How do stars shine?

(According to Physicists.)

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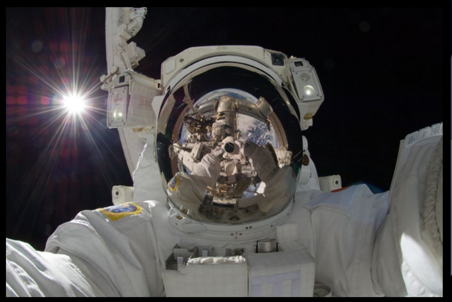
Oral Portion of the Doctoral Candidacy Exam

The Ohio State University
CENTER FOR COSMOLOGY AND ASTROPARTICLE PHYSICS

Motivation



Humanity has been fascinated with stars since antiquity, but a detailed understanding of how they shine has only been possible in the last century. A theoretical description of stars involves fascinating interdisciplinary physics which have far-reaching applications outside of astronomy.

Outline

01 Fundamental Assumptions

02 Stellar Structure

03 Stellar Energy Generation

04 Energy Transport in Stars

05 Applications of Stars

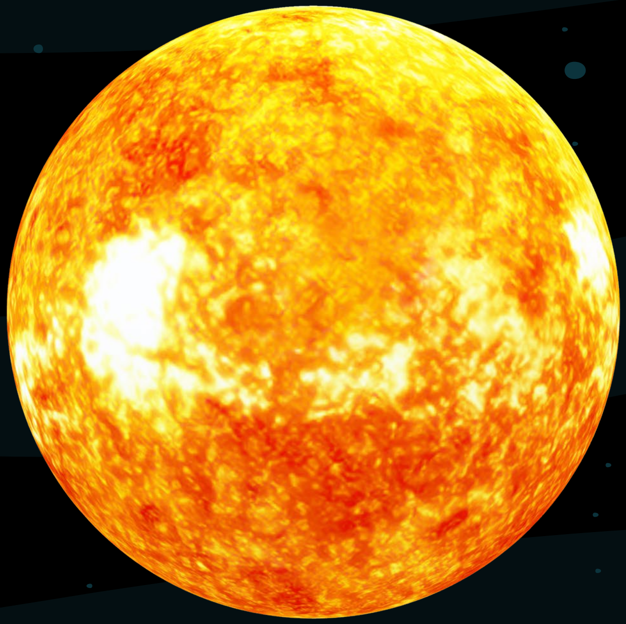
In this talk, I address the central question of "How do stars shine?" in the context of the Sun and from the perspective of a theoretical physicist.

01

Fundamental Assumptions

A Theorist's Simple Star

Stars are rich environments requiring a broad application of many kinds of physics. To make answering the central question tractable, I focus the theoretical description by considering a following toy model of a star.



Spherically Symmetric
At Constant Composition
Non-rotating
Non-magnetic
In Thermodynamic Equilibrium
Companionless
Sun-like on the Main Sequence

Physics of a Simple Star

I further specialize my discussion by focusing on the most relevant physics in the Sun:

Nuclear Fusion	Electromagnetic Radiation	Equation of State
Consider • Hydrogen-fusing reactions **Omit** • Helium-burning and other processes	**Consider** • Blackbody radiation fields **Omit** • Non-thermal emission processes	**Consider** • Monatomic ideal gasses **Omit** • Relativistic and degenerate physics

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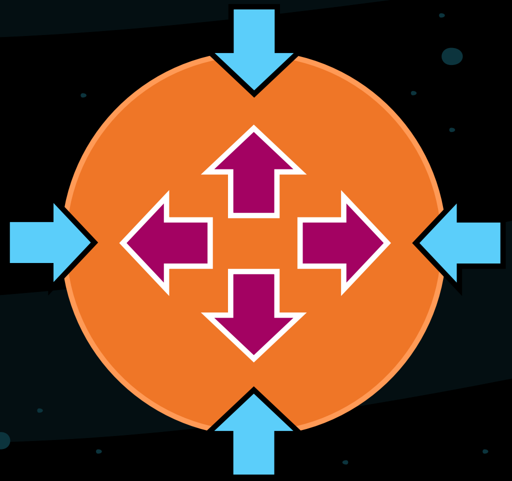
02

Stellar Structure

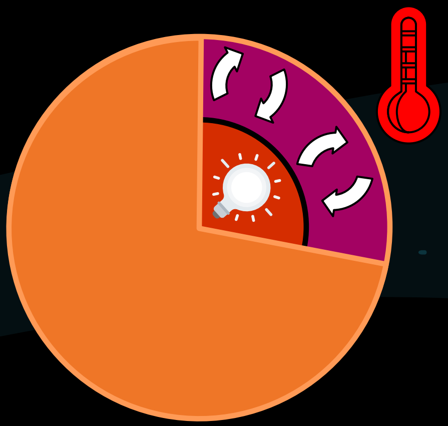
Equations of Stellar Structure

To frame the question of “How do stars shine?”, consider three canonical equations describing the structure of stars:

