Uncertainty Analysis in Physics with *Mathematica*

MBKocic, SK180N KTH, 2010-10-22

This document presents UCAnalysis framework for *Mathematica* for both symbolic and numerical uncertainty analysis.

UC-Analysis Package Revisions

v1.0	2010-10-22	Initial release
v2.0	2011-08-25	Converted m-package to notebook
v2.1	2011-09-15	Isolated private symbols; changed symbol names
v2.2	2011-09-21	NumberBox and DigitsOf rewritten from scratch
v2.3	2011-09-30	Package divided into UCQuantity and UCAnalysis
v2.4	2011-10-11	Added CODATA package

Propagation of Uncertainty

Rules of Significant Figures

It is usual that, in problems in physics textbooks, uncertainties of input quantities are implicitly quoted with significant figures rather than explicitly with the best estimate, uncertainty and specified level of confidence; however, identifying the correct number of significant figures in input quantities sometimes can be problematic, for, first, there is an inherent ambiguity to determine the number of significant figures in quantities ending with one or more zeros but without decimal point (like 100 °C), and second, one should not miss to recognize all *defined* input quantities having the infinite number of significant figures (like 25.4 mm/inch or 273.15 in T/K = t/°C + 273.15).

When relying on significant figures, estimation of the uncertainty of a calculated result may be done by using some simple rules for addition and multiplication during calculations. These rules can be stated as follows:

- *For multiplication and division*: the number of significant figures that are reliably known in a product or quotient is the same as the smallest number of significant figures in any of the original numbers.
- *For addition and subtraction*: the result should be rounded off to the last decimal place reported for the least precise number.

While these rules are an efficient means of propagating uncertainty in many simple calculations, they are not valid whenever non-linear mathematical functions are involved (like exponentials, logarithms and trigonometric functions) or when there are many steps throughout the calculation, which is often the case. Further sections demonstrate other, more reliable methods for uncertainty analysis.

Combined Standard Uncertainty

Let $Y = f(X_1, X_2, \ldots X_n)$ be a functional relationship between input quantities X_i (conventionally expressed as $X_i = x_i \pm u_i$) and the result Y.

If input quantities X_i are independent (uncorrelated) and their uncertainties are standard deviations of measured data (called "standard uncertainties" u_i of the input estimates x_i), we can determine a combined standard uncertainty u_c of the output estimate $y = f(x_1, x_2, \ldots x_n)$ using the law of propagation of uncertainties (commonly called the "root-sum-of-squares" or RSS method):

$$u_c = \sqrt{\sum_{i=1}^{n} \left(\frac{\partial f(x_1, x_2, \ldots x_n)}{\partial x_i} \right)^2 u_i^2} \quad \text{where} \quad Y = y \pm u_c, \quad \text{so} \tag{1}$$

$$Y = f(x_1, x_2, \ldots x_n) \pm \sqrt{\sum_{i=1}^{n} \left(\frac{\partial f(x_1, x_2, \ldots x_n)}{\partial x_i} \right)^2 u_i^2} \tag{2}$$

When this is done, the combined standard uncertainty u_c should be equivalent to the standard deviation of the result, making this uncertainty value correspond with a 68.27 % (1σ) confidence interval. If a wider confidence interval is desired, the uncertainty can be multiplied by a coverage factor (usually $k = 2$ or 3) to provide an uncertainty interval $U = k\, u_c$ (called *expanded uncertainty*) that is believed to include the true value with a confidence of 95.45 % or 99.73 % respectively.

Note again that the previous expressions are valid only if input quantities are estimated by standard uncertainties, i.e. before combination, all uncertainty contributions must be expressed as standard deviations. This may involve conversion from some other measure of dispersion. The following rules give some guidance for converting an uncertainty component to a standard deviation.

- If the uncertainty component was evaluated experimentally from the dispersion of repeated measurements, it can readily be expressed as a standard deviation.
- If an uncertainty estimate is derived statistically from previous results and data, it may already be expressed as a standard deviation. However, where a confidence interval is given with a level of confidence, in the form $\pm U$ at p%, then we should divide the value expanded uncertainty U by the appropriate percentage point of the normal distribution for the level of confidence given to calculate the standard deviation.

If limits $\pm\delta$ are given without a confidence level and there is reason to expect that extreme values are likely, e.g. when reading from digital instruments or when using significant figures, it is normally appropriate to assume an uniform distribution, with a standard deviation of $\delta/\sqrt{3}$.

- If limits $\pm\delta$ are given without a confidence level, but there is reason to expect that extreme values are unlikely, e.g. when reading from analog instruments, it is normally appropriate to assume a triangular distribution, with a standard deviation of $\delta/\sqrt{6}$.

