P 1.3. a) Assuming a flat $\Omega_M = 1$ universe, show that the probability that a light beam from a point source at "z" crossing an intervening (foreground) galaxy is given by:

$$P = \frac{2}{3}\frac{c}{H_0}\pi R_{\rm gal}^2 n_0[(1+z)^{3/2} - 1]\,, \tag{3.1}$$

where, $R_{\rm gal} \approx 10$ kpc [1] is the average effective radius of a galaxy and $n_0 \approx 0.02\ {\rm Mpc}^{-3}$ is the comoving number density of galaxies at $z = 0$ ($n(z) = n_0(1+z)^3$).

b) At which redshift does this expression become unity? Assume a Hubble constant value of $H_0 = 68\ {\rm km\,s^{-1}\,Mpc^{-1}}$.

Hint: The probability to intersect a galaxy with redshift in the range $[z, z+{\rm d}z]$ is

$$ {\rm d}P = \pi R_{\rm gal}^2 n\,{\rm d}l\,, \tag{3.2}$$

where ${\rm d}l$ is the distance traveled by the light ray in the redshift interval ${\rm d}z$.

Solution

Part a) The distance ${\rm d}l$ traveled by the light ray in the redshift interval ${\rm d}z$ is actually the proper distance covered by the light in the same redshift interval along the path which corresponds to the lookback time. Hence, ${\rm d}l$ in a world time ${\rm d}t$ is simply (cf. Eq. (4.77) in the book),

$$ {\rm d}l = c\,{\rm d}t = c\frac{{\rm d}a}{a}\frac{a}{{\rm d}a}{\rm d}t = c\frac{{\rm d}a}{a}\frac{a}{\dot a} = \frac{c\,{\rm d}z}{1+z}\frac{1}{H(z)} = H_0^{-1}\frac{c\,{\rm d}z}{(1+z)E(z)}\,, \tag{3.3}$$

where the expansion rate is given by,

$$E(z) = \frac{H_0}{H(z)} = \sqrt{\Omega_\gamma(1+z)^4 + \Omega_M(1+z)^3 + \Omega_K(1+z)^2 + \Omega_\Lambda}\,. \tag{3.4}$$

Now, assuming a flat $\Omega_M = 1$ universe, the last expression reduces to $E(z) = (1+z)^{3/2}$, which, after substituted back into Eq. (3.3), gives an expression for the distance traveled by the light ray solely in terms of the redshift z,

$$ {\rm d}l = \frac{c\,{\rm d}z}{H_0(1+z)}\cdot\frac{1}{(1+z)^{3/2}} = \frac{c\,{\rm d}z}{H_0(1+z)^{5/2}}\,. \tag{3.5}$$

Thus, the intersection probability per unit redshift interval, given by Eq (3.2), becomes,

$$ {\rm d}P = \pi R_{\rm gal}^2 n(z)\frac{c\,{\rm d}z}{H_0(1+z)^{5/2}} = \frac{c\pi R_{\rm gal}^2}{H_0}\frac{n_0(1+z)^3}{(1+z)^{5/2}}{\rm d}z = \frac{c\pi R_{\rm gal}^2 n_0}{H_0}(1+z)^{1/2}{\rm d}z\,, \tag{3.6}$$

and the probability to intersect a galaxy with the redshift z is consequently,

$$P = \frac{c\pi R_{\rm gal}^2 n_0}{H_0}\int_0^z (1+z')^{1/2}{\rm d}z' = \frac{c\pi R_{\rm gal}^2 n_0}{H_0}\left(\frac{2}{3}(1+z')^{3/2}\right)\Big|_0^z\,, \tag{3.7}$$

or, finally,

$$\boxed{P = \frac{2\pi c R_{\rm gal}^2 n_0}{3H_0}\left((1+z)^{3/2} - 1\right).} \tag{3.8}$$

[1] 1 pc = 3.261 light years = 3.086×10^{16} m

Part b) In case of the full coverage of the sky, when every light beam ends up on some galaxy, the probability becomes unity,

$$\frac{2\pi c R_{\mathrm{gal}}^2 n_0}{3H_0}\left((1+z)^{3/2}-1\right)=1\,. \tag{3.9}$$

From the last equation we can easily obtain the redshift z for which the effective temperature of remote light is redshifted down (out of the visible region, for example),

$$(1+z)^{3/2}-1=\frac{3H_0}{2\pi c R_{\mathrm{gal}}^2 n_0}\,, \tag{3.10}$$

$$z=\left(1+\frac{3H_0}{2\pi c R_{\mathrm{gal}}^2 n_0}\right)^{2/3}-1\,, \tag{3.11}$$

