

Why does neural net work?  
Why does gradient descent work?  
How can I trust Deep Learning?

# So why deep learning works?

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# Overview

## 1 Why does neural net work?

- Universal Approximation Theorem
- Deep Visualization
- Manifold Hypothesis
- Finite Approximation

## 2 Why does gradient descent work?

- Global Minimum
- Generalizability

## 3 How can I trust Deep Learning?

- Verifying Neural Network
- Testing Neural Network

# Universal Approximation Theorem-1

Neural Network is dense in function space, so we can approximate any function with neural network.

What Neural Network, What function space, What topology?

## Universal Approximation Theorem-2

We say a function  $\sigma : \mathbb{R} \rightarrow \mathbb{R}$  is sigmoidal if

$$\sigma(t) \rightarrow \begin{cases} 1 & t \rightarrow +\infty \\ 0 & t \rightarrow -\infty \end{cases}$$

For  $y_i \in \mathbb{R}^n$  and  $\alpha_i, \theta_i \in \mathbb{R}$  ( $i = 1, \dots, N$ ), a neural network is a function from  $\mathbb{R}^n$  to  $\mathbb{R}$

$$G(x) = \sum_{i=1}^N \alpha_i \sigma(y_i^T x + \theta_i)$$

## Universal Approximation Theorem-3

### Theorem

*Let  $\sigma$  be any continuous sigmoidal function. Then the single layer neural networks*

$$G(x) = \sum_{i=1}^N \alpha_i \sigma(y_i^T x + \theta_i)$$

*are dense in  $C(I_n)$ .*

*In other words, given any  $f \in C(I_n)$  and  $\epsilon > 0$ , there is a neural network  $G(x)$  such that*

$$|G(x) - f(x)| < \epsilon$$

*for every  $x \in I_n$ . [Cyb89]*

## Universal Approximation Theorem-4

We have other results also. Followings hold.

Function Types	Activation	Function space
$\sigma(y^T x + \theta)$	sigmoidal	$C(I_n)$
$\sigma(y^T x + \theta)$	$L^1(\mathbb{R})$ , $\int \sigma(t) dt \neq 0$	$L^1(I_n)$
$\sigma(y^T x + \theta)$	continuous, sigmoidal	$L^2(I_n)$
$\sigma(Ux + y)$	Indicator of a rectangle	$L^1(I_n)$
$\sigma(tx + y)$	$L^1(\mathbb{R}^n)$ , $\int \sigma(t) dt \neq 0$	$L^1(\mathbb{R}^n)$

Figure: Table of UAT variations

Meaning, given activation function, then finite linear sum of function types are dense in function spaces.

## Universal Approximation Theorem-5

We also have arbitrary depth version, with bounded width.

### Theorem

*For any Bochner-Lebesgue  $p$ -integrable function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  and  $\epsilon > 0$ , there exists a fully-connected ReLU network  $F$  of width exactly  $\max(n + 1, m)$  and arbitrary depth that*

$$\int_{\mathbb{R}^n} ||f(x) - F(x)||^p dx < \epsilon$$

*Moreover, there exists a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  that is not approximable by fully-connected ReLU network  $F$  of width less than  $\max n + 1, m$ . [Par+21]*

## Universal Approximation Theorem-6

But we usually have finite samples from function, not function itself. Here is finite sample version of universal approximation theorem.

### Theorem

*There exists a two-layer neural network with ReLU activations and  $2n + d$  weights that can represent any function on a sample of size  $n$  in  $d$  dimensions. [Zha+16]*

## Universal Approximation Theorem-7

What about other networks?

**CNN** Continuous function on compact subset of  $\mathbb{R}^n$  can be approximated by CNN w.r.t. supnorm. [Zho20]

**RNN** Dynamical system with continuous transition function can be approximated by RNN. [SZ06]

**Generative Network** For source and target probability measure  $\pi, p_z$  there is a neural network whose feed forward measure of  $\pi$  approximates  $p_z$ . (Constructive) [LL20]

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# Deep Visualization-1

- Is there more intuitive way of understanding neural network?
- Can we visualize what neural network learned?
- What do each neurons do?

