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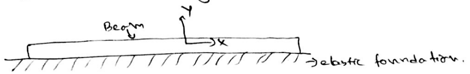
Answer to Problem II

$u(x,t) \rightarrow$ vertical vibrations

$L \rightarrow$ beam length



(1) $$\frac{\delta^2 u}{\delta t^2} = -c^2 \frac{\delta^4 u}{\delta x^4}$$

$$\frac{\delta u}{\delta t} = 0, \quad u(x,0) = x(L-x)$$

where $c^2 = \dfrac{EI}{\rho A}$

E : Young's Modulus of elasticity
I : Moment of inertia
ρ : Density
A : Cross Sectional Area, 1

Boundary conditions,

$$u(0,t) = u(L,t) = 0$$

$$u_{xx}(0,t) = u_{xx}(L,t) = 0$$

From, $u(x,t) = F(x)\,G(t)$

$$F G'' = -c^2 F^{IV} G$$

$$\frac{G''}{-c^2 G} = \frac{F^{IV}}{F} = \lambda^4$$

λ^4 is a constant

When $\lambda = 0$,

$$\frac{F^{IV}}{F} = 0$$

$$\therefore \; F(x) = ax^3 + bx^2 + cx + d$$

$$F(0)\, G(t) = 0$$

$$F(L)\, G(t) = 0$$

$$F(0) = F(L) = 0 \;, \quad G(t) = 0$$

$$F''(0)G(t) = 0$$

$$F''(L)\, G(t) = 0$$

$$F''(0) = F''(L) = 0$$

$$F(0) = d = 0$$

$$F'(x) = 3ax^2 + 2bx + c$$

$$F''(x) = 6ax + 2b$$

$$F''(0) = 2b = 0$$

$$\Rightarrow b = 0$$

$$F(L) = aL^3 + cL = 0$$

$$F''(L) = 6aL$$

$$\therefore \; a = 0$$

$$c = 0$$

Hence $F(x) = 0$.

For, $\lambda \neq 0$,

$$\frac{F^{IV}}{F} = \lambda^4$$

$$F^{IV} - \lambda^4 F = 0$$

$$F(x) = A_1 \cos \lambda x + A_2 \sin \lambda x + A_3 e^{\lambda x} + A_4 e^{-\lambda x}$$

$$= A_1 \cos \lambda x + A_2 \sin \lambda x + \left(\frac{A_5 + A_6}{2}\right) e^{\lambda x}$$

$$+ \left(\frac{A_5 - A_6}{2}\right) e^{-\lambda x}$$

$$= A_1 \cos \lambda x + A_2 \sin \lambda x + A_5 \left(\frac{e^{\lambda x} + e^{-\lambda x}}{2}\right)$$

$$+ A_6 \left(\frac{e^{\lambda x} - e^{-\lambda x}}{2}\right)$$

$$F(x) = A_1 \cos \lambda x + A_2 \sin \lambda x + A_5 \cosh \lambda x$$

$$+ A_6 \sinh \lambda x$$

$$F'(x) = -A_1 \lambda \sin(\lambda x) + A_2 \lambda \cos(\lambda x) + A_5 \lambda \sinh(\lambda x)$$

$$+ A_6 \lambda \cosh(\lambda x)$$

$$F''(x) = -A \lambda^2 \cos(\lambda x) - A_2 \lambda^2 \sin(\lambda x)$$

$$+ A_5 \lambda^2 \sinh(\lambda x) + A_6 \lambda^2 \cosh(\lambda x)$$

When $x = 0$,

$F(0) = A_1 + A_5 = 0$

$F''(0) = -A_1\lambda^2 + A_5\lambda^2 = 0$

Hence $A_1 = A_5 = 0$

When, $x = L$

$F(L) = A_2 \sin(\lambda L) + A_6 \sinh(\lambda L)$

$= 0$

$F''(L) = -A_2\lambda^2 \sin(\lambda L) + A_6\lambda^2 \sinh(\lambda L)$

$= 0$

$\therefore D = 0$

$\therefore A_2 \sin(\lambda L) = 0$

$\sin(\lambda L) = 0, \quad A_2 \neq 0$

$\lambda = \frac{n\pi}{L}$ where $n = 1, 2, 3 \ldots \infty$

$F_n(x) = A_2 \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3 \ldots \infty$

$$G'' + c^2 \lambda^4 G = 0$$

$$G_n(t) = a_n \cos(c\lambda^2 t) + b_n \sin(c\lambda^2 t)$$

$$\lambda = \frac{\pi n}{L}$$

$$u_n(x,t) = F_n(x)\, G_n(t)$$

$$= \sin\left(\frac{n\pi x}{L}\right)\left(A_2 a_n \cos(c\lambda^2 t) + A_2 b_n \sin(c\lambda^2 t)\right)$$

