

# Quantum mechanics I : tutorial solutions

2020.11.19



week 8

1)

1. (a) Write the operators corresponding to the three components  $p_x$ ,  $p_y$ , and  $p_z$  of linear momentum. Use them to derive the operators  $\hat{L}_x$ ,  $\hat{L}_y$  and  $\hat{L}_z$  for the components of angular momentum.
- (b) State the commutation relations between the operators  $\hat{L}_x$ ,  $\hat{L}_y$  and  $\hat{L}_z$  and also between the operator  $\hat{L}^2$  and any one of  $\hat{L}_x$ ,  $\hat{L}_y$  and  $\hat{L}_z$ . Explain the consequences of these relations for the measurements made of the angular momentum of a quantum particle. Compare the results with similar measurements on a classical system.
- (c) State the eigenvalue equations for  $\hat{L}^2$  and  $\hat{L}_z$  in terms of the eigenfunction  $Y_{\ell m}(\theta, \phi)$ . Calculate the magnitude of the angular momentum of a particle in the state with  $\ell = 2$  and  $m = 1$ .

$$(a) \quad \underline{p}_i = -i\hbar \mathcal{D}_i, \quad i = x, y, z$$

$$\text{in CLASSICAL mechanics, } \underline{L} = \underline{r} \times \underline{p}$$

thanks to the CORRESPONDENCE principle of  
QUANTUM mechanics, we obtain  $\hat{\underline{L}}$  by  
replacing the variables with operators,

$$\hat{\underline{L}} = \hat{\underline{r}} \times \hat{\underline{p}} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & z \\ -i\hbar \mathcal{D}_x & -i\hbar \mathcal{D}_y & -i\hbar \mathcal{D}_z \end{vmatrix}$$

$$\begin{aligned} &= -i\hbar [ \hat{x} (y \mathcal{D}_z - z \mathcal{D}_y) \\ &\quad - \hat{y} (x \mathcal{D}_z - z \mathcal{D}_x) \\ &\quad + \hat{z} (x \mathcal{D}_y - y \mathcal{D}_x) ] \end{aligned}$$

thus, we have

$$\hat{L}_x = -i\hbar(y\gamma_z - z\gamma_y)$$

$$\hat{L}_y = -i\hbar(z\gamma_x - x\gamma_z)$$

$$\hat{L}_z = -i\hbar(x\gamma_y - y\gamma_x)$$

which is also

$$\hat{L}_x = y\hat{P}_z - z\hat{P}_y$$

$$\hat{L}_y = z\hat{P}_x - x\hat{P}_z$$

$$\hat{L}_z = x\hat{P}_y - y\hat{P}_x$$

(b)

the commutation relations are :

$$[\hat{L}_i, \hat{L}_j] = i\hbar \epsilon_{ijk} \hat{L}_k$$

$$[\hat{L}^2, \hat{L}_i] = 0, \forall i$$

These commutation relations tell us that

I can measure with arbitrary precision the magnitude of the total angular momentum  $\hat{L}^2$  simultaneously with any component  $\hat{L}_i$ ,

but I can only have finite precision

when measuring 2 components,  $\hat{L}_i \& \hat{L}_j$ .

As usual, this is not the case for a classical system for which every precision is allowed.

(c)

the eigenvalue equations are:

$$\hat{L}^2 Y_{\ell,m}(\theta, \phi) = \hbar^2 l(l+1) Y_{\ell,m}(\theta, \phi)$$

$$\hat{L}_z Y_{\ell,m}(\theta, \phi) = \hbar m Y_{\ell,m}(\theta, \phi)$$

where  $\hat{L}^2$  &  $\hat{L}_z$  in spherical coordinates are

$$\hat{L}^2 = -\hbar^2 \left[ \frac{1}{\sin\theta} \partial_\theta (\sin\theta \partial_\theta) + \frac{1}{\sin^2\theta} \partial_\phi^2 \right]$$

$$\hat{L}_z = -i\hbar \partial_\phi$$

for a particle with  $l=2$ ,  $m=1$ , we find:

$$L^2 = \hbar^2 l(l+1) = 6\hbar^2$$

$$L_z = \hbar m = \hbar$$

2)

2. The operators  $\hat{L}_+ = \hat{L}_x + i\hat{L}_y$  and  $\hat{L}_- = \hat{L}_x - i\hat{L}_y$  are the angular momentum raising and lowering operators respectively (c.f. the raising and lowering operators for the harmonic oscillator). Prove the following commutation relations:

$$\begin{aligned}(a) \quad & [\hat{L}_z, \hat{L}_+] = \hbar \hat{L}_+ \\(b) \quad & [\hat{L}_z, \hat{L}_-] = -\hbar \hat{L}_- \\(c) \quad & [\hat{L}^2, \hat{L}_+] = [\hat{L}^2, \hat{L}_-] = 0\end{aligned}$$

Carry out the following steps to show that the effect of  $\hat{L}_+$  is to transform the eigenvalue equation for  $\hat{L}_z$  with eigenvalue  $m$ , into the eigenvalue equation with eigenvalue  $m + 1$ .