## Deep Visualization-2

Recall the definition of neural network, where we have

$$x_{i,j} = \sigma \left( \sum_{k=1}^{N_{i-1}} w_{k,j} x_{i-1,k} + b_{i,j} \right)$$

Here each  $x_{i,j}$  is called **activation value**  $\alpha_{i,j}(x)$  of  $j$ -th node in layer  $i$  for input  $x$ .

If we find some input  $x$  that  $\alpha_{i,j}(x)$  is large enough, we can check what feature this node represents.

## Deep Visualization-3

To increase  $\alpha_{i,j}(x)$ , we use gradient ascent, but regularized.  
Initial input is defined as average of all input.

$$x_0 = \frac{1}{|D|} \sum_{x \in D} x$$

Then it is updated by gradient ascent, and some regularization.

$$x_{t+1} = \text{Regularize} \left( x_t + \eta \frac{\partial \alpha_{i,j}(x)}{\partial x} \right)$$

After enough repeat, we have visualization of node. [Yos+15]

## Deep Visualization-4

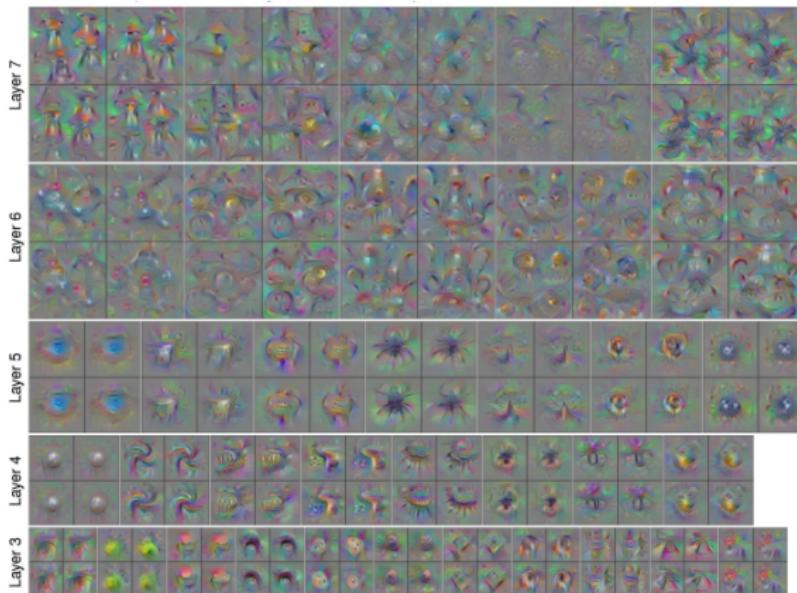


Figure: Visualization of middle nodes

## Deep Visualization-5

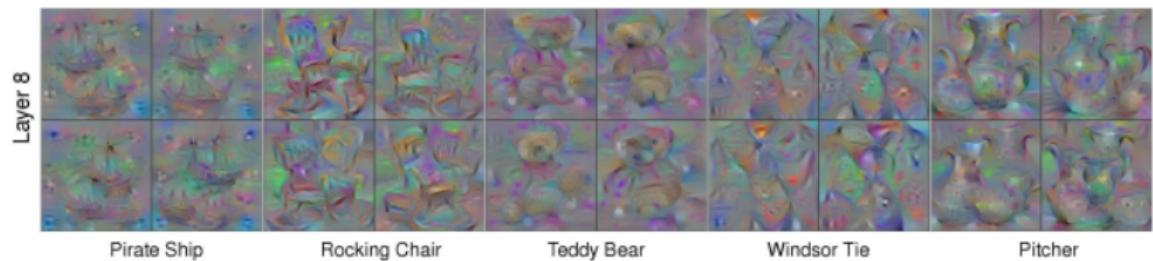


Figure: Visulization of classification nodes

## Deep Visualization-6

What does visualization means?

- Earlier layer learns simple edges.
- Latter layer learns combination.
- Each neuron learns feature of image.
- Finally classification node gives silhouette.

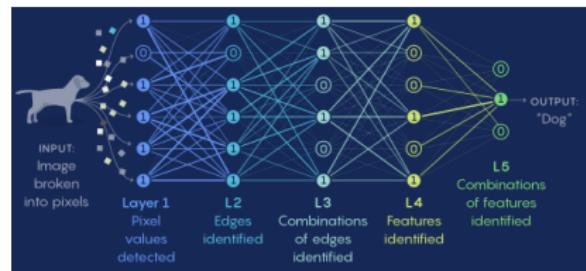


Figure: Explaining the neural network [Har19]

## Deep Visualization-7

Recently, new method called Attention mechanism is introduced. Intuitively, this mechanism computes ‘attention’ on each pixel or words.