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left(A_2 a_n \cos\left(c\left(\frac{n\pi}{L}\right)^2 t\right) + A_2 b_n \sin\left(c\left(\frac{n\pi}{L}\right)^2 t\right)\right)$$

$$\left.\frac{\delta u}{\delta t}\right|_{t=0} = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) A_2 b_n\, c\left(\frac{n\pi}{L}\right)^2 = 0$$

$$A_2 b_n = 0$$

$$u(x,0) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) A_2 a_n$$

$$\Rightarrow f(x) = x(L-x)$$

$$\therefore A_2 a_n = \frac{2}{L}\int_0^L x(L-x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{L}\int_0^L x L \sin\left(\frac{n\pi x}{L}\right) - \frac{2}{L}\int_0^L x^2 \sin\left(\frac{n\pi x}{L}\right)dx$$

$$= I_1 - I_2$$

$$\int uv = u\int v - \int u' \int v$$

$$I_1 = \frac{2}{L}\cdot L\left[-x.\frac{\cos\left(\frac{n\pi x}{L}\right)}{\frac{n\pi}{L}} + \frac{\sin\left(\frac{n\pi x}{L}\right)}{\frac{n^2\pi^2}{L^2}}\right]_0^L$$

$$= \frac{2}{L}\left[-\frac{L^3}{n\pi}\cos(n\pi)\right]$$

$$I_2 = \frac{2}{L}\left[-x^2\frac{\cos\left(\frac{n\pi x}{L}\right)}{\frac{n\pi}{L}} + \frac{2x\sin\left(\frac{n\pi x}{L}\right)}{\frac{n^2\pi^2}{L^2}}\right.$$

$$\left. + \frac{2\cos\left(\frac{n\pi x}{L}\right)}{\frac{n^3\pi^3}{L^3}}\right]_0^L$$

$$= \frac{2}{L}\left[-\frac{L^3}{n\pi}\cos(n\pi) + \frac{2L^3}{n^3\pi^3}\cos(n\pi) - \frac{2L^3}{n^3\pi^3}\right]$$

$$\therefore A_2\, a_n = \frac{2}{L}\left[\frac{2L^3}{n^3\pi^3} - \frac{2L^3}{n^3\pi^3}\cos(n\pi)\right]$$

$$= \frac{4L^2}{n^3\pi^3} - \frac{4L^2}{n^3\pi^3}\cos(n\pi)$$

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[\frac{4L^2}{n^3\pi^3} - \frac{4L^2}{n^3\pi^3}\cos(n\pi)\right]\cos\left(c\left(\frac{n\pi}{L}\right)^2 t\right)$$

(2) $$\frac{\delta^2 u}{\delta t^2} = -c^2\frac{\delta^4 u}{\delta x^4}$$

$$\frac{\delta u}{\delta t} = 0, \quad u(x,0) = x(L-x)$$

$$u(0,t) = u(L,t) = 0$$

$$\therefore u_{xx}(0,t) = u_{xx}(L,t) = 0$$

$$\frac{\delta u(t)}{\delta t} = v(t)$$

$$\frac{\delta^2 u(t)}{\delta t^2} = \frac{\delta v(t)}{\delta t} = -\frac{c^2\delta^4 u(t)}{\delta x^4}$$

$$\frac{\delta u}{\delta t}\Big|_{t=0} = 0$$

$$v(0) = 0$$

$$V(0) = v_0 = 0$$

$$u(x,0) = u(0)$$

$$\Rightarrow x(L-x) = u_0$$

$$\frac{\delta u(t)}{\delta t} = v(t)$$

$$\frac{\delta^2 y}{\delta t^2} = \frac{\delta v(t)}{\delta t} = -c^2 \frac{\delta^4 u(t)}{\delta x^4}$$

$$F(u(t)) = -c^2 \frac{\delta^4 u(t)}{\delta x^4}$$

(3) $\begin{vmatrix} A_\tau & 0 \\ 0 & A_\tau \end{vmatrix} \begin{vmatrix} u_\tau \\ v_\tau \end{vmatrix} = \begin{vmatrix} f_\tau \\ g_\tau \end{vmatrix}$

$\tau = \frac{1}{3}$, $0 \leq x \leq L$

$\cancel{X} = \begin{pmatrix} u \\ v \end{pmatrix}$

$$\frac{d\cancel{X}}{dt} = \begin{vmatrix} \frac{\delta u}{\delta t} \\ \frac{\delta v}{\delta t} \end{vmatrix}$$

$$= \begin{vmatrix} v \\ -c^2 \frac{\delta^4 u}{\delta x^4} \end{vmatrix}$$

$$\therefore \frac{X(n) - X(n-1)}{\tau} = \begin{vmatrix} \frac{u(n) - u(n-1)}{\tau} \\ \frac{v(n) - v(n-1)}{\tau} \end{vmatrix} = \begin{vmatrix} v(n-1) \\ -\frac{c^2 \delta^4 u(n-1)}{\delta x^4} \end{vmatrix}$$