- (d) Write down the eigenvalue equation for  $\hat{L}_z$ .
- (e) Pre-multiply the equation by  $\hat{L}_+$ .
- (f) Use the commutation relation in (a) above to reverse the order of the operators and hence show that  $(\hat{L}_z \hat{L}_+ - \hbar \hat{L}_+) Y_{\ell m} = m \hbar \hat{L}_+ Y_{\ell m}$ .

Rearranging this gives the eigenvalue equation  $\hat{L}_z (\hat{L}_+ Y_{\ell m}) = (m + 1) \hbar (\hat{L}_+ Y_{\ell m})$  as required.

$$\begin{aligned}(a) \quad [\hat{L}_z, \hat{L}_+] &= \hat{L}_z \hat{L}_+ - \hat{L}_+ \hat{L}_z \\&= \hat{L}_z (\hat{L}_x + i\hat{L}_y) - (\hat{L}_x + i\hat{L}_y) \hat{L}_z \\&= \hat{L}_z \hat{L}_x - \hat{L}_x \hat{L}_z + i (\hat{L}_z \hat{L}_y - \hat{L}_y \hat{L}_z) \\&= [\hat{L}_z, \hat{L}_x] - i [\hat{L}_y, \hat{L}_z] \\&= i\hbar \hat{L}_y + \hbar \hat{L}_x \\&= \hbar \hat{L}_+\end{aligned}$$

$$\begin{aligned}(b) \quad [\hat{L}_z, \hat{L}_-] &= \hat{L}_z \hat{L}_- - \hat{L}_- \hat{L}_z \\&= \hat{L}_z (\hat{L}_x - i\hat{L}_y) - (\hat{L}_x - i\hat{L}_y) \hat{L}_z \\&= [\hat{L}_z, \hat{L}_x] + i [\hat{L}_y, \hat{L}_z] \\&= i\hbar \hat{L}_y + i(\hbar) \hat{L}_x \\&= -\hbar (\hat{L}_x - i\hat{L}_y) \\&= -\hbar \hat{L}_-\end{aligned}$$

$$\begin{aligned}
(c) \quad [\hat{L}^2, \hat{L}_z] &= [\hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2, \hat{L}_x \pm i \hat{L}_y] \\
&= [\hat{L}_x^2, \hat{L}_x] \pm i [\hat{L}_x^2, \hat{L}_y] \\
&\quad + [\hat{L}_y^2, \hat{L}_x] \pm i [\hat{L}_y^2, \hat{L}_y] \\
&\quad + [\hat{L}_z^2, \hat{L}_x] \pm i [\hat{L}_z^2, \hat{L}_y] \\
&= 0 \pm i (\hat{L}_x [\hat{L}_x, \hat{L}_y] + [\hat{L}_x, \hat{L}_y] \hat{L}_x) \\
&\quad + \hat{L}_y [\hat{L}_y, \hat{L}_x] + [\hat{L}_y, \hat{L}_x] \hat{L}_y \pm 0 \\
&\quad + \hat{L}_z [\hat{L}_z, \hat{L}_x] + [\hat{L}_z, \hat{L}_x] \hat{L}_z \\
&\quad \pm i (\hat{L}_z [\hat{L}_z, \hat{L}_y] + [\hat{L}_z, \hat{L}_y] \hat{L}_z) \\
&= \pm i (i\hbar) (\hat{L}_x \hat{L}_z + \hat{L}_z \hat{L}_x) \\
&\quad - i\hbar (\hat{L}_y \hat{L}_z + \hat{L}_z \hat{L}_y) \\
&\quad + i\hbar (\hat{L}_z \hat{L}_y + \hat{L}_y \hat{L}_z) \\
&\quad \pm i (-i\hbar) (\hat{L}_z \hat{L}_x + \hat{L}_x \hat{L}_z) \\
&= 0
\end{aligned}$$

(d) eigenvalue eq. for  $\hat{L}_z$

$$\hat{L}_z + (l, m) = \hbar m + (l, m)$$

premultiply by  $\hat{L}_+$

$$\hat{L}_+ \hat{L}_z + (l, m) = \hbar m \hat{L}_+ + (l, m)$$

using the commutation relation we found in (a)

$$[\hat{L}_z, \hat{L}_+] = \hat{L}_z \hat{L}_+ - \hat{L}_+ \hat{L}_z = \hbar \hat{L}_+$$

$$\hat{L}_+ \hat{L}_z = \hat{L}_z \hat{L}_+ - \hbar \hat{L}_+$$

we have

$$\hat{L}_z \hat{L}_+ + (l, m) - \hbar \hat{L}_+ + (l, m) = \hbar m \hat{L}_+ + (l, m)$$

$$\hat{L}_z \hat{L}_+ + (l, m) = \hbar (m+1) \hat{L}_+ + (l, m)$$

3)