Bounds of Uncertainty and Absolute Maximum Uncertainty

When the origin or correlation of input quantities are not known, to account for worst-case situations, the only we can state is that uncertainty interval of the calculated result is between local minimum f_{min} and local maximum f_{max} of the function $f(X_1, X_2, \ldots X_n)$ on the domain $\left\{\wedge_{i=1}^{n} x_i - \varepsilon_i \leq X_i \leq x_i + \varepsilon_i\right\}$ of input quantities. In this case, the output interval $[f_{\text{min}}; f_{\text{max}}]$ is almost never symmetrical around the central value $y = f(x_1, x_2, \ldots x_n)$.

Alternative way similar to RSS is to, based on first-order Taylor series approximation of f, calculate an absolute maximum uncertainty ε_{max} by adding up all maximum absolute uncertainty contributions linearly, rather than as squared like in RSS:

$$\varepsilon_{\text{max}} = \sum_{i=1}^{n} \left| \frac{\partial f(x_1, x_2, \ldots x_n)}{\partial x_i} \right| \varepsilon_i \tag{3}$$

where ε_i is the absolute maximum unceraintу of the input quantity X_i. Since an arithmetic sum can never be smaller than a geometric one, linear addition guarantees that the correct total uncertainty, calculated using RSS-like method, is under no circumstances larger than the uncertainty ε_{max}, i.e. inequality $u_c \leqslant \varepsilon_{\text{max}}$ always holds.

Monte Carlo Method

As it is often difficult to calculate the combined standard uncertainty in complex systems, Monte Carlo simulation can be used as an experimental probabilistic method since computers can easily simulate a large number of experimental trials that have random outcomes. When applied to uncertainty estimation, random numbers are used to randomly sample uncertainty intervals of input quantities automatically taking care of correlation between input variables and possible different statistical distributions.

This approach involves the following steps:

- Select a distribution $\mathcal{D}_i$ to describe possible values of input quantity X_i based on available information (Table 1 in JCGM 101:2008, p. 21), e.g. uniform distribution if lower and upper limits are known, or normal distribution if best estimate and associated standard uncertainty are known.
- Generate random data from this distribution $x_i = \text{rand}(\mathcal{D}_i)$.
- Use the generated data as possible values the model to produce output estimates $y = f(x_1, x_2, \ldots x_n)$.
- Calculate expected value and standard deviation of produced output.

How to Use `UCAnalysis.m` Package

1. Setup Function Relationship and Input Quantities

1.1 Load the package first:

```
<< UCAnalysis`
```

1.2 Provide the functional relationship *f* between input quantities by using the operator ↦ in the form:

$$f \mapsto X$$

where *X* is a table with symbolic names of input quantities in the first column, best estimate ± uncertainty values in the second column (± uncertainty part might be omitted for exact values) and type of distribution (normal, uniform or triangular depending on the type of uncertainty evaulation of input quantities) in the third column; like in the following example:

$$a + b\,\Delta c \quad \mapsto \begin{pmatrix} a & 2.264 \pm 0.008 & \text{Normal}\mathcal{D} \\ b & 4.1 \pm 0.05 & \text{Triangular}\mathcal{D} \\ \Delta c & 50\times 10^{-6} \pm 0.5\times 10^{-6} & \text{Uniform}\mathcal{D} \end{pmatrix}$$

1.3 Display a brief information about input quantities and evaluated functional relationship between them:

```
♀FunctionalRelationship
```

$y = x_1 + x_2\, x_3$

```
♀InputQuantities
```

Quantity		Estimate ± Uncertainty	Units	Distribution	$\lvert \partial f/\partial x_i \rvert$
x_1	a	$(2.264 \pm 0.008)\times 10^{0}$		Normal, 1σ	1.
x_2	b	$(4.10 \pm 0.05)\times 10^{0}$		Triangular	$5.\times 10^{-5}$
x_3	Δc	$(5.00 \pm 0.05)\times 10^{-5}$		Uniform	4.1

1.4 The previous setup procedure creates an analysis environment with global symbols that may be used for further uncertainty analysis. Available symbols are:

```
f[x]
```

$x_1 + x_2\, x_3$

```
♀VerboseArgs
```

$\{x_1 \to a,\ x_2 \to b,\ x_3 \to \Delta c,\ u_1 \to u_a,\ u_2 \to u_b,\ u_3 \to u_{\Delta c},\ \varepsilon_1 \to \varepsilon_a,\ \varepsilon_2 \to \varepsilon_b,\ \varepsilon_3 \to \varepsilon_{\Delta c}\}$

```
f[x] //. ♀VerboseArgs
```

