymbol{k} = \partial_r$, the metric for any vacuum, aligned null electromagnetic or pure radiation *pp*-wave space-time can be written in the Brinkmann form

$$\mathrm{d}s^2 = -2\,\mathrm{d}u\,\mathrm{d}r - 2H(\zeta,\bar{\zeta},u)\,\mathrm{d}u^2 + 2\,\mathrm{d}\zeta\,\mathrm{d}\bar{\zeta}, \tag{17.1}$$

where the complex coordinate ζ spans the wave surfaces, and $H(\zeta,\bar{\zeta},u)$ is an arbitrary real function.[1] With the additional null vectors $\boldsymbol{l} = \partial_u - H\partial_r$ and $\boldsymbol{m} = \partial_{\bar{\zeta}}$, the only non-zero components of the curvature tensor are

$$\Psi_4 = H_{,\zeta\zeta}, \qquad \Phi_{22} = H_{,\zeta\bar{\zeta}}. \tag{17.2}$$

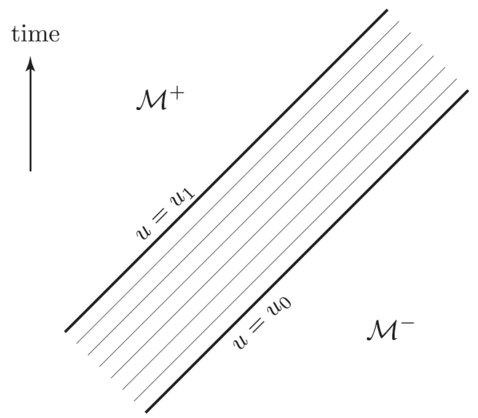


Fig. 17.2 A sandwich wave in which the function $H(\zeta,\bar{\zeta},u)$ is non-trivial only for $u_0 \le u \le u_1$ is joined before and after to two different regions of Minkowski space.