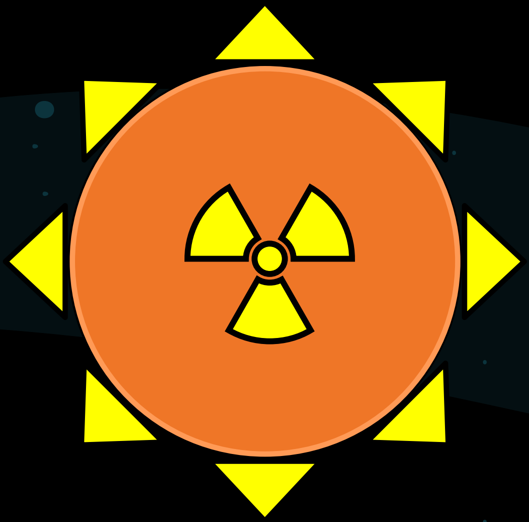
Hydrostatic Equilibrium



Thermal Structure



Energy Conservation

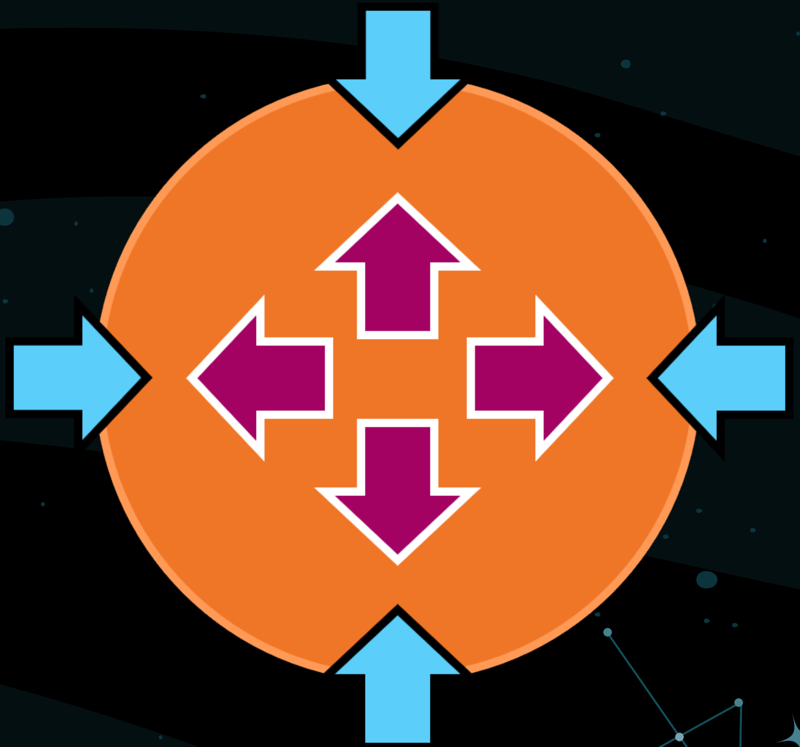


Hydrostatic Equilibrium

$$\frac{\partial P}{\partial r} = -\frac{Gm\rho}{r^2}$$

- Describes the relationship between pressure, radial distance, enclosed mass, and density

- Balances the mechanical structure of stellar interiors



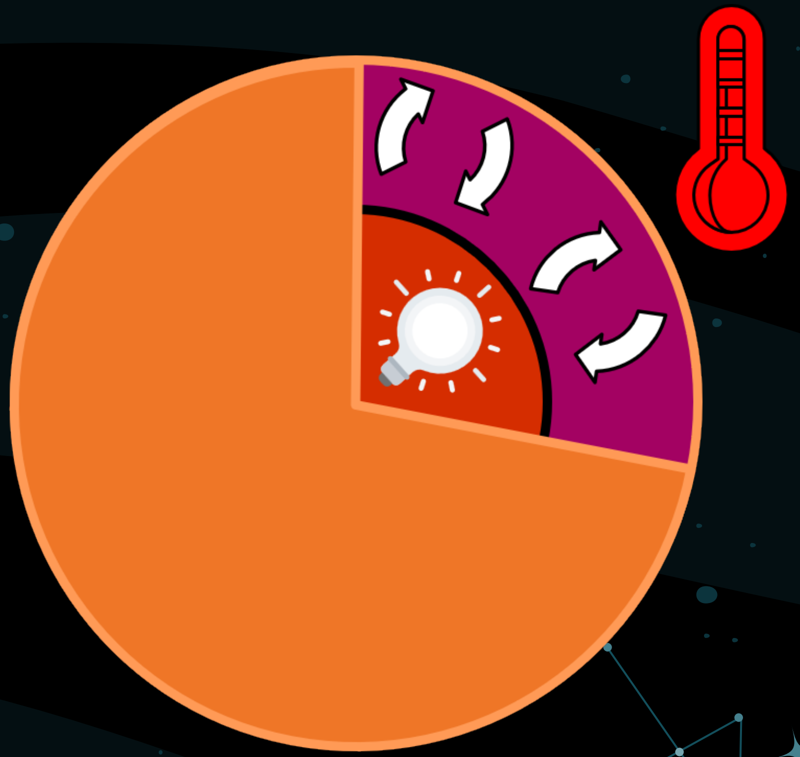
Thermal Structure

- Describes the relationship between temperature, radial distance, enclosed mass, pressure, and density

- Balances the temperature profile within stellar interiors

- Encodes energy transport mechanisms via $\nabla = \frac{P}{T}\frac{\partial T}{\partial P}$

$$\frac{\partial T}{\partial r} = -\frac{Gm\rho}{r^2}\frac{T}{P}\nabla$$

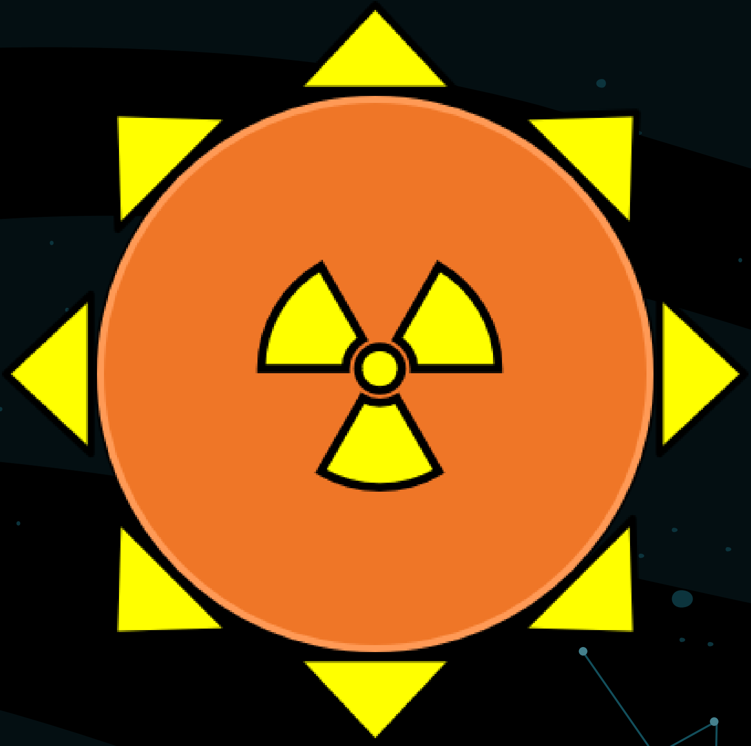


Energy Conservation

$$\frac{\partial l}{\partial r} = 4\pi r^2 \rho \epsilon$$

- Describes the relationship between enclosed luminosity, radial distance, density, and energy content