After substituting the given numerical values: $H_0=68\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$, $R_{\mathrm{gal}}\approx 10\ \mathrm{kpc}$, $n_0\approx 0.02\ \mathrm{Mpc^{-3}}$ and $c=299\,792\,458\ \mathrm{m\,s^{-1}}$, we obtain,

$$z=\left(1+\frac{3\times\left(68\times10^3\,\mathrm{m\,s^{-1}\,Mpc^{-1}}\right)}{2\pi\times\left(299\,792\,458\,\mathrm{m\,s^{-1}}\right)\times\left(10\times10^{-3}\,\mathrm{Mpc}\right)^2\times\left(0.02\,\mathrm{Mpc^{-3}}\right)}\right)^{2/3}-1\,, \tag{3.12}$$

$$z=\left(1+\frac{3\times68\times10^3}{2\pi\times299\,792\,458\times(10\times10^{-3})^2\times0.02}\cdot\frac{\mathrm{m\,s^{-1}\,Mpc^{-1}}}{\mathrm{m\,s^{-1}\,Mpc^2\,Mpc^{-3}}}\right)^{2/3}-1\,, \tag{3.13}$$

Assuming that the input data was specified by at least two significant digits (which is generous, as both R_{gal} and n_0 have only one significant digit!), we get,

$$z\approx 13\,. \tag{3.14}$$

According to a more thorough uncertainty analysis (see the appendix), we get the value,

$$\boxed{z\approx 13.5\pm 5.7\,,} \tag{3.15}$$

where the number following the symbol $\pm$ is the numerical value of a combined standard uncertainty with a level of confidence of approximately 68 %.

Appendix: P1.3 Uncertainty Analysis

UCAnalysis Package v2.5, © 2011-2013 by MBKocic

$$\left(1+\frac{3\,H_0}{2\,\pi\,c\,(R_{gal})^2\,n_0}\right)^{2/3}-1 \mapsto \begin{pmatrix} H_0 & 67\,800\pm770 & \text{Normal}\mathcal{D} & (m/s)/MPc \\ c & \text{CODATA}[c] & & \\ R_{gal} & 0.010\pm0.005 & \text{Uniform}\mathcal{D} & Mpc \\ n_0 & 0.02\pm0.005 & \text{Uniform}\mathcal{D} & Mpc^{-3} \end{pmatrix}$$

Assumptions

(1) As of 21 March 2013, the Hubble constant, as measured by the Planck Mission, is 67.80 ± 0.77 (km/s)/Mpc (combined Planck + WP + highL + BAO 68% limits).

(2) The average effective radius of a galaxy R_{gal} is assumed to be given with 1 signficant digit.

(3) The comoving number density of galaxies n_0 is assumed to be given with 1 significant digit.

Result (1σ coverage)

$z = 13.4608 \pm 5.633\,98$; Normal$\mathcal{D}$, $k = 1\sigma$

$\in [7.826\,87;\ 19.0948]$

$\approx (1.35 \pm 0.57)\times10^{1} = 1.35(57)\times10^{1}$

Uncertainty Analysis

$$y = -1 + \left(1+\frac{3\,x_1}{2\,\pi\,x_2\,x_3^2\,x_4}\right)^{2/3}$$

Quantity		Estimate ± Uncertainty	Units	Distribution	$\lvert \partial f/\partial x_i \rvert$
x_1	H_0	$(6.780 \pm 0.077)\times10^{4}$	$m\,s^{-1}\,MPc^{-1}$	Normal, 1σ	1.39606×10^{-4}
x_2	c	299 792 458 (exact)	$m\,s^{-1}$	–	3.15727×10^{-8}
x_3	R_{gal}	$(1.0 \pm 0.5)\times10^{-2}$	Mpc	Uniform	1.89305×10^{3}
x_4	n_0	$(2.0 \pm 0.5)\times10^{-2}$	Mpc^{-3}	Uniform	4.73263×10^{2}

y	13.460 848 083 285 334 613 033 274 934 2	
u_c	5.633 978 024 840 062 514 352 474 239 73	= 41.9 %
$y \pm u_c$	$(1.35 \pm 0.57)\times10^{1}$	$= 1.35(57)\times10^{1}$
ε_{max}	12.154 055 346 395 523 654 381 490 573 8	= 90.3 %
$y \pm \varepsilon_{max}$	$(1.3 \pm 1.2)\times10^{1}$	$= 1.3(12)\times10^{1}$
y_{min}	6.255 318 59	$= y - 7.20553$
y_{max}	41.711 953 26	$= y + 28.2511$

```
φUC[ φfEstimate ± φfMaximumUncertainty , 2, 12 ]
```

13.460 848 083 3 ± 12.1541

$\in [1.3068;\ 25.615]$

$\approx (1.3 \pm 1.2)\times10^{1} = 1.3(12)\times10^{1}$

```
φUC[ φfEstimate ± φfStandardUncertainty, 2, 12 ]
```

13.460 848 083 3 ± 5.633 98

$\in [7.826\,87;\ 19.0948]$

$\approx (1.35 \pm 0.57)\times10^{1} = 1.35(57)\times10^{1}$

```
φUC[ φMonteCarlo[ 10^6 ], 3, 10 ]
```

$z = 15.928\,576\,29 \pm 7.505\,15$; Normal$\mathcal{D}$, $k = 1\sigma$

$\in [8.423\,423;\ 23.433\,73]$

$\approx (1.593 \pm 0.751)\times10^{1} = 1.593(751)\times10^{1}$