Büchi Automata

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1 Definition

Deterministic Büchi automaton is tuple A $= (Q, \Sigma, \delta, q_0, F)$, such that

- Q is a finite set. Elements of Q are called the states of A.
- Σ is a finite set called alphabet of A.
- $\delta : Q \times \Sigma \leftarrow Q$ is a function, called transition function of A.
- $q_0 \in Q$ is initial state of A.
- $F \subset Q$ is acceptance condition.

A accepts exactly those runs in which at least one of state in F occurs infinitely often.

Nondeterministic Büchi automaton is one where transition function δ is replaced with a transition relation $\Delta \subset Q \times \Sigma \times Q$, initial state replaced by set $I \in Q$.

2 Closure Properties

Let $A = (Q_A, \Sigma, \Delta_A, I_A, F_A), B = (Q_B, \Sigma, \Delta_B, I_B, F_B)$ be Büchi automata, and $C = (Q_C, \Sigma, \Delta_C, I_C, F_C)$ be a finite automaton. Assume Q_A, Q_B, Q_C are mutually disjoint.

2.1 Union

There is Büchi automaton that recognizes the language $L(A) \cup L(B)$

Proof. $(Q_A \cup Q_B, \Sigma, \Delta_A \cup \Delta_B, I_A \cup I_B, F_A \cup F_B)$ □

2.2 Intersection

There is Büchi automaton that recognizes the language $L(A) \cap L(B)$

Proof.

- $Q' = Q_A \times Q_B \times \{1, 2\}$
- $\Delta' = \Delta_1 \cup \Delta_2$
  $\Delta_1 = \{((q_A, q_B, 1), a, ({q'}_A, {q'}_B, i)) : (q_A, a, {q'}_A) \in \Delta_A$ and $(q_B, a, {q'}_B) \in \Delta_B$ and if $q_A \in F_A$ then $i = 2$ else $i = 1\}$
  $\Delta_2 = \{((q_A, q_B, 2), a, ({q'}_A, {q'}_B, i)) : (q_A, a, {q'}_A) \in \Delta_A$ and $(q_B, a, {q'}_B) \in \Delta_B$ and if $q_B \in F_B$ then $i = 1$ else $i = 2\}$
- $I' = I_A \times I_B \times \{1\}$
- $F' = \{(q_A, q_B, 2) : q_B \in F_B\}$

$r' = (q_A^0, q_B^0, i^0), (q_A^1, q_B^1, i^1), \ldots$ is run of automaton A' on input word w if $r_A = q_A^0, q_A^1, \ldots$ is run of A on w, $r_B = q_B^0, q_B^1, \ldots$ is run of B on w.
And r_A, r_B is accepting if 1-state and 2-state alternatively happens.
□

2.3 Concatenation

There is Büchi automaton that recognizes the language $L(C) + L(A)$

Proof. $A' = (Q_C \cup Q_A, \Sigma, \Delta', I', F_A)$ recognizes $L(C) + L(A)$, where $\Delta' = \Delta_A \cup \Delta_C \cup \{(q, a, q') : q' \in I_A$ and $\exists f \in F_C.(q, a, f) \in \Delta_C\}$, and if $I_C \cap F_C$ is empty, $I' = I_C$, else $I' = I_C \cup I_A$ □

2.4 ω-closure

If $\epsilon \notin L(C)$, then there is a Büchi automaton that recognizes the language $L(C)^\omega$

Proof. Let $A' = (Q_C \cup \{q_{\text{new}}, \Sigma, \Delta'_A = \Delta_A \cup \{(q_{\text{new}}, a, q') : \exists q \in I_C.(q, a, q') \in \Delta_C\}\})$ Then $A'' = (Q_C \cup \{q_{\text{new}}\}, \Sigma, \Delta'_A \cup \{(q, a, q_{\text{new}}) : \exists q' \in F_C.(q, a, q') \in \Delta'\})$ $\square$

2.5 Complementation

Let $A = (Q, \Sigma, \Delta, Q_0, F)$ be a Büchi automaton.

Proof. Define equivalence relation $\sim_A$ over elements of Σ^+ as $v \sim_A w$ if for all $p, q \in Q$ A has run from p to q over v iff this is also possible over w, and if A has a run via F from p to q over v iff this is possible over w. Then now, mapping $f : Q \leftarrow 2^Q \times 2^Q$ defines class of $\sim_A$, L_f. we interpret this definition as following, $w \in L_f$ iff for each state $p \in Q, (Q_1, Q_2) = f(p)$ w can move automaton A from p to each state in Q_1 and to each state in Q_2 via state in F. Note that $Q_2 \in Q_1$.