- Encodes where and how energy is produced within stellar interiors



03

Stellar Energy Generation

Big Picture

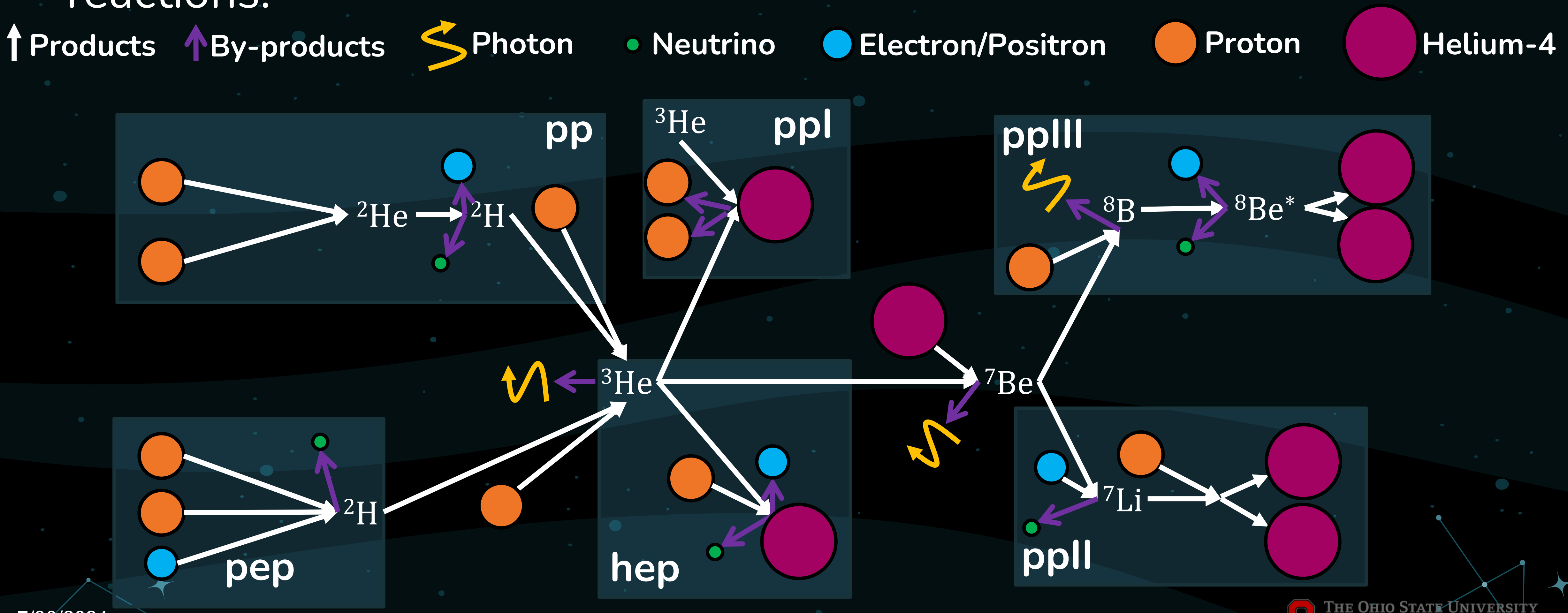
Main sequence stars are powered by the nuclear fusion of Hydrogen, during which four protons combine to form one Helium nucleus, releasing nuclear binding energy.

$$4\mathrm{p} \longrightarrow {}^{4}\mathrm{He} + 27.4\ \mathrm{MeV}$$

In reality, this energy is released over many reactions via two distinct processes within the Sun.

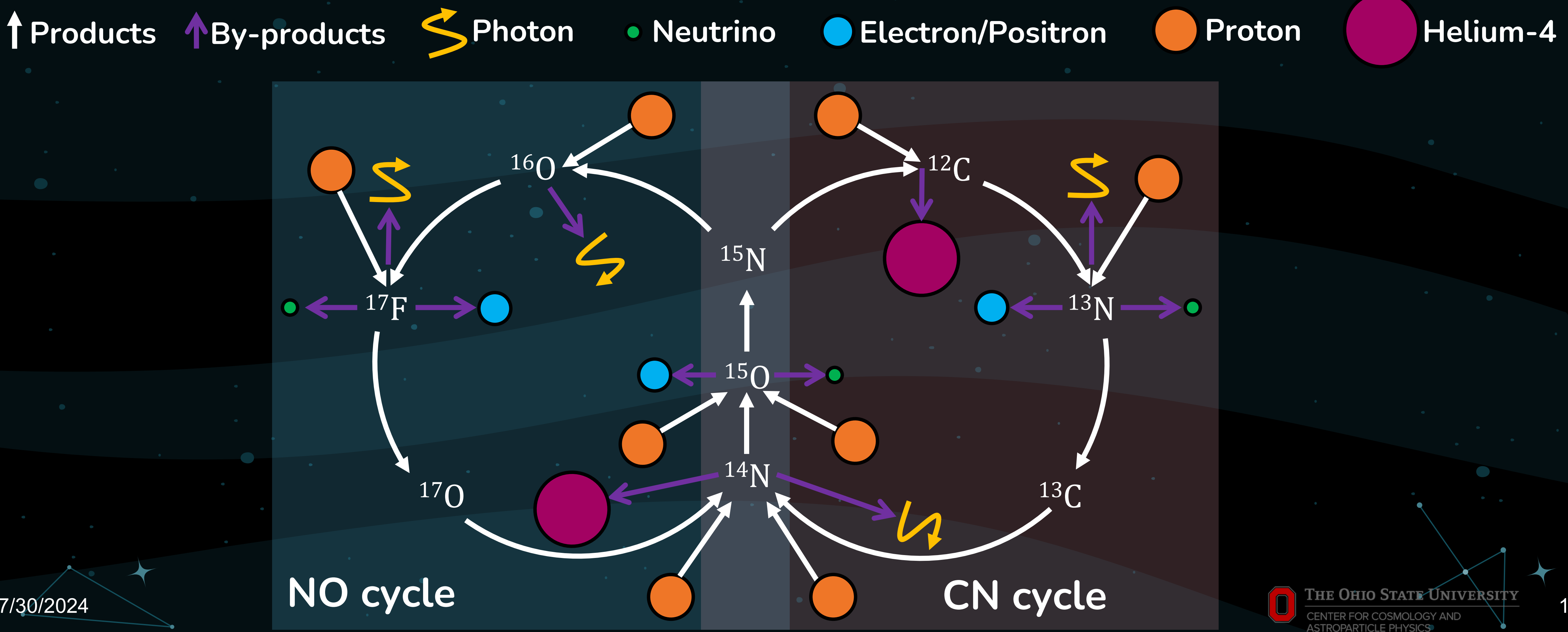
The Proton-Proton (pp) Chain

Energy generation in Sun-like stars is dominated by the following reactions:



