LOGIC, NOTION, DEFINITION, AND PROOFS

Sets and notation

- a set is a collection of distinct elements (spimplied definition – Russell's paradox)

  name ($A = \{\text{even integers}\}$ ⟶ infinite # of elements
  $B = \{4, 5, 6\}$ ⟶ complete list of elements

- Notation

$\in$	is an element of
$\notin$	isn't an element of
$\subseteq$	is a subset of & may equal to
$\subsetneq$	is a subset of & may not equal to
$\cup$	union of
$\cap$	intersection of
$\emptyset$	null (empty) set

$\mathbb{N}$	naturals set	$= \{0, 1, 2, 3 \dots\}$
$\mathbb{Z}$	integerals set	$= \{\dots -3, -2, -1, 0, 1, 2, 3 \dots\}$
$\mathbb{Q}$	rationals set	$= \{$quotients of integers (fractions)$\}$
$\mathbb{R}$	real # set	$= \{$#s with a decimal expansion$\}$

Summary: $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$

Set-building notation

$$\overset{\text{name}}{A} = \overbrace{\{x \in \mathbb{Z} \overset{\text{such that}}{:} x^2 < 6\}}^{\text{description}}$$

$$A = \{\underbrace{x \in \mathbb{Z}}_{\text{take the element from}} \mid \underbrace{x^2 < 6}_{\text{extra constraints}}\}$$

Ex.

$A = \{-2, -1, 0, 1, 2\}$
$B = \{2x : x \in A\}$
$B = \{-4, -2, 0, 2, 4\}$

- Interval

  let $a, b \in \mathbb{R}$

  1. $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$
  2. $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$
  3. $(-\infty, b] = \{x \in \mathbb{R} \mid x \leq b\}$

Quantifiers

- $\forall$ – for all / every
- $\exists$ – there exists / is (at least one)

$\forall\ x \in \mathbb{R},\ x^2 \geq 0$ True ; $\forall\ x \in \mathbb{Z},\ x > \pi$ False

$\exists\ x \in \mathbb{R}$ (such that) $x^2 = 5$ True (Ex. $x = -\sqrt{5}$) ; $\exists\ x \in \mathbb{R}\ x^2 = -1$ False

$x^2 = 5$ meaningless (needs context)

- Double Quantifiers
  - Order of quantifers matters

1. $\forall\ x \in \mathbb{Z},\ \exists\ y \in \mathbb{Z}$ s.t. $\boxed{x < y}$ [different y for different x] "Every integers is smaller than some other one" True

2. $\exists\ y \in \mathbb{Z}$, s.t. $\boxed{\forall\ x \in \mathbb{Z}\ x < y}$ [same y for different x] "There's an integer (y) greater than all integers" False

- Simple proofs
  - Thm 1 let $A = \{2, 3, 4\}$ $\forall\ x \in A,\ x > 0$ pf: $2 > 0$ ✓, $3 > 0$ ✓, $4 > 0$ ✓ ■ (end of proof)
  - Thm 2 $\forall\ x \in \mathbb{Z},\ x > 0$ [prof with negation] : $\exists\ x \in \mathbb{Z}$, st. $x \leq 0$ pf: Take $x = -6$, $-6 \in \mathbb{Z}$ ✓, $-6 \leq 0$ ■ ✓
  - Thm 3. $\forall\ x \in \mathbb{Z},\ \exists\ y \in \mathbb{Z}$ s.t. $x < y$ pf.

    ① Fix[①] an arbitary $x \in \mathbb{Z}$ ⟶ • Let $x \in \mathbb{Z}$
    ② Choose y in term of x ⟶ • Take $y = x + 1$
    ③ Verify $y \in \mathbb{Z}$, $y > x$ ⟶ • $\because x \in \mathbb{Z}$, $y \in \mathbb{Z}$ ✓, $y > x$ ✓ ■

① In the Thm's statement, x is a quantified variable, ∴ it must be given an intrinsic value before using it as a independent variable in step ②.

- Empty Set
  - For $\emptyset$ sets, statements start with "$\forall$" is true, statements start with "$\exists$" is false.

1) $\forall\ x \in \emptyset$. $\boxed{x > 0}$ * (don't matter) True

   To prove ① is T, need to verify all elements in $\emptyset$ satisfy ⊛ ✓

   Can check every element because there's no element

2) $\exists\ x \in \emptyset$, s.t. $x > 0$ False

   To prove ① is F, need to verify one elements in $\emptyset$ satisfy ⊛ ✗

   cannot check one because there's none.

- To prove if an statement is T, needs to varify all element in the set satisfy that condition
  - Go through one-by-one, or use variables to prove for all
- To prove an statement is false, needs to verify (find) at least one element that does not satisfy the condition

Conditionals

- IF P, THEN Q ; $P \Rightarrow Q$; P implies Q
  - └ Whenever P is true, then Q must be true as well ; Whenever P is false, we don't care
- $\boxed{P \Rightarrow Q}$ & $\boxed{\text{not } Q \Rightarrow \text{not } P}$ is the same
- If $P \Rightarrow Q$ & $Q \Rightarrow P$ at the same time ; then $P \Leftrightarrow Q$ "if and only if" = "IFF" (P & Q are equivalent)
  - └ must be both true or both false

Ex. Let $A \subseteq \mathbb{R}$. assume $x \in A \Rightarrow x > 0$

- $x \notin A$ No. conclution
- $x > 0$ No conclution (x is still allowed to be positive , but not an element of A.
- $x \leq 0 \Rightarrow x \notin A$

Ex. True or False : $\overset{F}{0 > 1} \Rightarrow \overset{?}{\overbrace{1035\,74\,289 \text{ is prime}}}$

└ The full statement is true because the "if" is false , so it must be true because the condition can be anything

Negation of Conditionals

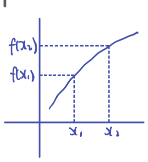
- Negation of if then, is never an if then
- Many conditions have a implicit (hidden) quantifier (eg. $x \in \mathbb{R}$ in the case below), when writing negation, must find out that quantifier, and write it explicitly

Ex. Negate : Let $A \subseteq \mathbb{R}$, $\boxed{\text{If } x \in A, \text{ then } x > 0}$ ⊛

└ ⊛ means $\underset{\text{└hidden}}{\forall x \in \mathbb{R}}$ $\begin{cases} (x \in A \text{ and } x > 0) \text{ OR} \\ (x \notin A \text{ and } x > 0) \text{ OR} \\ (x \notin A \text{ and } x \leq 0) \end{cases}$

└ $\exists x \in \mathbb{R}$ s.t. ($x \in A$ & $x \leq 0$)

Rigorous, mathematical definition



Def : A function f is increasing on :
an interval I when
$\forall x_1, x_2 \in I$,
$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

① (introduce f(x))
②, ⓑ (define domain , "for some" or "for all")
ⓒ (set condition)
ⓓ (Compare 2 points of the x-interval)

Prof of Definition

- Prove $f(x) = 3x + 7$ is increasing on $\mathbb{R}$ directly from the definition

WTS: $\underline{\forall x_1, x_2 \in \mathbb{R}}$, $\underline{x_1 < x_2 \Rightarrow f(x_1) < f(x_2)}$

① Call the definition (WTS) → write it explicitly (reader know)

Pf:
- Let $x_1, x_2 \in \mathbb{R}$
- Assume $x_1 < x_2$
  $3x_1 < 3x_2$
  $f(x_1) = (3x_1 + 7) < (3x_2 + 7) = f(x_2)$
- Shown $f(x_1) < f(x_2)$ ∎

② fix an arbitrary values for x (generic)

③ Assume the "if", prove the "then"

- Bad prof:

Ex. "Thm" : $\sqrt{xy} \leq \frac{x+y}{2}$

"Prof:" $xy \leq \left(\frac{x+y}{2}\right)^2$ ①
$4xy \leq x^2 + 2xy + y^2$
$0 \leq x^2 - 2xy + y^2$
$0 \leq (x-y)^2$

Error: ① started with assuming WTS
└ forbidden
② What's x & y ? (not defined)
└ didn't show x & y need to be (+)

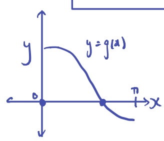
Fix:
Pf:
- Since a square is always non-negative
  $0 \leq (x-y)^2 = x^2 - 2xy + y^2$
  $xy \leq \left(\frac{x+y}{2}\right)^2$
- Since both sides are non-negative :
  $\sqrt{xy} \leq \sqrt{\left(\frac{x+y}{2}\right)^2} \rightarrow \left|\frac{x+y}{2}\right|$
  $\sqrt{xy} \leq \frac{x+y}{2}$, because $x, y \geq 0$ ∎

Proof a false case from definition

- Prove $g(x) = \cos x$ is **not** increasing on $[0, \pi]$ directly from the definition
  - Definition of increasin :
    No($[\forall x_1, x_2 \in I,\ x_1 < x_2 \Rightarrow g(x_1) < g(x_2)]$)
  - WTS: $\boxed{\exists x_1, x_2 \in I \text{ s.t. } x_1 < x_2 \Rightarrow g(x_1) \geq g(x_2)}$

① Find negation (if = T ; then = F)

② Present WTS.



Pf: Take $\begin{cases} x_1 = 0 \\ x_2 = \frac{\pi}{2} \end{cases}$

$0 < \frac{\pi}{2}$ ✓

$g(0) = 1 \geq 0 = g(\frac{\pi}{2})$ ✓ ∎

③ Take an arbitory value x (choose any, easy to show)

④ Verify if ⇒ then

Proof of a theorem

- Prove that the sum of increasing functions is increasing
  - Let f, g be functions on the interval I
    let $h = f + g$ } set-up
  - IF: f, g increasing on I } hypothesis (assume)
  - THEN: h is increasing on I } conclusion (WTS)

① Re-write formal theorem

Pf: WTS: $\forall x_1, x_2 \in I,\ x_1 < x_2 \Rightarrow h(x_1) < h(x_2)$

- let $x_1, x_2 \in I$, assume $x_1 < x_2$
- Since f is increasing on I, $f(x_1) < f(x_2)$
  Since g is increasing on I, $g(x_1) < g(x_2)$
- Add both inequalities
  $f(x_1) + g(x_2) < f(x_2) + g(x_2)$
  $h(x_1) < h(x_2)$ ∎

② Fix value for x

③ Use assuptions

④ Manipulate assumptions

Works well with theorems that depends on a NATURAL #

Proof by induction

- Prove that $\forall n \geq 4,\ n! > 2^n \rightarrow S_n$
  - prove S_4
  - prove $\forall n \geq 4,\ S_n \Rightarrow S_{n+1}$ $\begin{cases} S_4 \Rightarrow S_5 \\ S_5 \Rightarrow S_6 \\ S_6 \Rightarrow S_7 \end{cases}$

① Figure out base case

② Induction step (does not prove anything by itself need base case)
- Induction does not prove the statement,
- It proves that the statement for n **IMPLIES** S_n

Pf: (by induction on n)

Base Case ($n=4$), WTS $4! > 2^4$

$4! = 24 > 2^4 = 16$ ✓

Induction step
- let $n \geq 4$
- Assume $n! > 2^n$, WTS $(n+1)! > 2^{n+1}$ [induction hypothesis]

$\boxed{(n+1)!} = (n+1)n! > (n+1) \cdot 2^n > 2 \cdot 2^n = \boxed{2^{n+1}}$

↳ by induction hypothesis ↳ ∵ $n \geq 4$, then $n+1 > 2$ ∎

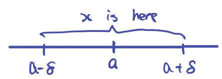
LIMITS AND CONTINUITY

Tips for solving absolute values:
- break into the 2 cases

Absolut Values & Distances

- Absolute value algebric definition : For every $x \in \mathbb{R}$, $|x| \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$
- Geometric interpretation: $|x|$ is the distance between x & 0 (e.g. $|x-a| \Rightarrow$ the distance between x & a)
- Common Properties : $|xy| = |x||y|$; $|x+y| \leq |x| + |y|$

Ex. Equivalent expression: $|x-a| < \delta$
  └ The distance between x and a is smaller than δ
  └ $|x-a| < \delta$
   $-\delta < x-a < \delta$
  $a-\delta < x < a+\delta$

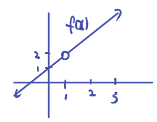


Ex. $0 \overset{①}{<} |x-a| \overset{②}{<} \delta$
① $0 < |x-a|$, $x < a$ or $x > a \Rightarrow x \neq a$
② $|x-a| < \delta$, $a-\delta < x < a+\delta$
∴ $0 < |x-a| < \delta$ is:
 $a-\delta < x < a+\delta$, $x \neq a$.
 $(a-\delta, a) \cup (a, a+\delta)$

The idea of limit

- what is a limit [$\lim_{x \to a} f(x) = L$] * L has to be in $\mathbb{R}$.

