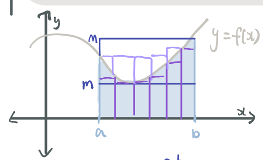
THE DEFINITION OF INTEGRAL

Preview of the definition



$\int_a^b f(x)\,dx$ = Area of shaded region $\Rightarrow$ Definite integral

"The integral from a to b of f(x) with respect to x

- $m(b-a) \le \int_a^b f(x)dx \le M(b-a)$ or Area of rectangles $\le \int_a^b f(x)dx \le$ Area of rectangles
- Plan to compute the area (get the integral)
  1. Cut region into slices
  2. Under or Over-estimate each slice with a rectangle
  3. Some sort of limit with rectangles

Σ (sigma) Notation of Sums

Ex. $\sum_{i=3}^{7} \frac{1}{i} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}$ (sum of $\frac{1}{i}$ from $i=3$ to 7)

- $\sum_{i=1}^{N} a_i = a_1 + a_2 \dots a_N$ (sum of a_i from $i=1$ to N)
- $\sum_{i=1}^{N} (c \cdot a_i) = c \sum_{i=1}^{N} a_i$; $\sum_{i=1}^{N} (a_i + b_i) = \sum_{i=1}^{N} a_i + \sum_{i=1}^{N} b_i$

- Summation index is a "dummy index" [doesn't mean anything]

① $\sum_{i=1}^{3} \frac{i}{k} = \frac{1}{k} + \frac{2}{k} + \frac{3}{k} = \frac{6}{k}$ ③ $\sum_{n=1}^{3} \frac{i}{k} = \frac{i}{k} + \frac{i}{k} + \frac{i}{k} = \frac{3i}{k}$

② $\sum_{k=1}^{3} \frac{i}{k} = \frac{i}{1} + \frac{i}{2} + \frac{i}{3} = \frac{11}{6}i$

Supremum & Infimum

Ex. 2 is the maximum of the set $[0,2]$, 2 is the supremum of $(0,2)$

- A number $c \in \mathbb{R}$ is the maximum of a set A when: $c \in A \ \wedge \ \forall x \in A, x < c$

DF Let $A \subseteq \mathbb{R}$. Let $c \in \mathbb{R}$

- c is an upper bound of A means $\forall x \in A, x \le c$
- c is the least upper bound (lub) or **supremum (sup)** of A means:
  - c is a upper bound of A & If b is a upper bound of A, $c \le b$
- If the supremum of A is in A, it is called maximum
- A is bounded above means it has (at least) one upper bound

DF Let $A \subseteq \mathbb{R}$, let $c \in A$

- c is an lower bound of A means $\forall x \in A, x \ge c$
- c is the infimum of A means c is a lower bound of A and If b is a lower bound of A, $c \ge b$
- If the infimum of A is in A, it's called minimum
- A is bounded below means it has (at least) one lower bound

great lower bound

- "A set is bound" - both bound above & below

THEOREM: [The L.U.B. Principle]

- Let $A \subseteq \mathbb{R}$ IF A is bounded above and not empty THEN A has a least upper bound

- Supremum of a function == supremum of its range

Supremum of a function f (on a domain I) $= \sup \{ f(x) \mid x \in I \} = \sup_{x \in I} f(x)$; Infimum of a function f (on a domain I) $= \inf_{x \in I} f(x)$

range

range

THEOREM: [The consequence of the LUB Priciple]

Let f be a function defined on a domin $I \neq \emptyset$ IF f is bounded above on I THEN f has a supremum on I

THEOREM: [EVT]

Let $a < b$. Let f be a function defined on $[a,b]$ IF f is continuous on $[a,b]$ THEN f has a max & min on $[a,b]$

Integral definition 2

1. Let f be a function with domain $(0,1)$. Assume f is increasing. Assume f is bounded. Let us call $L = \sup f$. Prove that $L = \lim_{x \to 1^-} f(x)$.

   Note: Drawing a picture will help. You need to prove that the number $L = \sup f$ satisfies the definition of limit. In your proof, you will probably need to use the definition of sup. Beware of assuming that $\lim_{x \to 1^-} f(x)$ exists as part of your proof.

   Solution: We want to prove the definition of the limit $L = \lim_{x \to 1^-} f(x)$, which is

   $$\forall \varepsilon > 0, \exists \delta > 0 \colon\ 1 - \delta < x < 1 \Rightarrow |L - f(x)| < \varepsilon.$$

   Fix $\varepsilon > 0$ arbitrary.

   - Since the supremum of f on $(0,1)$ is L, the number $L - \varepsilon$ is not an upper bound of f on $(0,1)$, and hence there exists $x_0 \in (0,1)$ such that

     $$L - \varepsilon < f(x_0).$$

   - Set $\delta = 1 - x_0$.

   Next I need to verify that

   $$1 - \delta < x < 1 \Rightarrow |L - f(x)| < \varepsilon.$$

   Let x be any number such that $1 - \delta < x < 1$.

   - Since f is increasing on $(0,1)$,

     $$x_0 = 1 - \delta < x < 1 \Rightarrow f(x_0) < f(x).$$

   - In addition, since L is the supremum of f on $(0,1)$:

     $$f(x) \leq L$$

   - Putting all the above equations together:

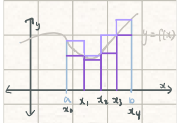
     $$L - \varepsilon < f(x_0) < f(x) \leq L < L + \varepsilon$$

   and hence

   $$|f(x) - L| < \varepsilon$$

   as we wanted.

The definition of integral



- Let $a<b$. Let f be a bounded function on $[a,b]$

1. Cut the region into slices – Take some partition $\{x_0, x_1, x_2, x_3, x_4\}$ where $x_0 = a < b = x_4$

DF1 A partition of the interval $[a,b]$ is a set P such that is finite and $P \subseteq [a,b]$ $a \in P \wedge b \in P$ (endpoints & a few more)

↳ $P = \{x_0, x_1, x_2 \dots x_N\}$ implicitly means $a = x_0 < x_1 < x_2 \dots < x_N = b$

↳ Take a partition $P = \{x_0, x_1, x_2, x_3, x_4\}$ of the interval $[a,b]$

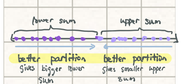
2. Under or Over-estimate the area with rectangles

L Underestimate with m rectangles : $m_1(x_1 - x_0) + m_2(x_2 - x_1) + \dots = \sum_{i=1}^{4} m_i (x_i - x_{i-1})$

m_1 = infimum of $[x_0, x_1]$; m_2 = infimum of $[x_1, x_2]$ ...

L Overestimate with m rectangles : $\sum_{i=1}^{4} M_i (x_i - x_{i-1})$

M_1 = supremum of $[x_0, x_1]$; M_2 = supremum of $[x_1, x_2]$ ...



DF2 Let f be a bounded function on $[a,b]$. Let $P = \{x_0, x_1, \dots x_N\}$ be a partition of $[a,b]$

For each $i = 1 \dots N$ let

L m_i be the infimum of f on $[x_{i-1}, x_i]$

L M_i be the supremum of f on $[x_{i-1}, x_i]$

L $\Delta x_i = x_i - x_{i-1}$

- P- **lower sum** of f is the number $L_P(f) = \sum_{i=1}^{N} m_i \cdot \Delta x$ [underestimates]
- P- **upper sum** of f is the number $U_P(f) = \sum_{i=1}^{N} M_i \cdot \Delta x$ [overestimates]

Depends on 3 things
1. function
2. interval
3. partition

■ Properties of lower/upper sums

1. For every partition $(\forall)$ P of $[a,b]$: $L_P(f) \le U_P(f)$
2. Let P and Q be $(\exists)$ 2 partition of $[a,b]$ } IF $P \subseteq Q$ THEN $L_P(f) \le L_Q(f) \wedge U_Q(f) \le U_P(f)$

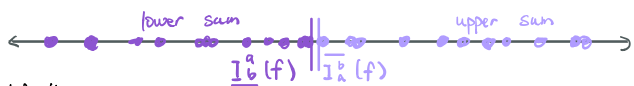
   DF Q is finer than P when $P \subseteq Q$
3. Let P and Q be any $(\forall)$ 2 partition of $[a,b]$ THEN $L_P(f) \le U_Q(f)$ [every lower sum ≤ every upper sum]

   L Pf Call $R = P \cup Q$ Then $P \subseteq R$ and $Q \subseteq R$:

   $$L_P(f) \underset{\text{by property 2}}{\le} L_R(f) \underset{\text{by property 1}}{\le} U_R(f) \underset{\text{by property 3}}{\le} U_Q(f)$$

3. Define the integral

- Assume: Lemma [Every lower sum ≤ every upper sum]



DF3 Let f be a bounded function on $[a,b]$

lower integral of f from a to b $= \underline{I}_a^b(f) = \sup\{\text{lower sums of } f\}$ $\le$ upper integral of f from a to b $= \overline{I}_a^b(f) = \inf\{\text{lower sums of } f\}$

DF4 when $\underline{I}_a^b(f) = \overline{I}_a^b(f)$, f is integrable on $[a,b]$

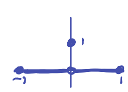
$$\int_a^b f(x)\,dx = \underline{I}_a^b(f) = \overline{I}_a^b(f)$$

DF5 when $\underline{I}_a^b(f) \ne \overline{I}_a^b(f)$, f is non-integrable on $[a,b]$

$\int_a^b f(x)\,dx$ is undefined

THEOREM : IF f is continuous function on $[a,b]$ THEN f is integrable on $[a,b]$

Ex. $f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \ne 1 \end{cases}$ Is f integrable on $[-1,1]$?