The Carbon-Nitrogen-Oxygen (CNO) Bi-cycle

The fusion of Hydrogen into Helium via metal catalysts dominates in stars more massive than the Sun, but still plays a minor role:



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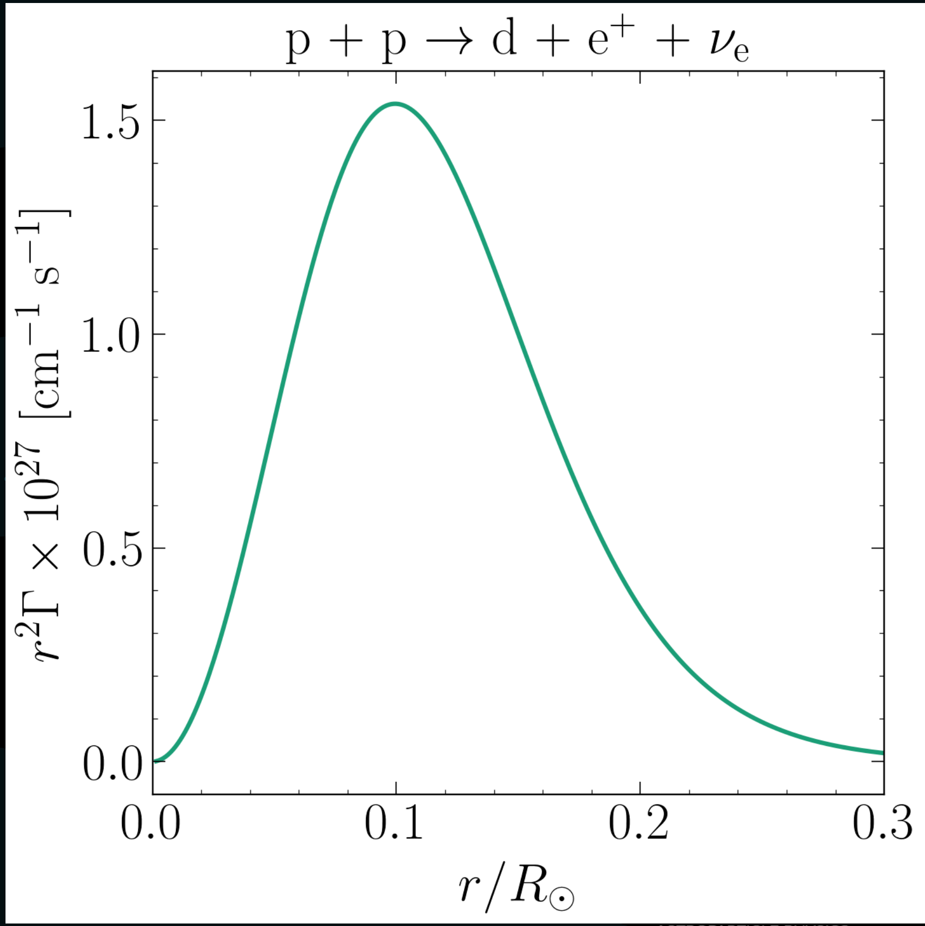
Nuclear Reaction Rates

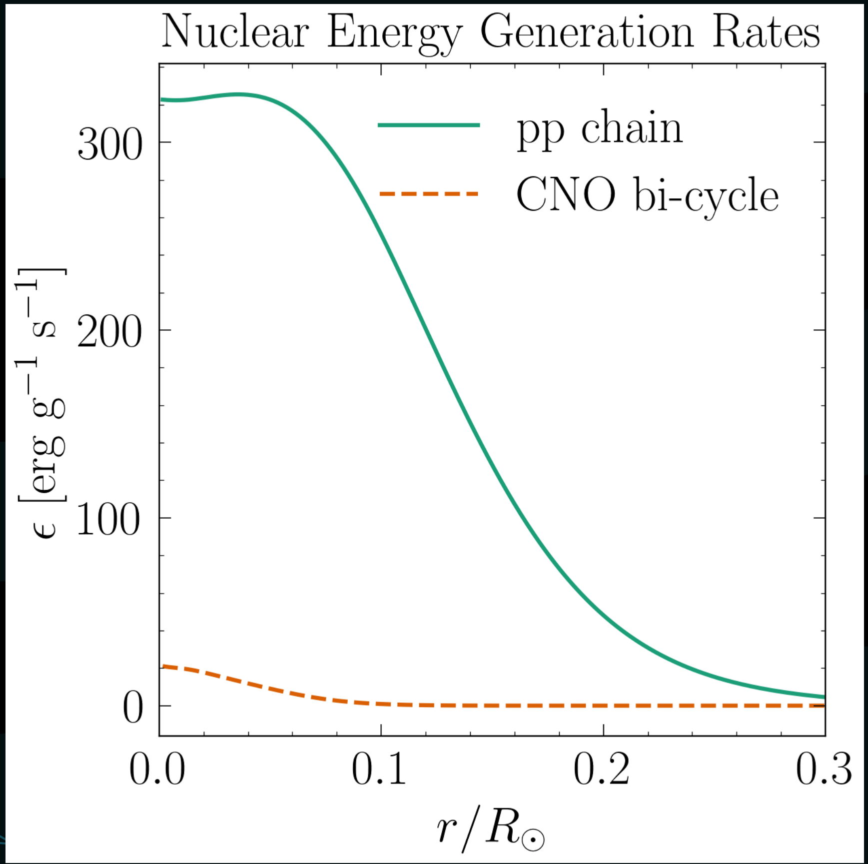
The rate at which nuclear reactions occur can by calculated as a product between the number densities of the reactants and the reaction rate per particle pair:

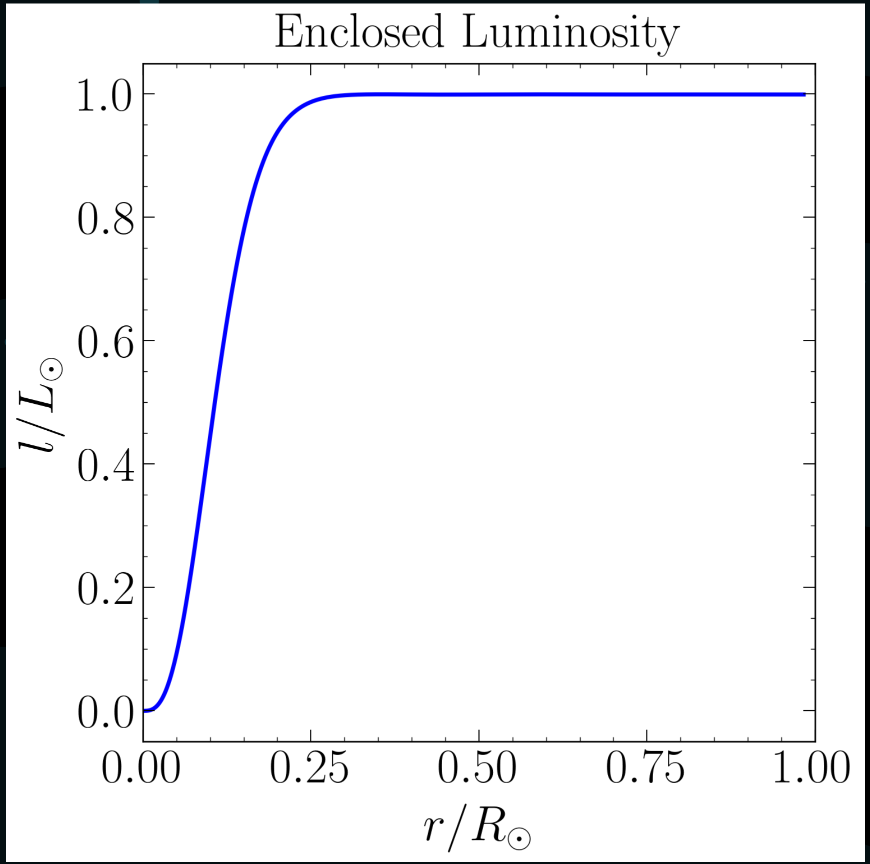
$$\Gamma_{ab} = \frac{n_a n_b}{1 + \delta_{ab}} \langle \sigma v \rangle$$



Energy Generation Rates

Calculating the rate at which energy is produced involves carefully analyzing the specifics of all reactions. It is customary to use approximations of the energy content.





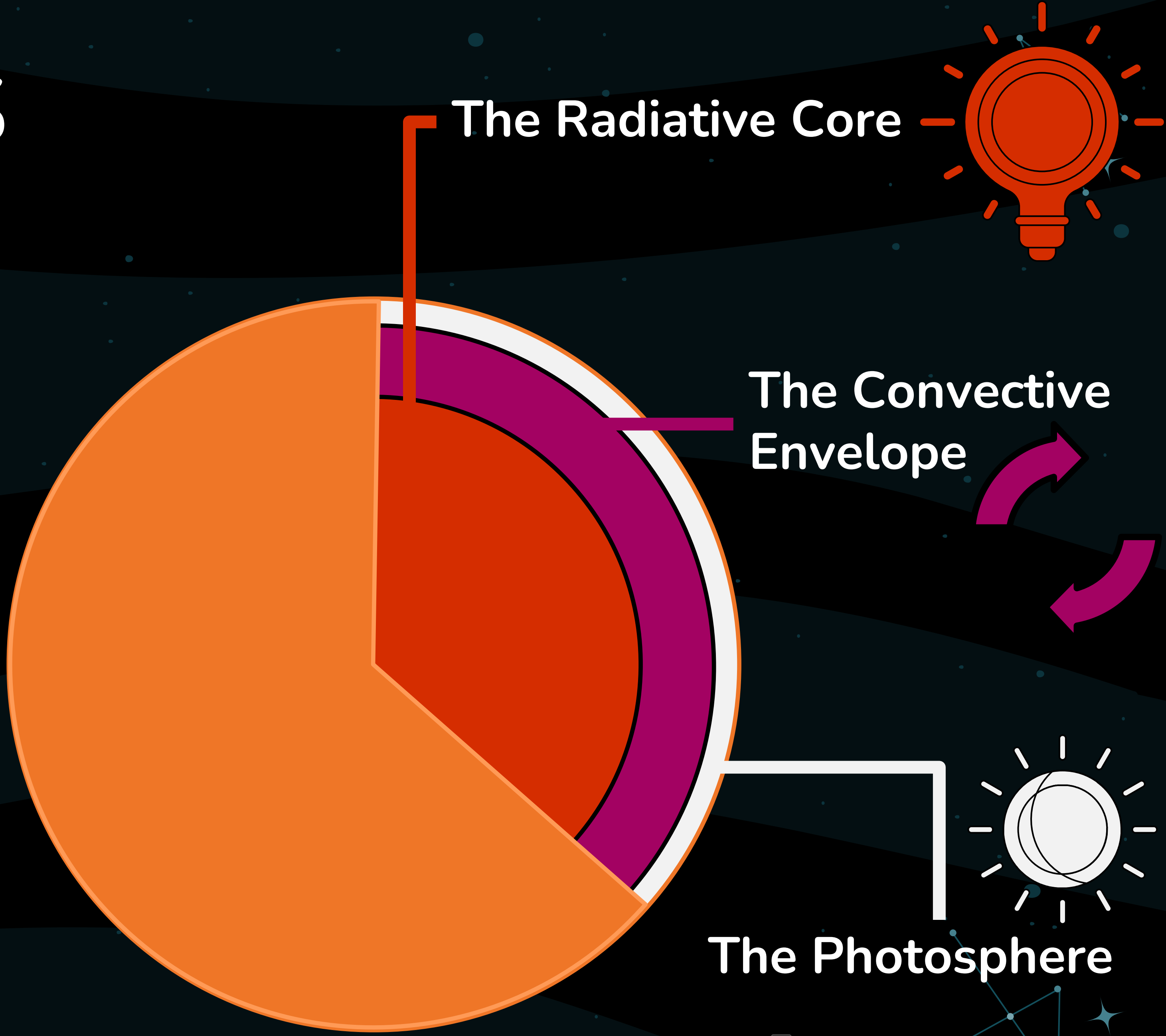
04

Energy Transport in Stars

Stellar Zones

The thermal structure of stars is influenced by the mechanisms by which energy is transported throughout the inner layers.

I consider three distinct zones within the Sun.