Ex. $\begin{cases} f(x) = \dfrac{x^2-1}{x-1} , & f(1) = \text{undefined} \\ g(x) = x+1 , & g(1) = 2 \end{cases}$ > $f(x) = g(x)$ whenever $x \neq 1$



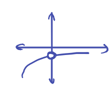
$\Rightarrow \lim_{x \to 1} f(x) = 2$.

- Idea of limit: If x is very close to 1, but not 1, then $f(x)$ is very close to 2

Ex. $f(x) = \dfrac{1-\sqrt{x+1}}{x}$, $f(0)$ is undefined ,

x	1	0.1	0.01	0.001
y	-0.414	-0.488	-0.498	-0.499

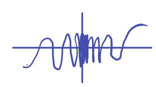
, $\Rightarrow \lim_{x \to 0} f(x) = 0.5$



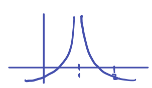
- When limit Do not exist (DNE)

Ex. $h(x) = \sin \frac{\pi}{2x}$, $h(0)$ is undefined

x	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$
y	1	0	-1	0	1	0	-1



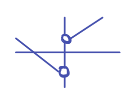
Ex. $F(x) = \dfrac{1}{(x-1)^2}$, $F(1)$ is undefined



$\lim_{x \to 1} F(x) = \infty$ $\Rightarrow$ the limit does not exist

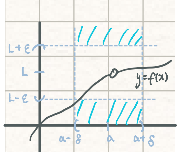
- Side limits

Ex. $G(x) = \dfrac{x^2+x}{|x|}$, $G(0)$ is undefined, $\begin{cases} x > 0, & |x| = x & G(x) = \dfrac{x(x+1)}{x} = x+1 \\ x < 0, & |x| = -x & G(x) = \dfrac{x(x+1)}{-x} = -x-1 \end{cases}$



└ $\lim_{x \to 0^+} G(x) = 1$, $\lim_{x \to 0^-} G(x) = -1$, $\lim_{x \to 0} G(x)$ DNE.

The formal definition of limit



- x is close to a $\Rightarrow$ the distance between x & a is small $\Rightarrow$ $|x-a|$ is small $\Rightarrow$ $|x-a| < \delta$
  └ add $x \neq a$, $\Rightarrow$ $0 < |x-a| < \delta$
- $f(x)$ is close to $L \Rightarrow$ $|f(x) - L| < \varepsilon$

∴ $0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

- (ε is very very small) → $\forall \varepsilon > 0, \exists \delta > 0$ + implicity $\forall x \in \mathbb{R}$

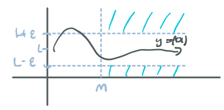
Df
- let a, L be real numbers
- let f be a function defined, at least on the interval centered @ a, except maybe @ a.

$\lim_{x \to a} f(x) = L$, when $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $(\forall x \in \mathbb{R})$ $0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

Limit at ∞

Ex. $\lim_{x\to\infty} \frac{x+1}{x} = \lim_{x\to\infty} 1+\frac{1}{x}$, as $x\to\infty$, $\frac{1}{x}\approx 0$ $\therefore \lim_{x\to\infty} 1+\frac{1}{x} = 1$

$\lim_{x\to\infty} f(x) = L$ $\quad \underbrace{x \text{ large}}_{x>M} \Rightarrow \underbrace{f(x) \text{ close to } L}_{|f(x)-L|<\varepsilon}$



(ε needs to be small)

PF Let $L\in\mathbb{R}$

Let f be a function defined, at least, on the interval of the form (p,∞) for some $p\in\mathbb{R}$

$\lim_{x\to\infty} f(x) = L$ when $\forall\varepsilon>0, \exists M\in\mathbb{R}$ s.t. $x>M \Rightarrow |f(x)-L|<\varepsilon$

Prove a function has a limit from definition

Ex. $\lim_{x\to 3} (2x+1) = 7$

Definition

$\forall\varepsilon>0, \exists\delta>0$ s.t. $0<|x-3|<\delta \Rightarrow |(2x+1)-7|<\varepsilon$

1. let (or fix) $\varepsilon>0$
2. Take (or set) $\delta = \frac{\varepsilon}{2}$

Draft work:

want to end $|(2x+1)-7| = |2x-6| = 2|x-3| < 2\delta$

Take: $2\delta=\varepsilon$ or $\delta=\frac{\varepsilon}{2}$ ($\delta\le\frac{\varepsilon}{2}$ will work)

3. Let $x\in\mathbb{R}$. Assume $0<|x-3|<\delta$

then $|(2x+1)-7| = |2x-6| = 2|x-3| < 2\delta = \varepsilon$

4. Concluded $|(2x+1)-7|<\varepsilon$ ∎

Ex. $\lim_{x\to 4}(x^2+1) = 17$

WTS: $\forall\varepsilon>0, \exists\delta>0$ s.t. $0<|x-4|<\delta \Rightarrow |(x^2+1)-17|<\varepsilon$

Draft work (find $\delta=?$)

$|(x^2+1)-17| = |x^2-16| = |x+4||x-4| < |x+4|\delta$ (don't use $\delta=\frac{\varepsilon}{|x+4|}$ δ don't depend on x)

- If make $|x+4|<C$ for some C. then $\delta\le\frac{\varepsilon}{C}$ will work. } make sure such δ exist.
- Choose $\delta\le 1$ (put constrains on δ)

  then $|x-4|<1$, $3<x<5 \Rightarrow 7<x+4<9 \Rightarrow |x+4|<9$.
- So $\delta\le 1$ & $\delta\le\frac{\varepsilon}{9}$ at the same time, take $\delta=\min(1,\frac{\varepsilon}{9})$

pf:

- let $\varepsilon>0$
- Take $\delta=\min\{1,\frac{\varepsilon}{9}\}$, therefore $\delta\le 1$ and $\delta\le\frac{\varepsilon}{9}$
- Let $x\in\mathbb{R}$, Assume $0<|x-4|<\delta$, this implies:

  $|x-4|<\frac{\varepsilon}{9}$ & $|x-4|<1$ hence: $3<x<5$ and $7<x+4<9$,

  Thus $|(x^2+1)-17| = |x^2-16| = |x+4||x+4| < \frac{\varepsilon}{9}(9) = \varepsilon$
- Proven $|(x^2+1)-17|<\varepsilon$ as needed ∎

Ex. $h(x)=\cos(\pi x)$ $\quad \lim_{x\to\infty} h(x)$ does not exist

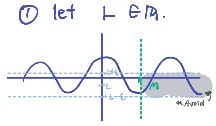
- If x is even, $h(x)=1$; if x is odd, $h(x)=-1$

Definition: $\lim_{x\to\infty} h(x)=L$: $\forall\varepsilon>0, \exists M\in\mathbb{R}$ s.t. $(\forall x\in\mathbb{R})$ $x>M \Rightarrow |h(x)-L|<\varepsilon$

- Negate: $\lim_{x\to\infty} h(x)\ne L$ means: $\exists\varepsilon>0$ s.t. $\forall M\in\mathbb{R}$ s.t. $\exists x\in\mathbb{R}$, $x>M \Rightarrow |h(x)-L|\ge\varepsilon$
- $\lim_{x\to\infty} h(x)$ DNE means $\forall L\in\mathbb{R}$, $\lim_{x\to\infty} h(x)\ne L$

$\therefore$ WTS: $\forall L\in\mathbb{R}, \exists\varepsilon>0$ s.t. $\forall M\in\mathbb{R}, \exists x\in\mathbb{R}$ s.t. $x>M$ and $|h(x)-L|\ge\varepsilon$

① let $L\in\mathbb{R}$. ② Take $\varepsilon=???$ ③ Let $M\in\mathbb{R}$ ④ Take $x=???$ } Draft work.



- Pick $\varepsilon\le 1$. Then 1 or $-1\notin(L-\varepsilon, L+\varepsilon)$
- Pick $x\in\mathbb{Z}$, Then $h(x)=1$ or -1

Pf
- Let $L \in \mathbb{R}$, Take $\varepsilon = \frac{1}{2}$, Let $M \in \mathbb{R}$
- At least one of the following must be true
  - Case A $1 \notin (L-\varepsilon, L+\varepsilon)$ ⇒ Choose any $x \in \mathbb{Z}$, even satisfying $x > M$ ⇒ $h(x) = 1$
  - Case B $-1 \notin (L-\varepsilon, L+\varepsilon)$ ⇒ Choose any $x \in \mathbb{Z}$, odd satisfying $x > M$ ⇒ $h(x) = -1$
- Either way, it satisfies $x > M$ and $|h(x) - L| \geq \varepsilon$

Limit Laws

- Prove basic Limits → Limit laws only apply if the initial limit exist
  - $\lim_{x \to a} x = a$ $\lim_{x \to a} c = c$
- Prove limit Laws

  Assume $\lim_{x \to a} f(x) = L$ & $\lim_{x \to a} g(x) = M$,

  then $\lim_{x \to a} [f(x) + g(x)] = L + M$ $\lim_{x \to a} [f(x) g(x)] = LM$ $\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{L}{M}$, assume $M \neq 0$

- **Sums**
  - let $a, L, M \in \mathbb{R}$; let f & g be a function defined at least on a interval centered at a, except maybe @ a.
  - IF $\lim_{x \to a} f(x) = L$ & $\lim_{x \to a} g(x) = M$ THEN $\lim_{x \to a} [f(x) + g(x)] = L + M$
  - Call $h(x) = f(x) + g(x)$

⇒ WTS: $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < |x-a| < \delta$ ⇒ $|h(x) - (L+M)| < \varepsilon$

$|h(x) - (L+M)| = |f(x) + g(x) - L - M| = |[f(x) - L] + [g(x) - M]| \leq \boxed{|f(x) - L| + |g(x) - M|}$

- make $|f(x) - L| < \frac{\varepsilon}{2}$, (we know $\lim_{x \to a} f(x) = L$
- $\exists \delta_1 > 0$ s.t. $0 < |x-a| < \delta_1$ ⇒ $|f(x) - L| < \frac{\varepsilon}{2}$; $\exists \delta_2 > 0$ s.t. $0 < |x-a| < \delta_2$ ⇒ $|g(x) - M| < \frac{\varepsilon}{2}$
- Take $\delta = \min\{\delta_1, \delta_2\}$

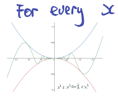
Assume (alredy know)
$\lim_{x \to a} f(x) = L$ $\lim_{x \to a} g(x) = M$
- Can choose any ε
- There exist a δ that works

Pf:
- let $\varepsilon > 0$,
- use $\frac{\varepsilon}{2}$ in the definition of $\lim_{x \to a} f(x) = L$, $\exists \delta_1 > 0$. st. $0 < |x-a| < \delta_1$ ⇒ $|f(x) - L| < \frac{\varepsilon}{2}$
- use $\frac{\varepsilon}{2}$ in the definition of $\lim_{x \to a} g(x) = M$, $\exists \delta_2 > 0$. st. $0 < |x-a| < \delta_2$ ⇒ $|g(x) - M| < \frac{\varepsilon}{2}$
- Take $\delta = \min\{\delta_1, \delta_2\}$
- Let $x \in \mathbb{R}$, Assume $0 < |x-a| < \delta$, ⇒ $\begin{cases} 0 < |x-a| < \delta_1, & \text{Thus } |f(x) - L| < \frac{\varepsilon}{2} \\ 0 < |x-a| < \delta_2 & \text{Thus } |g(x) - M| < \frac{\varepsilon}{2} \end{cases}$
- Then $|h(x) - (L+M)| = |[f(x) - L] + [g(x) - M]| \leq |f(x) - L| + |g(x) - M| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$
- Haven proven $|h(x) - (L+M)| < \varepsilon$. ∎