- $\underline{I}_{-1}^{1}(f) = \sup\{\text{lower sum of } f\} = \overline{I}_{-1}^{1}(f) = \inf\{\text{upper sum of } f\}$ ⟺ f is integrable in $[-1,1]$
- $\sup\{0\} = 0 == \inf(0,2] = 0$ ∴ $\int_{-1}^{1} f(x)dx = 0$

Ex. $g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$ Is g integrable on $[0,1]$

- $\underline{I}_0^1(g) = \sup\{\text{lower sum of } g\} == \overline{I}_0^1(g) = \inf\{\text{upper sum of } f\}$ ⟺ g is integrable in $[0,1]$
- Let $a, b \in \mathbb{R}$, s.t. $a<b$, $\exists t \in [a,b]$ s.t. $t \in \mathbb{Q}$, $\exists s \in [a,b]$ s.t. $s \notin \mathbb{Q}$ (contains both rational & non-irrational)
- $\forall$ partition of $[0,1]$, $\inf(g) = 0$, $\sup(g) = 1$
- $\underline{I}_0^1(g) = \sup\{0\} = 0 \ne 1 = \inf\{1\} = \overline{I}_0^1(g)$ ∴ $\int_0^1 g(x)dx$ is undefined

Question 27

8. *[6 points]* Let $a < b$. Let f be a bounded function on $[a, b]$. We want you to write the definition of the statement

"f is integrable on $[a, b]$".

However, to define "integrable" we first need to define a few other concepts. Assume you are writing this for somebody who does not know what a partition is or anything else involving partitions, but who already knows everything about sigma notation, suprema, and infima. Define all the necessary concepts, in the correct order, so that you can conclude with the definition of "integrable". You do not need to prove anything, or list properties, or give examples; you just need to **make sure that every concept and every piece of notation you use in your definition has been defined previously**.

- A partition P of $[a, b]$ is a finite set of points of $[a, b]$ that include the endpoints. Equivalently, a partition of $[a, b]$ is a finite subset $P \subseteq [a, b]$ such that $a \in P$ and $b \in P$.
- Let
$$P = \{a = x_0 < x_1 < \ldots < x_n = b\}.$$
be one such partition. Let us denote by m_j and M_j the infimum and supremum, respectively, of the function f on the interval $[x_{j-1}, x_j]$:
$$m_j = \inf_{x \in [x_{j-1}, x_j]} f(x), \qquad M_j = \sup_{x \in [x_{j-1}, x_j]} f(x)$$
Then we define the upper and lower sums of f for the partition P of $[a, b]$:
$$U_P(f) := \sum_{j=1}^{n} M_j(x_j - x_{j-1}), \qquad L_P(f) := \sum_{j=1}^{n} m_j(x_j - x_{j-1})$$
- Finally, for the function f, consider the infimum $\inf_P U_P(f)$ of upper sums over all partitions P, and consider the supremum $\sup_P L_P(f)$ of lower sums over all partitions P. Then we say the function f is integrable if and only if these two values coincide:
$$\inf_P U_P(f) = \sup_P L_P(f).$$

Integral as limits

- Let $a<b$, let f be a bounded function on $[a,b]$

DF Let $P=\{x_0, x_1, x_2 \dots x_n\}$ be a partition of $[a,b]$. For each i, let $\Delta x_i = x_i - x_{i-1}$

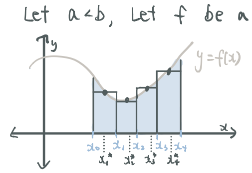
The **norm** of P: $\|P\| = \max\{\Delta x_1, \Delta x_2 \dots \Delta x_n\}$ (the largest length of all sub intervals.

THEORM 1 $\underline{I}_a^b(f) = \lim_{\|P\|\to 0} L_P(f)$ $\begin{cases} \forall \varepsilon>0,\ \exists \delta>0,\ \forall \text{ partition } P \text{ of } [a,b] \\ \|P\|<\delta \Rightarrow |\underline{I}_a^b(f) - L_P(f)| < \varepsilon \end{cases}$

THEORM 2 $\begin{cases} \text{Pick a sequence of partitions } P_1, P_2, P_3 \dots \text{ satisfying } \lim_{n\to\infty} \|P_n\| = 0 \\ \text{THEN } \quad \underline{I}_a^b(f) = \lim_{n\to\infty} L_{P_n}(f) \quad \overline{I}_a^b(f) = \lim_{n\to\infty} U_{P_n}(f) \end{cases}$ [ex. P_n: break $[a,b]$ into n equal length subintervals]

Riemann Sums

- Let $a<b$, Let f be a **bounded & Integrable** function on $[a,b]$ (continuous $\Rightarrow$ bounded & integrable)



Area of rectangles $= \sum_{i=1}^{N} f(x_i^*)\cdot\Delta x_i \to$ a Riemann Sum

definition for f and P

DF Let f be a bounded function on the interval $[a,b]$. Let $P=\{x_0, x_1, x_2 \dots x_n\}$ be a partition of $[a,b]$

For each $i = 1,2,\dots n$ $\begin{cases} \text{Let } \Delta x_i = x_i - x_{i-1} \\ \text{Choose a number } x_i^* \in [x_{i-1}, x_i] \end{cases}$ THEN: $S_P^*(f) = \sum_{i=1}^{N} f(x_i^*)\cdot\Delta x_i$

THEORM Let f be a bounded function on the interval $[a,b]$. Assume f is intergrable on $[a,b]$

- Pick a sequence of partitions $P_1, P_2, P_3 \dots$ of $[a,b]$ s.t. $\lim_{n\to\infty}\|P_n\|=0$
  - └ Pick P_n to break the interval $[a,b]$ into n subintervals of equal length)
- On each subinterval of each partition, pick $x_i^* \in [x_{i-1}, x_i]$ (ex. take $x^* = x_i$)
- THEN $\int_a^b f(x)\,dx = \lim_{N\to\infty} S^*_{P_n}(f) = \lim_{n\to\infty}\left[\sum_{i=1}^{n} f(x_i^*)\cdot\Delta x_i\right]$

Pf

- $\lim_{n\to\infty} L_{P_n}(f) = \underline{I}_a^b(f) = \int_a^b f(x)\,dx$
- $\lim_{n\to\infty} U_{P_n}(f) = \overline{I}_a^b(f) = \int_a^b f(x)\,dx$

} because f is intergrable on $[a,b]$

- $L_{P_n}(f) \le S_{P_n}^*(f) \le U_{P_n}(f) \xRightarrow{\text{be squeeze theorm}} \lim_{n\to\infty} S_{P_n}^*(f) = \int_a^b f(x)\,dx$ ∎

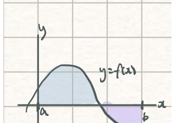
Ex. Calculat $\int_0^1 x\,dx$ using Riemann Sums

- $f(x)=x$ is continuous on $[0,1]$
- Choose P_n: break $[0,1]$ into n equal subintervals $P_n = \{0, \frac{1}{n}, \frac{2}{n} \dots 1\}$
- On each subinterval, choose the right endpoint $x_i^* = \frac{i}{n}$

$$\int_0^1 x\,dx = \lim_{n\to\infty} S_{P_n}^*(f) = \lim_{n\to\infty}\left[\sum_{i=1}^{n}\frac{i}{n}\cdot\frac{1}{n}\right] = \lim_{n\to\infty}\left[\frac{1}{n^2}\sum_{i=1}^{n} i\right] = \lim_{n\to\infty}\left[\frac{1}{n^2}\cdot\frac{n(n-1)}{2}\right] = \lim_{n\to\infty}\frac{n+1}{2n} = \frac{1}{2}$$

Properties of definite integral

- $a<b$, f as a bounded function on $[a,b]$
- f & g are integrable on $[a,b]$, $[b,c]$



- Definite integrals are linear

1. $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
2. $\int_a^b [c f(x)]\,dx = c \cdot \int_a^b f(x)\,dx$
3. $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

DF $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ & $\int_a^a f(x)\,dx = 0$

- Geometric interpretation: $\int_a^b f(x)\,dx = (\text{area above } x\text{-axis}) - (\text{area below } x\text{-axis})$
  - └ Area is ALWAYS (+), but integral may not have to be
- IF $\forall x \in [a,b]$, $f(x) \le g(x)$ THEN $\int_a^b f(x) \le \int_a^b g(x)$

THE FUNDAMENTAL THEOREM OF CALCULOUS

Antidrivatives

1. Definite integral : $\int_a^b f(x)dx$
   - A number ; geometrically it measures area

2. Indefinite integral : $\int f(x)dx$
   - A family of functions ; means antiderivative

Df Let f be a function defined on an interval

- A **antiderivative** of f is any function F such that $F' = f$
- The collection of all antiderivatives of f is denoted $\int f(x)dx$

Ex. $\int x^2 dx$

- Guess $F(x) = \frac{1}{3}x^3$ is one antiderivative $\left(\frac{d}{dx}\left[\frac{1}{3}x^3\right] = x^2\right)$
- For any constant C, $F(x) = \frac{1}{3}x^3 + C$ [This is all of $F(x)$ be def of MVT : $f'(x) = g'(x)$ in $[a,b]$ $\Rightarrow f(x) - g(x) = C$]