$u(n) = u(n-1) + \tau v(n-1) = f_\tau$

$v(n) = v(n-1) - \tau c^2 \frac{\delta^4 u(n-1)}{\delta x^4} = g_\tau$

$u(n) = u_\tau$ $\quad\quad A_\tau = 1.$

$v(n) = v_\tau$

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} u(n) \\ v(n) \end{vmatrix} = \begin{vmatrix} u(n-1) + \tau v(n-1) \\ v(n-1) - \dfrac{\tau c^2 \delta^4 u(n-1)}{\delta x^4} \end{vmatrix}$$

$A\tau = 1$

Answer to Problem (III)

(i) Feasible region V,

- Space of real valued functions,

w on $\bar{\Omega} = \Omega \cup \delta\Omega$

such that

w & $\nabla . w$ are integrable on Ω

$V = \{ w : \bar{\Omega} \to \mathbb{R} \mid w$ & ∇w are L^2 on Ω & $w = 0$ on $\partial\Omega \}$

$V = H_0^2(w) = C_0'(\Omega)$, zero boundary condition.

(ii) $J(v) = \frac{1}{2} \int_\Omega \left(\frac{d^2 v}{\delta x^2}\right)^2 dx - \int_\Omega f v \, dx$

$J(u) = \min_{v \in V} J(v)$

The variational equation $\langle J'(u), v\rangle = 0$

$\forall v \in V$

$$\left.\frac{J(u+\lambda v) - J(u)}{\lambda}\right|_{\lambda>0}$$

$$= \frac{1}{\lambda}\left[\frac{1}{2}\int_\Omega \left(\frac{d^2(u+\lambda v)}{dx^2}\right)^2 dx - \int_\Omega f(u+\lambda v)\,dx - \frac{1}{2}\int_\Omega \left(\frac{d^2 u}{dx^2}\right)^2 dx + \int_\Omega f u\,dx\right]$$

$$= \frac{1}{\lambda}\left\{\frac{1}{2}\int_\Omega \left(\frac{d^2 u}{dx^2} + \lambda\frac{\delta^2 v}{\delta x^2}\right)^2 dx - \int_\Omega f u\,dx - \lambda\int_\Omega f v\,dx - \frac{1}{2}\int_\Omega \left(\frac{d^2 u}{dx^2}\right)^2 dx + \int_\Omega f u\,dx\right]$$

$$= \int_\Omega \Delta u . \Delta v\,dx + \frac{\lambda}{2}\int_\Omega \left(\frac{d^2 v}{dx^2}\right)^2 dx - \int f v\,dx$$

$$\left.\frac{J(u+\lambda v) - J(u)}{\lambda}\right|_{\lim_{\lambda\to 0^+}} = \int_\Omega \Delta u . \Delta v\,dx - \int f v\,dx = 0$$

$\therefore \int_\Omega \Delta u \cdot \Delta v \, dx = \int_\Omega f v \, dx, \quad \forall v \in V$

Boundary value problem,

$\Delta^2 u = f$ on Ω.

$u = \Delta u = 0$ on $\delta\Omega$.

Integrating by parts : $v \times [\Delta^2 u = f v]$

$\int_\Omega \Delta u \cdot \Delta v \, dx = \int_\Omega f v \, dx$

(iii) $u(x) = \frac{f_0}{24}(x^4 - 2Lx^3 + L^3 x)$

$u'(x) = \frac{f_0}{24}(4x^3 - 6Lx^2 + L^3)$

$u''(x) = \frac{f_0}{24}(12x^2 - 12Lx)$

$u'''(x) = \frac{f_0}{24}(24x - 12L)$

$u^{iv}(x) = \frac{f_0}{\cancel{24}} \cdot \cancel{24} = f_0$

$$u(0) = \frac{f_0}{24} \cdot 0 = 0 \quad , \quad u(L) = \frac{f_0}{24} \cdot (4L^3 - 6L^3 + L^3)$$
$$= 0$$

$$u''(0) = \frac{f_0}{24} \cdot 0 = 0 \quad , \quad u''(L) = \frac{f_0}{24} (12L^2 - 12L^2)$$
$$= 0$$

$$\therefore \quad u^{IV}(x) = f(x)$$
$$u(0) = u(L) = 0$$
$$u''(0) = u''(L) = 0.$$