Squeeze Theorem

Ex. $\lim_{x \to 0} x^2 \sin\frac{1}{x}$

- $\lim_{x \to 0} [x^2 \sin\frac{1}{x}] = [\lim_{x \to 0} x^2] \cdot [\lim_{x \to 0} \sin\frac{1}{x}] = 0 \cdot [\lim_{x \to 0} \sin\frac{1}{x}] = 0$ ✗ This is incorrect
  - ↳ Product limit law has a condition ⇒ the limit of a & limit of b has to exist in order for $a \cdot b = ab$. $(a, b \in \mathbb{R})$
- For every $x \neq 0$; $-1 \leq \sin\frac{1}{x} \leq 1$ ⇒ $-x^2 \leq \sin\frac{1}{x}(x^2) \leq x^2$



  - The graph of $x^2 \sin\frac{1}{x}$ will be squeezed between $-x^2$ & x^2
  - ∴ The limit of $x^2 \sin\frac{1}{x}$ will exist ($\lim_{x \to 0} x^2 \sin\frac{1}{x} = 0$)

∴ The squeezed theorem says that if a function that is SQUEEZED in between other 2 functions, no matter what it behaves, if the 2 functions both have a limit L at a, the the squeezed function must also have a limit L at a

DF:
- Let $a, L \in \mathbb{R}$; Let f, g, h be functions defined near a, except possibly @ a
- For x close to a but not a, ($\exists p > 0$, s.t. $0 < |x-a| < p$ ⇒ $h(x) \leq g(x) \leq f(x)$)

IF $h(x) \leq g(x) \leq f(x)$; $\lim_{x \to a} f(x) = L$; $\lim_{x \to a} h(x) = L$ THEN $\lim_{x \to a} g(x) = L$

Solution to $\lim_{x \to 0} x^2 \sin\frac{1}{x}$,
- for every $x \neq 0$, $-x^2 \leq x^2 \sin\frac{1}{x} \leq x^2$
- $\lim_{x \to 0} (-x^2) = 0$; $\lim_{x \to 0} x^2 = 0$ ∴ By squeezed theorem, $\lim_{x \to a} [x^2 \sin\frac{1}{x}] = 0$.

- Writing the proof for the squeezed theorem
  - (WTS) $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < |x-a| < \delta \Rightarrow |g(x) - L| < \varepsilon$
    - └ Fix an arbitary ε, find a value of δ that works for ε
    - └ KNOW: $\lim_{x\to a} f(x) = L$ — choose (freely) a value of ε, & there has to be a value of δ that works
    - Manipulate : $|g(x) - L| < \varepsilon \Leftrightarrow \boxed{L-\varepsilon} < g(x) < \boxed{\varepsilon + L}$
    - It known that $0 < |x-a| < p \Rightarrow \boxed{h(x)} \leq g(x) \leq \boxed{f(x)}$
    - $\lim_{x\to a} f(x) = L$ (take the same ε as in $\lim_{x\to a} g(x)$), conclude that $\exists \delta_1 > 0$ s.t. $0 < |x-a| < \delta_1 \Rightarrow L - \varepsilon < f(x) < L + \varepsilon$
    - $\lim_{x\to a} h(x) = L$ (take the same ε as in $\lim_{x\to a} g(x)$), conclude that $\exists \delta_1 > 0$ s.t. $0 < |x-a| < \delta_2 \Rightarrow L - \varepsilon < h(x) < L + \varepsilon$
    - Take $\delta = \min\{\delta_1, \delta_2, p\}$ (then $0 < |x-a| < \delta \Rightarrow 0 < |x-a| < \delta \leq \delta_1 \leq \delta_2 \leq p$)

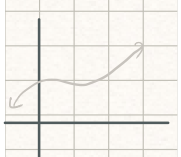
pf: · DF: Let $a, L \in \mathbb{R}$, let f, g, h be functions, $\exists p > 0$, s.t. $0 < |x-a| < p \Rightarrow h(x) \leq g(x) \leq f(x)$

- Let $\varepsilon > 0$,
  - └ use the same value in the definition of $\lim_{x\to a} f(x) = L$, $\exists \delta_1 > 0$ s.t. $0 < |x-a| < \delta_1 \Rightarrow 0 < |f(x) - L| < \varepsilon \Rightarrow \underline{f(x) < L + \varepsilon}$
  - └ use the same value in the definition of $\lim_{x\to a} h(x) = L$, $\exists \delta_2 > 0$ s.t. $0 < |x-a| < \delta_2 \Rightarrow 0 < |h(x) - L| < \varepsilon \Rightarrow \underline{L - \varepsilon < h(x)}$
- Take $\delta = \min\{\delta_1, \delta_2, p\}$
- Let $x \in \mathbb{R}$, Assume $0 < |x-a| < \delta$, this implies

$$\left.\begin{array}{ll} 0 < |x-a| < \delta_1, & \text{Thus } f(x) < L + \varepsilon \\ 0 < |x-a| < \delta_2, & \text{Thus } L - \varepsilon < h(x) \\ 0 < |x-a| < p & \text{Thus } h(x) \leq g(x) \leq f(x) \end{array}\right\} \text{Therefore } L - \varepsilon < g(x) < L + \varepsilon$$

- Equivalently, it's proven that $|g(x) - L| < \varepsilon$, as need ■

Continuity

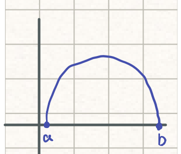


- The function is continous means "the graph can be sketched without lifting the pen from the paper" (idea)
- f is continuous at a when $\lim_{x\to a} f(x) = f(a)$
  - └ means that $\lim_{x\to a} f(x)$ exist (a number)
  - └ $f(a)$ is defined
  - └ $\lim_{x\to a} f(x) = f(a)$

DF:

Let $a \in \mathbb{R}$; Let f be a function defined, at least, on an interval centered at a.

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\overset{\text{not needed}}{0 <} |x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$



- Continous @ a point (c) : $\lim_{x\to c} f(x) = f(c)$
- Continous on the open interval (a,b) : $\forall c \in (a,b)$ f is continous @ c.
- Continous on the closed interval $[a,b]$: $\lim_{x\to a^+} f(x) = f(a)$, $\forall c \in (a,b)$ is in continous @ c, $\lim_{x\to b^-} f(x) = f(b)$

Main Continuity Theorem

- Any function can be constructed with sum, product, quotient, and composition of polynomials, roots, trigonometric functions, exponentials, logarithms, and absolute value is continuous (on its domain).
  - └ call elementary functions.
- Steps to prove it
  - └ Prove basic functions are continuous $[f(x) = c,\ f(x) = x,\ f(x) = e^x,\ f(x) = \ln x\ \ f(x) = \sin x,\ f(x) = |x|]$
  - └ Prove the sum, product, quotient and composition of continous functions is continuous (use limit laws)

Limits & Composition

- Composition functions does not respect limits, but does respect continuity.

→ IF $\lim_{x\to a} f(x) = L \wedge \lim_{y\to L} g(y) = m$ ~~THEN $\lim_{x\to a} g(f(x)) = m$~~

- IF f continuous at a $\wedge$ g continuous at $f(a)$ THEN $g \circ f$ is continuous at a.
  - └ x close to a $\Rightarrow f(x)$ close to $f(a)$; y close to $f(a)$ $g(y)$ close to $g(f(x))$
  - └ x close to a $\Rightarrow g(f(x)) \Rightarrow g(f(a))$

$\lim_{x\to a} f(x) = L$ means:

$\left\{\begin{array}{c} x \text{ close to } a \\ x \neq a \end{array}\right\} \Rightarrow f(x)$ close to L

f continuous at a mens:

x close to a $\Rightarrow f(x)$ close to $f(a)$

- limit's input & output lack symmetry, therefore composition does not work well with limits.

① IF $\lim_{x\to a} f(x) = L$ $\wedge$ $f(x) \neq L$ for x on an interval centered @ a, except maybe at a. and $\lim_{y\to L} g(y) = M$

THEN $\lim_{x\to a} g[f(x)] = M$

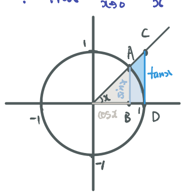
② If $\lim_{x\to a} f(x) = L$ $\wedge$ g is continuous at L, $\lim_{x\to a} g(f(x)) = g(L)$

Discontinuits

- Removable discontinuity (has a hole)
  $\lim_{x\to a} f(x)$ exists, but $\lim_{x\to a} f(x) \neq f(a)$
  - ↳ can be fixed by filling the hole (define a new function that "fills" the hole when that $= L$ when $\lim_{x\to a} f(x) = L$.
    - ↳ $h(x) = \frac{\sin x}{x}$; $\lim_{x\to 0} h(x) = 1$, $h(0)$ undefined } can be fixed: $H(x) \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$
- Non-removable continuity (jump, verticle asymptotes, oscillation)
  - ↳ $\lim_{x\to a} f(x)$ DNE.

Geometric proof for a trig. limit

. Prove $\lim_{x\to 0} \frac{\sin x}{x} = 1$



Area of OAB $\leq$ Area of OAD $\leq$ Area OCD

$\frac{1}{2} \cdot \cos x \cdot \sin x \leq \frac{1}{2} x \cdot 1^2$ ① $\leq \frac{1}{2} \cdot 1 \cdot \tan x$

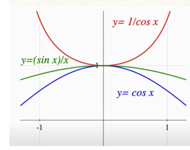
$\cos x \cdot \sin x \leq x \leq \tan x \left(\frac{\sin x}{\cos x}\right)$

↳ $\frac{\sin x}{x} \leq \frac{1}{\cos x}$, $\cos x \leq \frac{\sin x}{x}$

$\cos x \leq \frac{\sin x}{x} \leq \frac{1}{\cos x}$ for $0 < |x| < \frac{\pi}{2}$

① (sector R, x), angle x, area $S = ?$, | Full circle: $x = 2\pi$, $S = \pi R^2$

$S = k \cdot x$ (for some constant k) | $(\pi R^2) = k \cdot (2\pi) \Rightarrow k = \frac{\pi R^2}{2\pi} = \frac{1}{2} R^2$