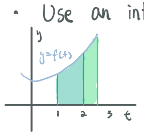
- The main integration technique: Guess & Check

  Ex. $\int (2x+7)^{11} dx$; guess: $\frac{d}{dx}[(2x+7)^{12}] = 12(2x+7)^{11}\cdot 2 = 24(2x+7)^{11}$ $\quad \therefore \int (2x+7)^{11}dx = \frac{1}{24}(2x+7)^{12} + C$

Functions defined as Integrals

DO NOT EVER WRITE $F(x) = \int_a^x f(x)dx$

- Use an integral to define a function: $F(x) = \int_a^x f(t)\,dt$



- $\int_1^2 f(t)dt$
- $\int_1^{2.5} f(t)dt$

fixing the first point (a) and use the second point (x) as variable of a new function $\Rightarrow F(x)$

- Let I be an interval, let $a \in I$, let f be a function integrable on I
- Then for each value $x \in I$, $F(x) = \int_a^x f(t)\,dt$ returns a number (area)

Fundemental Theorm of Calculus

- FTC: connections between definite integrals and indefinit integrals (antiderivatives)
  - Part 1: What can say about the function $F(x) = \int_a^x f(t)dx$
  - Part 2: How can we quickly compute $\int_a^b f(x)dx$ quickly

PART 1:

<u>THEORM I</u>: Let I be an interval. Let $a \in I$. Let f be a function on I, $F(x) = \int_a^x f(t)\,dt$

<u>IF</u> f is continuous <u>THEN</u> F is differentiable and $F' = f$

- i.e. 'a function defined as an integral is an antiderivative." $\left[F'(x) = f(x) \Rightarrow \frac{d}{dx}\int_a^x f(t)\,dt = f(x)\right]$

Calculate $F'(x)$:
- sub in upper bound (x)
- muliply by the derivative of the upper bound

Ex. Let $F(x) = \int_a^x e^{-t^2}dt$. Calculate $F'(2)$ $\quad F'(x) = e^{-x^2}$, $F'(2) = e^{-4}$

Ex. Construct a function g st. $\forall x \in \mathbb{R}$ $\quad g'(x) = (1 + x^2 + x^{10})^{-1}$

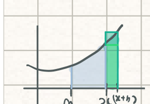
Start with defining $g(x) = \int_2^x (1+t^2+t^{10})^{-1}dt + C$ (now $g(2) = C$ $\therefore C = 5$) $\Rightarrow g(x) = \int_2^x (1+t^2+t^{10})^{-1}dx + 5$

Ex. Let $G(x) = \int_{-4}^{x^2} \frac{\sin t}{t}dt$. Calculate $G'(x)$

$F(x) = \int_{-4}^{x} \frac{\sin t}{t}dx$, $f(x) = \frac{\sin t}{t}$, $\quad G(x) = F(x^2)$ $\quad \therefore g(x) = f(x^2) = \frac{\sin x^2}{x^2}$

Ex. Let $H(x) = \int_{x^3+1}^{x^2+2x} e^{-t^2}dt$, Calculate $H'(x)$

$$H(x) = \int_{x^3+1}^{x^2+2x} e^{-t^2}dt = \int_0^{x^2+2x} e^{-t^2}dt - \int_0^{x^3-1} e^{-t^2}dt \Rightarrow H'(x) = e^{-(x^2+2x)^2} - e^{-(x^3-1)^2}$$



$\int_x^{x+h} f(t)dt \approx f(t)h$

$\frac{1}{h}\int_x^{x+h} f(t)dt \approx f(t)$

pf Fix $x \in I$ $\quad$ WTS $F'(x) = f(x)$

- $F'(x) = \lim_{h\to 0}\frac{F(x+h) - F(x)}{h} = \lim_{h\to 0}\left[\frac{1}{h}\left(\int_a^{x+h} f(t)dt - \int_a^x f(t)dt\right)\right] = \lim_{h\to 0}\left[\frac{1}{h}\left(\int_x^{x+h} f(t)dt\right)\right]$ WTS " " $= f(x)$

- Assume $h > 0$. Call $\begin{cases} M_h = \sup \text{ of } f \text{ on } [x, x+h] \\ m_h = \inf \text{ of } f \text{ on } [x, x+h] \end{cases}$, THEN $\begin{cases} m_h \cdot h \le \frac{1}{h}\int_x^{x+h} f(t)dt \le M_h \cdot h \\ m_h \le \int_x^{x+h} f(t)dt \le M_h \end{cases}$ ①

- f is continuous. By MVT, it has a max on $[x, x+h]$
- M_h = max of f on $[x, x+h]$: $\forall h > 0$, $\exists C_h \in [x, x+h]$ s.t. $M_h = f(C_h)$
- $x \le C_h \le x+h$ $\quad$ As $h \to 0$, $C_h \to x$ $\quad$ THEN: $M_h = f(C_h) \to f(x)$ because f is continuou

THEN: $\lim_{h\to0} m_h = \lim_{h\to0} M_h = f(x)$ ②

- Using ①, ②, by Squeeze Theorm: $\lim_{h\to0}\left[\frac{1}{h}\int_x^{x+h} f(t)\,dt\right] = f(x)$ ∎

PART 2:

THEORM 2 Let $a<b$, let f be a continuous function on $[a,b]$

Let G be any antiderivative G of f, THEN $\int_a^b f(x)\,dx = G(b) - G(a)$

- Notation: $G(b) - G(a) = G(x)\Big|_{x=a}^{x=b} = G(x)\Big|_a^b$

Ex. $\int_1^2 x^2\,dx = \frac{x^3}{3}\Big|_{x=1}^{x=2} = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$

Pf
- $G' = f$ WTS $\int_a^b f(x)\,dx = G(b) - G(a)$
- Define $F(x) = \int_a^x f(t)\,dt$ WTS $F(b) = G(b) - G(a)$
- Since f is continuous, by FTC 1, $F' = f$
- Due to MVT, $F' = G'$ so $F - G$ is a constant
- $\exists\, c \in \mathbb{R}$ s.t. $F(x) = G(x) + C$
- Choose $x = a$ to figure out the constant: $0 = F(a) = G(a) + C$ Thus $C = -G(a)$
- Thus, for every $x \in [a,b]$, $F(x) = G(x) - G(a)$
- In particular $F(b) = G(b) - G(a)$ ∎

Calculating Areas:
- $\int(\text{上} - \text{下})\,dx$
- $\int(\text{右} - \text{左})\,dy$

Summary: 3 Notions of Integral

1. Definite Integral $\int_a^b f(x)\,dx$	2. Indefinite Integral $\int f(x)\,dx$	3. Function defined by an Integral $F(x) = \int_a^x f(t)\,dt$
• A number	• A set of functions	• One single function
Formal def • $\overline{I}_a^b(f) = \inf\{U_P(f) \mid P \text{ is a partition of } [a,b]\}$ • $\underline{I}_a^b(f) = \sup\{L_P(f) \mid P \text{ is a partition of } [a,b]\}$ • If equal: $\int_a^b f(x)\,dx = \overline{I}_a^b(f) = \underline{I}_a^b(f)$	Formal def: • A collection of all antiderivatives of	Formal Def • If f is integrable, then this is a valid way to define a new function F • According to FCP I: f is continuous $\Rightarrow$ F is an antiderive of f
Geometric interpretation: • area under the curve		
Computation: • According to FTC 2, • Find one antiderivative G of f $\int_a^b f(x)\,d(x) = G(b) - G(a)$ Ex. $\int_1^2 x^2\,dx = \frac{x^3}{3}\Big\vert_1^2 = \frac{7}{3}$	Computation • According theorm as consequence of MVT • Find one antiderivative G of f $\int f(x)\,dx = G(x) + C$ Ex. $\int x^2\,dx = \frac{x^3}{3} + C$	Ex. $F(x) = \int_3^x e^{-t^2}\,dt \Rightarrow \begin{cases} \forall x \in \mathbb{R}\ F'(x) = e^{-x^2} \\ F(3) = 0 \end{cases}$

INTEGRATION METHODS

Integration by Substitution

$F'(x) = f(x)$

$(F \circ g)'(x) = f(g(x)) \cdot g'(x)$

THEOREM [Substitution Rule or backwards Chain Rule]

- Let g, f, F be functions, Assume F and g are differentiable

IF $\int f(x)\,dx = F(x) + C$ THEN $\int f(g(x))\,g'(x)\,dx = F(g(x)) + C$

- Assume $\int f(x)\,dx = F(x) + C$
- $\int f(g(x))\,g'(x)\,dx$ let u rep. $g(x)$, $du = g'(x)\,dx$:

$= \int f(u)\,du = F(u) + C$, sub. u back $= F(g(x)) + C$

Ex. Indefinite Integral Substitutions.

