

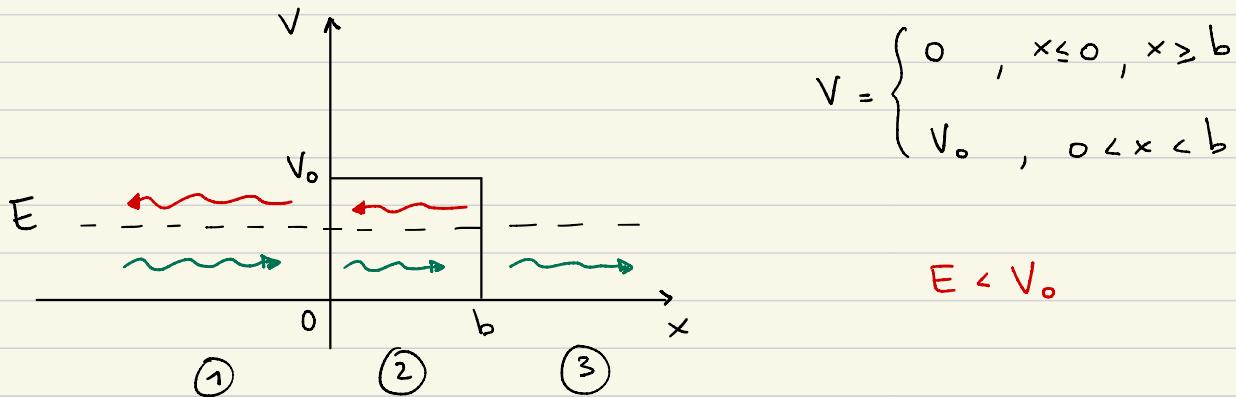
Quantum mechanics I : tutorial solutions

2021.10.28

/ self-study pack 3

TUNNELLING

We consider a particle "coming from the left". 



the wave function is :

$$\textcircled{1} \quad u_1 = A e^{ikx} + B e^{-ikx}$$

$$\textcircled{2} \quad u_2 = C e^{qx} + D e^{-qx}$$

$$\textcircled{3} \quad u_3 = F e^{ikx}$$

As always, we use the boundary conditions to find the coefficients A, B, C, D, F

The boundary conditions have to be applied in

$$x = 0 \rightarrow u_1(0) = u_2(0)$$

$$u_1'(0) = u_2'(0)$$

$$x = b \rightarrow u_2(b) = u_3(b)$$

$$u_2'(b) = u_3'(b)$$

$$1) \quad \left\{ \begin{array}{l} u_1(0) = A + B = C + D = u_2(0) \\ x=0 \quad u_1'(0) = ik(A - B) = q(C - D) = u_2'(0) \end{array} \right.$$

$$x=b \quad \left\{ \begin{array}{l} u_2(b) = Ce^{qb} + De^{-qb} = Fe^{ikb} = u_3(b) \\ u_2'(b) = q(Ce^{qb} - De^{-qb}) = ikF e^{ikb} = u_3'(b) \end{array} \right.$$

$$2) \quad \text{Find} \quad T = F/A$$

$$\left\{ \begin{array}{l} A = C + D - B \\ A - B = -i \frac{q}{k} (C - D) \\ Ce^{qb} + De^{-qb} = Fe^{ikb} \\ -i \frac{q}{k} (Ce^{qb} - De^{-qb}) = Fe^{ikb} \end{array} \right.$$

$$\text{with} \quad \alpha \equiv i \frac{q}{k}$$

In matrix form :

NB : look at the Mathematica code on the website, for this resolution!