With Attention, we can visualize reason of classification. [LBH15]

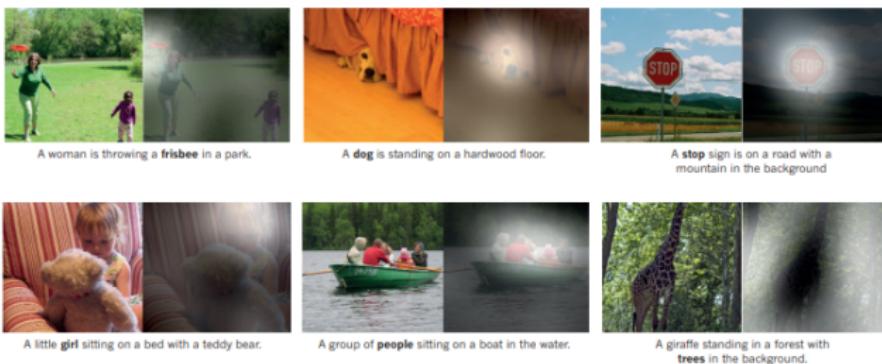


Figure: Image captioning with attention on image

## Deep Visualization-8

Attention is used in seq2seq like machine translation, where we can see attention matrix in translation.

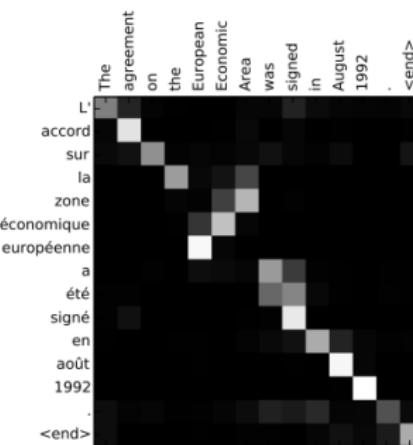


Figure: Attention matrix in machine translation [BCB16]

## Deep Visualization-9

- Intuitive
- Not theoretical
- Only applicable to image data
- Attention is applied to various area, including Computer Vision, Natural Language Processing, Program Processing
- But does not explain neural network itself

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# Manifold Hypothesis-1

Many deep learning researches relies on hypotheses, like distributional hypothesis, naturalness hypothesis, etc.

These hypotheses assumes dataset will behave good, so that designed architecture works well.

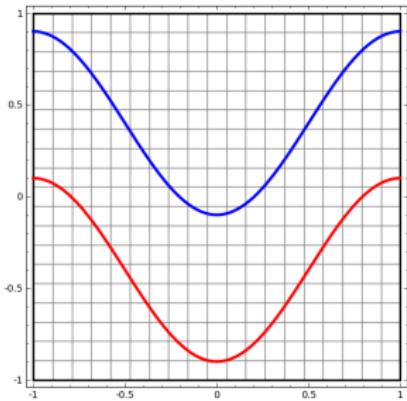
Manifold Hypothesis is one of such hypotheses, that dataset forms a (relatively) low dimensional manifold, and neural network learns structure of that manifold.

## Definition

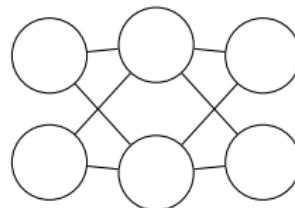
We say some space is  $n$ -dimensional manifold if it is locally homeomorphic to  $\mathbb{R}^n$ . For simplicity, we will not consider specific type of manifolds. You can assume that it is differentiable manifold.

## Manifold Hypothesis-2

Let's begin with simple example.

