Path Integrals: Perturbation expansion for the ϕ^4-theory

by M.B.Kocic – Version 1.06 (2016-03-28) – FK8017 HT15

Lorentzian Path Integral

$$\int \mathrm{d}^4x \equiv \int \mathrm{d}x^0 \cdots \int \mathrm{d}x^3, \qquad \int \mathrm{d}^4p \equiv \int \mathrm{d}p^0 \cdots \int \mathrm{d}p^3,$$
$$\Box \equiv \partial_0^2 - \nabla^2 = \partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2,$$
$$x^2 \equiv (x^0)^2 - \mathbf{x}^2, \quad p^2 \equiv (p^0)^2 - \mathbf{p}^2, \quad p\cdot x \equiv p^0x^0 - \mathbf{p}\cdot\mathbf{x}$$

Action

$$\mathcal{L}[\phi] = \mathcal{L}_0[\phi] + \mathcal{L}_{\mathrm{int}}[\phi], \qquad \mathcal{L}_0[\phi] = \tfrac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \tfrac{1}{2}m^2\phi^2, \qquad \mathcal{L}_{\mathrm{int}}[\phi] = -\tfrac{1}{4!}\lambda\,\phi^4,$$
$$S[\phi] = S_0[\phi] + S_{\mathrm{int}}[\phi], \qquad S_0[\phi] = \int \mathrm{d}^4x\,\mathcal{L}_0[\phi], \qquad S_{\mathrm{int}}[\phi] = \int \mathrm{d}^4x\,\mathcal{L}_{\mathrm{int}}[\phi]\,.$$

The Wick Rotation $\mathrm{i}S_{\mathrm{L}} = -S_{\mathrm{E}}$

$$x^0 \equiv -\mathrm{i}x^4, \quad x^4 \equiv \mathrm{i}x^0, \quad p^0 \equiv \mathrm{i}p^4, \quad p^4 \equiv -\mathrm{i}p^0,$$
$$\partial_0 = \mathrm{i}\partial_4, \quad \mathrm{d}x^0 = -\mathrm{i}\mathrm{d}x^4, \quad \Box_{\mathrm{L}} = -\Box_{\mathrm{E}},$$
$$\int \mathrm{d}^4x_{\mathrm{L}} = -\mathrm{i}\int \mathrm{d}^4x_{\mathrm{E}}, \quad \int \mathrm{d}^4p_{\mathrm{L}} = \mathrm{i}\int \mathrm{d}^4p_{\mathrm{E}},$$
$$x_{\mathrm{L}}^2 = -x_{\mathrm{E}}^2, \quad p_{\mathrm{L}}^2 = -p_{\mathrm{E}}^2,$$
$$p^0x^0 = p^4x^4, \quad (p\cdot x)_{\mathrm{L}} \neq (p\cdot x)_{\mathrm{E}},$$
$$\delta(x^0) \text{ using FT def.} = \mathrm{i}\delta(x^4)\,.$$

Euclidean Path Integral

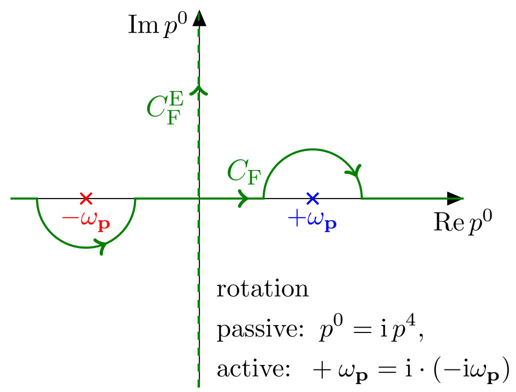
$$\int \mathrm{d}^4x \equiv \int \mathrm{d}x^1 \cdots \int \mathrm{d}x^4, \qquad \int \mathrm{d}^4p \equiv \int \mathrm{d}p^1 \cdots \int \mathrm{d}p^4,$$
$$\Box \equiv \partial_4^2 + \nabla^2 = \partial_1^2 + \partial_2^2 + \partial_3^2 + \partial_4^2,$$
$$x^2 \equiv \mathbf{x}^2 + (x^4)^2, \quad p^2 \equiv \mathbf{p}^2 + (p^4)^2, \quad p\cdot x \equiv \mathbf{p}\cdot\mathbf{x} + p^4x^4$$