$a + b\,\Delta c$

```
Table[{x_i, "⟶", N@fDist[i]}, {i, 1, n}] //. ♀VerboseArgs // TableForm
```

```
a    ⟶  NormalDistribution[2.264, 0.008]
b    ⟶  TriangularDistribution[{4.05, 4.15}]
Δc   ⟶  UniformDistribution[{0.0000495, 0.0000505}]
```

```
♀AsStandardUncertainty //. ♀VerboseArgs // N
```

$\{a \to 2.264,\ u_a \to 0.008,\ b \to 4.1,\ u_b \to 0.0204124,\ \Delta c \to 0.00005,\ u_{\Delta c} \to 2.88675\times 10^{-7}\}$

ϙAsMaximumUncertainty //. ϙVerboseArgs // N

$\{a \to 2.264,\ \varepsilon_a \to 0.024,\ b \to 4.1,\ \varepsilon_b \to 0.05,\ \Delta c \to 0.00005,\ \varepsilon_{\Delta c} \to 5.\times 10^{-7}\}$

ϙfDomain //. ϙVerboseArgs // N

$\{2.24 \le a \le 2.288,\ 4.05 \le b \le 4.15,\ 0.0000495 \le \Delta c \le 0.0000505\}$

ϙfMinValue //. ϙVerboseArgs // N

$\{2.2402,\ \{a \to 2.24,\ b \to 4.05,\ \Delta c \to 0.0000495\}\}$

ϙfMaxValue //. ϙVerboseArgs // N

$\{2.28821,\ \{a \to 2.288,\ b \to 4.15,\ \Delta c \to 0.0000505\}\}$

2. Analyze Propagated Uncertainty

ϙAnalysisResult

y	2.264 205		
u_c $y \pm u_c$	0.008 000 000 152 656 248 543 504 361 288 99 $(2.2642 \pm 0.0080)\times 10^0$	= =	0.353 % 2.2642(80)
ε_{max} $y \pm \varepsilon_{max}$	0.024 004 55 $(2.264 \pm 0.024)\times 10^0$	= =	1.06 % 2.264(24)
y_{min} y_{max}	2.240 200 475 2.288 209 575	= =	y − 0.0240045 y + 0.0240046

2.1 Calculate Absolute Maximum Uncertainty

ϙfEstimate ± ϙfMaximumUncertainty // ϙUC

2.264 205 ± 0.024 004 5

∈ [2.2402; 2.288 21]

≈ **$(2.264 \pm 0.024)\times 10^0$ = 2.264(24)**

2.1.1 Define $\varepsilon_{\max}$ as $\sum_{i=1}^{n}\left|\frac{\partial f(x)}{\partial x_i}\right|\varepsilon_i$:

$$\varepsilon_{max} = \sum_{i=1}^{n} |\partial_{x_i} f[x]|\ \varepsilon_i; \quad \varepsilon_{max} \text{ // ϙVerbose}$$

$\varepsilon_{max} == \varepsilon_a + \mathrm{Abs}[\Delta c]\ \varepsilon_b + \mathrm{Abs}[b]\ \varepsilon_{\Delta c}$

2.1.2 Calculate $f(x) \pm \varepsilon_{\max}$ and display the result using **ϙUCE** function:

f[x] ± ε_{max} // ϙUCE

2.264 205 ± 0.024 004 5

∈ [2.2402; 2.288 21]

≈ **$(2.264 \pm 0.024)\times 10^0$ = 2.264(24)**

2.2 Calculate Combined Standard Uncertainty

ϙfEstimate ± ϙfStandardUncertainty // ϙUC

2.264 205 ± 0.008

∈ [2.256 205; 2.272 205]

≈ **$(2.2642 \pm 0.0080)\times 10^0$ = 2.2642(80)**

2.2.1 Define propagated standard uncertainty u_c as $\sqrt{\sum_{i=1}^{n}\left(\frac{\partial f(x)}{\partial x_i}\right)^2 u_i^2}$:

```
u_c = (Sum[(∂_{x_i} f[x])^2 u_i^2, {i, 1, n}])^(1/2) ;   u_c // ϙVerbose
```

$$u_c == \sqrt{u_a^2 + \Delta c^2\, u_b^2 + b^2\, u_{\Delta c}^2}$$

2.2.2 Calculate $f(x) \pm u_c$ and display the result using **ϙUCA** function:

```
f[x] ± u_c // ϙUCA
```

```
2.264 205 ± 0.008
   ∈  [2.256 205; 2.272 205]
   ≈  (2.2642 ± 0.0080) × 10^0  = 2.2642(80)
```