$$\begin{bmatrix} 1 & 1 & -1 & -1 & 0 \\ 1 & -1 & \alpha & -\alpha & 0 \\ 0 & 0 & e^{qb} & e^{-qb} & -e^{ikb} \\ 0 & 0 & -\alpha e^{qb} & \alpha e^{-qb} & -e^{ikb} \end{bmatrix}$$

$$R_2 = R_2 - R_1$$

$$\begin{bmatrix} 1 & 1 & -1 & -1 & 0 \\ 0 & -2 & 1+\alpha & 1-\alpha & 0 \\ 0 & 0 & e^{qb} & e^{-qb} & -e^{ikb} \\ 0 & 0 & -\alpha e^{qb} & \alpha e^{-qb} & -e^{ikb} \end{bmatrix}$$

$$R_4 = R_4 + \alpha R_3$$

$$\begin{bmatrix} 1 & 1 & -1 & -1 & 0 \\ 0 & -2 & 1+\alpha & 1-\alpha & 0 \\ 0 & 0 & e^{qb} & e^{-qb} & -e^{ikb} \\ 0 & 0 & 0 & 2\alpha e^{-bq} & -e^{ikb}(1+\alpha) \end{bmatrix}$$

$$R_3 = R_3 - \frac{1}{2\alpha} R_4$$

$$\begin{bmatrix} 1 & 1 & -1 & -1 & 0 \\ 0 & -2 & 1+\alpha & 1-\alpha & 0 \\ 0 & 0 & e^{qb} & 0 & \frac{(1-\alpha)}{2\alpha} e^{ikb} \\ 0 & 0 & 0 & 2\alpha e^{-bq} & -e^{ikb}(1+\alpha) \end{bmatrix}$$

$$R_2 = R_2 - \frac{1+\alpha}{e^{bq}} R_3$$

$$\begin{bmatrix} 1 & 1 & -1 & -1 & 0 \\ 0 & -2 & 0 & 1-\alpha & -\frac{(1+\alpha)(1-\alpha)}{2\alpha} e^{ikb-bq} \\ 0 & 0 & e^{qb} & 0 & \frac{(1-\alpha)}{2\alpha} e^{ikb} \\ 0 & 0 & 0 & 2\alpha e^{-bq} & -e^{ikb}(1+\alpha) \end{bmatrix}$$

$$R_2 = R_2 - \frac{1-\alpha}{2\alpha e^{-bq}} R_4 \left[\begin{array}{ccccc} 1 & 1 & -1 & -1 & 0 \\ 0 & -2 & 0 & 0 & \frac{(1-e^{2bq})(1-\alpha^2)}{2\alpha} e^{ikb-bq} \\ 0 & 0 & e^{qb} & 0 & \frac{(1-\alpha)}{2\alpha} e^{ikb} \\ 0 & 0 & 0 & 2\alpha e^{-bq} & -e^{ikb}(1+\alpha) \end{array} \right]$$

$$R_1 = R_1 - \frac{1}{2\alpha e^{-bq}} R_4 + \frac{1}{e^{bq}} R_3 + \frac{1}{2} R_2$$

$$\left[\begin{array}{cccc} 1 & 0 & 0 & 0 & \frac{[(1-\alpha)^2 - (1+\alpha)^2 e^{2bq}]}{4\alpha} e^{ikb-bq} \\ 0 & -2 & 0 & 0 & \frac{(1-e^{2bq})(1-\alpha^2)}{2\alpha} e^{ikb-bq} \\ 0 & 0 & e^{qb} & 0 & \frac{(1-\alpha)}{2\alpha} e^{ikb} \\ 0 & 0 & 0 & 2\alpha e^{-bq} & -e^{ikb}(1+\alpha) \end{array} \right]$$

The first line is the equation,

$$A = - \frac{[(1-\alpha)^2 - (1+\alpha)^2 e^{2bq}]}{4\alpha} e^{ikb-bq} F$$

thus the transmission coefficient is

$$T = \frac{F}{A} = - \frac{4\alpha e^{-ikb}}{(1-\alpha)^2 e^{-bq} - (1+\alpha)^2 e^{bq}}$$

with $\alpha = i q/k$

NB the transmission coefficient $T = \frac{F}{A}$ is the amount (i.e. the amplitude) of wave function which crosses the barrier.