Action

$$\mathcal{L}[\phi] = \mathcal{L}_0[\phi] + \mathcal{L}_{\mathrm{int}}[\phi], \qquad \mathcal{L}_0[\phi] = \tfrac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) + \tfrac{1}{2}m^2\phi^2, \qquad \mathcal{L}_{\mathrm{int}}[\phi] = \tfrac{1}{4!}\lambda\,\phi^4,$$
$$S[\phi] = S_0[\phi] + S_{\mathrm{int}}[\phi], \qquad S_0[\phi] = \int \mathrm{d}^4x\,\mathcal{L}_0[\phi], \qquad S_{\mathrm{int}}[\phi] = \int \mathrm{d}^4x\,\mathcal{L}_{\mathrm{int}}[\phi]\,.$$

Feynman propagator

$$\Delta_{\mathrm{F}}(x_1 - x_2) = \int_{C_{\mathrm{F}}} \frac{\mathrm{d}^4p}{(2\pi)^4}\,\mathrm{e}^{-\mathrm{i}p\cdot(x_1-x_2)}\,\frac{1}{p^2-m^2}$$
$$\mathrm{i}\Delta_{\mathrm{F}}(x_1 - x_2) = \langle 0|\,\mathrm{T}\{\,\hat\phi(x_1),\,\hat\phi(x_2)\,\}\,|0\rangle$$
$$\mathrm{i}\Delta_{\mathrm{F}}(x_1 - x_2) = \Delta_{\mathrm{F}}^{\mathrm{E}}(x_1 - x_2)$$

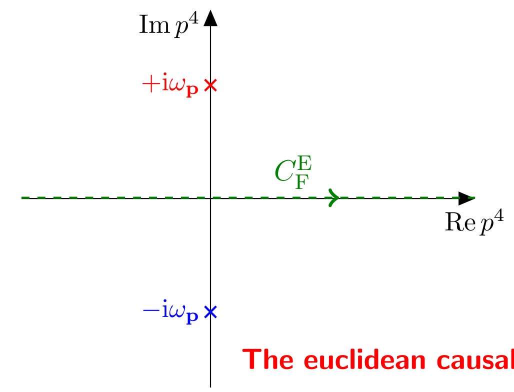


Note: The signs are chosen such that the product of time and energy, $x^0p^0 = x^4p^4$, is invariant. This is an arbitrary choice which ensures that the travelling direction of a plane wave remains unchanged when the coordinates are transformed.

For an arbitrary $f(p^2)$, it holds:

$$\int \frac{\mathrm{d}^4p}{(2\pi)^4}\,\mathrm{e}^{-\mathrm{i}p\cdot x}\,f(p^2) = \mathrm{i}\int \frac{\mathrm{d}^4p_{\mathrm{E}}}{(2\pi)^4}\,\mathrm{e}^{-\mathrm{i}p_{\mathrm{E}}\cdot x_{\mathrm{E}}}\,f(-p_{\mathrm{E}}^2)$$

Since $p\cdot x \neq p_{\mathrm{E}}\cdot x_{\mathrm{E}}$, the above relation requires the insensitivity under spatial reflections: $\mathbf{p} \to -\mathbf{p}$.



Feynman propagator

$$\Delta_{\mathrm{F}}^{\mathrm{E}}(x_1 - x_2) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\,\mathrm{e}^{-\mathrm{i}p\cdot(x_1-x_2)}\,\frac{1}{p^2+m^2}$$
$$\Delta_{\mathrm{F}}^{\mathrm{E}}(x_1 - x_2) = \langle 0|\,\mathrm{T}\{\,\hat\phi(x_1),\,\hat\phi(x_2)\,\}\,|0\rangle$$
$$\Delta_{\mathrm{F}}^{\mathrm{E}}(x_1 - x_2) = \mathrm{i}\Delta_{\mathrm{F}}(x_1 - x_2)$$

The euclidean causal propagator does not require the $\mathrm{i}\varepsilon$-prescription!