2.3 Monte Carlo Simulation

```
ϙMonteCarlo[10^6] // ϙUC
```

```
ϙY  =  2.264 196 2 ± 0.007 997 84
    ∈  [2.256 198 3; 2.272 194]; Normal𝒟, k = 1σ
    ≈  (2.2642 ± 0.0080) × 10^0  = 2.2642(80)
```

2.3.1 Generate random input data:

```
trials = 10^6 ;
randomInput = Table[RandomReal[fDist[i], {trials}], {i, 1, n}];
```

2.3.2 Apply specified functional relationship f on input data and get output data:

```
data = f @@ randomInput;
```

2.3.3 Show expected value and standard deviation of output data:

```
Mean[data] ± StandardDeviation[data] // ϙUCA
```

```
2.264 206 6 ± 0.007 999 6
   ∈  [2.256 207; 2.272 206 2]
   ≈  (2.2642 ± 0.0080) × 10^0  = 2.2642(80)
```

2.3.4 Finally, don't forget to purge temporary global variables. (You can also set up previous variables as local using modules or blocks.)

```
trials =.; randomInput =.; data =.;
```

Example: Uncertainty in Reading Digital Instruments

For a digital instrument, the reading uncertainty is ± one-half of the last digit (assuming that the instrument has been properly engineered to round a reading correctly on the display), so the value is equally likely to fall anywhere in between the upper and lower limits of uncertainty (including extreme values) i.e. we have Type B evaulated input quantity with uniform distribution between the upper and lower limits of uncertainty.

```
δ = 0.5;
v  ↦  (v    1 ± δ    Uniform𝒟)
```

Evaluated Functional Relationship

```
⚲InputQuantities
```

Quantity		Estimate ± Uncertainty	Units	Distribution	$\lvert \partial f / \partial x_i \rvert$
x_1	v	$(1.0 \pm 0.5) \times 10^0$		Uniform	1.

```
⚲AnalysisResult
```

y	1	
u_c	0.288 675 134 594 812 882 254 574 390 251	= 28.9 %
$y \pm u_c$	$(1.00 \pm 0.29) \times 10^0$	= 1.00(29)
ε_{max}	0.5	= 50. %
$y \pm \varepsilon_{max}$	$(1.0 \pm 0.5) \times 10^0$	= 1.0(5)
y_{min}	0.5	= y - 0.500000
y_{max}	1.5	= y + 0.500000

Standard Uncertainty with 1σ Level of Confidence

$$u_c = \left(\sum_{i=1}^{n} (\partial_{x_i} f[x])^2 u_i^2 \right)^{1/2}; \quad f[x] \pm u_c \; // \; \text{⚲UCA}$$

1 ± 0.288 675
∈ [0.711 325; 1.288 68]
≈ $(1.00 \pm 0.29) \times 10^0$ = 1.00(29)

Monte Carlo Simulation with Uniform Distribution of Input Quantities

The standard uncertainty for a uniform distribution is $\delta/\sqrt{3}$ where δ is the semi-range (or half-width) between the upper and lower limits.

```
Block[{data, trials = 10^6}, data = f @@ Table[RandomReal[fDist[i], {trials}], {i, 1, n}];
  StandardDeviation[data]] ~ (δ / N[√3])
```

0.288 89 ~ 0.288 675

```
δ =.
```

Example: Relativistic Addition of Velocities

This example shows how rules of significant figures can give an erroneous result.

Let us assume that we should add two relativistic velocities $\beta_1 = 0.8$ and $\beta_2 = 0.999$ by using expression $\frac{\beta_1+\beta_2}{1+\beta_1\,\beta_2}$.

Rules of Significant Figures

If we apply rules of significant figures to propagate uncertainty, we get:

$\text{step}_1 = \beta_1 + \beta_2$	$= 0.\underline{8} + 0.\underline{999}$	$= \underline{1.7}99$	*addition rule*
$\text{step}_2 = \beta_1\,\beta_2$	$= 0.\underline{8} \times 0.\underline{999}$	$= 0.\underline{7}992$	*multiplication rule*
$\text{step}_3 = 1 + \text{step}_2$	$= \underline{1.0000} + 0.\underline{7}992$	$= \underline{1.7}992$	*addition rule; 1 is a defined quantity*
$\text{step}_4 = \text{step}_1 / \text{step}_3$	$= \underline{1.7}99 / \underline{1.7}992$	$= 0.\underline{99}9889$	*division rule*
result	$= \text{Round}[\text{step}_4]$	$\simeq \underline{\mathbf{1.00}}$	*rounding off and final answer*

where significant figures are underlined. Note that extra digits are kept during intermediate steps to minimize round-off errors of the calculation. If we didn't keep those extra digits in steps 1 and 2, we would get even worse calculation in step 4, namely $\underline{1.8} / \underline{1.8} = \underline{1.0}$.