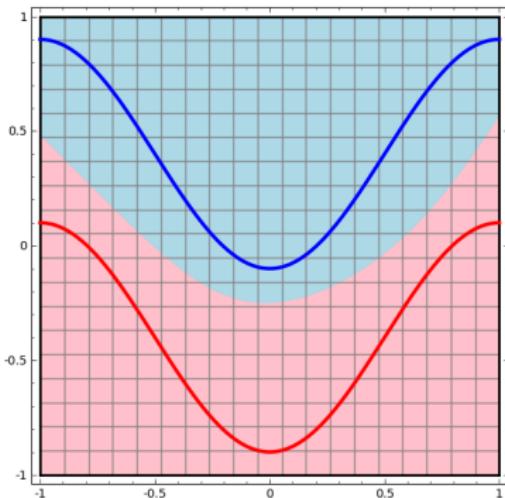


(a) Classification of two one-dimensional manifold [Ola14]

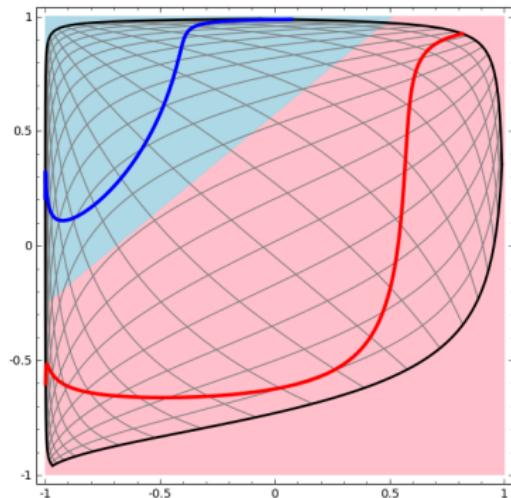


(b) Small neural network used for this problem

## Manifold Hypothesis-3



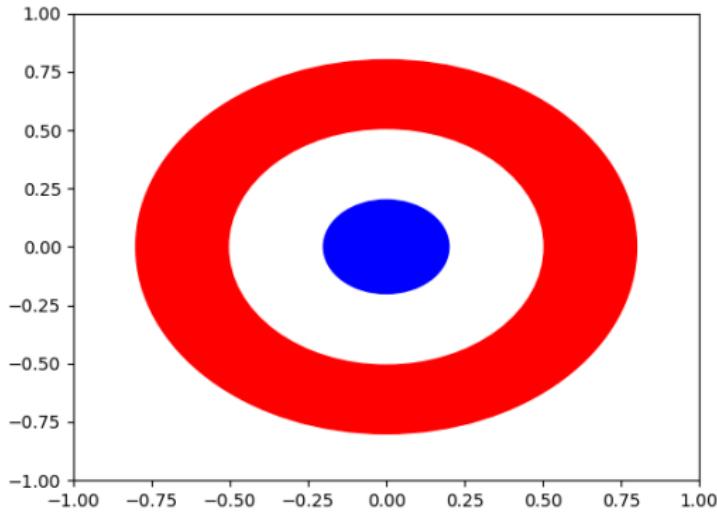
(a) Classification on first layer



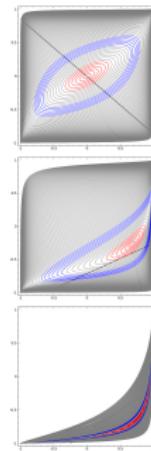
(b) Classification on second layer

## Manifold Hypothesis-4

Let's see more complex example.



(a) Data hard to classify



(b)  
Process of  
learning

## Manifold Hypothesis-5

- Why did preceding example not classified?
- Because they are linked
- Not classifiable by homeomorphism
- But possible in 3D

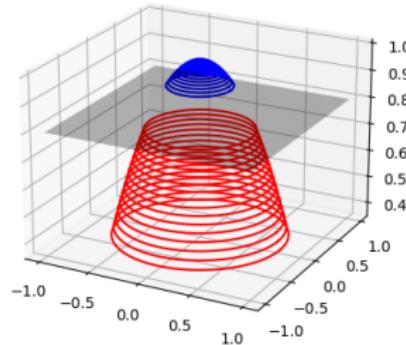


Figure: Classification on 3D

## Manifold Hypothesis-6

- Knowing dimension of data-manifold help creating efficient embedding
- Knowing structure of data-manifold help to know width required
- However, this is very complex. Even most simple MNIST dataset is still unknown.

### Definition

**Intrinsic Dimension** is the minimal number of coordinates which are necessary to describe its points without significant information loss.