$\lim_{x\to 0} \cos x = 1$, $\lim_{x\to 0} \frac{1}{\cos x} = 1$

By Squeeze theorem,

$\lim_{x\to 0} \frac{\sin x}{x} = 1$ ∎

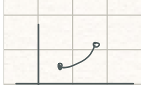
Computation

- Method 1 [the function is defined & continuous] Evaluate:
  - ↳ $\lim_{x\to 2} x^2 + 1 = 2^2 + 1 = 5$
- Method 2 [not, continuous], Algebraic manipulations to convert into continuous near a (not at a)
  - ↳ Factoring: $\lim_{x\to 2} \frac{x^2-4}{x^2-3x+2} = \lim_{x\to 2} \frac{(x-2)(x+2)}{(x-2)(x-1)} = \lim_{x\to a} \frac{x+2}{x-1} = \frac{2+2}{2-1} = 4$
  - ↳ Conjugate: $\lim_{x\to 0} \frac{1-\sqrt{1+x}}{x} = \lim_{x\to 0} \frac{[1-\sqrt{1+x}][1+\sqrt{1+x}]}{x[1+\sqrt{1+x}]} = \lim_{x\to 0} \frac{1-(1+x)}{x[1+\sqrt{1+x}]} = \lim_{x\to 0} \frac{-1}{1+\sqrt{1+x}} = \frac{-1}{1+\sqrt{1+0}} = \frac{-1}{2}$
- Method 2 Reduce to a limit that's already solved.
  - ↳ $\lim_{x\to 0} \frac{\sin(2x)}{x}$, (know $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$, sub. $\lim_{2x\to 0} \frac{\sin(2x)}{2x} = 1$) $\Rightarrow \lim_{x\to 0} \left[2 \cdot \frac{\sin(2x)}{2x}\right] = 2 \cdot \lim_{x\to 0} \left[\overset{=1}{\frac{\sin(2x)}{2x}}\right] = 2 \cdot 1 = 2$.
  - ↳ $\lim_{x\to 0} \frac{\sin(2x)}{\sin 3x}$, (know, $\lim_{x\to 0} \frac{\sin ax}{ax} = 1$ for any $a \neq 0$)
    - ↳ $\lim_{x\to 0} \frac{2x \cdot \frac{\sin(2x)}{2x} \to 1}{3x \cdot \frac{\sin(3x)}{3x} \to 1} = \frac{2}{3}$
  - ↳ $\lim_{x\to 0} \frac{1-\cos x}{x^2}$ (conjugate + know $\lim_{x\to 0} \frac{\sin ax}{ax} = 1$ + $1 - \cos^2 x = \sin^2 x$)
    $= \lim_{x\to 0} \frac{[1-\cos x][1+\cos x]}{x^2[1+\cos x]} = \lim_{x\to 0} \frac{1-\cos^2 x}{x^2[1+\cos x]} = \lim_{x\to 0} \frac{\sin^2 x}{x^2[1-\cos x]} = \lim_{x\to 0} \left[\left(\frac{\sin x}{x}\right)^2 \frac{1}{1+\cos x}\right]$
    $1^2 \cdot \left[\frac{1}{1+\cos 0}\right] = \frac{1}{2}$
  - ■ limit of rational functions @ Infinity
    - ↳ based on: ① $\lim_{x\to\infty} x = \infty$, ② $\lim_{x\to\infty} \frac{1}{x} = 0$

    ①: only the term with the largest exponent matters. (because biggest can be factored out)

    - ↳ $\lim_{x\to\infty} [2x^3 - 3x^2 + 7] = \infty$, because: $\lim_{x\to\infty} x^3 [2 - 3\frac{1}{x} + 7\frac{1}{x^3}] \Rightarrow \lim_{x\to\infty} 2x^3 \ldots \Rightarrow \infty$
    - ↳ $\lim_{x\to\infty} \frac{2x^2 + x - 1}{3x^2 + 10x + e} = \lim_{x\to\infty} \frac{x^2[2 + \frac{1}{x} - \frac{1}{x^2}]}{x^2[3 + 10\frac{1}{x} + e\frac{1}{x^2}]} = \lim_{x\to\infty} \frac{2 + \frac{1}{x} - \frac{1}{x^2}}{3 + 10\frac{1}{x} + e\frac{1}{x^2}} = \frac{2}{3}$
    - ↳ $\lim_{x\to\infty} \frac{\sqrt{x^4+1} + x^2}{\sqrt{2x^4+1} + 2x} = \lim_{x\to\infty} \frac{\sqrt{x^4(1 - 1/x^4)} + x^2}{\sqrt{x^4(2 + 1/x^4)} + 2x} = \lim_{x\to\infty} \frac{x^2\sqrt{1 - 1/x^4} + x^2}{x^2\sqrt{2 + 1/x^4} + 2x} = \lim_{x\to\infty} \frac{\cancel{x^2}[\sqrt{1 - 1/x^4} + 1]}{\cancel{x^2}[\sqrt{2 + 1/x^4} + 2/x]} = \frac{1+1}{\sqrt{2}} = \sqrt{2}$

Extreme Value Theorem



no max (on its domain)

- 'Under certain conditions, a function is guaranteed to have a max. & a min' (They exist, not how to find it)

DF
- the function f has a Maximum on a set I: $\exists c \in I$ s.t. $\forall x \in I$, $f(x) \leq f(c)$ [max is $f(c)$, f has a max @ c]
  - ↳ if do not specify, 'the max of f' means 'the max of f on its domain'.
- Means needs a continuous function on a closed & bounded interval that includes both endpoint.

DT: IF f is a continuous function on an interval $[a,b]$, THEN f has a max & min on $[a,b]$

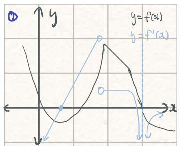
Intermediate Value Theorem

- Let f be a function defined on the interval $[a,b]$, $M \in \mathbb{R}$,

DT IF $f(a) < M < f(b)$, f is continuous on $[a,b]$. THEN $\exists c \in (a,b)$ s.t. $f(c) = M$

  - IF f is continuous on $[a,b]$, THEN, f takes all the values between $f(a)$ & $f(b)$

DERIVATIVES



drawing derivative graphs.

② Finding $f^{(n)}(x)$

& Proof from definition

differentiable $\Rightarrow$ continuous

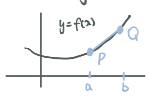
f' continuous $\Rightarrow$ f differentiable

f differentiable $\rightarrow$ $c \cdot f$ differ.

f differentiable $\not\Rightarrow$ $\frac{1}{f}$ differ.

Derivatives as slope

- Let f be a function with domain I, Let $a \in I$.
- Idea: $f'(a)$ = slope of line tangent to the graph of $y=f(x)$ at the point with $x=a$
- Using "limits" to define derivative



- as Q gets "closer" to P, the secant line between Q & P gets "closer" to the tangent line of P
- slope of secant line $\overline{QP}$: $\frac{f(b)-f(a)}{b-a}$; slope of tangent $\vec{P}$: $f'(a) = \lim_{b \to a} \frac{f(b)-f(a)}{b-a}$

DT
- Let $a \in \mathbb{R}$, Let f be a function defined, at least on an interval centred at a
- The derivative of f at a is a number: $f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ [change $h = x-a$: $f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$]
- "f is differentiable at a" when this limit exist
- "f is differentiable" means at all points on its domain

- The tangent line to the graph of $y=f(x)$ at the point with x-coordinate a is the line
  - through the point $(a, f(a))$
  - with slope $f'(a)$
  - $y = f(a) + f'(a)(x-a)$

Compute a derivative from the defination

Ex. Let $f(x) = 4x - x^2$, Compute $f'(1)$ directly from the definition

1. Write the definition of the derivative as the limit
   $f'(1) = \lim_{h \to 0} \frac{f(1+h)-f(1)}{h}$
2. Sub. $f(x)$ as expression in the limit function
   $f'(1) = \lim_{h \to 0} \frac{[4 \cdot (1+h) - (1+h)^2] - [4 \cdot 1 - 1^2]}{h} = \lim_{h \to 0} \frac{[4+4h-(1+2h+h^2)-3]}{h} = \lim_{h \to 0} \frac{2h-h^2}{h} = \lim_{h \to 0} [2-h]$
3. Evalute the continuous function's limit
   $\lim_{h \to 0} [2-h] = \boxed{2}$

Derivatives as rate of change

Ex. I am driving my car in the highway, Right now, my velocity is 180km/h, What does this mean?

- "If Δt is a very small time interval, in the next Δt, I will drive approximatly a distance of ($180 \text{ km/h} \cdot \Delta t$)."
- Let t = time, x = position
  - average velocity between t_1 & t_2 = $\frac{\Delta x}{\Delta t}$
  - instantaneous velocity at t_1 = $\lim_{t_2 \to t_1} \left[\begin{smallmatrix} \text{average velocity} \\ \text{between } t_1 \& t_2 \end{smallmatrix} \right] = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$ $\Rightarrow$ "the derivative of x with respect to t
- let $x = f(t)$
  $\boxed{\frac{dx}{dt}} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta d} = \lim_{t_2 \to t_1} \frac{f(t_2)-f(t_1)}{t_2 - t_1} = \boxed{f'}$
- Let physical quantities Q, x, Q depends on x,
  - average ROC : $\frac{\Delta Q}{\Delta x}$
  - Instantaneous ROC : $\lim_{\Delta x \to 0} \frac{\Delta Q}{\Delta x} = \frac{dQ}{dx}$

Question 1 (2020 winter TT2)

continuous

differe-ciable

6. Let $a \in \mathbb{R}$. Let f be a function that is differentiable at a. Assume $f(a) > 0$.
Define the function g by $g(x) = \dfrac{1}{f(x)}$.

(6a) (2 points) Give a short explanation why g is defined at and near a. No ε-δ proof is necessary.

- Note f is differentiable at a implies f is continuous at a.
- As f is continuous at a and $f(a) > 0$, there exists $\delta > 0$ such that $\forall x \in (a-\delta, a+\delta), f(x) > 0$.
- Hence, $g(x) = \dfrac{1}{f(x)}$ is defined for all $x \in (a-\delta, a+\delta)$. f near a is also >0

☆ Pr 必考题

(6b) (4 points) Prove that g is differentiable at a and $g'(a) = -\dfrac{f'(a)}{(f(a))^2}$.

Write a proof directly from the definition of the derivative. Do not use any differentiation rules, e.g. quotient rule or chain rule.

Common Errors

1. Refer to limit law of constants when extracting $-\frac{1}{f(a)}$

2. Has to state the ①&② limits exists

- From (6a), g is defined at and near a so it remains to show $g'(a) = \lim\limits_{x\to a} \dfrac{g(x) - g(a)}{x-a}$ exists and satisfies the desired formula.
- By definition,

$$g'(a) = \lim_{x\to a} \frac{\frac{1}{f(x)} - \frac{1}{f(a)}}{x-a} = \lim_{x\to a} \frac{f(a) - f(x)}{f(x)f(a)(x-a)}$$
$$= -\frac{1}{f(a)} \lim_{x\to a} \left[\frac{f(x)-f(a)}{x-a} \cdot \frac{1}{f(x)}\right] \quad \text{by limit law for constants}$$

① $\lim A \cdot B = \lim A \cdot \lim B$ iff $\lim A$ & $\lim B$ exists

- Since f is differentiable at a, we have f is continuous at a and also

①

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a).$$

- As $f(a) > 0$ and f is continuous at a, this implies that $\dfrac{1}{f}$ is continuous at a so

②

$$\lim_{x\to a} \frac{1}{f(x)} = \frac{1}{f(a)}.$$

①

- By the limit law for products, we conclude that g is differentiable at a and

$$g'(a) = -\frac{1}{f(a)} \cdot f'(a) \cdot \frac{1}{f(a)} = -\frac{f'(a)}{(f(a))^2}.$$

This completes the proof. $\square$

Differenciation Rules

much faster way to compute derivatives will need to obtain & prove them (just once) from the definition

- Goal: Compute derivative of common functions without using the definition evey time
- Lazy version of the rules, (let $f \cdot g$ be functions, $c \in \mathbb{R}$ is a constant

  - $\frac{d}{dx}[c] = 0$
  - $(f+g)' = f' + g'$
  - $(f \cdot g)' = f'g + fg'$
  - $\frac{d}{dx}[x^c] = cx^{c-1}$
  - $(cf)' = cf'$
  - $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$, $g^2 \neq 0$

☆ Can only be used if both function is differentiable

Ex. $f(x) = x^7 - 5x^3 + 2x + 11$, $f'(x) = \frac{d}{dx}(x^7) - \frac{d}{dx}(5x^3) - \frac{d}{dx}(2x) + \frac{d}{dx}(11) = 7x^6 - 15x^2 + 2$

Ex. $g(x) = \frac{x^2+1}{x^2+x}$, $g'(x) = \frac{2x(x^2+x) - (x^2+1)(2x+1)}{(x^2+x)^2} = \frac{2x^3+2x^2-(2x^3+2x+x^2+1)}{(x^2+x)^2} = \frac{x^2-2x-1}{(x^2+x)^2}$

Ex. $\frac{d}{dx}\left[\frac{1}{x^3}\right] = \frac{d}{dx}[x^{-3}] = -3x^{-4}$

Ex. $\frac{d}{dx}[\sqrt{x}] = \frac{d}{dx}[x^{\frac{1}{2}}] = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

DF Formal Theorem for the product Rule:

- Let $a \in \mathbb{R}$, Let f & g be functions defined at a & near a
- Defined the function $h(x) = f(x)g(x)$

IF f & g are differentiable at a THEN h is differentiable at a, and:

$h'(a) = f'(a)g(a) + f(a)g'(a)$

This is a IF-THEN theorem

Higher-order derivatives

- If f is a function, f' is another function, assuming f is differentiable at every point in the domain.
- The second derivative would be the derivative of f' $[(f')']$ ...
- Leibnize-notation: assume x, y are physical quantities, $y = f(x)$
  - units: let position $= x$ (m), time $= t$ (s); velociy $= \frac{dx}{dt}$ (m/s), acceleration $= \frac{d^2x}{dt^2}$ (m/s²)

derivative order	function-notation	name	Leibnize-notation			name
0	f	'function f'	$y = f(x)$			function y
1	f'	'f prime'	$\frac{d}{dx}(y)$	$=$	$\frac{dy}{dx}$	'derivative of y with respect to x'
2	f''	'f double prime'	$\frac{d}{dx}\left[\frac{d}{dx}(y)\right]$	$=$	$\frac{d^2y}{dx^2}$	'2-nd derivative of y with respect to x'
...	...	...	...			...
n	$f^{(n)}$ ①	'n-th derivative of f'	$\frac{d}{dx}\left[\frac{d}{dx} \dots \frac{d}{dx}(y) \dots\right]$	$=$	$\frac{d^ny}{dx^n}$	'n-th derivative of y with respect to x'

① n need to be in parentheses, not power

Prove of Product Rule & Power Rule

- Prof for product rule

DF
- Use the definition of $h(x)$ in $h'(a)$, recall $h(x) = f(x)g(x)$, $h'(a) = \lim_{x \to 0} \frac{h(x) - h(a)}{x-a} = \lim_{x \to a} \frac{f(x)g(x) - f(a)g(a)}{x-a}$
- Add & subtract some terms to factor

$$h'(a) = \lim_{x \to a} \frac{f(x)g(x) - f(a)g(x) + f(a)g(x) - f(a)g(a)}{x-a} = \lim_{x \to a}\left[g(x)\frac{f(x)-f(a)}{x-a} + f(a)\frac{g(x)-g(a)}{x-a}\right] = ①$$

$f'(a)$ differentiable @ a

$$① = \left[\lim_{x \to a} \frac{f(x)-g(a)}{x-a}\right]\left[\lim_{x \to a} g(x)\right] + f(a)\left[\lim_{x \to a} \frac{g(x)-g(a)}{x-a}\right] = f'(a)g(a) + f(a)g'(a)$$

$g'(a)$ differentiable @ a

- essencial that $f(x)$ & $g(x)$ are differentiable @ a
- essencial that $g(x)$ is continuous at a.

- Proof for power rule when $\{c \in \mathbb{N} : c > 0\}$ (proof by induction on c)

DF
- Base case $(c = 1)$ WTS $\frac{d}{dx}[x] = 1 \cdot x^0 = 1$

  Call $f(x) = x$, THEN $f'(x) = \lim_{h \to 0} \frac{f(h+x) - f(x)}{h} = \lim_{h \to 0} \frac{h+x-x}{h} = \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1$
- Induction step
  - Fix $c \geq 1$, assume $\frac{d}{dx}x^c = cx^{c-1}$, WTS $\frac{d}{dx}x^{c+1} = c+1\,x^c$

$\frac{d}{dx}x^{c+1} = \frac{d}{dx}[x \cdot x^c]$, by product rule

$= x \cdot \frac{d}{dx}[x^c] + \frac{d}{dx}[x] \cdot x^c$, by induction hypothesis

$= x \cdot (cx^{c-1}) + 1 \cdot (x^c)$

$= cx^c + x^c = x^c(c+1)$

implicit differenciation

1

Continuous, but not differentiable functions.

- f is differentiable $\Rightarrow$ f is continuous
- But f could be continuous, but not differentiable
  - └ f is continuous at a means $\lim_{x\to a} f(x) = f(a)$; f is differentiable at a means $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists

$\therefore$ f is not differentiable @ a means $\left[\lim_{x\to a} \frac{f(x)-f(a)}{x-a}\right]$ DNE & $\lim_{x\to a} f(x) = f(a)$

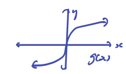
└ DNE because the side limits are different (Corner)

Ex. $f(x) = |x|$, $f'(0) = \lim_{x\to0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to0} \frac{|x|}{x}$

└ case ①: $\lim_{x\to0^+} \frac{x}{x} = 1$; case ②: $\lim_{x\to0^-} \frac{-x}{x} = -1$, DNE because $\lim_{x\to0^+} \neq \lim_{x\to0^-}$

└ DNE because the limit is $\pm\infty$ (Vertical tangent)

Ex $g(x) = x^{\frac{1}{3}}$, $g'(0) = \lim_{x\to0} \frac{g(x)-g(0)}{x-0} = \lim_{x\to0} \frac{x^{\frac{1}{3}}}{x} = \lim_{x\to0} \frac{1}{x^{\frac{2}{3}}} = \infty$ $\Rightarrow$ vertical tanget line



└ when $x \neq 0$, $g'(x) = {}^1/3x^{\frac{2}{3}}$, and $\lim_{x\to0} g'(x) = \lim_{x\to0} \left[{}^1/3x^{\frac{2}{3}}\right] = \infty$ $\therefore$ g' is not differentiable.

└ $\lim_{x\to a} g'(x) = \pm\infty \Rightarrow \lim_{x\to a} \frac{g(x)-g(a)}{x-a} = \pm\infty$ $\therefore$ $\lim_{x\to a} g'(x) = \pm\infty$ is a stronger definition for vertical tangent line

Vertical tangent $\neq$ Vertical asympote

Chain Rule

Ex. $\frac{d}{dx}\sqrt{x^3+1}$, sub. $\left(\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}\right)$

$= \frac{1}{2\sqrt{x^3+1}} \cdot \frac{d}{dx}(x^3+1)$

$= \frac{3x^2}{2\sqrt{x^3+1}}$

Ex. $\frac{d}{dx}[\sin(6x)]$, sub. $\frac{d}{dx}[\sin x] = \cos x$

$= \cos(6x) \cdot \frac{d}{dx}(6x)$

$= 6\cos(6x)$

Ex. $\frac{d}{dx}(x^2+1)^{100}$, sub. $\frac{d}{dx}x^{100} = 100x^{99}$

$= 100(x^2+1)^{99} \cdot \frac{d}{dx}(x^2+1)$

$= 200x(x^2+1)^{99}$

name	value
f	$f(x)$
$\sin$	$\sin(x)$
$f\circ g$	$f(g(x))$

Ex. $(f\circ g)(x) = f(g(x)) = \sqrt{x^3+1}$, $f(x) = \sqrt{x}$, $g(x) = x^3+1$, $f'(x) = \frac{1}{2\sqrt{x}}$, $g'(x) = 3x^2$ $\therefore$ $\frac{d}{dx}\sqrt{x^3+1} = \frac{1}{2\sqrt{x^3+1}}(3x^2)$

- $(f\circ g)'(x) = f'(g(x)) \cdot g'(x)$; $u = g(x)$, $y = f(u) = f(g(x))$ $\rightarrow$ $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$

Df: · Let $a \in \mathbb{R}$, let f, g be functions

IF $\begin{cases} \cdot\ g \text{ is differentiable at } a \\ \cdot\ f \text{ is differentiable at } g(a) \end{cases}$ THEN : $f\circ g$ is differentiable at a, and $(f\circ g)'(a) = f'(g(a))\, g'(a)$

~~Pf~~: WTS: $(f\circ g)'(a) = \lim_{x\to a} \frac{f(g(x)) - f(g(a))}{x-a}$

- Change the denominator: $\lim_{x\to a}\left[\frac{f(g(x)) - f(g(a))}{g(x)-g(a)} \cdot \frac{g(x)-g(a)}{x-a}\right]$, need $\begin{Bmatrix} x \text{ close to } a \\ x \neq a \end{Bmatrix} \Rightarrow g(x) \neq g(a)$
- Using the limit laws: $\lim_{x\to a}\left[\frac{f(g(x)) - f(g(a))}{g(x)-g(a)}\right]\left[\lim_{x\to a} \frac{g(x)-g(a)}{x-a}\right] \rightarrow g'(a)$, is ① $f'(g(a))$?

└ $f'(g(a)) = \lim_{u\to g(a)} \frac{f(u) - f(g(a))}{u - g(a)}$, is $u = g(a)$? $[x\to a \Rightarrow g(x) = u \rightarrow g(a)]$

$g(a)$ is just a #

Derivatives for Trig. functions

- $\frac{d}{dx}\sin x = \cos x$
- $\frac{d}{dx}\tan x = \sec^2 x$
- $\frac{d}{dx}\sec x = \sec x \tan x$
- $\frac{d}{dx}\cos x = \sin x$
- $\frac{d}{dx}\cot x = -\csc^2 x$
- $\frac{d}{dx}\csc x = -\csc x \cot x$

} can be derived from the first column

Get $\frac{d}{dx}$ for these:

1. Obtain $\frac{d}{dx}\sin x$ from the definition
2. Write $\cos x = \sin\left(\frac{\pi}{2} - x\right)$ to obtain $\frac{d}{dx}\cos x$
3. Obtain the other 4 functions from: ($\tan x = \frac{\sin x}{\cos x}$, $\sec x = \frac{1}{\cos x}$ ...)

Df · let $f(x) = \sin x$

$f'(x) = \lim_{h\to0} \frac{\sin(x+h) - \sin x}{h}$

$= \lim_{h\to0} \frac{[\sin x\cos h + \cos x \sin h] - \sin x}{h}$

$= \lim_{h\to0}\left[\sin x \cdot \frac{\cos h - 1}{h} + \cos x \frac{\sin h}{h}\right]$

$= \sin x\left[\lim_{h\to0} \frac{\cosh - 1}{h}\right] + \cos x\left[\lim_{h\to0} \frac{\sinh}{h}\right]$

$= \sin x\,(0) + \cos x\,(1)$

$= \cos x.$

Use the definition of derivative as the limit, use trig identities to expand variables with $[x]$ can be treated as constants, they don't depend on h.

Use limit laws, iff both limit here exists

Sub. $\lim_{h\to0} \frac{\cos x - 1}{x} = 0$, $\lim_{h\to0} \frac{\sin x}{x} = 1$

L'Hôpital's circle:

$\frac{d}{dx}(\sin x) = \cos x$

L'Hôpital's Rule

$\lim_{x\to0} \frac{\sin x}{x} = 1$

$\therefore$ Use geometric to prove first.

Implicite Differentiation

Ex. $3x^2 + 2y^2 - 4xy + 2x - 8y + 11 = 0$, find slope at $(1, 4)$

$\frac{d}{dx}[3x^2 + 2y^2 - 4xy + 2x - 8y + 11] = \frac{d}{dx}0$

$6x + 4y(y') - (4y + 4xy') + 2 - 8y' = 0$

$6 + 16y' - (16 + 4y') + 2 - 8y' = 0$

★ y is a function, therefore, need to use chain rule for $\frac{d}{dx}(y)$

Sub. $x = 1$, $y = 4$

$4y' = 8$

$y' = 2$. $\therefore$ The slope is 2.

TRANSCENDENTAL FUNCTIONS

Conversion (single variable calculus)
- L defined a function with just the rule
- L Assume domain as largest subset of $\mathbb{R}$ possible
- L Assume codomain $\forall y \in \mathbb{R}$

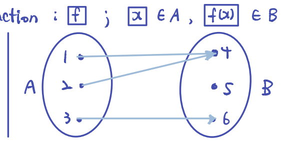
$f(x)$ is not the name

range doesn't need to be defined (its calculated)

What is a Function

DF A function f consists of
1. a domain - set of inputs (A)
2. a codomain - set of "potential" outputs (B)
3. a rule that matches each input $(x \in A)$ to exactly one output $[f(x) \in B]$

Name of function : $\boxed{f}$; $\boxed{x} \in A$, $\boxed{f(x)} \in B$



Notation: $f: A \to B$ (f is the name, A is the domain, B is the codomain)

- Codomain (set of potential outputs) vs. range (set of actual outputs) codomain $= \{4,5,6\}$, range $= \{4,6\}$
  - L functions with the same domain & range, but different codomain are different functions.