The following sections present more reliable uncertainty analysis of this classical textbook problem.

Setup Analysis Environment

Since input quantities originate from Type B evaluation without a specific knowledge about the possible values within the interval and with reason to expect that extreme values are likely, it is appropriate to assume an uniform distribution of input quantities:

$$\frac{\beta_1 + \beta_2}{1 + \beta_1\,\beta_2} \mapsto \begin{pmatrix} \beta_1 & 0.8\grave{}1 & \text{Uniform}\mathcal{D} \\ \beta_2 & 0.999\grave{}3 & \text{Uniform}\mathcal{D} \end{pmatrix}$$

Evaluated Functional Relationship

FunctionalRelationship

$$y = \frac{x_1 + x_2}{1 + x_1\,x_2}$$

InputQuantities

Quantity		Estimate ± Uncertainty	Units	Distribution	$\lvert \partial f / \partial x_i \rvert$
x_1	β_1	$(8.0 \pm 0.5) \times 10^{-1}$		Uniform	6.17524×10^{-4}
x_2	β_2	$(9.990 \pm 0.005) \times 10^{-1}$		Uniform	1.1121×10^{-1}

AnalysisResult

y	0.999 888 839 484 215 206 758 559 359 715	
u_c	0.000 036 720 806 315 761 214 676 693 952 385 2	= 36.7 ppm
$y \pm u_c$	$(9.998\,89 \pm 0.000\,37) \times 10^{-1}$	$= 9.998\,89(37) \times 10^{-1}$
ε_{max}	0.000 086 481 176 061 451 945 145 918 318 323 8	= 86.5 ppm
$y \pm \varepsilon_{max}$	$(9.998\,89 \pm 0.000\,87) \times 10^{-1}$	$= 9.998\,89(87) \times 10^{-1}$
y_{min}	0.999 785 576 4	$= y - 0.000103263$
y_{max}	0.999 959 450 1	$= y + 0.0000706107$

Absolute Maximum Uncertainty

$$\varepsilon_{max} = \sum_{i=1}^{n} |\partial_{x_i} f[x]|\, \varepsilon_i; \quad \varepsilon_{max} \,//\, \text{ϘVerbose}$$

$$\varepsilon_{max} == \text{Abs}\left[-\frac{\beta_2\,(\beta_1+\beta_2)}{(1+\beta_1\,\beta_2)^2} + \frac{1}{1+\beta_1\,\beta_2}\right] \varepsilon_{\beta_1} + \text{Abs}\left[-\frac{\beta_1\,(\beta_1+\beta_2)}{(1+\beta_1\,\beta_2)^2} + \frac{1}{1+\beta_1\,\beta_2}\right] \varepsilon_{\beta_2}$$

$$f[x] \pm \varepsilon_{max} \,//\, \text{ϘUCE}$$

```
0.999 888 839 ± 0.000 086 481 2
  ∈ [0.999 802 358; 0.999 975 321]
  ≈ (9.998 89 ± 0.000 87) × 10^-1 = 9.998 89(87) × 10^-1
```

```
ϘfEstimate ± ϘfMaximumUncertainty // ϘUC
```

```
0.999 888 839 ± 0.000 086 481 2
  ∈ [0.999 802 358; 0.999 975 321]
  ≈ (9.998 89 ± 0.000 87) × 10^-1 = 9.998 89(87) × 10^-1
```

Combined Standard Uncertainty

$$u_c = \left(\sum_{i=1}^{n} (\partial_{x_i} f[x])^2\, u_i^2\right)^{1/2}; \quad u_c \,//\, \text{ϘVerbose}$$

$$u_c == \sqrt{\left(-\frac{\beta_2\,(\beta_1+\beta_2)}{(1+\beta_1\,\beta_2)^2} + \frac{1}{1+\beta_1\,\beta_2}\right)^2 u_{\beta_1}^2 + \left(-\frac{\beta_1\,(\beta_1+\beta_2)}{(1+\beta_1\,\beta_2)^2} + \frac{1}{1+\beta_1\,\beta_2}\right)^2 u_{\beta_2}^2}$$

$$f[x] \pm u_c \,//\, \text{ϘUCA}$$

```
0.999 888 839 ± 0.000 036 720 8
  ∈ [0.999 852 119; 0.999 925 56]
  ≈ (9.998 89 ± 0.000 37) × 10^-1 = 9.998 89(37) × 10^-1
```

```
ϘfEstimate ± ϘfStandardUncertainty // ϘUC
```

```
0.999 888 839 ± 0.000 036 720 8
  ∈ [0.999 852 119; 0.999 925 56]
  ≈ (9.998 89 ± 0.000 37) × 10^-1 = 9.998 89(37) × 10^-1
```

```
ϘUCResult[ v, "c", UcPrecision → 2 ]
```

```
v = (0.999 888 839 ± 0.000 036 720 8) c
  ∈ [0.999 852 119; 0.999 925 56] c; Normal𝒟, k = 1σ
  ≈ (9.998 89 ± 0.000 37) × 10^-1 c = 9.998 89(37) × 10^-1 c
```