## Manifold Hypothesis-7

### Proposition

Let  $x_i$  be uniformly sampled on a manifold with intrinsic dimension  $d$  and let  $N$  be the total number of points. If  $r_i^{(1)}$  and  $r_i^{(2)}$  are distances of the first and the second neighbor of  $i$ -th data, respectively, then

$$\mu_i = \frac{r_i^{(2)}}{r_i^{(1)}} \sim \text{Pareto}(d+1)$$

we can compute likelihood of intrinsic dimension as [Ans+19]

$$P(d|\mu) = d^N \prod_{i=1}^N \mu_i^{-(d+1)}$$

## Manifold Hypothesis-8

One research proposed an algorithm testing the manifold hypothesis, given population  $P$  determines which one is true in

- There exists  $M \in G(d, CV, \frac{\tau}{C})$  such that  $L(M, P) \leq C\epsilon$
- There exists no  $M \in G(d, V/C, C\tau)$  such that  $L(M, P) \leq \frac{\epsilon}{C}$

with probability at least  $1 - \delta$  [FMN13].

Here,  $G(d, V, \tau)$  is the set of  $d$ -dimensional submanifolds whose volume is at most  $V$  and reach is at least  $\tau$ ,  $L(M, P)$  is a mean-squared distance of a random point from  $P$  to manifold  $M$ .

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# Finite Approximation-1

Let's consider following problem.

For neural network, what happens if we send each hyperparameters to infinite?

- Width
- Depth
- Number of steps
- Sample size

## Finite Approximation-2

### Definition

A time continuous stochastic process  $\{X_t\}_{t \in T}$  is **Gaussian** if for every finite indices  $t_1, \dots, t_k \in T$

$$X_{t_1, \dots, t_k} = (X_{t_1}, \dots, X_{t_k})$$

is a multivariate gaussian random variable.

### Theorem

*A infinite width with single layer is equivalent to a Gaussian Process. [HJ15]*

## Finite Approximation-3

A standard feedforward neural network is  $C^0$  function

$$x^{(l+1)} := f(x^{(l)}; \theta)$$

A residual network has the form

$$x^{(l+1)} := x^{(l)} + f(x^{(l)}; \theta)$$

where by sending depth  $L$  to infinity we achieve

$$x^{(l+1)} \simeq x^{(l)} + f(x^{(l)}; \theta) \Delta l$$

which is a  $C^1$  coordinate transform.

Similarly,  $C^k$  neural network defines a  $C^k$  coordinate transform, which becomes solving differential equation of degree  $k$ . [HR17]

## Finite Approximation-4

Generative Adversarial Network is an architecture that tries to generate data by training two adversarial network. Its loss

$$\mathcal{L}(\theta, \phi) := \mathbb{E}_{x \sim p(x)}[\log(D(x; \theta))] + \mathbb{E}_{z \sim p(z)}[\log(1 - D(G(z; \phi); \theta))]$$

which then becomes ordinary differential equation [Qin+20]

$$\begin{pmatrix} \frac{d\theta}{dt} \\ \frac{d\phi}{dt} \end{pmatrix} = -[\alpha \frac{\partial \mathcal{L}_D}{\partial \theta}, \beta \frac{\partial \mathcal{L}_G}{\partial \phi}]$$

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## Global Minimum-1

Now suppose we have ‘Good’ approximation for our desired function.

How can we find it? Moreover, can we find it by gradient descent?

- We may require too many steps.
- We may stuck in local minimum.
- We may stuck in saddle point.

## Global Minimum-2

### Definition

Gaussian random field is a stochastic process over euclidean space  $\{X_x\}_{x \in \mathbb{R}^n}$  that for every finite points  $x_1, \dots, x_k \in \mathbb{R}^n$

$$X_{x_1, \dots, x_k} = (X_{x_1}, \dots, X_{x_k})$$

is a multivariate gaussian random variable.

We can approximate loss space of neural network with gaussian random field.

## Global Minimum-3

### Theorem

*For a critical point with error  $\epsilon$  from the global minimum, are exponentially likely to be saddle points.*

*Conversely, all local minimum are likely to have an error very close to that of the global minimum. [Dau+14; BD07]*

*Intuitively in high dimensions the probability that all the partial derivative lead upward is exponentially small w.r.t. the number of dimensions.*

And through empirical evaluation, this also holds in neural network's loss space.

## Global Minimum-4

We can validate former result with random matrix theory.

According to Wigner's semicircular law, eigenvalue distribution of large Gaussian random matrix follows semicircle distribution with mean 0.