- $\int x^2 \sin(x^3+7)\,dx$, let $u = x^3 + 7$, $du = 3x^2\,dx$, then

$= \frac{1}{3}\int \sin(x^3+7)(3x^2\,dx) = \frac{1}{3}\int \sin(u)\,du = \frac{1}{3}(-\cos(u)) + C = -\frac{1}{3}\cos(x^3+7) + C$

- $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$ let $u = \cos x$ $du = -\sin x\,dx$, then

$= \int -\frac{1}{u}\,dx = -\ln|u| + C = -\ln|\cos x| + C$

- $\int x^3\sqrt{x^2+1}\,dx$ let $u = x^2 + 1$, $du = 2x\,dx$, then

$= \int \frac{1}{2}x^2\sqrt{x^2+1}\,2x\,dx = \frac{1}{2}\int (u-1)\sqrt{u}\,dx = \frac{1}{2}\int (u^{3/2} - u^{\frac{1}{2}})\,du = \frac{1}{2}\frac{u^{5/2}}{5/2} - \frac{1}{2}\frac{u^{3/2}}{3/2} + C = \frac{1}{5}(x^2+1)^{5/2} - \frac{1}{3}(x^2+1)^{3/2} + C$

THEOREM - Change of variable for definite integrals

- Let $a < b$, Let g be a function with continuous derivative on $[a,b]$, Let f be a continuous function
- Assume the range of g on $[a,b]$ is contained in the domain of f. THEN $\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)\,du$

Ex. Definite Integrals.

- $\int_1^2 x(x^2+1)^{100}dx$, let $u = x^2+1$, $du = 2x\,dx$. $[x=1 \Rightarrow u=1^2+1=2 \quad x=2 \Rightarrow u=2^2+1=5]$

  $= \frac{1}{2}\int_1^2 (x^2+1)^{100}(2x\,dx) = \frac{1}{2}\int_2^5 (u)^{100}du = \frac{1}{2}\left.\frac{u^{101}}{101}\right|_{u=2}^{u=5} = \frac{5^{101}-2^{101}}{202}$

Integration by Parts

THEOREM - Integration by parts, backwards product rule

- Let f, g be differentiable functions THEN $\int f(x)g'(x) = f(x)g(x) - \int f'(x)g(x)dx$
- let $\begin{cases} u = f(x) & du = f'(x)dx \\ v = g(x) & dv = g'(x)dx \end{cases} \Rightarrow \int u\,dv = uv - \int v\,du$

derivative du for $u=x$ is easier than the antiderivative $dv = x$, $v = \square$

Ex. - $\int xe^x dx$

  let $\begin{cases} u = x & du = dx \\ dv = e^x & v = e^x \end{cases} \Rightarrow \int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C$

- $\int x\cos x\,dx$

  let $\begin{cases} u = x & du = dx \\ dv = \cos x & v = \sin x dx \end{cases} \Rightarrow \int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C$

- $\int x^2 e^{-x}dx$

  let $\begin{cases} u = x^2 & du = 2x\,dx \\ dv = e^{-x} & v = -e^{-x} \end{cases} \Rightarrow -x^2e^{-x} + \int e^{-x}2x\,dx$, let $\begin{cases} u = 2x & du = 2dx \\ dv = e^{-x} & v = -e^{-x} \end{cases} \Rightarrow -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C$

- $\int 1\cdot\arctan x\,dx$

  parts $\begin{cases} u = \arctan x & du = \frac{1}{1+x^2}dx \\ dv = dx & v = x \end{cases} \Rightarrow x\arctan x - \int \frac{x}{1+x^2}dx$; sub $\begin{cases} t = 1+x^2 \\ dt = 2x\,dx \end{cases} \Rightarrow x\arctan x - \frac{1}{2}\ln|1+x^2| + C$

- $\int e^x \sin x\,dx = I$

  Parts $\begin{cases} u = e^x & du = e^x \\ dv = \sin x\,dx & v = -\cos x \end{cases} \Rightarrow -e^x\cos x + \int e^x\cos x\,dx$ Parts $\begin{cases} u = e^x \\ v = \cos x \end{cases} \Rightarrow -e^x\cos x + e^x\sin x - \int e^x \sin x\,dx$

  ↳ $I = -e^x\cos x + e^x\sin x - I \Rightarrow 2I = -e^x\cos x + e^x\sin x \Rightarrow I = \int e^x\sin x\,dx = \frac{1}{2}e^x(\sin x - \cos x) + C$

Integration of products of Trig. functions

- To compute $\int \sin^n x\cos^m x\,dx$ using $[\sin^2 x + \cos^2 x = 1]$
  - IF m odd then $\begin{cases} u = \sin x \\ du = \cos x\,dx \end{cases}$
  - IF n odd then $\begin{cases} u = \cos x \\ du = -\sin x\,dx \end{cases}$

Comput $\int \sec^n \tan^m dx$ using $\sec^2 x = 1 + \tan x^2$
  - IF m odd then use $\begin{cases} u = \sec x \\ du = \sec x\tan x\,dx \end{cases}$
  - IF n even then use $\begin{cases} u = \tan x \\ du = \sec^2 x\,dx \end{cases}$

- Half angle identities
  - $\sin^2 x = \frac{1-\cos(2x)}{2}$ $\qquad \cos^2 x = \frac{1+\cos(2x)}{2}$

- Integration by parts.

  $\sin^2 x + \cos^2 x = 1 \Rightarrow \int \sin^2 x\,dx + \int \cos^2 x\,dx = \int 1\,dx$

Integral of secant ($\int \sec dx$)

- Method 1

  $I = \int \sec dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}dx$ sub. $\begin{cases} u = \sec x + \tan x \\ du = (\sec x\tan x + \sec^2 x)dx \end{cases}$ $= \int \frac{du}{u} = \ln|\sec x + \tan x| + C$

- Method 2

  $I = \int \sec dx = \int \frac{1}{\cos x}dx$ | use $\int \sin^n x\cos^m x\,dx$ sub. $\begin{cases} u = \sin x \\ du = \cos x \end{cases}$

  $I = \int \frac{\cos x}{\cos^2 x}dx = \int \frac{\cos x\,dx}{1-\sin^2 x} = \int \frac{du}{1-u^2} \cdots$

$\frac{7}{5} = 1 + \frac{2}{5}$

Integration of Rational Functions

Ex. $\int \frac{(x+3)}{(x-1)(x+1)}\,dx$, know $\int \frac{1}{x-1} = \ln|x-1| + C$; $\int \frac{1}{x+1} = \ln|x+1| + C$ Goal: $\frac{(x+3)}{(x+1)(x-1)} = \frac{A}{(x+1)} + \frac{B}{(x-1)}$ numbers need to find out

└ $\left[\frac{x+3}{(x+1)(x-1)} = \frac{A(x+1)+B(x-1)}{(x+1)(x-1)} \Rightarrow x+3 = A(x+1) + B(x-1) \Rightarrow x+3 = (A+B)x + (A-B) \quad \begin{cases} 1 = A+B & A=2 \\ 3 = A-B & B=-1 \end{cases} \right]$

$= \int \left[\frac{2}{x-1} - \frac{1}{x+1} \right] dx = 2\ln|x-1| - \ln|x+1| + C = \ln\left|\frac{(x-1)^2}{x+1}\right| + C$

- $\int \frac{p(x)}{x^3-3x^2+2x}\,dx = q(x) + \frac{ax^2+bx+c}{x^3-3x^2+2x} = q(x) + \frac{ax^2+bx+c}{x(x-1)(x-2)} \to q(x) + \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2}$
  - └ Goal: degree of numerator < degree of denominator.

Ex. $\int \frac{3x+7}{x^2+4}\,dx = 3\int \frac{x}{x^2+4}\,dx \; ① + 7\int \frac{1}{x^2+4}\,dx$ sub. ① $\begin{cases} u = x^2+4 \\ du = 2x\,dx \end{cases}$ know ② $\left[\int \frac{1}{x^2+1}\,dx = \arctan x + C \right]$

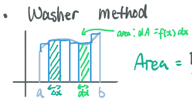
① $3\int \frac{\frac{1}{2}du}{u} = \frac{3}{2}\ln|u| + C$ ② $\int \frac{1}{4} \frac{1}{(\frac{x^2}{4}+1)}\,dx = \frac{1}{4}\int \frac{1}{(\frac{x}{2})^2+1}$ sub. $\begin{cases} u = \frac{x}{2} \\ du = \frac{1}{2}dx \end{cases}$ $= \int \frac{2\,du}{u^2+1} = \frac{1}{2}\arctan x + C$

$= \frac{3}{2}\ln|x^2+4| + \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C$

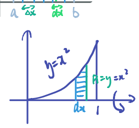
VOLUMES

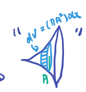
Computing Volumn as Integrals

- Washer method



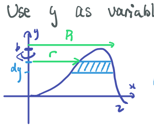
Area $= \lim_{x\to\infty} \sum_i f(x)\,\Delta x = \int_a^b dA = \int_a^b f(x)\,dx$

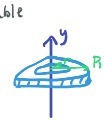




$V = \int_0^1 \pi (x^2)^2\,dx \;\ldots = \frac{\pi}{5}$

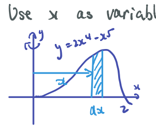
- Cylindrical shell method
  - └ Use y as variable

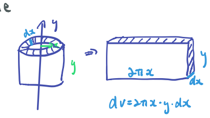




$dV = [\pi R^2 - \pi r^2]\,dy = \int_0^b [\pi R^2 - \pi r^2]\,dy$ (solve for x: $y = 2x^4 - x^5$)

  - └ Use x as variable





$V = \int_0^2 2\pi x \cdot (2x^4 - x^5)\,dx$

SEQUENCE

What is a Sequence

DI Sequence is a function with domain $\{n \in \mathbb{N} \mid n \geq n_0\} = \{n_0, n_0+1, n_0+2 \ldots\}$ for some fixed $n_0 \in \mathbb{Z}$