Monte Carlo Simulation with Uniform Distribution of Input Quantities

Monte Carlo simulation with uniform distribution of input quantities should give the same result as combined standard uncertainty.

```
Block[ { data, trials = 10^6 }, data = f @@ Table[RandomReal[fDist[i], {trials}], {i, 1, n}];
  Mean[data] ± StandardDeviation[data] ] // ϘUCA
```

```
0.999 888 603 ± 0.000 037 183 9
  ∈ [0.999 851 419; 0.999 925 787]
  ≈ (9.998 89 ± 0.000 37) × 10^-1 = 9.998 89(37) × 10^-1
```

```
ϘMonteCarlo[10^6] // ϘUC
```

$$v = (0.999\,888\,537 \pm 0.000\,037\,155\,2)\ c$$
$$\in [0.999\,851\,381;\ 0.999\,925\,692]\ c;\ \text{Normal}\mathcal{D},\ k = 1\sigma$$
$$\approx (9.998\,89 \pm 0.000\,37) \times 10^{-1}\ c = 9.998\,89(37) \times 10^{-1}\ c$$

Result

If we add two relativistic velocities $\beta_1 = 0.8$ and $\beta_2 = 0.999$, assuming that input velocities are given with an uncertainty implicitly quoted by significant figures from a Type B evaluation, we get resulting velocity 0.99989(4) where the number in parentheses is the numerical value of the combined standard uncertainty referred to the corresponding last digits of the quoted result. (Note that the result in this particular example has larger number of significant figures than input quantities!)

Example: Addition

$$a + b \quad \mapsto \quad \begin{pmatrix} a & 3.1 \pm 0.05 & \text{Uniform}\mathcal{D} \\ b & 4.125 \pm 0.0005 & \text{Uniform}\mathcal{D} \end{pmatrix}$$

Evaluated Functional Relationship

ϙFunctionalRelationship

$$y = x_1 + x_2$$

ϙInputQuantities

Quantity		Estimate ± Uncertainty	Units	Distribution	$\lvert \partial f / \partial x_i \rvert$
x_1	a	$(3.10 \pm 0.05) \times 10^0$		Uniform	1.
x_2	b	$(4.1250 \pm 0.0005) \times 10^0$		Uniform	1.

ϙAnalysisResult

y	7.225		
u_c	0.028 868 956 799 071 674 572 352 487 613 4	=	0.4 %
$y \pm u_c$	$(7.225 \pm 0.029) \times 10^0$	=	7.225(29)
ε_{max}	0.0505	=	0.699 %
$y \pm \varepsilon_{max}$	$(7.225 \pm 0.051) \times 10^0$	=	7.225(51)
y_{min}	7.1745	=	y − 0.0505000
y_{max}	7.2755	=	y + 0.0505000

Absolute Maximum Uncertainty

$\varepsilon_{max} = \sum_{i=1}^{n} \lvert \partial_{x_i} f[x] \rvert \, \varepsilon_i$; ε_{max} // ϙVerbose

$$\varepsilon_{max} == \varepsilon_a + \varepsilon_b$$

$f[x] \pm \varepsilon_{max}$ // ϙUCE

7.225 ± 0.0505

∈ [7.1745; 7.2755]

≈ $(7.225 \pm 0.051) \times 10^0$ = 7.225(51)

ϙfEstimate ± ϙfMaximumUncertainty // ϙUC

7.225 ± 0.0505

∈ [7.1745; 7.2755]

≈ $(7.225 \pm 0.051) \times 10^0$ = 7.225(51)

Standard Uncertainty

$u_c = \left(\sum_{i=1}^{n} (\partial_{x_i} f[x])^2 \, u_i^2 \right)^{1/2}$; u_c // ϙVerbose

$$u_c == \sqrt{u_a^2 + u_b^2}$$

```
f[x] ± u_c // ♀UCA
```

7.225 ± 0.028 869
∈ [7.196 131; 7.253 869]
≈ **$(7.225 \pm 0.029) \times 10^0$ = 7.225(29)**

```
♀fEstimate ± ♀fStandardUncertainty // ♀UC
```

7.225 ± 0.028 869
∈ [7.196 131; 7.253 869]
≈ **$(7.225 \pm 0.029) \times 10^0$ = 7.225(29)**