When dealing with large barriers ($b \rightarrow \infty$) or high barriers ($q \rightarrow \infty$),

$$\begin{aligned}
 T &= - \frac{4\alpha e^{-ikb}}{(1-\alpha)^2 e^{-bq} - (1+\alpha)^2 e^{bq}} \\
 &\quad \underbrace{\phantom{- \frac{4\alpha e^{-ikb}}{(1-\alpha)^2 e^{-bq} - (1+\alpha)^2 e^{bq}}}_{0}}_0 \\
 &= \frac{4\alpha e^{-ikb}}{(1+\alpha)^2 e^{bq}} \\
 &= \frac{4i q/k e^{-ikb}}{(1+i q/k)^2 e^{bq}} \\
 &= \frac{4i k q e^{-ikb-bq}}{(k+i q)^2}
 \end{aligned}$$

3) the TRANSMISSION PROBABILITY is

$$\begin{aligned}
 |T|^2 &= - \frac{4i k q e^{ikb-bq}}{(k-i q)^2} \frac{4i k q e^{-ikb-bq}}{(k+i q)^2} \\
 &= \frac{16 k^2 q^2 e^{-2bq}}{(k^2 + q^2)^2}
 \end{aligned}$$

$$|T|^2 = \frac{16 k^2 q^2 e^{-2bq}}{(k^2 + q^2)^2}$$

NB This result tells you that the probability of having a transmitted particle decays EXPONENTIALLY with the width of the barrier and the less energy the particle has.

4) We rearrange the expression of T to find the general expression of $|T^2|$

$$\begin{aligned} T &= - \frac{4\alpha e^{-ikb}}{(1-\alpha)^2 e^{-bq} - (1+\alpha)^2 e^{bq}} \\ &= - \frac{4i q/k e^{-ikb}}{(1-i q/k)^2 e^{-bq} - (1+i q/k)^2 e^{bq}} \\ &= - \frac{4ikq e^{-ikb}}{(k-iq)^2 e^{-bq} - (k+iq)^2 e^{bq}} \\ &= - \frac{4ikq e^{-ikb}}{(k^2 - q^2 - 2ikq) e^{-bq} - (k^2 - q^2 + 2ikq) e^{bq}} \\ T &= \frac{2ikq e^{-ikb}}{(k^2 - q^2) \sinh(bq) + 2ikq \cosh(bq)} \end{aligned}$$

$$\begin{aligned}
 |T|^2 &= \frac{2ikq e^{-ikb}}{(k^2 - q^2) \sinh(bq) + 2ikq \cosh(bq)} \\
 &\cdot \frac{-2ikq e^{ikb}}{(k^2 - q^2) \sinh(bq) - 2ikq \cosh(bq)} \\
 &= \frac{4k^2 q^2}{(k^2 - q^2)^2 \sinh^2(bq) + 4k^2 q^2 \cosh^2(bq)} \\
 &= \frac{4k^2 q^2}{(k^2 + q^2)^2 \sinh^2(bq) + 4k^2 q^2} \quad \text{1} + \sinh^2(bq)
 \end{aligned}$$