Hessian matrix of loss space behaves similar, but shifted by an amount determined by  $\epsilon/$

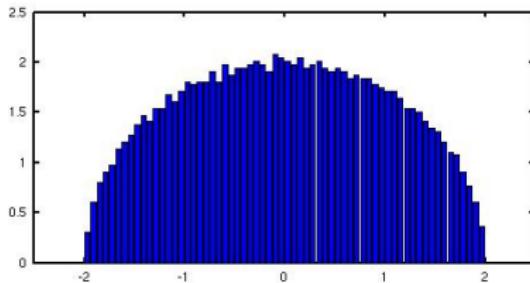


Figure: Wigner's semicircular law

## Global Minimum-5

- For global minimum, distribution is shifted right so that all eigen values are positive.
- As  $\epsilon$  increases, it includes more negative eigenvalues with high probability of zero eigenvalues.
- This indicates existence of plateau around saddle point with large error.
- So if we can escape plateau, we can escape saddle point.

## Global Minimum-6

There are research that analyzed behavior of local minimums in neural network. [HWT20]

- ① The loss surface of neural network with piecewise linear activation is partitioned into multiple smooth open cells.
- ② If we have single hidden layer with two-piece linear activation and convex loss, all local minima are global minimas in cells.
- ③ Moreover if we have strictly convex loss, all the local minima in each cells are path-connected.

## Global Minimum-7

Moreover, gradient descent ‘can’ find global minimum. Moreover, we can always find neural network.

### Theorem

*For three networks, fully-connected feedforward network, ResNet, convolutional ResNet with enough width  $m(n, \delta, K^{(H)}, \lambda_{min}, H)$  randomly initialized gradient descent satisfies following inequality with probability at least  $1 - \delta$ . [Du+18]*

$$L(\theta(k)) \leq \left(1 - \frac{\eta \lambda_{min}(K^{(H)})}{2}\right)^k L(\theta(0))$$

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## Generalizability-1

So far we checked

- There exists a neural network that approximates given function.
- Trained neural network shows reasons that they learn function well.
- And we can find global minimum with gradient descent.

But the problem is, loss zero does not mean that we have 100% accuracy. Neural network has to generalize its learning.

## Generalizability-2

We say a model is **overfitted** when it learns training dataset too closely, so that it may fail to fit other data or predict future observations.

To prevent overfitting, we use regularization, dropout, cross validation, etc.

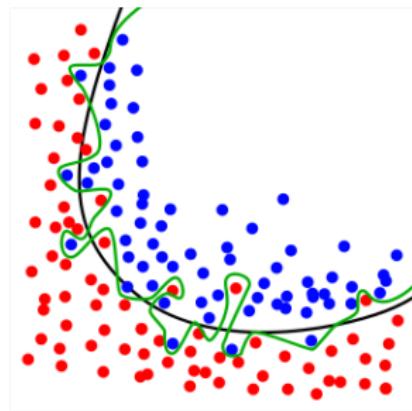


Figure: Black line represents reasonable classifier, where green line represents overfitted one.

[Wik21]

## Generalizability-3

Neural network has extreme over parametrization, since it contains enormous number of parameters.

One of famous research have shown that neural network can learn all following corrupted data. [Zha+16]

- All labels are changed randomly.
- Pixels are shuffled universally.
- Pixels are shuffled individually.
- Pixels are sampled from Gaussian distribution.

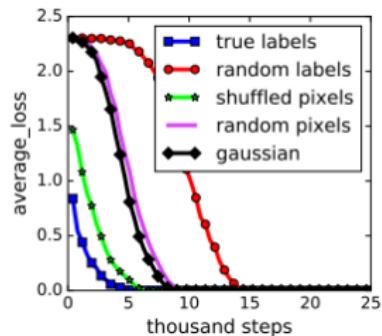


Figure: Risk curves for corrupted data

## Generalizability-4

Usually machine learning's risk curve w.r.t. model capacity looks like right one.

Recent research argues that it is actually a double descent risk curve. [Bel+19]

This is empirically tested with RFF model, fully connected neural network, random forests.

