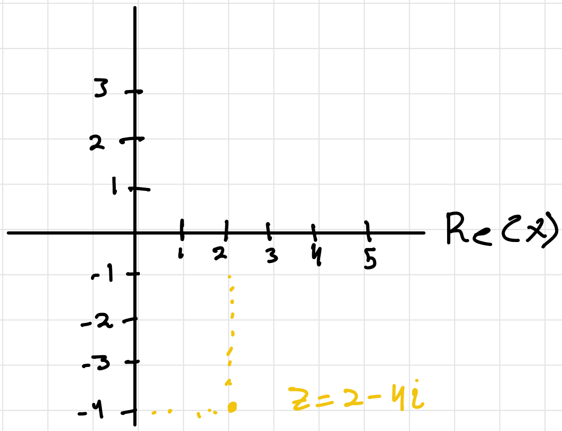
Q1) a)

$z = 2-4i$

$z^* = 2+4i$

$|z|^2 = (2-4i)(2+4i) = 20$

b)



c) $z = Ae^{i\theta}$

$A = \sqrt{|z|^2} \Rightarrow \sqrt{20} = \boxed{4.4721}$

$\theta = \tan^{-1}\left(\frac{Im(z)}{Re(z)}\right)$

$= \tan^{-1}\left(\frac{-4}{2}\right) = -1.1071$ radians

$\Rightarrow$ in $0 \leq \theta \leq 2\pi \Rightarrow \theta + 2\pi$

$= -1.1071 + 2\pi$

$\boxed{= 5.178 \text{ rad}}$

Q2) 4-points: $1, i, -1, -i$

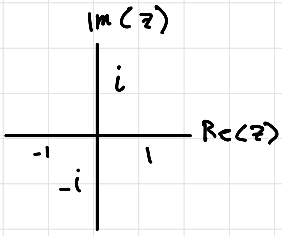
$A=1$

$1 = e^{i\cdot 0}$

$i = e^{i\frac{\pi}{2}}$

$-1 = e^{i\pi}$

$-i = e^{i\frac{3\pi}{2}}$



b)

$i \cdot 1 = i$

$i \cdot i = -1$

$i \cdot -1 = -i$

$i \cdot -i = 1$

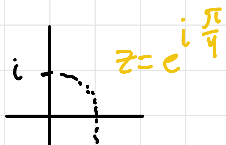
Q3) a)

$z = i e^{-i\pi/4}$

write i in front as $i = e^{i\frac{\pi}{2}}$

$z = e^{i\frac{\pi}{2}} e^{-i\frac{\pi}{4}} = \boxed{e^{+i\frac{\pi}{4}}}$

b)



• multiplying by i is like multiplying by $e^{i\pi/2}$ which add $\frac{\pi}{2}$ to θ = rotate counter-clock wise by $\pi/2$

Q4)

a) $\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1/2 \end{pmatrix} = \begin{pmatrix} 0 & -i/2 \\ -i & 0 \end{pmatrix}$

$AC = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix} = \begin{pmatrix} 4/5\, i \\ -3/5\, i \end{pmatrix}$

b) $\det(\lambda I - A) = \begin{vmatrix} \lambda - 0 & -i \\ i & \lambda - 0 \end{vmatrix} = \lambda^2 - 1 = 0$

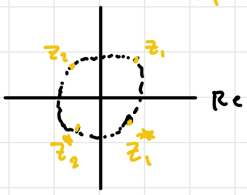
$\lambda = \pm 1$

c) CA

• we can't get the product since the number of columns of C doesn't equal the number of rows of A, according to matrix multiplications.

$CA = \begin{pmatrix} \cdot \\ \cdot \end{pmatrix}\begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}$

Q5) a) $z_1 = e^{i\pi/4}$, $z_2 = e^{i3\pi/4}$, $z_1^* = e^{-i\frac{\pi}{4}}$, $z_2^* = e^{-i3\pi/4}$



b)

$$z_1 z_2 = e^{i\pi/4} \cdot e^{i\pi/4} = e^{i\pi} = -1$$

$$z_2 / z_1 = \frac{e^{i3\pi/4}}{e^{i\pi/4}} = e^{i\pi/2} = i$$

$$z_2 \cdot z_1^* = e^{i3\pi/4} \cdot e^{-i\frac{\pi}{4}} = e^{i\pi/2} = i$$

* note $z^* = \frac{1}{z_1}$

c) $z_1 = \frac{1+i}{\sqrt{2}} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$, $z_2 = \frac{-1+i}{\sqrt{2}} = \frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$

$$z_1 z_2 = \left(\frac{1+i}{\sqrt{2}}\right)\left(\frac{-1+i}{\sqrt{2}}\right) = \frac{1}{2}(-1+i-i-1) = \underline{\underline{-1}}$$

Same as b)

Q6) a)

$z = a + bi$
$z^* = a - bi$

$\Big\}$ Add: $z + z^* = 2a \Rightarrow Re\{z\} = a = \frac{z+z^*}{2}$

Subtract: $z - z^* = 2bi \Rightarrow Im\{z\} = b = \frac{z-z^*}{2i}$

b) let $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i$

$$z_1 z_2 = (a_1 + b_1 i)(a_2 + b_2 i) = a_1 a_2 - b_1 b_2 + i(a_1 b_2 + b_1 a_2)$$

$$\Rightarrow Re\{z_1 z_2\} = a_1 a_2 - b_1 b_2 = Re\{z_1\} Re\{z_2\} - Im\{z_1\} Im\{z_2\}$$

$$Im\{z_1 z_2\} = a_1 b_2 + b_1 a_2 = Re\{z_1\} Im\{z_2\} + Im\{z_1\} Re\{z_2\}$$

c) $z = Ae^{i\theta} \Rightarrow |z|^2 = A^2$ but $z^2 = A^2 e^{2i\theta}$

$Re\{|z|^2\} = A^2$, $Re\{z^2\} = A^2 \cos 2\theta$ ← from Euler

• In another way: $z = a + bi \Rightarrow |z|^2 = a^2 + b^2$ but, $z^2 = (a^2 - b^2) + 2abi$

d) $z = Ae^{i\theta}$ $\quad Re\{z\} = A\cos\theta$ $\quad Im\{z\} = A\sin\theta$

$z^* = Ae^{-i\theta}$, $|z| = A$ $\quad$ results from Euler's formula

e) $e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

add: $e^{i\theta} + e^{-i\theta} = 2\cos\theta$

$\Rightarrow \cos\theta = \dfrac{e^{i\theta} + e^{i\theta}}{2}$

subtract:

$e^{i\theta} - e^{-i\theta} = 2i\sin\theta \Rightarrow \sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$