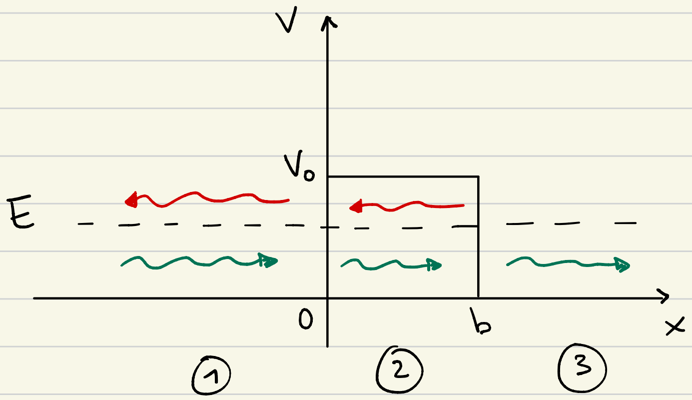
Quantum mechanics I : tutorial solutions

2020.10.29 self-study pack 3

- TUNNELLING

We consider a particle "coming from the left".



$$V = \begin{cases} 0 & , \ x \leq 0 \ , \ x \geq b \\ V_0 & , \ 0 < x < b \end{cases}$$

$$E < V_0$$

the wave function is :

1 $\quad u_1 = A e^{ikx} + B e^{-ikx}$

2 $\quad u_2 = C e^{qx} + D e^{-qx}$

3 $\quad u_3 = F e^{ikx}$

As always, we use the boundary conditions to find the coefficients A, B, C, D, F

The boundary conditions have to be applied in

$x = 0 \longrightarrow \quad u_1(0) = u_2(0)$

$\qquad\qquad\quad u_1'(0) = u_2'(0)$

$x = b \longrightarrow \quad u_2(b) = u_3(b)$

$\qquad\qquad\quad u_2'(b) = u_3'(b)$

1) $x=0$ $\begin{cases} u_1(0) = A + B = C + D = u_2(0) \\ u_1'(0) = ik(A-B) = q(C-D) = u_2'(0) \end{cases}$

$x=b$ $\begin{cases} u_2(b) = C e^{qb} + D e^{-qb} = F e^{ikb} = u_3(b) \\ u_2'(b) = q(C e^{qb} - D e^{-qb}) = ikF e^{ikb} = u_3'(b) \end{cases}$

2) Find $T = F/A$

$$\begin{cases} A = C + D - B \\ A - B = -i\,{}^{q}/_{k}\,(C-D) \\ C e^{qb} + D e^{-qb} = F e^{ikb} \\ -i\,{}^{q}/_{k}\left(C e^{qb} - D e^{-qb}\right) = F e^{ikb} \end{cases}$$

with $\alpha \equiv i\,{}^{q}/_{k}$

NB : look at the Mathematica code on the website, for this resolution!

In matrix form:

$$\begin{bmatrix} 1 & 1 & -1 & -1 & 0 \\ 1 & -1 & \alpha & -\alpha & 0 \\ 0 & 0 & e^{qb} & e^{-qb} & -e^{ikb} \\ 0 & 0 & -\alpha e^{qb} & \alpha e^{-qb} & -e^{ikb} \end{bmatrix}$$

$$R_2 = R_2 - R_1 \qquad \begin{bmatrix} 1 & 1 & -1 & -1 & 0 \\ 0 & -2 & 1+\alpha & 1-\alpha & 0 \\ 0 & 0 & e^{qb} & e^{-qb} & -e^{ikb} \\ 0 & 0 & -\alpha e^{qb} & \alpha e^{-qb} & -e^{ikb} \end{bmatrix}$$

$$R_4 = R_4 + \alpha R_3 \qquad \begin{bmatrix} 1 & 1 & -1 & -1 & 0 \\ 0 & -2 & 1+\alpha & 1-\alpha & 0 \\ 0 & 0 & e^{qb} & e^{-qb} & -e^{ikb} \\ 0 & 0 & 0 & 2\alpha e^{-bq} & -e^{ikb}(1+\alpha) \end{bmatrix}$$

$$R_3 = R_3 - \frac{1}{2\alpha} R_4 \qquad \begin{bmatrix} 1 & 1 & -1 & -1 & 0 \\ 0 & -2 & 1+\alpha & 1-\alpha & 0 \\ 0 & 0 & e^{qb} & 0 & \frac{(1-\alpha)}{2\alpha} e^{ikb} \\ 0 & 0 & 0 & 2\alpha e^{-bq} & -e^{ikb}(1+\alpha) \end{bmatrix}$$