Generating Functional (Lorentzian)

$$Z_0[J] = \int \mathcal{D}\phi\,\exp\left(\mathrm{i}\,S_0 + \mathrm{i}\int \mathrm{d}^4z\,J(z)\phi(z)\right)$$
$$Z[J] = \int \mathcal{D}\phi\,\exp\left(\boxed{\mathrm{i}\,S_{\mathrm{int}}} + \mathrm{i}\,S_0 + \mathrm{i}\int \mathrm{d}^4z\,J(x)\phi(z)\right)$$

$\Longrightarrow$

$$Z_0[J] = Z_0[0]\,\exp\left(-\frac{\mathrm{i}}{2}\int \mathrm{d}^4z_1\,\mathrm{d}^4z_2\,J(z_1)\Delta_{\mathrm{F}}(z_1-z_2)J(z_2)\right)$$
$$Z[J] = \exp\left(\boxed{\mathrm{i}\,S_{\mathrm{int}}}\left[\frac{\delta}{\mathrm{i}\,\delta J(y)}\right]\right)Z_0[J]$$
$$Z_0[0] = \int \mathcal{D}\phi\,\exp(\mathrm{i}S_0) = \big(\det(\Box + m^2)\big)^{-1/2}$$

Generating Functional (Euclidean)

$$Z_0[J] = Z_0[0]\,\exp\left(\frac{1}{2}\int \mathrm{d}^4z_1\,\mathrm{d}^4z_2\,J(z_1)\Delta_{\mathrm{F}}^{\mathrm{E}}(z_1-z_2)J(z_2)\right)$$
$$Z[J] = \exp\left(\boxed{-S_{\mathrm{int}}}\left[\frac{\delta}{\delta J(y)}\right]\right)Z_0[J]$$
$$Z_0[0] = \int \mathcal{D}\phi\,\exp(-S_0) = \big(\det(-\Box + m^2)\big)^{-1/2}$$

$\Longleftarrow$

$$Z_0[J] = \int \mathcal{D}\phi\,\exp\left(-S_0 + \int \mathrm{d}^4z\,J(z)\phi(z)\right)$$
$$Z[J] = \int \mathcal{D}\phi\,\exp\left(\boxed{-S_{\mathrm{int}}} - S_0 + \int \mathrm{d}^4z\,J(x)\phi(z)\right)$$

n-point correlation function (Lorentzian)

$$G^{(n)}(x_1,\dots,x_n) = \left(\frac{1}{Z[J]}\frac{\delta}{\mathrm{i}\delta J(x_1)}\cdots\frac{\delta}{\mathrm{i}\delta J(x_n)}Z[J]\right)_{J=0}$$

$$G^{(n)}(x_1,\dots,x_n) = G_{\mathrm{E}}^{(n)}(x_1,\dots,x_n),$$
$$G_0^{(2)}(x_1,x_2) = \mathrm{i}\Delta_{\mathrm{F}}(x_1-x_2) = \Delta_{\mathrm{F}}^{\mathrm{E}}(x_1-x_2) = G_{0,\mathrm{E}}^{(2)}(x_1,x_2)$$

n-point correlation function (Euclidean)

$$G_{\mathrm{E}}^{(n)}(x_1,\dots,x_n) = \left(\frac{1}{Z[J]}\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)}Z[J]\right)_{J=0}$$