Inverse Function

- Inverse functions' graph is a reflection with respect to the diagonal of the 1^{st} & 3^{rd} quadrants

- Idea: swap input & output

DF
- Let $f: A \to B$ be a function
- The inverse of f is another function $f^{-1}: B \to A$ [Domain $f = A$; Range $f = C$]
- $\forall x \in A, \forall y \in C$ $x = f^{-1}(y) \iff y = f(x)$, assuming f^{-1} is a function

DF
- Let $f: A \to B$ be a function
- f is surjective or onto when : Range $f = B$ (Calculus uses range, don't worry about it)

DF
- Let $f: A \to B$ be a function
- f is injective or one-to-one when : $\forall x_1, x_2 \in A$ $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$
  - L Equivalent to $\forall x_1, x_2 \in A$, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

Theorem: f has an inverse $\Updownarrow$ f is injective $\wedge$ surjective

Ex. $f(x) = 2x+1$ Domain $\in \mathbb{R}$, Range $\in \mathbb{R}$
- L $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}$, $x = f^{-1}(y) \iff y = f(x)$
- L $y = 2x+1 \iff x = \frac{y-1}{2}$ $\therefore f^{-1}(y) = \frac{y-1}{2}$

- $f^{-1}(f(x)) = \frac{f(x)-1}{2} = \frac{(2x+1)-1}{2} = x$
- $f(f^{-1}(y)) = 2f^{-1}(y) + 1 = 2\left(\frac{y-1}{2}\right) + 1 = y$

Ex. $E(x) = e^x$, $L(x) = \ln x$ ($\forall x \in \mathbb{R}, y > 0$ $x = \ln y \iff y = e^x$)
- L $\forall x \in \mathbb{R}$ $\ln(e^x) = x$; $\forall y > 0$ $e^{\ln y} = y$

Ex. $f(x) = x^2$ (is not one-to-one) $\Rightarrow$ no inverse
- L $g(x) = \sqrt{x}$, g is NOT the inverse function of f.
- L g is the inverse of the restriction of f to $[0, \infty)$

$a = \sqrt{b}$ $\begin{cases} b = a^2 \\ a \geq 0 \end{cases}$ | $\sqrt{x^2} = \begin{cases} x & \text{if } x \geq 0 \\ |x| & \forall x \in \mathbb{R} \end{cases}$

Derivative of the inverse of a function

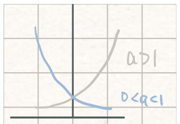
- It's possible for f to have a derivative while f^{-1} might not.
- Theorem
  - Let f be a function defined on an interval I.
  - IF ① f has an inverse , ② f is differentiable ③ $\forall x \in I$, $f'(x) \neq 0$
  - THEN f^{-1} is differentiable
- Write $(f^{-1})'$ in terms of f'
  - L Assume f, f^{-1} is differentiable

$$\frac{d}{dy}\left[f(f^{-1}(y))\right] = \frac{d}{dy}[y]$$

$$f'(f^{-1}(y)) \cdot (f^{-1})'(y) = 1$$ call $f^{-1}(y) = x$ $[y = f(x)]$

$$f'(x) \cdot (f^{-1})'(y) = 1$$

$$(f^{-1})'(y) = \frac{1}{f'(x)}$$



Derivative of exponentials & the number e

- Let $a>0$, $f(x)=a^x$

call this L_a

- $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{a^{x+h}-a^x}{h} = \lim_{h\to 0} \frac{a^x(a^h-1)}{h} = a^x \boxed{\lim_{h\to 0}\frac{a^h-1}{h}} = L_a a^x$

Df : $\boxed{e}$ is the only number st. $L_e = 1$; Equivalently, $\frac{d}{dx}e^x = e^x$

- Derivative of other exponentials (write $\frac{d}{dx}a^x$ in terms of $\frac{d}{dx}e^x = e^x$)

$a^x = (e^{\ln a})^x = e^{x\ln a}$

$\frac{d}{dx}a^x = \frac{d}{dx}e^{x\ln a} = e^{x\ln a}\cdot\frac{d}{dx}[x\ln a] = a^x\cdot\ln a$

Derivative of logarithm

- know $\frac{d}{dx}e^x = e^x$

$\frac{d}{dx}e^{\ln x} = \frac{d}{dx}x$ for $x>0$

$x\cdot\frac{d}{dx}\ln x = 1$

$\frac{d}{dx}\ln x = \frac{1}{x}$

Logarithmic differentiation

☆必考 : $\frac{d}{dx}f(x)^{g(x)}$

Ex. $f(x) = (\cos x)^{\sin x}$ (f is not just a power or exponential)

- Method 1 (use $a^b = e^{b\ln a}$)

$f(x) = (\cos x)^{\sin x} = e^{\sin x\ln(\cos x)}$

$f'(x) = e^{\sin x\ln(\cos x)}\cdot\left[\cos x\ln(\cos x) + \sin x\frac{-\sin x}{\cos x}\right]$

- Method 2 (Log. Diff.)

$\ln f(x) = \ln(\cos x)^{\sin x}$

$\frac{d}{dx}[\ln f(x)] = \frac{d}{dx}[\sin x\cdot\ln(\cos x)]$

$\frac{1}{f(x)}\cdot f'(x) = \cos x\cdot\ln(\cos x + \sin x\frac{-\sin x}{\cos x}$

$f'(x) = f(x)\cdot[\cos x\cdot\ln(\cos x) + \sin x\cdot\frac{-\sin x}{\cos x}$

For $c\in\mathbb{R}$, $x>0$, $x^c = e^{c\ln x}$

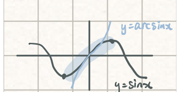
$\frac{d}{dx}[x^c] = \frac{d}{dx}[e^{c\ln x}]$

$= e^{c\ln x}\cdot\frac{d}{dx}[c\ln x]$

$= x^c\cdot\frac{c}{x}$

$= c\cdot x^{c-1}$

Arcsin



- Since $\sin x$ is not a one-to-one function, thus it has no inverse functions
- Restrict $\sin x$ so $[\sin x]$ becomes one-to-one [choose $(-\frac{\pi}{2},\frac{\pi}{2})$]

Df: arcsin is the inverse function of the restriction of sin to $[-\frac{\pi}{2},\frac{\pi}{2}]$

└ Domain: $(-1,1)$, Range $[-\frac{\pi}{2},\frac{\pi}{2}]$

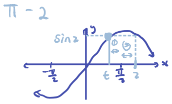
- $x = \arcsin y \iff y = \sin x$, for $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$, $-1\le y\le 1$

Ex. · What is $\arcsin\frac{1}{2}$?

$\arcsin\frac{1}{2} = t$ means $\begin{cases}\sin t = \frac{1}{2}\\ -\frac{\pi}{2}\le t\le\frac{\pi}{2}\end{cases}$ > thus. $\arcsin\frac{1}{2} = \frac{\pi}{6}$ $\Rightarrow$ $\arcsin(\sin x)$ for $-\frac{\pi}{2}\le x\frac{\pi}{2}$; $\sin(\arcsin y)$ for $-1\le y\le 1$

- $\sin(\arcsin 2)$ is undefined
- $\arcsin(\sin 2) = ???$ (cannot use identity above) $\pi - 2$

└ $\arcsin(\boxed{\sin 2}) = t \iff \begin{cases}\sin t = \sin 2\\ -\frac{\pi}{2}\le t\le\frac{\pi}{2}\end{cases}$



① = ②

$\frac{\pi}{2} - t = 2 - \frac{\pi}{2}$ $\Rightarrow$ $t = \pi - 2$

Derivative of arcsin

- $-1\le x\le 1$, $\frac{d}{dx}\sin(\arcsin x) = \frac{d}{dx}x$

$\cos(\arcsin x)\cdot\frac{d}{dx}(\arcsin x) = 1$ ← sub ①.

$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$

①: What is $\cos(\arcsin x)$, call $\theta = \arcsin x$

- know $\sin\theta = x$, want $\cos\theta$
- $\cos^2\theta = 1-\sin^2\theta$

$\cos\theta = \pm\sqrt{1-\sin^2\theta} = + \sqrt{1-x^2}$

for $-1\le x\le 1$
$-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$
$\cos\theta\ge 0$

$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}\arccos(x) = \frac{-1}{\sqrt{1-x^2}}$

$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$

Question 12

4. *[6 points]* The cotangent function is defined by $\cot(x) = \frac{1}{\tan(x)}$. Find an expression for

$$\cot(\arcsin(x))$$

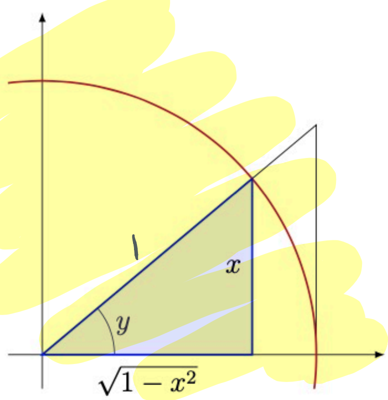
that involves no trigonometric functions. Also specify for which values of x your formula is valid.

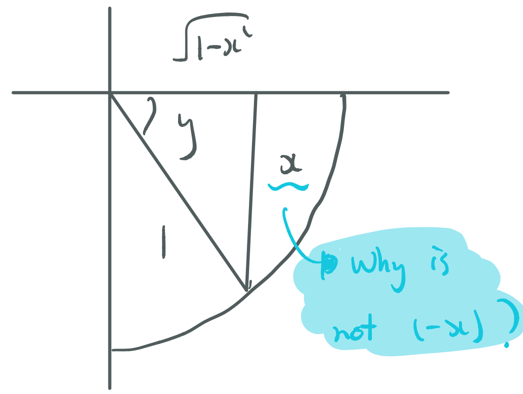
$\sqrt{1-x^2} = \cos y$ $\tan(y) = \frac{x}{\sqrt{1-x^2}}$

The function arcsin is defined for $x \in [-1, 1]$. Let $y = \arcsin(x)$. Then $x = \sin(y)$, and we can think of y as the angle formed by the horizontal axis and the line segment from $(0, 0)$ to $(\sqrt{1-x^2}, x)$ (see the picture).

SOH CAH TOA

$\frac{x}{1}$ $\frac{\sqrt{1-x^2}}{1}$ $\frac{x}{\sqrt{1-x^2}}$

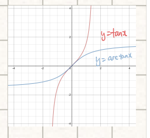


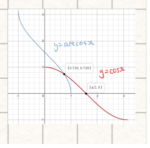


The picture for $x < 0$ is similar. From the picture, we can calculate

$$\cot(\arcsin(x)) = \cot(y) = \frac{\sqrt{1-x^2}}{x}.$$

This formula is valid for all $x \in [-1, 0) \cup (0, 1]$.





Derivative of arccos & arctan

DF arctan is the inverse function of the restriction of tan to $(-\frac{\pi}{2}, \frac{\pi}{2})$ → exclusive because of asymptotes

$x = \arctan y \Leftrightarrow y = \tan x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, $y \in \mathbb{R}$

DF arccos is the inverse function of the restriction of cos to $[0, \pi]$

$x = \arccos y \Leftrightarrow y = \cos x$ for $0 \le x \le \pi$, $-1 \le y \le 1$

- $\frac{d}{dx} \arccos x = \frac{-1}{\sqrt{1-x^2}}$ $\qquad$ $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$

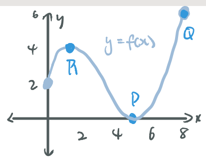
MEAN VALUE THEOREM

Extreme Value Theorem → Rolle's Theorem ← Local Extreme Value Theorem

Rolle's Theorem → Mean Value Theorem → Applications

The Local Extreme Value Theorem

IDEA:



- Domain $f = [0, 8]$
- f has min. of 0 (at $x=5$)
- f has max. of 6 (at $x=8$)
- f has local max 4 (at $x=1$)

- Extremum – maximum or minimum
- Global extremum – extremum of whole
- Plural: extrema, maxima, minima
- Endpoints donnot count as local extrem

DF: Let f be a function with domain I, let $c \in I$

- We say that f has a maximum at c when $\forall x \in I$, $f(x) \le f(c)$
- We say that f has a local maximum at c when (for every x close to c) $\exists \delta > 0$ s.t. $|x-c| < \delta \Rightarrow f(x) \le f(c)$
- We say that f has a min. at c when $\forall x \in I$, $f(x) \ge f(c)$
- We say that f has a local min at c when $\exists \delta > 0$ s.t. $|x-c| < \delta \Rightarrow f(x) \ge f(c)$

- Critical point – c is a critical point of f when c is an interior point of the domain f & $f'(c) = 0$ | DNE

DF Local Extrem Value Theorem

└ Let f be a function with domain an interval I, Let $c \in I$

IF f has a local extremum at c & c is an interior point (not an endpoint)

THEN $f'(c) = 0$ or DNE

c must be an interior point (needs 2 sides)

Pf Assume f has a local maximum at c $\left[f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \right]$, WTS this limit is 0 or DNE

Assume the above limit exists & I'll prove it is 0.