$$R_2 = R_2 - \frac{1+\alpha}{e^{bq}} R_3 \qquad \begin{bmatrix} 1 & 1 & -1 & -1 & 0 \\ 0 & -2 & 0 & 1-\alpha & -\frac{(1+\alpha)(1-\alpha)}{2\alpha} e^{ikb-bq} \\ 0 & 0 & e^{qb} & 0 & \frac{(1-\alpha)}{2\alpha} e^{ikb} \\ 0 & 0 & 0 & 2\alpha e^{-bq} & -e^{ikb}(1+\alpha) \end{bmatrix}$$

$$R_2 = R_2 - \frac{1-\alpha}{2\alpha e^{-bq}} R_4 \quad \begin{bmatrix} 1 & 1 & -1 & -1 & 0 \\ 0 & -2 & 0 & 0 & \frac{(1-e^{2bq})(1-\alpha^2)}{2\alpha} e^{ikb-bq} \\ 0 & 0 & e^{qb} & 0 & \frac{(1-\alpha)}{2\alpha} e^{ikb} \\ 0 & 0 & 0 & 2\alpha e^{-bq} & -e^{ikb}(1+\alpha) \end{bmatrix}$$

$$R_1 = R_1 - \frac{1}{2\alpha e^{-bq}} R_4 + \frac{1}{e^{bq}} R_3 + \frac{1}{2} R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{\left[(1-\alpha)^2 - (1+\alpha)^2 e^{2bq}\right] e^{ikb-bq}}{4\alpha} \\ 0 & -2 & 0 & 0 & \frac{(1-e^{2bq})(1-\alpha^2)}{2\alpha} e^{ikb-bq} \\ 0 & 0 & e^{qb} & 0 & \frac{(1-\alpha)}{2\alpha} e^{ikb} \\ 0 & 0 & 0 & 2\alpha e^{-bq} & -e^{ikb}(1+\alpha) \end{bmatrix}$$

The first line is the equation,

$$A = -\frac{\left[(1-\alpha)^2 - (1+\alpha)^2 e^{2bq}\right]}{4\alpha} e^{ikb-bq} \quad F$$

thus the transmission coefficient is

$$T = \frac{F}{A} = -\frac{4\alpha \, e^{-ikb}}{(1-\alpha)^2 e^{-bq} - (1+\alpha)^2 e^{bq}}$$

with $\alpha \equiv i \, q/k$

NB the transmission coefficient $T = \frac{F}{A}$ is the amount (i.e. the amplitude) of wave function which crosses the barrier.

When dealing with large barriers $(b \to \infty)$ or high barriers $(q \to \infty)$,

$$T = -\frac{4\alpha\, e^{-ikb}}{\underbrace{(1-\alpha)^2 e^{-bq}}_{\to 0} - (1+\alpha)^2 e^{bq}}$$

$$= \frac{4\alpha\, e^{-ikb}}{(1+\alpha)^2 e^{bq}}$$

$$= \frac{4i\, q/k\; e^{-ikb}}{(1+i\, q/k)^2 e^{bq}}$$

$$= \frac{4ikq\; e^{-ikb-bq}}{(k+iq)^2}$$

3)

the TRANSMISSION PROBABILITY is

$$|T|^2 = -\frac{4ikq\; e^{ikb-bq}}{(k-iq)^2}\,\frac{4ikq\; e^{-ikb-bq}}{(k+iq)^2}$$

$$= \frac{16\, k^2 q^2\, e^{-2bq}}{(k^2+q^2)^2}$$

$$|T|^2 = \frac{16\,k^2 q^2\, e^{-2bq}}{(k^2+q^2)^2}$$

NB This result tells you that the probability of having a transmitted particle decays EXPONENTIALLY with the width of the barrier and the less energy the particle has.

4) We rearrange the expression of T to find the general expression of $|T^2|$

$$T = -\frac{4\alpha\, e^{-ikb}}{(1-\alpha)^2 e^{-bq} - (1+\alpha)^2 e^{bq}}$$

$$= -\frac{4i\,q/k\; e^{-ikb}}{(1-i\,q/k)^2 e^{-bq} - (1+i\,q/k)^2 e^{bq}}$$

$$= -\frac{4ikq\; e^{-ikb}}{(k-iq)^2 e^{-bq} - (k+iq)^2 e^{bq}}$$

$$= -\frac{4ikq\; e^{-ikb}}{(k^2-q^2-2ikq)\,e^{-bq} - (k^2-q^2+2ikq)\,e^{bq}}$$

$$T = \frac{2ikq\; e^{-ikb}}{(k^2-q^2)\sinh(bq) + 2ikq\cosh(bq)}$$

$$|T|^2 = \frac{2ikq\, e^{-ikb}}{(k^2-q^2)\sinh(bq) + 2ikq\cosh(bq)} \cdot$$

$$\cdot \frac{-2ikq\, e^{ikb}}{(k^2-q^2)\sinh(bq) - 2ikq\cosh(bq)}$$

$$= \frac{4k^2q^2}{(k^2-q^2)^2\sinh^2(bq) + 4k^2q^2 \underbrace{\cosh^2(bq)}_{1+\sinh^2(bq)}}$$