First-order expansion of the 2-point correlation function $G^{(2)}(x_1,x_2)$ for the ϕ^4-theory in the euclidean framework: (for $\mathrm{E}\to\mathrm{L}$, replace $-\lambda \to \mathrm{i}\lambda$)

$$= \left.\frac{\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}Z[J]}{Z[J]}\right|_{J=0} = \left.\frac{\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}\exp\left(\boxed{-S_{\mathrm{int}}}\left[\frac{\delta}{\delta J(y)}\right]\right)Z_0[J]}{\exp\left(\boxed{-S_{\mathrm{int}}}\left[\frac{\delta}{\delta J(y)}\right]\right)Z_0[J]}\right|_{J=0} = \left.\frac{\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}\exp\left(-\frac{\lambda}{4!}\int \mathrm{d}^4y\,\frac{\delta^4}{\delta J(y)^4}\right)\left[Z_0[0]\exp\left(\frac{1}{2}\int \mathrm{d}^4z_1\,\mathrm{d}^4z_2\,J(z_1)\Delta_{\mathrm{F}}^{\mathrm{E}}(z_1-z_2)J(z_2)\right)\right]}{\exp\left(-\frac{\lambda}{4!}\int \mathrm{d}^4y\,\frac{\delta^4}{\delta J(y)^4}\right)\left[Z_0[0]\exp\left(\frac{1}{2}\int \mathrm{d}^4z_1\,\mathrm{d}^4z_2\,J(z_1)\Delta_{\mathrm{F}}^{\mathrm{E}}(z_1-z_2)J(z_2)\right)\right]}\right|_{J=0}$$

Now, Taylor expand the exponential function for the interaction to the required order in λ. Also, Taylor expand the exponential functions with the source to match the number of functional derivatives. Note that $J=0$ will cancel all unused source terms after the expansion (so they can be ignored).

$$= \left.\frac{\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}\left(1-\frac{\lambda}{4!}\int \mathrm{d}^4y\,\frac{\delta^4}{\delta J(y)^4}+\mathcal{O}(\lambda^2)\right)\left[(\text{ign.}\,\tfrac{1}{0!}) + \frac{1}{1!}\left(\frac{1}{2}\right)^1\int \mathrm{d}^4z_1\,\mathrm{d}^4z_2\,J(z_1)J(z_2)\,\Delta_{\mathrm{F}}^{\mathrm{E}}(z_1-z_2) + (\text{ign.}\,\tfrac{1}{2!}) + \frac{1}{3!}\left(\frac{1}{2}\right)^3\int \mathrm{d}^4z_1\cdots\mathrm{d}^4z_6\,J(z_1)\cdots J(z_6)\,\Delta_{\mathrm{F}}^{\mathrm{E}}(z_1-z_2)\Delta_{\mathrm{F}}^{\mathrm{E}}(z_3-z_4)\Delta_{\mathrm{F}}^{\mathrm{E}}(z_5-z_6)\right]}{\left(1-\frac{\lambda}{4!}\int \mathrm{d}^4y\,\frac{\delta^4}{\delta J(y)^4}+\mathcal{O}(\lambda^2)\right)\left[1 + (\text{ign.}\,\tfrac{1}{1!}) + \frac{1}{2!}\left(\frac{1}{2}\right)^2\int \mathrm{d}^4z_1\cdots\mathrm{d}^4z_4\,J(z_1)\cdots J(z_4)\,\Delta_{\mathrm{F}}^{\mathrm{E}}(z_1-z_2)\Delta_{\mathrm{F}}^{\mathrm{E}}(z_3-z_4)\right]}\right|_{J=0}$$

Note: $\frac{\delta^n}{\delta J(y)^n}J(z_1)\cdots J(z_n) = \\ = \sum_\sigma \delta(z_{\sigma(1)}-y)\cdots\delta(z_{\sigma(n)}-y)$, where σ runs over $n!$ permuations.