Monte Carlo Simulation

```
Block[{data, trials = 10^6}, data = f @@ Table[RandomReal[fDist[i], {trials}], {i, 1, n}];
  Mean[data] ± StandardDeviation[data]] // ♀UCA
```

7.224 958 ± 0.028 86
∈ [7.196 098; 7.253 818]
≈ **$(7.225 \pm 0.029) \times 10^0$ = 7.225(29)**

```
♀MonteCarlo[10^6] // ♀UC
```

♀Y = 7.225 015 ± 0.028 851 1
∈ [7.196 164; 7.253 866]; Normal𝒟, k = 1σ
≈ **$(7.225 \pm 0.029) \times 10^0$ = 7.225(29)**

Example: Multiplication

$$a\,b \mapsto \begin{pmatrix} a & 3.1 \pm 0.05 & \text{Uniform}\mathcal{D} \\ b & 4.125 \pm 0.0005 & \text{Uniform}\mathcal{D} \end{pmatrix}$$

Evaluated Functional Relationship

ϙFunctionalRelationship

$$y = x_1\, x_2$$

ϙInputQuantities

Quantity		Estimate ± Uncertainty	Units	Distribution	$\lvert \partial f / \partial x_i \rvert$
x_1	a	$(3.10 \pm 0.05) \times 10^0$		Uniform	4.125
x_2	b	$(4.1250 \pm 0.0005) \times 10^0$		Uniform	3.1

ϙAnalysisResult

y	12.7875	
u_c	0.119 081 855 600 814 909 930 462 442 572	= 0.931 %
$y \pm u_c$	$(1.279 \pm 0.012) \times 10^1$	= $1.279(12) \times 10^1$
ε_{max}	0.2078	= 1.63 %
$y \pm \varepsilon_{max}$	$(1.279 \pm 0.021) \times 10^1$	= $1.279(21) \times 10^1$
y_{min}	12.579 725	= $y - 0.207775$
y_{max}	12.995 325	= $y + 0.207825$

Absolute Maximum Uncertainty

$$\varepsilon_{max} = \sum_{i=1}^{n} \lvert \partial_{x_i} f[x] \rvert\, \varepsilon_i; \quad \varepsilon_{max} \;//\; \text{ϙVerbose}$$

$$\varepsilon_{max} == \text{Abs}[b]\, \varepsilon_a + \text{Abs}[a]\, \varepsilon_b$$

$f[x] \pm \varepsilon_{max}$ // ϙUCE

12.7875 ± 0.2078

$\in$ [12.5797; 12.9953]

$\approx$ $(1.279 \pm 0.021) \times 10^1$ = $1.279(21) \times 10^1$

ϙfEstimate ± ϙfMaximumUncertainty // ϙUC

12.7875 ± 0.2078

$\in$ [12.5797; 12.9953]

$\approx$ $(1.279 \pm 0.021) \times 10^1$ = $1.279(21) \times 10^1$

Standard Uncertainty

$$u_c = \left(\sum_{i=1}^{n} (\partial_{x_i} f[x])^2\, u_i^2 \right)^{1/2}; \quad u_c \;//\; \text{ϙVerbose}$$

$$u_c == \sqrt{b^2\, u_a^2 + a^2\, u_b^2}$$

```
f[x] ± u_c // ♀UCA
```

```
12.7875 ± 0.119082
  ∈  [12.66842; 12.90658]
  ≈  (1.279 ± 0.012) × 10^1  = 1.279(12) × 10^1
```

```
♀fEstimate ± ♀fStandardUncertainty // ♀UC
```

```
12.7875 ± 0.119082
  ∈  [12.66842; 12.90658]
  ≈  (1.279 ± 0.012) × 10^1  = 1.279(12) × 10^1
```

▪ Monte Carlo Simulation

```
Block[{data, trials = 10^6}, data = f @@ Table[RandomReal[fDist[i], {trials}], {i, 1, n}];
  Mean[data] ± StandardDeviation[data]] // ♀UCA
```

```
12.78762 ± 0.119058
  ∈  [12.66856; 12.90668]
  ≈  (1.279 ± 0.012) × 10^1  = 1.279(12) × 10^1
```

```
♀MonteCarlo[10^6] // ♀UC
```

```
♀Y  =  12.78737 ± 0.119098
    ∈  [12.66827; 12.90646]; Normal𝒟, k = 1σ
    ≈  (1.279 ± 0.012) × 10^1  = 1.279(12) × 10^1
```

Example: Power

$$a^b \mapsto \begin{pmatrix} a & 3.5 \pm 0.05 & \text{Uniform}\mathcal{D} \\ b & 100 & \text{Uniform}\mathcal{D} \end{pmatrix}$$