As $x \to c^+$ $\qquad$ As $x \to c^-$

$x - c > 0$ $\qquad$ $x - c < 0$

$f(x) - f(c) \le 0$ — from definition — $f(x) - f(c) \le 0$

$\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \le 0$ $\qquad$ $\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \ge 0$

∴ For the 2 side limits to exist and be equal, they must be 0. ∎

Ex. Find the extrema (& local extrema) of $f(x) = x^3 - 3x^2 - 9x + 35$ on the interval $[-4, 4]$

essential statement ⟶ ★ └ f is continuous on $[-4, 4]$. By EVT, it has a max & min

└ find endpoints & critical points $[f'(c) = 0$ or DNE$]$, compare, one would be max & one would be min

- Endpoints: $x = -4, 4$
- Critical points: $f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x+1)(x-3)$ $\quad$ ∴ C.P. @ $x = -1, 3$
- Compare: $f(-4) = -41$; $f(-1) = 40$; $f(3) = 8$; $f(4) = 15$

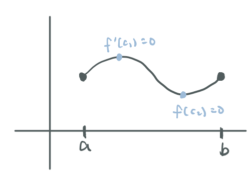
★必考 find max & min

∴ Max @ $x = -1$, Min @ $x = -4$

Rolle's Theorem

DF Let $a < b$. Let f be a function defined on $[a, b]$

IF $\begin{cases} ① \ f \text{ is continuous on } [a,b] \\ ② \ f \text{ is differentiable on } (a,b) \\ ③ \ f(a) = f(b) \end{cases}$ THEN: $\exists c \in (a, b)$ s.t. $f'(c) = 0$

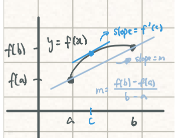


Pf f is continuous on $[a, b]$, By EVT, f must have a max & min on $[a, b]$

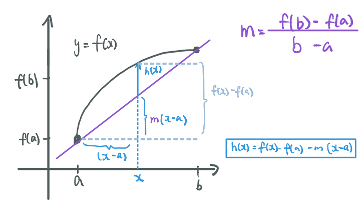
- Case 1 (If f has a max or min at some point $c \in (a,b)$)
  - └ then it's a local extremum
  - By Local EVT, $f'(c) = 0$ or DNE, since f is differentiable at c, then $f'(c) = 0$.
- Case 2 (if f has both max & min at end-points)
  - └ Since $f(a) = f(b)$, f must be constant
  - $\forall x \in (a,b)$, $f'(x) = 0$ ■

- How many zeros does a function have [Ex. $g(x) = x^6 + x^2 + x - 2$] $\because$ ① = ②, g has 2 zeros.
  1. Use IVT to prove it has at least n.
     - g is continuous on $\mathbb{R}$ $\{g(-2) = 64,\ g(0) = -2,\ g(1) = 1\}$
     - By IVT, g has at least 2 zeros ①
       - └ $-2 < x_1 < 0$ s.t. $g(x_1) = 0$ AND $0 < x_2 < 1$ s.t. $g(x_2) = 0$
  2. Use Rolle's Theorem to prove it has most n.
     - Assume $f(x_1) = f(x_2) = 0$, use Rolle's Theorem of f on $[x_1, x_2]$
     - There exists $x_1 < a < x_2$ s.t. $f'(a) = 0$
       - ① Between any 2 zeros of f, $\exists x$ s.t. $f'(x) = 0$
       - ② (# of zeros of f') + 1 $\geq$ # of zeros of f *
     - $g'(x) = 6x^5 + 2x + 1$; $g''(x) = 30x^4 + 2$ (It has no zeros)
       - Use Theorem (*) on g'': # zeros of $g' \leq$ # of zeros of g'' + 1 ≤ 1
       - Use Theorem (*) on g': # zeros of $g \leq$ # of zeros of g' + 1 ≤ 2 ②

The Mean Value Theorem



THEOREM
- Let $a < b$. Let f be a function defined on $[a,b]$
- IF ① f is continuous on $[a,b]$ $\wedge$ ② f is differentiable on (a,b)
- THEN $\exists c \in (a,b)$ s.t. $f'(c) = \dfrac{f(b) - f(a)}{b - a}$



pf
- Let $m = \dfrac{f(b) - f(a)}{b - a}$;
- Define a new function h on $[a,b]$: $h(x) = f(x) - f(a) - m(x-a)$

- h is continuous on $[a,b]$, because so is f
- h is differentiable on (a,b), because so is f
- $h(a) = 0$ & $h(b) = 0$

} verify the Rolle's hypotheses

$\therefore$ By Rolle's, $\exists c \in (a,b)$ s.t. $h'(c) = 0$ $\wedge$ $(h'(x) = f'(x) - m)$

$h'(c) = 0 \Rightarrow f'(c) = m = \dfrac{f(b) - f(a)}{b - a}$ ■

- Zero derivative implies constant
  - └ Theorem: let $a < b$, let f be a function defined on $[a,b]$
  - $\forall x \in (a,b)$, $f'(x) = 0$ $\wedge$ f is continuous on $[a,b]$ $\Rightarrow$ f is constant on $[a,b]$

pf WTS: $\forall x_1, x_2 \in [a,b]$, $f(x_1) = f(x_2)$
- Fix x_1, x_2 $[a,b]$ Assume $x_1 < x_2$
- Use MVT for f on $[x_1, x_2]$, verify the hypotheses:
  - └ f is continuous on $[a,b]$, so will also be continuous on $[x_1, x_2] \subseteq [a,b]$
  - └ f differentiable on (a,b), so will be differentiable on $(x_1, x_2) \subseteq (a,b)$
  - $\rightarrow$ $\exists c \in (x_1, x_2)$ s.t. $f'(c) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$, $f'(c) = 0$, so $f(x_1) = f(x_2)$ ■

Ex. prove $\arctan\sqrt{\frac{1-x}{1+x}} = C - \frac{1}{2}\arcsin x$ where C is a constant
- Let $F(x) = \arctan\sqrt{\frac{1-x}{1+x}} + \frac{1}{2}\arcsin x$
  - └ Domain of $F = (-1, 1]$, F in continuous on $(-1, 1]$
- $\forall x \in (-1, 1)$, $F'(x) = \ldots\ 0$.

$\therefore$ $\exists C \in \mathbb{R}$ s.t. $\forall x \in (-1,1)$, $F(x) = C$; F is continuous on $(-1, 1]$, $\forall x \in (-1, 1]$, $F(x) = C$

$\llcorner$ $C = F(0) = \arctan 1 + \frac{1}{2}\arcsin 0 = \frac{\pi}{4}$

$\rightarrow$ Conclusion: $\forall x \in [-1, 1]$, $\arctan\sqrt{\frac{1-x}{1+x}} = \frac{\pi}{4} - \frac{1}{2}\arcsin x$ ■

- Integrations Claim:

IF f satisfies $\forall x \in \mathbb{R}$, $f'(x) = x^2$ THEN, $\exists C \in \mathbb{R}$ s.t. $\forall x \in \mathbb{R}$, $f(x) = \frac{1}{3}x^3 + C$

Theorem ①: IF h has zero derivative on an open interval I, THEN h is constant on I

Corollary: IF f & g have same derivative on an open interval I THEN $f-g$ is constant on I

- Assume $f'(x) = x^2$ for all $x \in \mathbb{R}$ THEN $\frac{d}{dx}[f(x)] = \frac{d}{dx}\left[\frac{1}{3}x^3\right]$
- Therefore $f(x) - \frac{1}{3}x^3 = C$ $\therefore$ $f(x) = \frac{1}{3}x^2 + C$ are all the solutions to $f'(x) = x^2$

Monotonicity of function

increasing $\neq$ derivative > 1

DF Let f be a function defined on an interval I

- f is increasing on I when: $\forall x_1, x_2 \in I$ $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$
- f is non-increasing on I when: $\forall x_1, x_2 \in I$, $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$

- Theorem for increasing & derivatives
  - let $a < b$, let f be a function defined on $[a, b]$, f is continuous on $[a, b]$

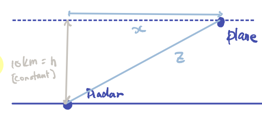
    IF $\forall x \in (a, b)$, $f'(x) > 0$ THEN f is increasing on (a, b)

APPLICATIONS OF DERIVATIVES

Related Rates

Ex. A plane, flying horizontally at an altitude of 10km, passes directly above a radar station. A bit later, the radar station measures that the distance between the plane and the station is 20km, and is increasing at a rate of 1,000 km/h. What is the speed of the plane?

① Modelling



[have] x, z depend on time, [relation] $z^2 = x^2 + h^2$

[want]: $\frac{dx}{dt}$, when $\begin{cases} z = 20\text{ km} \\ \frac{dz}{dt} = 1000\text{ km/h} \end{cases}$

② Calculus

$$\frac{d}{dt}z^2 = \frac{d}{dt}h^2 + x^2$$

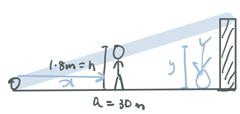
$$2z \cdot \frac{dz}{dt} = 0 + 2x \cdot \frac{dx}{dt}, \quad (h \text{ is constant})$$

$$\frac{dx}{dt} = \frac{z}{x} \cdot \frac{dz}{dt}, \quad (\text{sub in values, } x = \sqrt{z^2 - h^2})$$

$$= \frac{20\text{ km}}{\sqrt{20^2 - 10^2}} \cdot 1000\frac{\text{km}}{\text{h}} = \boxed{\frac{2000}{\sqrt{3}}\text{ km/h}}$$

$\rightsquigarrow$ speed of the plane

Ex. A prisoner is trying to escape at night. He runs in a straight line towards the wall of the prison. A guard points a spotlight at the prisoner. The spotlight is on the ground, 30 m away from the wall. The prisoner is 1.8 m tall and weights 85 kg. How fast is the height of the shadow on the wall changing when the prisoner is 20 m away from the wall and running at 8 m/s?



x, y depends on t

want: $\frac{dy}{dx}$, when $\begin{cases} \frac{dx}{dt} = 8\text{ m/s} \\ x = 10\text{ m} \quad (30-20=10) \end{cases}$

$\frac{h}{x} = \frac{y}{a}$

$$\frac{d}{dt}\frac{h}{x} = \frac{d}{dt}\frac{y}{a}$$

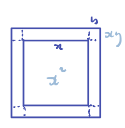
$$h \cdot \left(\frac{-1}{x^2}\right)\frac{dx}{dt} = \frac{1}{a} \cdot \frac{dy}{dt}$$

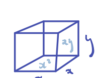
$$\frac{-ah}{x^2} \cdot \frac{dx}{dt} = \frac{dy}{dt} = \frac{-30(1.8)}{10^2} \cdot 8 = \boxed{-4.32\text{ m/s}}$$

(height is decreasing)

Optimization

Ex. I want to build an open box (four side walls and a bottom, but no top) with a square base. The volume of the box must be $500cm^3$. I want to spend as little as possible on the materials. What dimensions should I choose?