The Radiative Core

Energy is transported via the diffusion of photons in the Sun's core. The radiative temperature gradient is controlled by opacity, enclosed luminosity, pressure, enclosed mass, and temperature:

$$\nabla_{\mathrm{rad}} = \frac{3\kappa l P}{16\pi\sigma_{\mathrm{SB}} G m c T^4}$$

The Convective Envelope

The Sun's envelope is characterized by energy transport via adiabatically rising gas parcels. The adiabatic temperature gradient is parameterized by pressure, the coefficient of expansion of gasses, temperature, density, and specific heat at constant pressure:

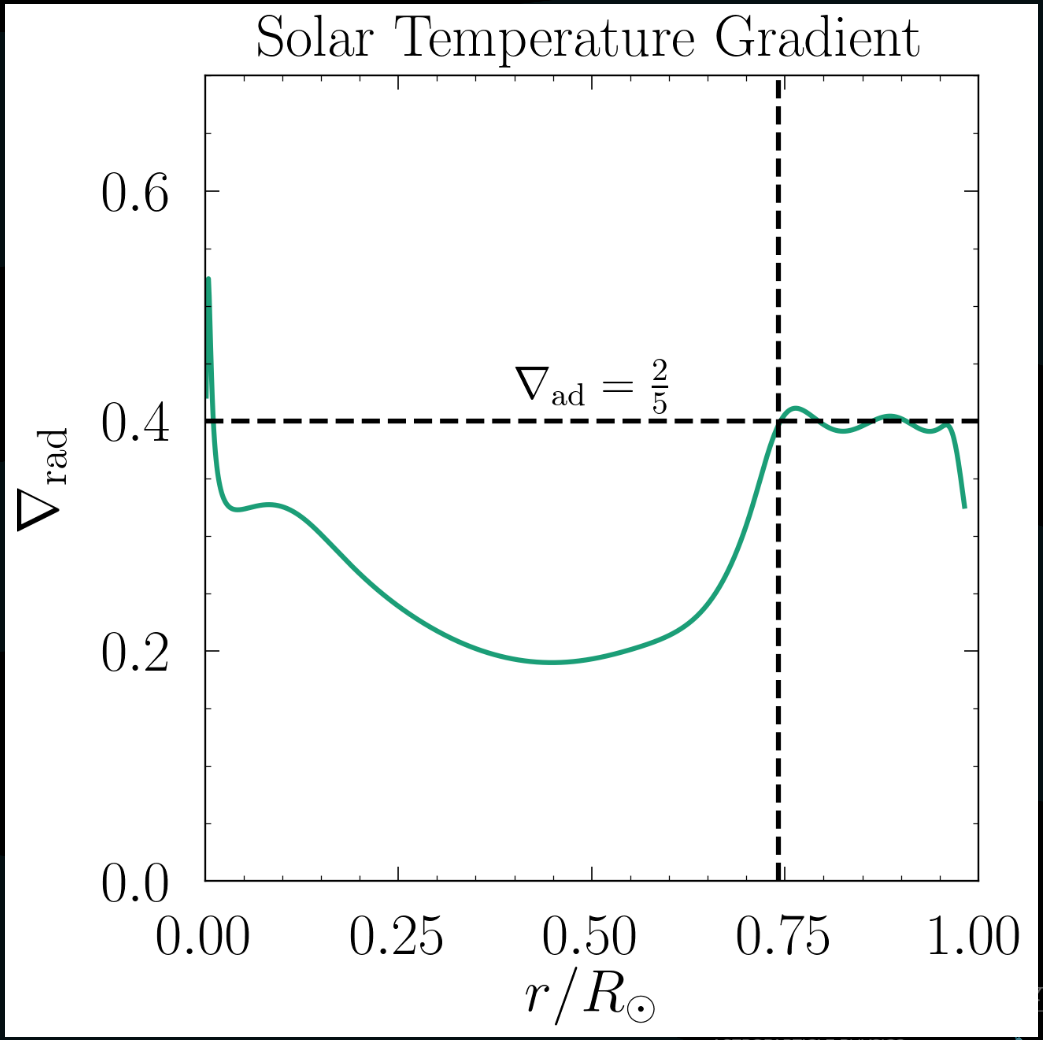
$$\nabla_{\mathrm{ad}} = \frac{P\delta}{T\rho c_P} = \frac{2}{5}$$

Convective Stability

Energy transport at some point in a star proceeds via the most efficient mechanism at that point. One appraises a stability condition to determine how energy is transported.

Convection sets in when

$$\nabla_{\mathrm{ad}} < \nabla_{\mathrm{rad}}$$



The Photosphere

Above the convective zone, conditions relax such that radiative transport takes over and the Sun's layers become more transparent. At last, the Sun shines.

Using the Stefan-Boltzmann law, one can determine the temperature at the surface of the Sun:

$$L_{\odot} = 4\pi {R_{\odot}}^2 \sigma_{\mathrm{SB}} {T_{\mathrm{eff}}}^4$$

$$T_{\mathrm{eff}} \approx 5770\ \mathrm{K}$$

05

Applications of Stars

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Stars are fascinating!

Hopefully, you are convinced that stars are rich in physical phenomena and are fascinating subjects of research.

I close this talk with a very short discussion on the applications of stars in other areas of physics and astronomy.

Uncertainties in Stellar Structure

- While stellar models are well-developed, there remain physical mechanisms that are difficult to implement
- The composition of metals in the Sun is not well understood, with models producing predictions that are at odds with observations
- Observations of seismic activity in stars can help constrain stellar structure and microphysics

Labs for Nuclear and Particle Physics

- Neutrinos produced by nuclear reactions help probe physics of the weak sector as well as constrain solar structure

- There exists a gap between theory and experiment in nuclear physics and astrophysics in the context of fusion

- The Sun is a multi-messenger source providing ample opportunity for modeling and measurement

Stars are Everywhere

- Distant stars improve our understanding of the Sun and vice versa
- The history of star formation has consequences for a better understanding of cosmology
- The evolution of stars is physically complicated but opens a window to the very-high-energy universe

Thank You!

I express gratitude towards Prof. J. Beacom for proposing an exciting and intellectually stimulating topic for my Candidacy Exam, as well as thank Profs. G. Cochran, K. Hughes, and A. Peter for serving on my Advisory Committee as approachable and helpful resources.

Thank you to everyone who attended my practice talk and to those who offered feedback and suggestions on this presentation.