Lemma 2.1. *$\sim_A$ has finitely many equivalent classes and each class is a regular language.*

Proof. There are only finitely many $f : Q \leftarrow 2^Q \times 2^Q$, so it has finitely many equivalent classes.
Claim : L_f is regular language.
For $p, q \in Q, i \in \{0, 1\}$ let $A_{i,p,q} = (\{0, 1\} \times Q, \Sigma, \Delta_1 \cup \Delta_2, \{(0, p)\}, \{(i, q)\})$ be nondeterministic finite automaton, where $\Delta_1 = \{((0, q_1), (1, q_2)) : q_1 \in F \wedge (q_1, q_2) \in \Delta\} \cup \{((1, q_1), (1, q_2)) : (q_1, q_2) \in \Delta\}, \Delta_2 = \{((0, q_1), (1, q_2)) : q_1 \in F \wedge (q_1, q_2) \in \Delta\}$.
Let $\alpha_{p,Q'} = \cap\{L(A_{1,p,q}) : q \in Q'\}$, which is set of words that can move A from p to all the states in Q' via some state in F. Let $\beta_{p,Q'} = \cap\{L(A_{0,p,q} - L(A_{1,p,q} - \epsilon)) : q \in Q'\}$, which is set of non-empty words that can move A from p to all the states in Q' and does not have a run that passes through any state in F. Let $\gamma_{p,Q'} = \cap\{\Sigma^+ - L(A_{0,p,q}) : q \in Q'\}$, which is set of non-empty words that can not move A from p to any of the states in Q'.
Then by definition, $L_f = \cap\{\alpha_{p,Q_2} \cap \beta_{p,Q_1 - Q_2} \cap \gamma_{p,Q-Q_1} : (Q_1, Q_2) = f(p) \wedge p \in Q\}$. $\square$

Lemma 2.2. *For each $w \in \Sigma^\omega$, there are $\sim_A$ classes L_f and L_g such that $w \in L_f(L_g)^\omega$*

Proof. Let $w = a_0 a_0 \ldots$ and $w(i,j) = a_i \ldots a_{j-1}$. Color $\mathbb{N}$, set of natural numbers by equivalence classes of $\sim_A$ on size 2 subsets, color of $\{i,j\}$ is equivalence class in which $w(i,j)$ occurs.
By infinite ramsey theorem [1] let $X \subset \mathbb{N}$ be infinite subset such that every size 2 subsets has same color. Then with $0 < i_0 < i_1 < \ldots \in X$ let f be equivalence class of $w(0, i_0) \in L_f$, g be equivalence class of $w(i_{j-1}, i_j) \in L_g$. Then $w \in L_f(L_g)^\omega$. □

Lemma 2.3. *For equivalence classes of* $\sim_A$ L_f, L_g, $L_f(L_g)^\omega \subset L(A)$ *or* $L_f(L_g)^\omega \cap L(A) =$

Proof. Suppose $w \in L(A) \cap L_f(L_g)^\omega$, r be the accepting run of A over input w. Now we need to show that every $w' \in L_f(L_g)^\omega$ is in $L(A)$, that there exists run r' of A over input w' such that states in F occurs in r' infinitely often.
Since $w \in L_f(L_g)^\omega, w = w_0 w_1 w_2 \ldots$ for $w_0 \in L_f, w_i \in L_g$ for $i > 0$. Let s_i be state in r after consuming $w_0 w_1 \ldots w_i$, and I be set of indices such that $i \in I$ iff segment of r from s_i to s_{i+1} contains a state from F. As $w \in L(A)$, I is infinite set. And split w' similarly. Now we will construct r'.
Let first state of r' be same as r. By definition of L_f, we can choose a run segment on word w'_0 to reach s_0. Then next at s_i by definition of L_g we can choose a run segment on word w'_i to reach s'_{i+1} and visiting state in F. Then this r' contains infinitely many state from F, then r' is accepting run. □

Then now by listing all $L_f(L_g)^\omega$, we can represent $\Sigma^\omega - L(A)$ as finite union of $L_f(L_g)^\omega$ which is Büchi Automaton acceptable. this construction is doubly exponential in terms of size of A. □

3 Algorithm

3.1 Emptiness

$\{A$ is an Büchi Automaton : $L(A) = \}$ is decidable.

Proof. First, by using any pathfinding algorithm, list all reachable final states from initial states. Then, for each of these reachable

[1] Let X be infinite set. If we color elements of $X^{(n)}$ in c different colors, then there exists some infinite subset M of X such that the size n subsets of M all have the same color.

final states again by using pathfinding algorithm, check if has path to coming back to it. $\square$