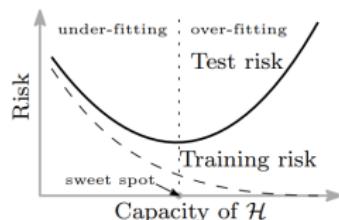


Figure: Classical Learning Curve

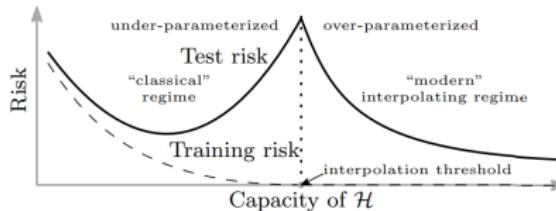


Figure: Modern Learning Curve

## Generalizability-5

We can define generalization gap as

$$\mathcal{R}[f_{A(S)}] - \mathcal{R}_S[f_{A(S)}]$$

where  $S$  is samples,  $f_{A(S)}$  is a function that algorithm  $A$  makes given sample  $S$ ,  $\mathcal{R}$  and  $\mathcal{R}_S$  is a expected risk and empirical risk, resp.

Our goal is to find upper bound of generalization gap for neural network, with

$$\mathcal{R}[f_{A(S)}] - \mathcal{R}_S[f_{A(S)}] \leq \sup_{f \in F} \mathcal{R}[f] - \mathcal{R}_S[f] = \mathcal{M}$$

## Generalizability-6

Here I list some of bounds that holds with probability at least  $1 - \delta$ .

$$\mathcal{M} \leq 2\mathfrak{R}_m(\mathcal{L}_f) + \sqrt{\frac{\log 1/\delta}{2m}}$$

where  $\mathfrak{R}_m(\mathcal{L}_f)$  is the Rademacher complexity.

$$\mathcal{M} \leq 2\beta + (4m\beta + M)\sqrt{\frac{\log 1/\delta}{2m}}$$

where  $\beta$  is a uniform stability.

$$|\mathcal{M}| \leq \zeta(S) + M\sqrt{\frac{2\Omega \log 2 + 2 \log 1/\delta}{m}}$$

where algorithm is  $(\Omega, \zeta(\cdot))$ -robust. [KKB20]

## Generalizability-7

Another comes from observation that the more complex the model is, it has less generalizability.

So by analyzing complexity of decision boundary, we can measure generalizability. [GL20]

By creating adversarial set as following, we can sample from decision boundary.

Then applying PCA to adversarial set, and compute Shannon entropy of eigen values.

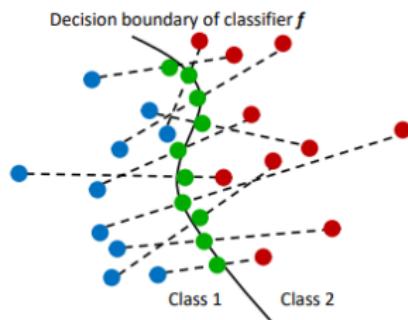


Figure: Findinf decision boundary by interpolation

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## Verification-1

In programming language theory, we verify that programs are correct with formal methods. For formal methods, we include

- Proof Assistant
- Static Analysis
- Model checking

We don't want following adversarial examples not happen.

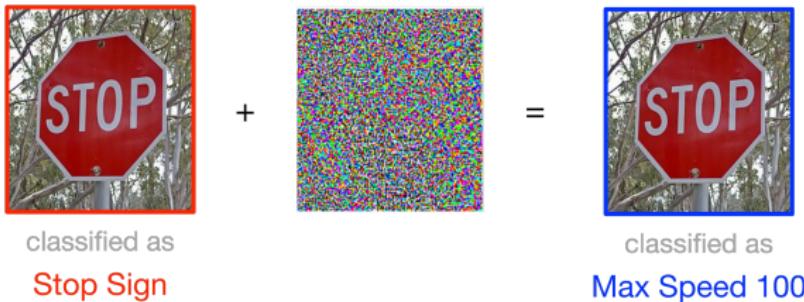


Figure: Adversarial example misclassified [Jie19]

## Verification-2

Usually our goal is to verify that there is no adversarial example.

### Definition

We say a classifier  $F$  is  $\delta$ -locally-robust at data point  $x$  if for every  $x'$  that  $\|x - x'\|_\infty \leq \delta$  we have  $F(x) = F(x')$ .

For regression function  $F$ , we can use Lipschitz constant  $L$  which satisfies following

$$\|F(x) - F(x')\| \leq L\|x - x'\|$$

## Verification-3

The first idea is simple. We ‘solve’ the linear equation that neural network defines.