Evaluated Functional Relationship

ϕFunctionalRelationship

$$y = x_1^{x_2}$$

ϕInputQuantities

Quantity		Estimate ± Uncertainty	Units	Distribution	$\lvert \partial f/\partial x_i \rvert$
x_1	a	$(3.50 \pm 0.05) \times 10^0$		Uniform	7.29015×10^{55}
x_2	b	100 (exact)		-	3.19649×10^{54}

ϕAnalysisResult

y	$2.551\,552\,067\,298\,685\,292\,412\,115\,015\,14 \times 10^{54}$	
u_c	$2.104\,484\,675\,580\,345\,811\,952\,700\,715\,82 \times 10^{54}$	$= 82.5\,\%$
$y \pm u_c$	$(2.6 \pm 2.1) \times 10^{54}$	$= 2.6(21) \times 10^{54}$
ε_{max}	$3.645\,074\,381\,855\,264\,703\,445\,878\,593\,06 \times 10^{54}$	$= 143.\,\%$
$y \pm \varepsilon_{max}$	$(2.6 \pm 3.7) \times 10^{54}$	$= 2.6(37) \times 10^{54}$
y_{min}	$6.052\,147\,55 \times 10^{53}$	$= y - 1.94634 \times 10^{54}$
y_{max}	$1.053\,987\,12 \times 10^{55}$	$= y + 7.98832 \times 10^{54}$

Absolute Maximum Uncertainty

$$\varepsilon_{max} = \sum_{i=1}^{n} \lvert \partial_{x_i} f[x] \rvert \, \varepsilon_i; \quad \varepsilon_{max} \text{ // ϕVerbose}$$

$$\varepsilon_{max} == \text{Abs}\left[a^{-1+b}\, b\right] \varepsilon_a + \text{Abs}\left[a^b \,\text{Log}[a]\right] \varepsilon_b$$

$f[x] \pm \varepsilon_{max}$ // ϕUCE

$$2.551\,55 \times 10^{54} \pm 3.645\,07 \times 10^{54}$$
$$\in \left[-1.0935 \times 10^{54};\, 6.1966 \times 10^{54}\right]$$
$$\approx (2.6 \pm 3.7) \times 10^{54} = 2.6(37) \times 10^{54}$$

ϕfEstimate ± ϕfMaximumUncertainty // ϕUC

$$2.551\,55 \times 10^{54} \pm 3.645\,07 \times 10^{54}$$
$$\in \left[-1.0935 \times 10^{54};\, 6.1966 \times 10^{54}\right]$$
$$\approx (2.6 \pm 3.7) \times 10^{54} = 2.6(37) \times 10^{54}$$

Standard Uncertainty

$$u_c = \left(\sum_{i=1}^{n} \left(\partial_{x_i} f[x]\right)^2 u_i^2\right)^{1/2}; \quad u_c \text{ // ϕVerbose}$$

$$u_c == \sqrt{a^{-2+2b}\, b^2\, u_a^2 + a^{2b}\, \text{Log}[a]^2\, u_b^2}$$

```
f[x] ± u_c // ⚲UCA
```

$2.55155\times10^{54} \pm 2.10448\times10^{54}$

$\in \left[4.4707\times10^{53};\ 4.656\times10^{54}\right]$

$\approx$ **$(2.6 \pm 2.1)\times10^{54} = 2.6(21)\times10^{54}$**

```
⚲fEstimate ± ⚲fStandardUncertainty // ⚲UC
```

$2.55155\times10^{54} \pm 2.10448\times10^{54}$

$\in \left[4.4707\times10^{53};\ 4.656\times10^{54}\right]$

$\approx$ **$(2.6 \pm 2.1)\times10^{54} = 2.6(21)\times10^{54}$**

■ Monte Carlo Simulation

```
Block[{data, trials = 10^6}, data = f @@ Table[RandomReal[fDist[i], {trials}], {i, 1, n}];
  Mean[data] ± StandardDeviation[data]] // ⚲UCA
```

$3.49809\times10^{54} \pm 2.70638\times10^{54}$

$\in \left[7.9171\times10^{53};\ 6.2045\times10^{54}\right]$

$\approx$ **$(3.5 \pm 2.7)\times10^{54} = 3.5(27)\times10^{54}$**

```
⚲MonteCarlo[10^6] // ⚲UC
```

⚲Y $= 3.49895\times10^{54} \pm 2.70626\times10^{54}$

$\in \left[7.9269\times10^{53};\ 6.2052\times10^{54}\right]$; Normal$\mathcal{D}$, $k = 1\sigma$

$\approx$ **$(3.5 \pm 2.7)\times10^{54} = 3.5(27)\times10^{54}$**

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