└ function = function with domain an interval ; (use 'x' as the variable) ; $f(x)$ = value of function f at x

└ sequence = function with domain $\mathbb{N}$. ; (use 'n' as the variable) ; a_n = value of the sequence a at n

Ex. $f(x) = \frac{3}{x+1}$ $a_n = \frac{3}{n+1} = \{\frac{3}{1}, \frac{3}{2}, \frac{3}{3} \ldots\}$

Ways to describe a sequence

1. $a_n = \frac{2^n n!}{n+1}$ (with equations)
2. $\{b_n\} = \{1, 2, 4, 8 \ldots\}$ (with the first few values.) [never fully define the sequence]
3. With words : p_n = n-th prime
4. Recurrance relation Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ by $F_0 = 1$, $F_1 = 1$, $\forall n \geq 2$, $F_n = F_{n-1} + F_{n-2}$

Notation:

- Most complete way : $\{a_n\}_{n=0}^{\infty}$
- Other way: $\{a_n\}_{n\geq 0}$, $\{a_n\}_n^{\infty}$, $\{a_n\}_n$, $\{a_n\}_0^{\infty}$, $\{a_n\}$ (a_n)

Ex. precise: $\left\{\frac{n}{k}\right\}_{k=2}^{\infty} = \left\{\frac{n}{2}, \frac{n}{3}, \frac{n}{4} \dots\right\}$

The limit of a Sequence

Ex. $\left\{\frac{n}{n+1}\right\}_{n=0}^{\infty} = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4} \dots\right\}$ $\cdot 0$ a_1 $a_2 a_3 \dots$ $\Rightarrow$ $\lim_{n\to\infty} \frac{n}{n+1} = 1$

- $\lim_{n\to\infty} a_n = L$ means

a 'tail' of the sequence must stay inside

$L-\varepsilon$ L $L+\varepsilon$

$\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ s.t.

$\forall n \in \mathbb{N}$, $n \geq n_0$ $\Rightarrow$ $L-\varepsilon < a_n < L+\varepsilon$

DF The sequence $\{a_n\}_{n=0}^{\infty}$ converges to the number $L \in \mathbb{R}$ means $= \lim_{n\to\infty} a_n = L$ or $a_n \to L$

$\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$, $n \geq n_0$ $\Rightarrow$ $L-\varepsilon < a_n < L+\varepsilon$ $|L - a_n| < \varepsilon$

(Every open interval centered at L contains a 'tail' of the sequence [all term after the first few ones])

- To determine a sequnce has a limit or not, it only matters what happens to the tail of the sequences.
- Convergent - a sequence has a limit
- Divergent - when it doesn't have a limit
  - divergent to ∞ $\{n^2\}$
  - " to $-\infty$ $\{-n\}$
  - " Oscillating $\{(-1)^n\}$

Properties of limits of Sequence

- Similar with limit of functions
  - limit law apply
  - Squeeze theorm apply
- Different with function limit
  - L'Hôpital's Rule does not apply
- When a sequence comes from a function

Let $c \in \mathbb{Z}$, Let f be a function defined on $[c, \infty)$. Define the sequence $\{a_n\}_{n=c}^{\infty}$ by $a_n = f(n)$

IF $\lim_{x\to\infty} f(x) = L$ THEN $\lim_{n\to\infty} a_n = L$; IF $\lim_{x\to\infty} f(x) = \infty$ THEN $\lim_{n\to\infty} a_n = \infty$

IF $\lim_{x\to\infty} f(x)$ DNE THEN $\lim_{n\to\infty} a_n$ may or may not exist

- Limit of Composition

THEOREM Let $\{a_n\}$ be a sequence. Let f be a function. Let $L \in \mathbb{R}$

IF $\begin{cases} a_n \to L \\ f \text{ is continuous at } L \end{cases}$ THEN $f(a_n) \to f(L)$

Ex. $\lim_{n\to\infty} e^{1/n!}$ $1/n! \to 0$; the function $f(x) = e^x$ is continuous ; $\therefore$ $e^{1/n!} \to e^0 = 1$

Mononic & Bounded Sequence

- A sequence $\{a_n\}_{n=1}^{\infty}$ is
  - increasing when $\forall n \in \mathbb{N}$, $a_n < a_{n+1}$ $(\forall n, m \in \mathbb{N},\ n < m \Rightarrow a_n < a_m)$
  - decreasing when $\forall n \in \mathbb{N}$, $a_n > a_{n+1}$ $(\forall n, m \in \mathbb{N},\ n > m \Rightarrow a_n > a_m)$
  - non-increasing when $\forall n \in \mathbb{N}$, $a_n \geq a_{n+1}$; non-decreasing $\forall n \in \mathbb{N}$, $a_n \leq a_{n+1}$
  - monotonic if its any one of the above
- A sequence is eventually decreasing when $\exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$, $n \geq n_0$ $\Rightarrow$ $a_n > a_{n+1}$

Ex. Is $\{n^3 e^{-n}\}_{n=0}^{\infty}$ monotonic

- This sequence comes from a function with domain $\mathbb{R}$.
- Let $f(x) = x^3 e^{-x}$ $f'(x) = x^2(3-x)e^{-x}$
- f is decreasing on $[3, \infty)$ $\Rightarrow$ $\{n^3 e^{-n}\}_{n=3}^{\infty}$ is decreasing
- This sequence is eventually dereasing

- Bound sequence $\{a_n\}_{n=0}^{\infty}$ is
  - bounded below $\exists A \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}, A \le a_n$
  - bounded above $\exists B \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}, a_n \le B$
  - bounded when both bounded below & above

THEOREM 1 IF a sequence is convergent THEN it is bounded

Because only tail matters

THEOREM 2 IF a sequence is (eventually) monotonic & bounded THEN it is convergent [The monotone Convergence theorm of sequence]
- " Increasing & bounded above or " " decreasing & bounded below ...

THEOREM 3 IF a sequence is (eventually) increasing & NOT bounded above THEN it's divergent to ∞

- An increasing sequence may be $\begin{cases} \text{convergent} \\ \text{divergent to } \infty \end{cases}$

Every convergent sequence is bounded

- THEOREM Let $\{a_n\}_{n=0}^{\infty}$ be a sequence
  IF $\{a_n\}_{n=0}^{\infty}$ is convergent THEN $\{a_n\}_{n=0}^{\infty}$ is bounded
- $\{a_n\}_{n=0}^{\infty}$ is bounded means there exists $A, B \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}$, $A \le a_n \le B$
- $\{a_n\}_{n=0}^{\infty}$ is convergent means there exists $L \in \mathbb{R}$ s.t. $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$, $n_0 \ge n \Rightarrow |a_n - L| < \varepsilon$
  - all terms of the sequence, except finitely many are close to L
- Structure
  0. WTS: $\exists A, B \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}$, $A \le a_n \le B$
  1. Assume sequence is convergent
  2. Take $A = ?$, $B = ?$
  3. Let $n \in \mathbb{N}$ be arbitary
  4. Proven
- Draft work
  - WTS: $\forall n \in \mathbb{N}$, $A \le a_n \le B$
  - From the def of $\lim_{n \to \infty} a_n = L$ $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}, n \ge n_0 \Rightarrow$ $\underline{|a_n - L| < \varepsilon}$ ($L - \varepsilon < a_n < L + \varepsilon$)
  - Need $A \le L - \varepsilon$ (need to fix ε first)
  - Also need $A \le a_0$, $A \le a_1$ ... $A \le a_{n_0 - 1}$ (definite set)
  - Take $A = \min\{L - \varepsilon, a_0, a_1, a_2 \dots a_{n_0 - 1}\}$ $\Rightarrow$ take $B = \max\{L + \varepsilon, a_0, a_1, a_2 \dots a_{n_0 - 1}\}$

pf **Assume the sequence $\{a_n\}_{n=0}^{\infty}$ is convergent. Let L be the limit**

choose any num. for ε

- Choose $\varepsilon = 1$ in the def of $L = \lim_{n \to \infty} a_n$, then know:
  $\exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$, $n \ge n_0 \Rightarrow L - 1 < a_n < L + 1$ (1)
- **Take** $A = \min\{L-1, a_0, a_1 \dots a_{n_0 - 1}\}$; $B = \max\{L+1, a_0, a_1 \dots a_{n_0 - 1}\}$
- WTS $\forall n \in \mathbb{N}$, $A \le a_n \le B$
- **Let** $n \in \mathbb{N}$
  1. Case one: (If $n \ge n_0$ then from (1), $L - 1 < a_n < L + 1$) $\Rightarrow$ $A \le L - 1 < a_n < L + 1 \le B$
  2. Case two: (If $n < n_0$) the from def of A&B: $A \le a_n \le B$ ∎

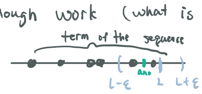
The monotone Convergence Thorem for Sequences

- THEOREM IF a sequence is (eventually) monotonic and bounded, THEN it is convergent
  - Will prove: IF a sequence is increasing & bounded above, THEN its convergent
- Structure
  0. Assume $\{a_n\}_{n=0}^{\infty}$ is increasing & bounded above
     WTS: $\exists L \in \mathbb{R}$ s.t. $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}, n \ge n_0 \Rightarrow L - \varepsilon < a_n < L + \varepsilon$
  1. Take $L = ?$
  2. Fix an arbitery $\varepsilon > 0$
  3. Take $n_0 = ?$