For each variable, we only record its maximum/minimum values.

Since each layer is linear, and ReLU is piecewise linear, we can safely track maximum/minimum values.

By feedforwarding initial square  $\{x' : \|x - x'\|_\infty \leq \delta\}$  through neural network, we can check whether they have same label.

This procedure is automated with SMT solver. [Kat+17]

## Verification-4

### Definition

Abstract Interpretation is a family of sets on  $\mathbb{R}^n$ , that is used to over-approximation, which( can be used to statically check numerical problem.

For some function  $F$ , if we have overapproximation ( $X \subseteq A$ ) of input set  $X$  by  $A$  /  $A$ , we can feed forward  $A$  into  $F$  to overapproximate  $F(X) \subseteq \bar{F}(A)$ .

We can use various abstract interpretation for neural network verification, like simplex, zonotope, polyhedra.

Simplex was not enough to verify all network, and polyhedra took many days even in simplest network. [Geh+18]

## Verification-5

Following theorem gives another approach. [RHK18]

### Theorem

Suppose we need to minimize  $w(x)$  for  $x \in [a, b]^n$ .

If  $w$  is Lipschitz continuous, there exists some function  $h$  that

$$\forall x, y \in [a, b]^n, h(x, y) \leq w(x) \text{ and } h(x, x) = w(x)$$

Moreover, there is sequence of  $Y_i$  that  $Y_i \subset Y_j$  for  $i < j$  with

$$\inf_{x \in [a, b]^n} \max_{y \in Y_i} h(x, y) \rightarrow \inf_{x \in [a, b]^n} w(x)$$

Since neural network is Lipschitz continuous with operator norm of linear layers, we can approximate minimum/maximum value of neural network with sequence.

## Verification-6

Another way is approximating the neural network, and verify the approximation. This research verifies reinforcement learning by guarding. [Zhu+19]

- ① Approximate neural network with piecewise linear function  $P_\theta$ .
- ② Find invariant  $\Phi$  that
  - Every initial state is in  $\Phi$
  - No danger state is in  $\Phi$
  - If  $s \in \Phi$ , then  $P_\theta(s) \in \Phi$
- ③ Now when using neural network, if its step violates  $\Phi$ , we use  $P_\theta$  instead.

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## Testing-1

In software engineering, we create test cases to check whether software is correct.

We usually check coverage as measure how test cases tests well, like statement coverage, condition coverage, mutation coverage, etc.

We'd like to define coverage for neural network, and find good test examples.

## Testing-2

Recall the definition of activation value.

$$x_{i,j} = \sigma\left(\sum_{k=1}^{N_{i-1}} w_{k,j} x_{i-1,k} + b_{i,j}\right)$$

We say a neuron is covered if its activation value is larger than threshold,  $x_{i,j} > \epsilon$ , like 0.3. [Pei+17]

Given  $n$  Neural Networks  $NN_1, \dots, NN_n$ , our goal is to find data that they disagree and neuron coverage is large enough by maximizing following loss.

$$\sum_{i \neq j} NN_i(x)[c] - \lambda_1 NN_j(x)[c] + \lambda_2 \alpha_{i,j}(x)$$

## Testing-3

There are other notions of coverage

- $k$ -multisection Neuron Coverage [Ma+18]
- Neuron Boundary Coverage [Ma+18]
- Strong Neuron Activation Coverage [Ma+18]
- $\text{Sign}(\text{Value})$ - $\text{Sign}(\text{Value})$  Coverage [SHK18]
- Surprise Adequacy [KFY18]

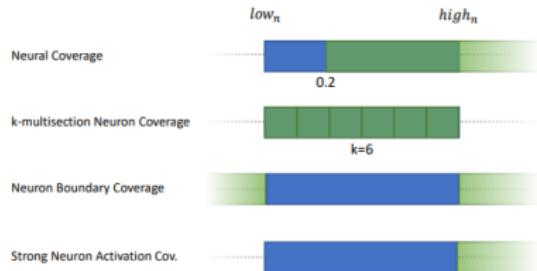


Figure: Visualization of neuron coverages

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Throughout the survey, following surveys helped a lot on finding papers. [HT21; Gui+18; UM21; 18]

Large portion of papers are accepted to following conferences, including ICML, ICLR, NIPS, IJCAI.



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