$$= \frac{4k^2q^2}{(k^2+q^2)^2\sinh^2(bq) + 4k^2q^2}$$

$$|T|^2 = \frac{1}{\frac{(k^2+q^2)^2}{4k^2q^2}\sinh^2(bq) + 1}$$

5) Remember that

$$k^2 = \frac{2mE}{\hbar^2} \quad , \quad q^2 = \frac{2m(V_0-E)}{\hbar^2}$$

and replacing these expressions in $|T|^2$ we obtain

$$|T|^2 = \frac{1}{\frac{V_0^2}{4E(V_0-E)}\sinh^2(qb) + 1}$$

shape of barrier

particle energy

potential of barrier

6) In the limit of large b, we found

$$|T|^2 = \frac{16\,k^2 q^2\, e^{-2bq}}{(k^2+q^2)^2}$$

while now, we found

$$|T|^2 = \frac{4k^2q^2}{(k^2+q^2)^2 \underbrace{\sinh^2(bq)}_{\frac{1}{2}(\cosh(2bq)-1)} + 4k^2q^2}$$

$$= \frac{e^{2bq} + e^{-2bq} - 2}{4}$$

$$= \frac{16k^2q^2}{(k^2+q^2)^2\,(e^{2bq} + e^{-2bq} - 2) + 16k^2q^2}$$

that, in the limit of large b gives,

$$= \frac{16\,k^2q^2\; e^{-2bq}}{(k^2+q^2)^2\,(1 + \underbrace{e^{-4bq}}_{0} - \underbrace{2e^{-2bq}}_{0}) + \underbrace{16\,k^2q^2\,e^{-2bq}}_{0}}$$

$$= \frac{16\,k^2q^2}{(k^2+q^2)^2}\, e^{-2bq}$$

ok

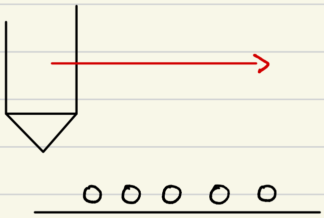
7) Quantum tunnelling examples.

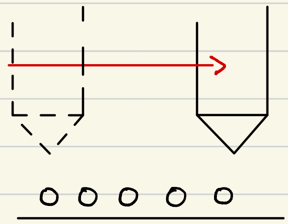
How does it contributes to the problem?

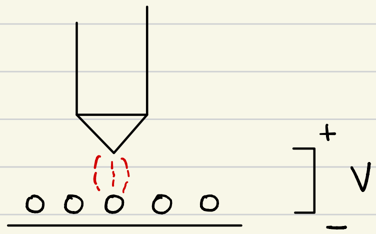
- SCANNING TUNNELLING MICROSCOPE

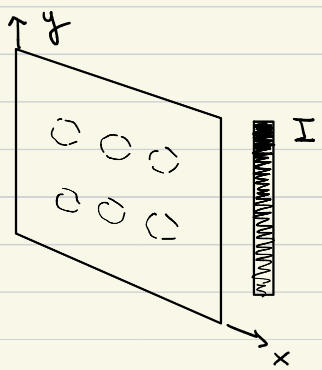
(watch https://www.youtube.com/watch?v=HE2yE8SvHmA)

- Voltage between the single-atom tip & the platform.
- The tip scans through the sample.
- When is in proximity of an atom, some electrons cross the potential barrier and flow through the tip, giving a current.
- Measuring this current give the image of the sample
- Technique used also for spectroscopy (see video)



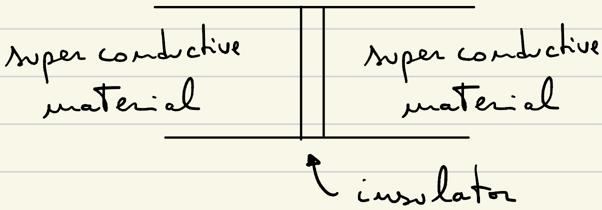






- SUPERCONDUCTING TUNNELLING JUNCTION

  - Josephson junction :



  - DC Josephson effect consists of a direct current crossing the "barrier" (insulator) when no e.m. field is applied.

  - This happens when the materials are brought below their critical temperature.