4. Fix an arbitary $n \in \mathbb{N}$

5. Assume $n \geq n_0$, Show $L - \varepsilon < a_n < L + \varepsilon$

- Rough work (what is L)



L = sup of the sequence
$L - \varepsilon$ is not an upper bound

$\exists n_0 \in \mathbb{N}$ s.t. $L - \varepsilon < a_{n_0} \leq L$
$\forall n \geq n_0$, $a_{n_0} \leq a_n \leq L$

pf Let $\{a_n\}_{n=0}^{\infty}$ be an increasing, and bounded above sequence

- Consider the set $A = \{a_n \mid n \in \mathbb{N}\}$, it's non-empty, & bounded above $\Rightarrow$ has a supremum
- Take $L = \sup A$, prove $L = \lim_{n\to\infty} a_n$
- Fix an arbitary $\varepsilon > 0$
- By def of supremum, $\exists n_0 \in \mathbb{N}$ s.t. $L - \varepsilon < a_{n_0}$
  - Take this value of n_0
- Fix $n \in \mathbb{N}$, Assume $n \geq n_0$ WTS $L - \varepsilon < a_n < L + \varepsilon$
  - known $L - \varepsilon < a_{n_0}$
  - Because sequence is increasing. $a_{n_0} \leq a_n$
  - By def of supremum, $a_n \leq L$
  - Thus, $L - \varepsilon < a_{n_0} \leq a_n < L < L + \varepsilon$ ■

The Big Theorem

DF Let $\{a_n\}$, $\{b_n\}$, be positive sequences: $a_n \ll b_n$ means $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$
- a_n is much smaller than b_n; b_n is much larger then a_n.

Big THEOREM: $\ln n \ll \underset{(a>0)}{n^a} \ll \underset{(c>1)}{c^n} \ll n! \ll n^n$ for every $a > 0$, $c > 1$

Ex. $\lim_{n\to\infty} \frac{e^n + 2n^{100}}{\ln n + 5e^n}$, $n^{100} \ll e^n$, $\ln n \ll 5e^n$ $\Rightarrow \lim_{n\to\infty} \frac{e^n}{5e^n} = \frac{1}{5}$

Proof of the Big Theorem

- Need to prove all 4 claims
  1. $\lim_{n\to\infty} \frac{\ln n}{n^a} = 0$ 2. $\lim_{n\to\infty} \frac{n^a}{c^n} = 0$ 3. $\lim_{n\to\infty} \frac{c^n}{n!} = 0$ 4. $\lim_{n\to\infty} \frac{n!}{n^n} = 0$
  - first two can be proven with L'Hôpital's Rule

3) Proof of $\lim_{n\to\infty} \frac{c^n}{n!} = 0$ for $c > 1$

- Fix $c > 1$ Call $p_n = \frac{c^n}{n!}$
- key observation $p_{n+1} = \frac{c^{n+1}}{(n+1)!} = \frac{c}{n+1} \cdot \frac{c^n}{n!} = \frac{c}{n+1} p_n$

1. $\{p_n\}$ is eventually decreasing $\forall n \geq c$, $p_{n+1} < p_n$
2. $\{p_n\}$ is bounded below by 0
3. By MCT $\{p_n\}$ is convergent Call $L = \lim_{n\to\infty} p_n$

$$\left[\lim_{n\to\infty} p_{n+1}\right] = \left[\lim_{n\to\infty} \frac{c}{n+1}\right] \cdot \left[\lim_{n\to\infty} p_n\right]$$

$$L = 0 \cdot L$$, Thus $L = 0$.

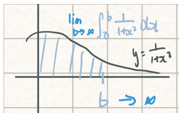
Need to know limit of sequence exist.

4) Proof of $\lim_{n\to\infty} \frac{n!}{n^n} = 0$

- $\frac{n!}{n^n} = \frac{1}{n} \cdot \frac{2}{n} \cdot \frac{3}{n} \cdots \frac{n-1}{n} \cdot \left(\frac{n}{n}\right) = \frac{1}{n} \cdot \left[\frac{2}{n} \cdot \frac{3}{n} \cdots \frac{n-1}{n} \cdot \frac{n}{n}\right] \leq \frac{1}{n}$
- Thus for all n $0 \leq \frac{n!}{n^n} \leq \frac{1}{n}$
- $\lim_{n\to\infty} 0 = \lim_{n\to\infty} \frac{1}{n} = 0$
- By Squeeze Theorm $\lim_{n\to\infty} \frac{n!}{n^n} = 0$ ■

IMPROPER INTEGRALS

Defining Improper Integrals



Ex. $\int_0^\infty \frac{1}{1+x^2}dx = \lim_{b\to\infty}\int_a^b \frac{1}{1+x^2}dx = \lim_{b\to\infty}\left[\arctan x \Big|_a^b\right] = \lim_{b\to\infty}\left[\overbrace{\arctan b}^{=\frac{\pi}{2}} - \overbrace{\arctan 0}^{=0}\right] = \frac{\pi}{2}$

Type I:

DF Let $a \in \mathbb{R}$, Let f be a continuous function on $[a,\infty)$, define the integral of f from 'a' to '∞'

$\int_a^\infty f(x)dx = \lim_{b\to\infty}\left[\int_a^b f(x)dx\right]$ Since the function is continuous, the integral always exists. assuming this limit exist

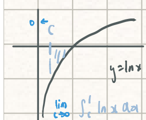
└ Convergent - when limit exist ; Divergent when limit DNE.

- The Special "P" family

$I_p = \int_1^\infty \frac{1}{x^p}dx = \lim_{b\to\infty}\left[\int_1^b \frac{1}{x^p}dx\right]$ $\begin{cases}\text{Convergent} & \text{iff } p>1\\ \text{divergent} & \text{iff } p\le 1\end{cases}$

└ p is not 1 $I_p = \lim_{b\to\infty}\left[\frac{x^{-p+1}}{-p+1}\Big|_{x=1}^{x=b}\right] = \lim_{b\to\infty}\left[\frac{b^{-p+1}}{-p+1} - \frac{1}{-p+1}\right]$ $\begin{cases}\infty & \text{if } p<1 \quad (\text{div})\\ \frac{1}{p-1} & \text{if } p>1 \quad (\text{conv})\end{cases}$

└ $p=1$ $I_p = \lim_{b\to\infty}\left[\ln|x|\Big|_1^b\right] = \ldots = \infty$ (div)



- Vertical asymptote

Ex. $\int_0^1 \ln x\,dx = \lim_{c\to 0^+}\left[\int_c^1 \ln x\,dx\right] = \lim_{c\to 0^+}\left[(1\cdot\ln 1 - 1) - (\boxed{c\ln c} - c)\right] = \lim_{c\to 0^+}\left[(1\cdot 0 - 1) - (0-0)\right] = -1$

$\lim_{c\to 0^+}[c\cdot\ln c] = \lim_{c\to 0^+}\frac{\ln c}{1/c} \overset{LH}{=} \lim_{c\to 0^+}\frac{1/c}{-1/c^2} = \lim_{c\to 0^+}[-c] =$ 0

Type II:

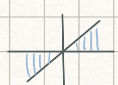
DF Let $a<b$, Let f be a continuous function on $(a,b]$, define the integral of f from a to b :

$\int_a^b f(x)dx = \lim_{c\to a^+}\left[\int_c^b f(x)dx\right]$ assume the limit exist

└ Convergent - when limit exist ; Divergent when limit DNE.

Doubly improper integrals

1. Break the domain into pieces so that the integral over each point is improper only @ one endpoint
2. Both piece are convergent ⇒ convergent as a whole
3. Any piece is divergent ⇒ divergent as a whole (not a number)



Ex. $I = \int_{-\infty}^\infty x\,dx = \int_{-\infty}^0 x\,dx + \int_0^\infty x\,dx = \left[\lim_{R\to\infty}\int_{-R}^0 x\,dx\right] + \left[\lim_{R\to\infty}\int_{-R}^0 x\,dx\right]$ ⇒ Divergent

Basic Comparison Test

- Let f be continuous on $[a,\infty)$, let $A = \int_a^\infty f(x)dx$

$A \begin{cases}\text{convergent (a number)}\\ \text{divergent} \begin{cases}\text{to } \infty\\ \text{to } -\infty\\ \text{oscillating}\end{cases}\end{cases}$

Assume $\forall x \ge a$, $f(x)>0$, then A:

$A \begin{cases}\text{convergent}\\ \text{divergent to } \infty\end{cases}$

Pf
- Call $F(b) = \int_a^b f(x)dx$
- Then $A = \int_a^\infty f(x)dx = \lim_{x\to\infty} F(b)$
- By FTC $F'(b) = f(b) > 0$ (assumption), so F is increasing
  - └ If F is bounded above, then $\lim_{b\to\infty} F(b)$ exists (Montone Convergent Theorem)
  - └ If F is is not bounded above, then $\lim_{b\to\infty} F(b) = \infty$
- Notation (only for improper integrals of positive functions)
  - $\int_a^\infty f(x)dx = \infty$ means Divergent
  - $\int_a^\infty f(x)dx < \infty$ means Convergent