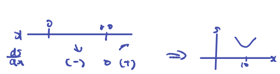


Want min $x^2 + 4xy$ with $x^2y = 500$ $\begin{cases} x > 0 \\ y > 0 \end{cases}$

y depend on x $\begin{cases} V: & y = \frac{500}{x^2} \\ S: & S = x^2 + 2000x, \quad x > 0 \end{cases}$

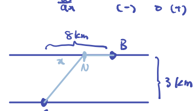
$$\frac{dS}{dx} = 2x - \frac{2000}{x^2} = \frac{2(x^3 - 1000)}{x^2}$$

$$\frac{dS}{dx} = 0 \iff x = 10\text{ cm}$$



$\therefore$ S has a min @ $x = 10$ cm ; $y = \frac{500}{x^2} = 5$ cm

Ex. A woman gets on her boat at point A on one side of a straight river, 3 km wide. She wants to get to point B, 8 km downstream on the opposite side. She could row the boat directly across the river and then run for 8 km; she could row directly from A to B; or she could row to some intermediate point and then run. If she can row 6 km/h and run 8 km/h, where should she land to reach B as soon as possible?



run: 8km/h
row: 6km/h

want min $T = \frac{1}{8}(8 - x) + \frac{1}{6}\sqrt{x^2 + 9}$, $0 \leq x \leq 8$

$T = \frac{1}{6}\sqrt{x^2+9} + 1 - \frac{x}{8}$

- By MVT, T must have a min $\begin{cases} \text{critical point} \\ \text{endpoint} \end{cases}$

$\frac{dT}{dx} = \frac{1}{12} \cdot \frac{1}{\sqrt{x^2+9}} \cdot 2x - \frac{1}{8}$ ... $\frac{dT}{dx} = 0 \iff x = \frac{9}{\sqrt{7}}$ in $0 \leq x \leq 8$

x	T
0	1.5 h
$9/\sqrt{7}$	≈ 1.3 h
8	≈ 1.42 h

$\therefore$ run $8 - 9/\sqrt{7}$, row: $\sqrt{\left(\frac{9}{\sqrt{7}}\right)^2 + 9}$

Assume $\begin{cases} \lim_{x\to a} f(x) = 1 \\ \lim_{x\to a} g(x) = 0 \end{cases}$

$\lim_{x\to 0} \frac{f(x)}{g(x)}$ $\begin{cases} \infty & \text{if } g(x) > 0 \\ -\infty & \text{if } g(x) < 0 \end{cases}$

1^{∞} could be anything

$\lim_{x\to 0^+} \left(e^{x}\right)^{\frac{c}{x}) \pm\infty} = e^{c}$ (any real)

L'Hôpital's Rule

- Indetermed limits
  - $\frac{0}{0}$, $\frac{\pm\infty}{\pm\infty}$, $0\cdot\infty$, $\infty-\infty$, 0^0, ∞^0, $1^{\pm\infty}$

THEORM:

- Let f, g be continuous. Let $a \in \mathbb{R}$. Want $\lim_{x\to a} \frac{f(x)}{g(x)}$ (also works for $a = \pm\infty$ or side limits)

IF
① f, g are differentiable as $x \to a$
② g and g' are never 0 as $x \to a$
③ The limit $\lim_{x\to a} \frac{f(x)}{g(x)}$ is an indetermine form of type $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$
④ The limit $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists or is ∞ or $-\infty$

THEN $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

Ex. $\lim_{x\to\infty} \frac{x}{\ln x} = \frac{\infty}{\infty} \overset{(L'H)}{=} \lim_{x\to\infty} \frac{1}{\frac{1}{x}} = \infty$

Ex. $\lim_{x\to 0} \frac{\cos x - \cos(2x)}{xe^{x} - x} = \frac{0}{0} \overset{L'H}{=} \frac{-\sin x + 2\sin(2x)}{e^{x} + xe^{x} - 1} = \frac{0}{0} \overset{L'H}{=} \frac{-\cos + 4\cos(2x)}{1 + e^{x} + xe^{x}} = \frac{3}{2}$

Ex. $\lim_{x\to 1} \frac{x^3 - 2x + 1}{x^2 + 3x + 2} = \frac{0}{6} = 0.$

$\to \infty \cdot 0$

Ex. $\lim_{x\to\infty} x\left(1 - e^{\frac{2}{x}}\right) = \infty\cdot 0 \quad \left(x = \frac{1}{\frac{1}{x}}\right) \Rightarrow = \lim_{x\to\infty} \frac{1 - e^{\frac{2}{x}}}{\frac{1}{x}} = \frac{0}{0} \overset{L'H}{=} \frac{e^{\frac{2}{x}} \cdot \frac{-2}{x^2}}{\frac{-1}{x^2}} = -2e^{2x} = -2$

$\to \infty - \infty$

Ex. $\lim_{x\to\infty} \left[\sqrt{x^2 - x} - x\right] = \infty - \infty$

- Method 1 (factor biggest possible term)

$\lim_{x\to\infty} \left[\sqrt{x^2 - x} - x\right] = \lim_{x\to\infty} \left[\sqrt{x^2}\sqrt{1 - \frac{1}{x}} - x\right] = \lim_{x\to\infty} \underbrace{x\left[\sqrt{1 - \frac{1}{x}} - 1\right]}_{\infty\cdot 0} = \overbrace{\lim_{x\to\infty} \frac{\sqrt{1 - \frac{1}{x}} - 1}{\frac{1}{x}}}^{\frac{0}{0}} \overset{L'H}{=} \frac{\frac{1}{2}\sqrt{1 - \frac{1}{x}}\left(\pm\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{-1}{2\sqrt{1 - \frac{1}{x}}} = \frac{-1}{2}$

$\to \sqrt{x^2} = |x| = x$

- Method 2 (conjugate)

$= \lim_{x\to\infty} \frac{\left[\sqrt{x^2 - x} - x\right]\left[\sqrt{x^2 - x} + x\right]}{\sqrt{x^2 - x} + x} = \lim_{x\to\infty} \frac{\left(\sqrt{x^2 - x}\right)^2 - x^2}{\sqrt{x^2 - x} + x} = \lim_{x\to\infty} \frac{-x}{\sqrt{x^2 - x} + x} = \lim_{x\to\infty} \frac{-x}{x\left[\sqrt{1 - \frac{1}{x}} + 1\right]} = \frac{-1}{2}$

- Indeterminate forms of exponential type $(0^0, \infty^0, 1^{\pm\infty})$

Ex. $\lim_{x\to 0^+} (1 - x)^{\frac{1}{x}} = 1^{\infty}$

- call $f(x) = (1 - x)^{\frac{1}{x}}$ $\quad \ln f(x) = \ln (1 - x)^{\frac{1}{x}} = \frac{1}{x} \ln(1 - x)$

① $\lim_{x\to 0^+} [\ln f(x)] = \lim_{x\to 0^+} \frac{\ln(1 - x)}{x} \left(\frac{0}{0}\right) \overset{L'H}{=} \frac{-1/(1 - x)}{1} = \lim_{x\to 0^+} \frac{-1}{1 - x} = -1$

② $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{\ln f(x)} = e^{-1} = \boxed{\frac{1}{e}}$

Concavity

- Concave up: $\smile$: when secant segments are above the graph / when tangent are below the graph / slope increases
- Concave down: $\frown$: when secant segments are below the graph / when tangent are below the graph / slope decreases

DF let f be differentiable function defined on an interval I
- f is concave-up on I when f' is increasing on I
- f is concave-down on I when f' is decreasing on I
- let c be an interior point on I, f has a inflection point at c when "f changes concavity at c"

Open interval
$f' > 0 \Rightarrow \nearrow$ $\quad f'' > 0 \Rightarrow \smile$
$f' < 0 \Rightarrow \searrow$ $\quad f'' < 0 \Rightarrow \frown$

THEOREM 1:
- let I be an open interval, f is twice differentiable function defined on I

$\forall x \in I,\ f''(x) > 0 \Rightarrow f$ is concave up on I ; $\forall x \in I,\ f''(x) < 0 \Rightarrow f$ is concave down on I

Close interval
- endpoints requires continuity

THEOREM 2:
- let I be an open interval, f is twice differentiable function defined on I, let $c \in I$

f has an inflection point at $c \Rightarrow f''(c) = 0$ or DNE

Points
- extremum $f'(c) = 0$ / DNE
- inflection $f''(c) = 0$ / DNE

Ex. let $f(x) = x^6 (x+3)^3$, Sketch the graph $\quad [f'(x) = 9x^5 (x+3)^2 (x+2)$; $f''(x) = 18x^4 (x+3)(2x+3)(2x+5)]$

① Intervals when f is increasing or decreasing

② Local extrema

③ intervals where f is concave up or concave down

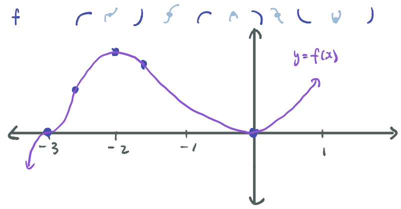
④ Inflection points

x: -3, $-5/2$, -2, $-3/2$, 0

f': $+$ 0 $+$ $+$ 0 (L.max) $-$ $-$ 0 (L.min) $+$ ⇒ increasing : $(-\infty,-2] \cup [0,\infty)$; decresing : $[-2,0]$; L.max : $(-2,64)$; L.min : $(0,0)$

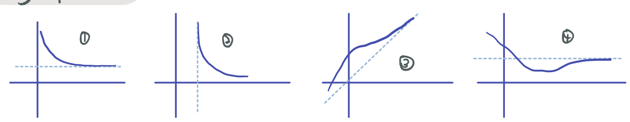
f'': $-$ 0 $+$ 0 $-$ $-$ 0 $+$ 0 $+$ ⇒ c.down : $(-\infty,-3] \cup [-5/2,-3/2]$; c.up : $[-3,-5/2] \cup [-3/2,\infty)$ inflection $(-3,0)$, $(-5/2, 30.5)$, $(-3/2, 38.4)$

f



Asymptotes

Ex.



horizontal asymptotes is a special case of →

- Vertical asymptotes :
  - Let f be a function, let $a \in \mathbb{R}$
  - The vertical line $x=a$ is an asymptote of f when : $\lim_{x \to a^{\pm}} f(x) = \pm\infty$

Ex. ② or $f(x) = \frac{x+1}{x^3-x^2}$; $f(x) = \tan x$; $f(x) = e^{\frac{1}{x}}$; $f(x) = \frac{\cos x}{e^x - 1}$

- Horizontal asymptotes :
  - Let f be a function, let $L \in \mathbb{R}$
  - The horizontal line $y = L$ is an asymptote of f when : $\lim_{x \to \pm\infty} f(x) = L$

Ex. ① & ④ or $f(x) = \frac{x^3-1}{x^3-x^2}$; $f(x) = \arctan x$; $f(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}}$; $f(x) = x - \sqrt{x^2 - x}$

- Slant asymptotes :
  - Let f be a function, let $m, b \in \mathbb{R}$
  - The line $y = mx + b$ is an asymptote of f when : $\lim_{x \to \pm\infty} [f(x) - (mx+b)] = 0$

Ex. ③ or $f(x) = \frac{x^3+1}{x^2+1}$; $f(x) = x \arctan x$; $f(x) = x + 1 + e^{\frac{1}{x^2}}$; $f(x) = x^2 \sin \frac{1}{x}$

Ex. Find asymptotes of $f(x) = \frac{x^3+1}{x^3-2x^2}$, stetch the graph $f(x) = \frac{x^3-1}{x^2(x-2)}$

L VA. $\lim_{x \to 0^+} \frac{x^3+1}{x^2(x-2)} = -\infty = \lim_{x \to 0^-} \frac{x^3+1}{x^2(x-2)}$; $\lim_{x \to 2^+} \frac{x^3+1}{x^2(x-2)} = \infty$ $\lim_{x \to 2^-} \frac{x^3+1}{x^2(x-2)} = -\infty$

L H.A $\lim_{x \to \pm\infty} \frac{x^3+1}{x^3-2x^2} = \lim_{x \to \pm\infty} \frac{\cancel{x^3}[1 + 1/x^3]}{\cancel{x^3}(1 - 2/x)} = 1$