Appendix

Ideal Gas Law

$$\mu = \left[\frac{1}{\mu_\mathrm{I}} + \frac{1}{\mu_\mathrm{e}}\right]^{-1}.$$

$$\mu_\mathrm{e} = \left[\sum_i \frac{Z_i X_i y_i}{A_i}\right]^{-1}$$

$$\mu_\mathrm{I} = \left[\sum_i \frac{X_i}{A_i}\right]^{-1}$$

$$P = nk_\mathrm{B}T = \frac{\mathcal{R}}{\mu}\rho T = \frac{\rho}{\mu m_a} k_B T$$

The isothermal compressibility coefficient for an ideal gas is

$$\alpha = \frac{P}{\rho}\left(\frac{\partial \rho}{\partial P}\right)_{T,\mu} = \frac{\mathcal{R}\rho T/\mu}{\rho}\left(\frac{\mu}{\mathcal{R}T}\right) = 1.$$

The coefficient of expansion for an ideal gas is

$$\delta = -\frac{T}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_{T,\mu} = -\frac{T}{P\mu/(\mathcal{R}T)}\left(-\frac{P\mu}{\mathcal{R}T^2}\right) = 1$$

First Law of Thermodynamics

The first law of thermodynamics relates the heat dq added per unit mass,

$$dq = du + P dv, \tag{4.1}$$

to the internal energy u and the specific volume $v = 1/\varrho$ (both also defined per unit mass).

We now assume rather general equations of state, $\varrho = \varrho(P, T)$ and $u = u(\varrho, T)$. Usually they will also depend on the chemical composition, but here this is assumed to be fixed. With the derivatives defined as

$$\alpha := \left(\frac{\partial \ln \varrho}{\partial \ln P}\right)_T = -\frac{P}{v}\left(\frac{\partial v}{\partial P}\right)_T, \tag{4.2}$$

$$\delta := -\left(\frac{\partial \ln \varrho}{\partial \ln T}\right)_P = \frac{T}{v}\left(\frac{\partial v}{\partial T}\right)_P, \tag{4.3}$$

the equation of state can be written in the form $d\varrho/\varrho = \alpha dP/P - \delta dT/T$.

We also need the specific heats:

$$c_P := \left(\frac{dq}{dT}\right)_P = \left(\frac{\partial u}{\partial T}\right)_P + P\left(\frac{\partial v}{\partial T}\right)_P, \tag{4.4}$$

$$c_v := \left(\frac{dq}{dT}\right)_v = \left(\frac{\partial u}{\partial T}\right)_v. \tag{4.5}$$

R. Kippenhahn et al., *Stellar Structure and Evolution*, Astronomy and Astrophysics Library, DOI 10.1007/978-3-642-30304-3_4, © Springer-Verlag Berlin Heidelberg 2012 25

With

$$du = \left(\frac{\partial u}{\partial v}\right)_T dv + \left(\frac{\partial u}{\partial T}\right)_v dT \tag{4.6}$$

and with (4.1) we find the change $ds = dq/T$ of the specific entropy to be

$$ds = \frac{dq}{T} = \frac{1}{T}\left[\left(\frac{\partial u}{\partial v}\right)_T + P\right] dv + \frac{1}{T}\left(\frac{\partial u}{\partial T}\right)_v dT. \tag{4.7}$$

Since ds is a total differential form, $\partial^2 s/\partial T \partial v = \partial^2 s/\partial v \partial T$ and

$$\frac{\partial}{\partial T}\left[\frac{1}{T}\left(\frac{\partial u}{\partial v}\right)_T + \frac{P}{T}\right] = \frac{1}{T}\frac{\partial^2 u}{\partial T \partial v}, \tag{4.8}$$

which after the differentiation on the left is carried out gives

$$\left(\frac{\partial u}{\partial v}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_v - P. \tag{4.9}$$

Next we derive an expression for $(\partial u/\partial T)_P$, taking P, T as independent variables. From (4.6) it follows that

$$\frac{du}{dT} = \left(\frac{\partial u}{\partial T}\right)_v + \left(\frac{\partial u}{\partial v}\right)_T \frac{dv}{dT}, \tag{4.10}$$

and therefore

$$\begin{aligned}\left(\frac{\partial u}{\partial T}\right)_P &= \left(\frac{\partial u}{\partial T}\right)_v + \left(\frac{\partial u}{\partial v}\right)_T \left(\frac{\partial v}{\partial T}\right)_P \\ &= \left(\frac{\partial u}{\partial T}\right)_v + \left(\frac{\partial v}{\partial T}\right)_P \left[T\left(\frac{\partial P}{\partial T}\right)_v - P\right],\end{aligned} \tag{4.11}$$

where we have made use of (4.9). From the definitions (4.4), (4.5) and from (4.11) we write

$$\begin{aligned}c_P - c_v &= P\left(\frac{\partial v}{\partial T}\right)_P + \left(\frac{\partial u}{\partial T}\right)_P - \left(\frac{\partial u}{\partial T}\right)_v \\ &= \left(\frac{\partial v}{\partial T}\right)_P \left(\frac{\partial P}{\partial T}\right)_v T.\end{aligned} \tag{4.12}$$

On the other hand, the definitions (4.2) and (4.3) for α and δ imply that

$$\left(\frac{\partial P}{\partial T}\right)_v = -\frac{\left(\frac{\partial v}{\partial T}\right)_P}{\left(\frac{\partial v}{\partial P}\right)_T} = \frac{P\delta}{T\alpha}, \tag{4.13}$$

and therefore

$$c_P - c_v = T\left(\frac{\partial v}{\partial T}\right)_P \frac{P\delta}{T\alpha} = \frac{P\delta^2}{\varrho T\alpha}, \tag{4.14}$$

where we have made use of $T(\partial v/\partial T)_P = v\delta = \delta/\varrho$; hence we arrive at the basic relation

$$c_P - c_v = \frac{P\delta^2}{\varrho T\alpha}. \tag{4.15}$$

For a perfect gas this equation reduces to the well-known relation $c_P - c_v = \Re/\mu$ [see (4.33)].