The Basic Comparison Test

- Let $a \in \mathbb{R}$, Let f, g be continuous functions on $[a,\infty)$
- Assume for every $x \ge a$ $0 \le f(x) \le g(x)$
- Then $\begin{cases}1.\ \text{If } \int_a^\infty f(x)dx = \infty & \text{Then } \int_a^\infty g(x)dx = \infty\\ 2.\ \text{If } \int_a^\infty g(x)dx < \infty & \text{Then } \int_a^\infty f(x)dx < \infty\end{cases}$ (Smaller ⇒ big, big ⇒ small)

Recall
$\int_1^\infty \frac{dx}{x^p} < \infty \Leftrightarrow p>1$
$\int_1^\infty \frac{dx}{x^p} = \infty \Leftrightarrow p \le 1$

Ex. • $\int_1^\infty \frac{\sin^2 x}{x^2}dx$ for $x \ge 1$, $0 \le \frac{\sin^2 x}{x^2} \le \frac{1}{x^2}$, then $\int_1^\infty \frac{1}{x^2}dx < \infty$ } By BCT, $\int_1^\infty \frac{\sin^2 x}{x^2}dx < \infty$

- $\int_1^\infty \frac{\sin^2 x}{x}\,dx$ for $x \geq 1$, $0 \leq \frac{\sin^2 x}{x} \leq \frac{1}{x}$, while $\int_1^\infty \frac{1}{x}\,dx = \infty$ } BCT does not help
- $\int_1^\infty \frac{\ln x}{x^2}\,dx$, Big Theorm $\ln \ll x^{1/2}$ ($\lim_{x\to\infty} \frac{\ln x}{x^{1/2}} = 0$) [large value of x], $\frac{\ln x}{x^{1/2}} < 1$, $\ln x < x^{1/2}$, then $0 \leq \frac{\ln x}{x^2} < \frac{x^{1/2}}{x^2} = \frac{1}{x^{3/2}}$, $\int_1^\infty \frac{1}{x^{3/2}}\,dx < \infty$ } By BCT, $\int_1^\infty \frac{\ln x}{x^2}\,dx < \infty$

Limit Comparison Test

Limit Comparison Test

- Let $a \in \mathbb{R}$, let f, g be positive, continuous functions on $[a, \infty)$
- IF the limit $L = \lim_{x\to\infty} \frac{f(x)}{g(x)}$ exists (is a number), and $L > 0$
- THEN $\int_a^\infty f(x)\,dx$ & $\int_a^\infty g(x)\,dx$ are both convergent or both divergent

Ex. $\int_1^\infty \frac{x^2+3x}{\sqrt{x^5}+1}\,dx$, call it $f(x)$. $\underset{x\to\infty}{\sim} \frac{x^2}{x^{5/2}} = \frac{1}{x^{1/2}} = g(x)$ | $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\frac{x^2+3x}{\sqrt{x^5}+1}}{\frac{1}{x^{1/2}}} \ldots = 1$

L $\int_1^\infty g(x)\,dx = \int_1^\infty \frac{1}{x^{1/2}}\,dx = \infty \Rightarrow$ By LCT, $\int_1^\infty f(x)\,dx = \infty$

Ex. $\int_1^\infty \sin \frac{1}{x^2}\,dx$, as $x \to \infty$ $\frac{1}{x^2} \to 0$, $\lim_{t\to 0} \frac{\sin t}{t} = 1$, Call $f(x) = \sin \frac{1}{x^2}$, $g(x) = \frac{1}{x^2}$ | $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\sin \frac{1}{x^2}}{\frac{1}{x^2}} = \ldots = 1$

L $\int_1^\infty g(x)\,dx = \int_1^\infty \frac{1}{x^2}\,dx$ is conv. $\Rightarrow$ By LCT, $\int_1^\infty f(x)\,dx$ is conv.

- Proof rough work
  - L $\lim_{x\to\infty} \frac{f(x)}{g(x)}\,dx = L$ means for large x, $\frac{f(x)}{g(x)} \approx L$
  - L Use def of limit: $\forall \varepsilon > 0$, $\exists M \in \mathbb{R}$ s.t. $x \geq M \Rightarrow \boxed{\left|\frac{f(x)}{g(x)} - L\right|} < \varepsilon$ $L - \varepsilon < \frac{f(x)}{g(x)} < \varepsilon + L$ ①

    ① $\begin{cases} f(x) < (L+\varepsilon)\,g(x) \\ (L-\varepsilon)\,g(x) < f(x) \end{cases}$
  - L Use BCT in both directions
  - L Pick $\varepsilon = \frac{L}{2}$ so that $L - \varepsilon > 0$

Pf

- Since f, g are possitive, EACH $\int_a^\infty f(x)\,dx$ & $\int_a^\infty g(x)\,dx$ is convergent or ∞
- Take $\varepsilon = \frac{L}{2} > 0$ in the def of $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$: $\exists M \in \mathbb{R}$ s.t. $x \geq M \Rightarrow \left|\frac{f(x)}{g(x)} - L\right| < \frac{L}{2}$
- Then $\forall x \geq M$, $\frac{L}{2} < \frac{f(x)}{g(x)} < \frac{3L}{2}$
- Notice that $\int_a^\infty \ldots = \int_a^M \ldots + \int_M^\infty \ldots$, Thus $\int_a^\infty \ldots$ is convergent iff $\int_M^\infty$ is convergent

1. Assume $\int_M^\infty g(x)$ is convergent,

   For all $x > M$ $0 < f(x) < \frac{3L}{2} g(x)$, by BCT, $\int_M^\infty f(x)\,dx$ is also convergent
2. Assume $\int_M^\infty f(x)$ is convergent,

   For all $x > M$ $0 < \frac{L}{2} g(x) < f(x)$, by BCT, $\int_M^\infty g(x)\,dx$ is also convergent ■

- Expanded LCT
  - Let $a \in \mathbb{R}$, let f, g be positive, continuous functions on $[a, \infty)$
  - IF the limit $L = \lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$ or ∞
  - THEN …

SERIES

serious - infinit sum

$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 \cdots$

sequence - an infinte list

$\{a_n\}_{n=1}^{\infty} = a_1, a_2, a_3 \ldots$

Infinite Sums (series)

- Series - a set of infinite sums
  - L It's convergent when it equals to a number; otherwise, just leave in the sum format
- $S = \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \ldots$

  $S_1 = \frac{1}{2}$, $S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ $S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$

  $S_k = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \ldots = \sum_{n=1}^{k} \frac{1}{2^n} = 1 - \frac{1}{2^k}$

  $\therefore$ $S = \lim_{k\to\infty} S_k = 1$

Df To sum the series $\sum_{n=1}^{\infty} a_n$:

1. Construct the sequence of partial sums $\{S_k\}_{k=1}^{\infty}$ ($S_1 = a_1$, $S_2 = a_1 + a_2$ ... $S_k = a_1 + ... + a_n = \sum_{n=1}^{k} a_n$)
2. Compute the limit $\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} S_k$
   - Convergent when the limit exist ; Divergent otherwise

Ex. $S = \sum_{n=1}^{\infty} \frac{1}{n^2+n}$

method ①
- $S_1 = \frac{1}{1^2+1} = \frac{1}{2}$; $S_2 = S_1 + \frac{1}{2^2+2} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$; $S_3 = S_2 + \frac{1}{3^2+3} = \frac{2}{3} + \frac{1}{12} = \frac{3}{4}$
- Conjecture $\forall k \geq 1$, $S_k = \frac{k}{k+1}$; need pf (By induction)

method ②
- $\sum_{n=1}^{k} \frac{1}{n(n+1)} = \sum_{n=1}^{k} \left[\frac{1}{n} - \frac{1}{n+1}\right] = \left[1 - \frac{1}{2}\right] + \left[\frac{1}{2} - \frac{1}{3}\right] + \left[\frac{1}{3} - \frac{1}{4}\right] + \ldots \left[\frac{1}{k} - \frac{1}{k+1}\right] = \frac{k}{k+1}$
- $\sum_{n=1}^{\infty} \frac{1}{n^2+n} = \lim_{k\to\infty} S_k = \lim_{k\to\infty} \frac{k}{k+1} = 1$ (convergent)

Ex. $S = \sum_{n=1}^{\infty} 1$
- $S = \lim_{k\to\infty} S_k = \lim_{k\to\infty} \left[\sum_{n=1}^{k} 1\right] = \lim_{k\to\infty} k = \infty$ (Divergent)

Ex. $\sum_{n=0}^{\infty} (-1)^n$
- $S = \lim_{k\to\infty} S_k$, but $S_k \begin{cases} 0 & \text{if } k \text{ odd} \\ 1 & \text{if } k \text{ even} \end{cases}$; $\lim_{k\to\infty} S_k$ DNE (Divergent)

Geometric Series

- Let $x \in \mathbb{R}$, $S = \sum_{n=0}^{\infty} x^n = \lim_{k\to\infty} S_k$ where $S_k = \sum_{n=0}^{k} x^n$
  - $S_k = 1 + x + x^2 + \ldots + x^k$
  - $x \cdot S_k = x + x^2 + x^3 + \ldots x^{k+1}$

  $\Big\}$ $S_k - xS_k = 1 - x^{k+1} \Rightarrow S_k = \frac{1-x^{k+1}}{1-x}$ (if $x \neq 1$) → $\sum_{n=0}^{\infty} 1 = \infty$, so if $x=1$ series will be divergent
  - $\lim_{k\to\infty}\left[\frac{1-x^{k+1}}{1-x}\right] = \frac{1-(\lim_{k\to\infty} x^{k+1})}{1-x}$ $\left(\lim_{k\to\infty} x^{k+1} \begin{cases} 0 & \text{if } -1<x<1 \\ \infty \text{ or DNE} & \text{if otherwise} \end{cases}\right)$