$$= \frac{\frac{1}{1!}\left(\frac{1}{2}\right)^1\times 2\times\Delta_{\mathrm{F}}^{\mathrm{E}}(x_1-x_2) + \frac{1}{3!}\left(\frac{1}{2}\right)^3\times 6\times 4!\times\left(-\frac{\lambda}{4!}\int \mathrm{d}^4y\right)\Delta_{\mathrm{F}}^{\mathrm{E}}(x_1-x_2)\Delta_{\mathrm{F}}^{\mathrm{E}}(y-y)\Delta_{\mathrm{F}}^{\mathrm{E}}(y-y) + \frac{1}{3!}\left(\frac{1}{2}\right)^3\times 6\times 4\times 4!\times\left(-\frac{\lambda}{4!}\int \mathrm{d}^4y\right)\Delta_{\mathrm{F}}^{\mathrm{E}}(x_1-y)\Delta_{\mathrm{F}}^{\mathrm{E}}(y-x_2)\Delta_{\mathrm{F}}^{\mathrm{E}}(y-y) + \mathcal{O}(\lambda^2)}{1 + \frac{1}{2!}\left(\frac{1}{2}\right)^2\times 4!\times\left(-\frac{\lambda}{4!}\int \mathrm{d}^4y\right)\Delta_{\mathrm{F}}^{\mathrm{E}}(y-y)\Delta_{\mathrm{F}}^{\mathrm{E}}(y-y) + \mathcal{O}(\lambda^2)}$$

Here, the disconnected loops in the denominator cancel out. The general result is the linked-cluster theorem which states that the vacuum diagrams cancel exactly out to all orders in perturbation theory. (The theoremis valid for all the n-point functions and for any local theory.)

$$= \frac{\Delta_{\mathrm{F}}^{\mathrm{E}}(x_1-x_2) + 3\times\Delta_{\mathrm{F}}^{\mathrm{E}}(x_1-x_2)\left(-\frac{\lambda}{4!}\int \mathrm{d}^4y\right)\Delta_{\mathrm{F}}^{\mathrm{E}}(y-y)\Delta_{\mathrm{F}}^{\mathrm{E}}(y-y) + 12\times\left(-\frac{\lambda}{4!}\int \mathrm{d}^4y\right)\Delta_{\mathrm{F}}^{\mathrm{E}}(x_1-y)\Delta_{\mathrm{F}}^{\mathrm{E}}(y-x_2)\Delta_{\mathrm{F}}^{\mathrm{E}}(y-y) + \mathcal{O}(\lambda^2)}{1 + 3\times\left(-\frac{\lambda}{4!}\int \mathrm{d}^4y\right)\Delta_{\mathrm{F}}^{\mathrm{E}}(y-y)\Delta_{\mathrm{F}}^{\mathrm{E}}(y-y) + \mathcal{O}(\lambda^2)} = \Delta_{\mathrm{F}}^{\mathrm{E}}(x_1-x_2) - \frac{\lambda}{2}\int \mathrm{d}^4y\,\Delta_{\mathrm{F}}^{\mathrm{E}}(x_1-y)\Delta_{\mathrm{F}}^{\mathrm{E}}(y-x_2)\Delta_{\mathrm{F}}^{\mathrm{E}}(y-y) + \mathcal{O}(\lambda^2)\,,$$

$$\Longrightarrow\; G^{\mathrm{E}}(p) = G_0^{\mathrm{E}}(p) - \frac{\lambda}{2}\,G_0^{\mathrm{E}}(p)^2\Delta_{\mathrm{F}}^{\mathrm{E}}(0) + \mathcal{O}(\lambda^2) = \frac{1}{p^2+m^2}\left(1 - \frac{\lambda}{2}\frac{1}{p^2+m^2}\int \frac{\mathrm{d}^4q}{(2\pi)^4}\frac{1}{q^2+m^2}\right) + \mathcal{O}(\lambda^2) = \frac{1}{p^2+m^2+\frac{\lambda}{2}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{1}{q^2+m^2}} + \mathcal{O}(\lambda^2),$$

with hard cut-off: $\delta m^2 = \dfrac{\lambda}{2}\displaystyle\int_0^\Lambda \frac{\mathrm{d}^4q}{(2\pi)^4}\frac{1}{q^2+m^2} = \frac{\lambda}{32\pi^2}\left(\Lambda^2 - m^2\log\frac{\Lambda^2+m^2}{m^2}\right).$