3. The classical expression for the kinetic energy of a rigid rotating body is

$$T = \frac{1}{2} \left( \frac{L_x^2}{I_x} + \frac{L_y^2}{I_y} + \frac{L_z^2}{I_z} \right)$$

where  $L_x, L_y$  and  $L_z$  are the components of angular momentum and  $I_x, I_y$  and  $I_z$  are the principal moments of inertia.

- (a) Set up the corresponding Hamiltonian operator for the quantum mechanical situation.
- (b) Determine the eigenvalues of this operator for the case in which the body is spherical, i.e.  $I_x = I_y = I_z$ . Give an expression for the degeneracy of the eigenstate with total angular momentum quantum number  $\ell$ .
- (c) Find an expression for the eigenvalues of the Hamiltonian operator if the body has  $I_x = I_y = 2I_z$  and calculate the eigenvalues and degeneracies for the states with  $\ell = 2$ .

(a) for the quantum version, we substitute the variables of the system w/ operators,

$$\hat{H} = \frac{1}{2} \left( \frac{\hat{L}_x^2}{I_x} + \frac{\hat{L}_y^2}{I_y} + \frac{\hat{L}_z^2}{I_z} \right)$$

(b) If we consider the spherically symmetric case, we have

$$\hat{H} = \frac{1}{2I} (\hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2) = \frac{\hat{L}^2}{2I}$$

the eigenvalue equation for the operator above for a particle with the quantum number of total angular momentum  $\ell$  is

$$\hat{H} + (l, m) = \frac{\hat{L}^2}{2I} + (l, m) = \frac{\hbar^2 l(l+1)}{2I} + (l, m)$$

Since

$$m \in \{-l, -l+1, \dots, 0, \dots, l-1, l\}$$

$2l+1$

the degeneracy is  $2l+1$

(c) If the body has  $I_x = I_y = 2I_z$   
the  $\hat{H}$  operator is

$$\begin{aligned}\hat{H} &= \frac{1}{2} \left( \frac{\hat{L}_x^2}{2I_z} + \frac{\hat{L}_y^2}{2I_z} + \frac{\hat{L}_z^2}{I_z} \right) \\ &= \frac{1}{4I_z} (\hat{L}_x^2 + \hat{L}_y^2 + 2\hat{L}_z^2) \\ &= \frac{1}{4I_z} (\hat{L}^2 + \hat{L}_z^2)\end{aligned}$$

the corresponding eigenvalue equation is

$$\begin{aligned}\hat{H} \psi(l, m) &= \frac{1}{4I_z} (\hat{L}^2 + \hat{L}_z^2) \psi(l, m) \\ &= \frac{1}{4I_z} (\hbar^2 l(l+1) + \hbar^2 m^2) \psi(l, m) \\ &= \frac{\hbar^2 (l(l+1) + m^2)}{4I_z} \psi(l, m)\end{aligned}$$

for a particle w/  $l = 2$ , we have

$$\text{degeneracy } 2l+1 = 5$$

the possible states are

$$(l=2, m=0) \quad E_{2,0} = \frac{3}{2} \frac{\hbar^2}{I_z}, \quad \text{deg} = 1$$

$$(l=2, m=\pm 1) \quad E_{2,\pm 1} = \frac{7}{4} \frac{\hbar^2}{I_z}, \quad \text{deg} = 2$$

$$(l=2, m=\pm 2) \quad E_{2,\pm 2} = \frac{5}{2} \frac{\hbar^2}{I_z}, \quad \text{deg} = 2$$

• try to evaluate the commutator  $[ab, cd]$

we compare our answers in the next tutorial.

• COMMUTATOR PROPERTIES

$$[a, b] = ab - ba \quad \text{when in doubt, use the definition}$$

$$[a, a] = 0$$

$$[a, b] = -[b, a]$$

$$[a+b, c] = [a, c] + [b, c]$$

$$[a, [b, c]] + [b, [c, a]] + [c [a, b]] = 0 \quad \text{Jacobi identity}$$

$$[\alpha a, b] = \alpha [a, b] \quad \text{where } \alpha \text{ is a constant}$$

$$[ab, c] = a\underbrace{[b, c]}_{\text{watch out where they go out}} + \underbrace{[a, c]b}_{\text{go out}}$$