5) Remember that

$$k^2 = \frac{2mE}{\hbar^2}, \quad q^2 = \frac{2m(V_0 - E)}{\hbar^2}$$

and replacing these expressions in $|T|^2$
we obtain

$$|T|^2 = \frac{1}{\frac{V_0^2}{4E(V_0 - E)} \sinh^2(qb) + 1}$$

↓ ↗
 particle energy shape of barrier
 potential of barrier

6) In the limit of large b , we found

$$|T|^2 = \frac{16 k^2 q^2 e^{-2bq}}{(k^2 + q^2)^2}$$

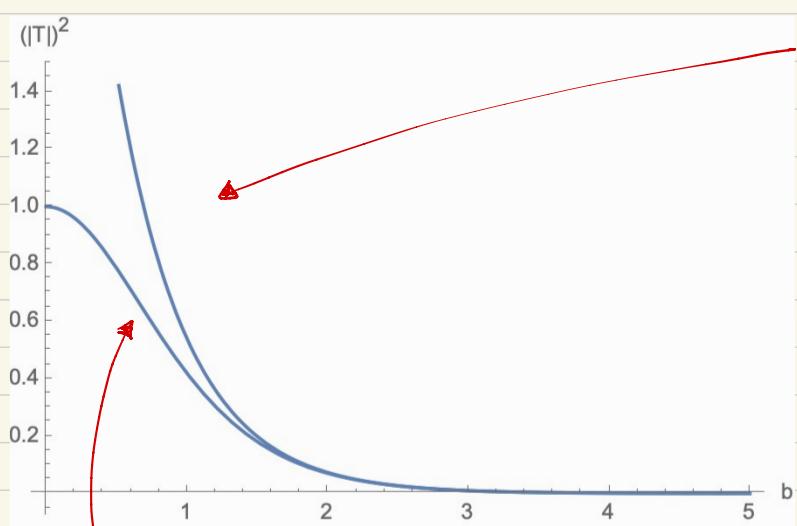
while now, we found

$$\begin{aligned} |T|^2 &= \frac{4 k^2 q^2}{(k^2 + q^2)^2 \sinh^2(bq) + 4 k^2 q^2} \\ &\quad \underbrace{\frac{1}{2}(\cosh(2bq) - 1)}_{=} \\ &= \frac{e^{2bq} + e^{-2bq} - 2}{4} \\ &= \frac{16 k^2 q^2}{(k^2 + q^2)^2 (e^{2bq} + e^{-2bq} - 2) + 16 k^2 q^2} \end{aligned}$$

that, in the limit of large b gives,

$$\begin{aligned} &= \frac{16 k^2 q^2 e^{-2bq}}{(k^2 + q^2)^2 (1 + e^{-4bq} - 2 e^{-2bq}) + 16 k^2 q^2 e^{-2bq}} \\ &= \frac{16 k^2 q^2}{(k^2 + q^2)^2} e^{-2bq} \quad \text{OK} \end{aligned}$$

from the Mathematica code



$$|T|^2 = \frac{16 k^2 q^2 e^{-2bq}}{(k^2 + q^2)^2}$$

approx. large b

$$|T|^2 = \frac{4 k^2 q^2}{(k^2 + q^2)^2 \sinh^2(bq) + 4 k^2 q^2}$$

complete expression

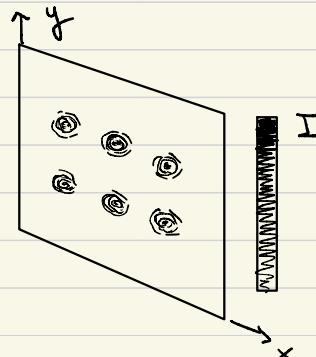
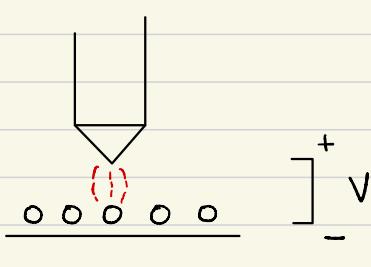
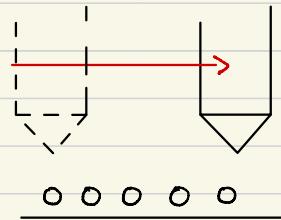
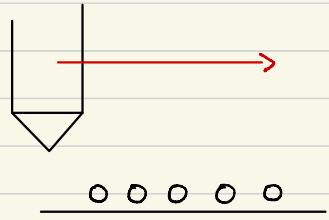
7) Quantum tunnelling examples .

How does it contributes to the problem?

- SCANNING TUNNELLING MICROSCOPE

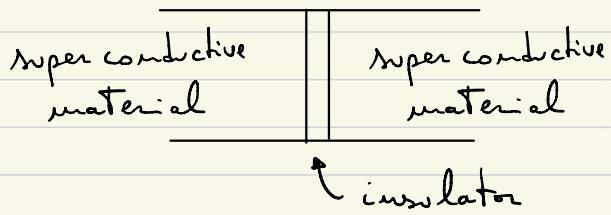
(watch <https://www.youtube.com/watch?v=HE2yE8SvHmA>)

- Voltage between the single-atom tip & the platform.
- The tip scans through the sample.
- When is in proximity of an atom, some electrons cross the potential barrier and flow through the tip, giving a current.
- Measuring this current give the image of the sample
- Technique used also for spectroscopy (see video)



- SUPERCONDUCTING TUNNELLING JUNCTION

- Josephson junction :



- DC Josephson effect consists of a direct current crossing the "barrier" (insulator) when no e.m. field is applied.
- This happens when the materials are brought below their critical temperature.