$\therefore$ $S = \lim_{k\to\infty} \sum_{n=0}^{\infty} x^n \begin{cases} \frac{1}{1-x} & \text{if } -1<x<1 \\ \text{Divergent} & \text{if otherwise} \end{cases}$

Linearity of Series

- $\sum_{n=0}^{\infty} (a_n + b_n) = \sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n$

  IF $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty}$ are both convergent THEN $\sum_{n=0}^{\infty} (a_n + b_n)$ is also convergent AND $\sum_{n=0}^{\infty} (a_n + b_n) = \sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n$
- $\sum_{n=0}^{\infty} (c\, a_n) = c \sum_{n=0}^{\infty} a_n$

  Let $c \in \mathbb{R}$ IF $\sum_{n=0}^{\infty} a_n$ is convergent THEN $\sum_{n=0}^{\infty} (c\, a_n)$ is also convergent AND $\sum_{n=0}^{\infty} (c\, a_n) = c \sum_{n=0}^{\infty} a_n$

Pf THEOREM I
- $\sum_{n=0}^{\infty} a_n = \lim_{k\to\infty} S_k$ where $S_k = \sum_{n=0}^{k} a_n$
- $\sum_{n=0}^{\infty} b_n = \lim_{k\to\infty} T_k$ where $T_k = \sum_{n=0}^{k} b_n$
- $\sum_{n=0}^{\infty} (a_n + b_n) = \lim_{k\to\infty} R_k$ where $R_k = \sum_{n=0}^{k} (a_n + b_n)$
- By properties of finite sums $R_k = S_k + T_k$
- By hypothesis, using limit laws $\lim_{k\to\infty} R_k = \lim_{k\to\infty} S_k + \lim_{k\to\infty} T_k$

Structure:

① Write series as limit of infinite sums

② Use properties of sequences

③ Apply to series with help of the hypothesis

The tail of Series

THEOREM I

$\sum_{n=0}^{\infty} a_n$ is convergent $\Leftrightarrow$ $\sum_{n=1}^{\infty} a_n$ is convergent

$\left[\sum_{n=0}^{\infty} a_n\right] = a_0 + \left[\sum_{n=1}^{\infty} a_n\right]$

- Notation $\sum^{\infty} a_n$ is convergent / divergent is correct
- Typical theorem
  - IF for all $n \in \mathbb{N}$ [something about a_n] THEN the series $\sum^{\infty} a_n$ is convergent
  - Due to THEOREM I, ∴ IF $\exists n_0 \in \mathbb{N}$ st. $\forall n \in \mathbb{N}$ $n > n_0 \Rightarrow$ [something about a_n] THEN $\sum a_n$ is convergent

$\exists n_0 \in \mathbb{N}$, $\forall n \in \mathbb{N}$ $n > n_0$ indicates the tail

Necessary Condition Test for Convergence

Recall :

series conv. $\Rightarrow$ sequence conv.

THEOREM [N.C.T.]

IF the series $\sum_{n=0}^{\infty} a_n$ is convergent **THEN** $\lim_{n\to\infty} a_n = 0$

- IF $\lim_{n\to\infty} a_n \neq 0$ THEN $\sum_{n=0}^{\infty} a_n$ is divergent (contrapositive)

Pf Assume the series $\sum_{n=0}^{\infty} a_n$ is convergent WTS $\lim_{n\to\infty} a_n = 0$

- This means that $\left[S = \lim_{k\to\infty} S_k \text{ where } S_k = \sum_{n=0}^{k} a_n \right]$ exist
- Notice $\forall n \geq 1$ $a_n = S_n - S_{n-1}$
- Use limit laws : $\lim_{n\to\infty} a_n = \left[\lim_{n\to\infty} S_n\right] - \left[\lim_{n\to\infty} S_{n-1}\right] = S - S = 0$ — exist and are equal

Positive Series

DF **Positive series** : $\forall n \in \mathbb{N},\ a_n > 0$ for $\sum_{n=0}^{\infty} a_n$; Negative Series $\forall n \in \mathbb{N},\ a_n < 0$ for $\sum_{n=0}^{\infty} a_n$

Non-positive series : $\forall n \in \mathbb{N},\ a_n \leq 0$ " " Non-negative series $\forall n \in \mathbb{N},\ a_n \geq 0$ " "

- Assume a series is positive, then the series would be increasing with $S_{n+1} - S_n = a_{n+1}$

  L Using MCT $\begin{cases} \text{An increasing \& bounded sequence is convergent} & \left(\sum_{n=0}^{\infty} a_n < \infty\right) \\ \text{An increasing \& unbounded sequence is divergent} & \left(\sum_{n=0}^{\infty} a_n = \infty\right) \end{cases}$

The Integral Test

- Series : $\sum_{n=1}^{\infty} f(n) = \lim_{k\to\infty} \sum_{n=1}^{k} f(n)$; integral : $\int_1^{\infty} f(x)dx = \lim_{b\to\infty} \int_1^{b} f(x)dx$
- f is positive - thus $\sum_{n=1}^{\infty} f(n)$ & $\int_1^{\infty} f(x)dx$ are convergent or ∞ AND f is decreasing

THEOREM

- Let $a \in \mathbb{N}$. Let f be continuous, positive, decreasing function on $[a, \infty)$, THEN

$$\int_a^{\infty} f(x)dx \text{ is convergent} \iff \sum_{n}^{\infty} f(n) \text{ is convergent}$$

$$\left(\sum_{n=2}^{\infty} f(n) \leq \int_1^{\infty} f(x)dx \leq \sum_{n=0}^{\infty} f(n) \right)$$

- Notation $\int_a^{\infty} f(x)dx \sim \sum_{n}^{\infty} f(n)$ (same convergency)
- Let $f(x) = \frac{1}{x^p}$ for $x \geq 1$, f is continuous, positive, decreasing

  L By Integral Test $\sum_{n=1}^{\infty} \frac{1}{n^p} \sim \int_1^{\infty} \frac{1}{x^p} dx$. Know this integral is convergent iff $p > 1$

  L Thus $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent iff $p > 1$
- Let $f(x) = \frac{1}{x \ln x}$. It's continuous, positive, decreasing

  L By Integral Test $\sum_{n=2}^{\infty} \frac{1}{x \ln x} \sim \int_2^{\infty} \frac{1}{x \ln x} dx$. Know $\lim_{b\to\infty} \int_2^{b} \frac{1}{x \ln x} = \lim_{b\to\infty} \left[\ln(\ln x) \Big|_2^b \right] = \lim_{b\to\infty} \left[\ln(\ln(b)) - \ln(\ln(2)) \right] = \infty$

  L Thus $\sum_{n=2}^{\infty} \frac{1}{x \ln x}$ is divergent to ∞

Comparison Test for series

- Positive series may only be convergent or divergent to ∞
- To prove a positive series is convergent , only need to show it does not approach ∞

BCT

THEOREM Let $\sum_{n}^{\infty} a_n$ & $\sum_{n}^{\infty} b_n$ be 2 series , Assume $\forall n \in \mathbb{N}$, $0 \leq a_n \leq b_n$

- **IF** $\sum_{n}^{\infty} a_n = \infty$ **THEN** $\sum_{n}^{\infty} b_n = \infty$

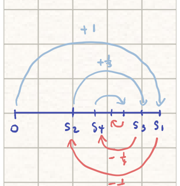
LCT

THEOREM Let $\sum_{n}^{\infty} a_n$ & $\sum_{n}^{\infty} b_n$ be 2 positive series.

- **IF** $L = \lim_{n\to\infty} \frac{a_n}{b_n}$ **THEN** $\sum_{n}^{\infty} a_n$ & $\sum_{n}^{\infty} b_n$ are both convergent or divergent

Alternating Series

DF A series $\sum_{n}^{\infty} a_n$ is alternating when $\forall n,\ a_n a_{n+1} < 0$ (the terms "alternate" between (+) & (-) values)



$S_2 < S_4 < S_6 \ldots < S_5 < S_3 < S_1$

Ex. $S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \ldots$ is an alternating series

Pf
- WTS (inequalities from diagram on the left)
- $\{S_{2n}\}_n$ is increasing and bounded above (by S_1) , Call $A = \lim_{n\to\infty} S_{2n}$
- $\{S_{2n+1}\}_n$ is decreasing and bounded above (by S_2) , Call $B = \lim_{n\to\infty} S_{2n+1}$
- $\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} \frac{1}{2n+1} \Rightarrow B = A + 0$

LAMMA Let $\{C_n\}_n^\infty$ be a sequence

- IF the sequence of even and odd terms $\{C_{2n}\}_n^\infty$ and $\{C_{2n+1}\}_n^\infty$ are convergent to the same limit
- THEN the full sequence $\{C_n\}_n^\infty$ is also convergent to the same limit

THEOREM [for series in the form : $\sum_n^\infty (-1)^n b_n$ OR $\sum_n^\infty (-1)^{n+1} b_n$

IF
① $\forall n,\ b_n > 0$
② the sequence $\{b_n\}_n^\infty$ is decreasing
③ $\lim_{b \to \infty} b_n = 0$

THEN the series is convergent

- Estimating values