We have now derived all the tools for rewriting (4.1) in terms of T and P. The first step is to write it in the form

$$\begin{aligned}dq = du + P dv &= \left(\frac{\partial u}{\partial T}\right)_v dT + \left[\left(\frac{\partial u}{\partial v}\right)_T + P\right] dv \\ &= \left(\frac{\partial u}{\partial T}\right)_v dT + T\left(\frac{\partial P}{\partial T}\right)_v dv\end{aligned} \tag{4.16}$$

by making use of (4.9), and then with (4.5) and (4.13) we have

$$\begin{aligned}dq &= c_v dT - \frac{T}{\varrho}\left(\frac{\partial P}{\partial T}\right)_v \frac{d\varrho}{\varrho} = c_v dT - \frac{P\delta}{\varrho\alpha}\frac{d\varrho}{\varrho} \\ &= c_v dT - \frac{P\delta}{\varrho\alpha}\left(\alpha\frac{dP}{P} - \delta\frac{dT}{T}\right) = \left(c_v + \frac{P\delta^2}{\varrho T\alpha}\right) dT - \frac{\delta}{\varrho} dP.\end{aligned} \tag{4.17}$$

The terms in parentheses in the last expression are, according to (4.15), simply c_P and therefore

$$dq = c_P dT - \frac{\delta}{\varrho} dP. \tag{4.18}$$

Next we define the adiabatic temperature gradient ∇_{ad}, a quantity often used in astrophysics, by

$$\nabla_{ad} := \left(\frac{\partial \ln T}{\partial \ln P}\right)_s, \tag{4.19}$$

where the subscript s indicates that the definition is valid for constant entropy. Since for adiabatic changes the entropy has to remain constant, i.e. $ds = dq/T = 0$, we can easily derive an expression for ∇_{ad} from (4.18), i.e.

$$0 = dq = c_P dT - \frac{\delta}{\varrho} dP \tag{4.20}$$

Specific Heats

The specific heats can be related starting from the second law of thermodynamics:

$$dq = T ds \tag{F.8}$$

After making the assumption that entropy is a function of temperature and pressure, we can write the second law as

$$dq = T\left(\left(\frac{\partial s}{\partial T}\right)_P dT + \left(\frac{\partial s}{\partial P}\right)_T dP\right). \tag{F.9}$$

If we also assume that pressure is a function of temperature and (specific) volume v, we can write

$$dP = \left(\frac{\partial P}{\partial T}\right)_v dT + \left(\frac{\partial P}{\partial v}\right)_T dv \tag{F.10}$$

Inserting the above equation into the second law of thermodynamics, we obtain

$$dq = T\left(\left(\frac{\partial s}{\partial T}\right)_T dT + \left(\frac{\partial s}{\partial P}\right)_T \left(\frac{\partial P}{\partial T}\right)_v dT + \left(\frac{\partial s}{\partial P}\right)_T \left(\frac{\partial P}{\partial v}\right)_T dv\right) \tag{F.11}$$

In looking for the specific heat at constant volume, we may set $dv = 0$ and divide the above equation by dT. This works out as

$$\left(\frac{\partial q}{\partial T}\right)_v = T\left(\frac{\partial s}{\partial T}\right)_P + T\left(\frac{\partial s}{\partial P}\right)_T \left(\frac{\partial P}{\partial T}\right)_v \tag{F.12}$$

From the definitions of thermodynamic quantities and the usual Maxwell's relations, we can identify three expressions in the above equation,

$$\left(\frac{\partial q}{\partial T}\right)_v = c_V \tag{F.13}$$

$$T\left(\frac{\partial s}{\partial T}\right)_P = c_P \tag{F.14}$$

$$\left(\frac{\partial s}{\partial P}\right)_T = -\left(\frac{\partial v}{\partial T}\right)_P \tag{F.15}$$

which we can then replace in Equation F.12 to obtain

$$c_P - c_V = T\left(\frac{\partial v}{\partial T}\right)_P \left(\frac{\partial P}{\partial T}\right)_v \tag{F.16}$$

The first differential can be rewritten in terms of the coefficient of expanison:

$$T\left(\frac{\partial v}{\partial T}\right)_P = v\delta = \frac{\delta}{\rho} \tag{F.17}$$

To rewrite the second differential, we again keep in mind that volume is kept constant:

$$dv = 0 = \left(\frac{\partial v}{\partial T}\right)_P + \left(\frac{\partial v}{\partial P}\right)_T \tag{F.18}$$

$$\left(\frac{\partial v}{\partial T}\right)_P = -\left(\frac{\partial v}{\partial P}\right)_T \tag{F.19}$$

$$\left(\frac{\partial P}{\partial T}\right)_v = -\left(\frac{\partial v}{\partial T}\right)_P \Big/ \left(\frac{\partial v}{\partial P}\right)_T \tag{F.20}$$

where, again, we can identify

$$\left(\frac{\partial v}{\partial T}\right)_P = \frac{\delta}{\rho T} \tag{F.21}$$

$$\left(\frac{\partial v}{\partial P}\right)_T = \alpha \tag{F.22}$$

to plug in Equation F.20 to obtain

$$\left(\frac{\partial P}{\partial T}\right)_v = \frac{P\delta}{T\alpha}. \tag{F.23}$$

When combining the above with Equation F.16, we arrive at

$$c_P - c_V = \frac{P\delta^2}{T\rho\alpha} \tag{F.24}$$

For an ideal gas, the equation of state can be expressed as

$$P = \frac{\rho}{\mu m_a} k_B T \tag{F.25}$$

Which, when substituted into Equation F.16, gives

$$c_P - c_V = \frac{k_B}{\mu m_a} = \mathscr{R}, \tag{F.26}$$

where $\mathscr{R}$ is the *molar* gas constant, used in the well-known $PV = N\mathscr{R}T$ equation for the ideal gas law.

Reintroducing the adiabatic index, we can write

$$c_P - c_V = \left(\frac{c_P}{c_V} - 1\right) = (\gamma - 1)\, c_V \tag{F.27}$$

which allows for finally obtaining expressions for the specific heats:

$$c_V = \frac{1}{\gamma - 1} \frac{k_B}{\mu m_a} \tag{F.28}$$

$$c_P = \frac{\gamma}{\gamma - 1} \frac{k_B}{\mu m_a} \tag{F.29}$$

Relativistic and Degenerate Physics

Coulomb interactions between ions within stellar interiors are potentially important in describing the mechanics of momentum exchange and gas pressure. A measure of the importance of such interactions can be constructed as a ratio between the average Coulomb potential energy and thermal energy of electrons:

$$\Gamma_e = \frac{e^2}{d_e k_B T} \tag{D.1}$$

where $d_e = \left(\frac{3}{4\pi n_e}\right)^{1/3}$ is the mean separation between electrons with number density n_e. This naturally leads to a discussion of electron degeneracy, the importance of which can be quantified by a degeneracy parameter

$$\zeta_e = \lambda_e^3 n_e \tag{D.2}$$

where $\lambda_e = h\,(2\pi m_e k_B T)^{-1/2}$ is the de Broglie wavelength of the electron. The importance of relativistic effects can be described by the ratio between the thermal energies and rest energies of electrons:

$$x_e = \frac{k_B T}{m_e c^2} \tag{D.3}$$

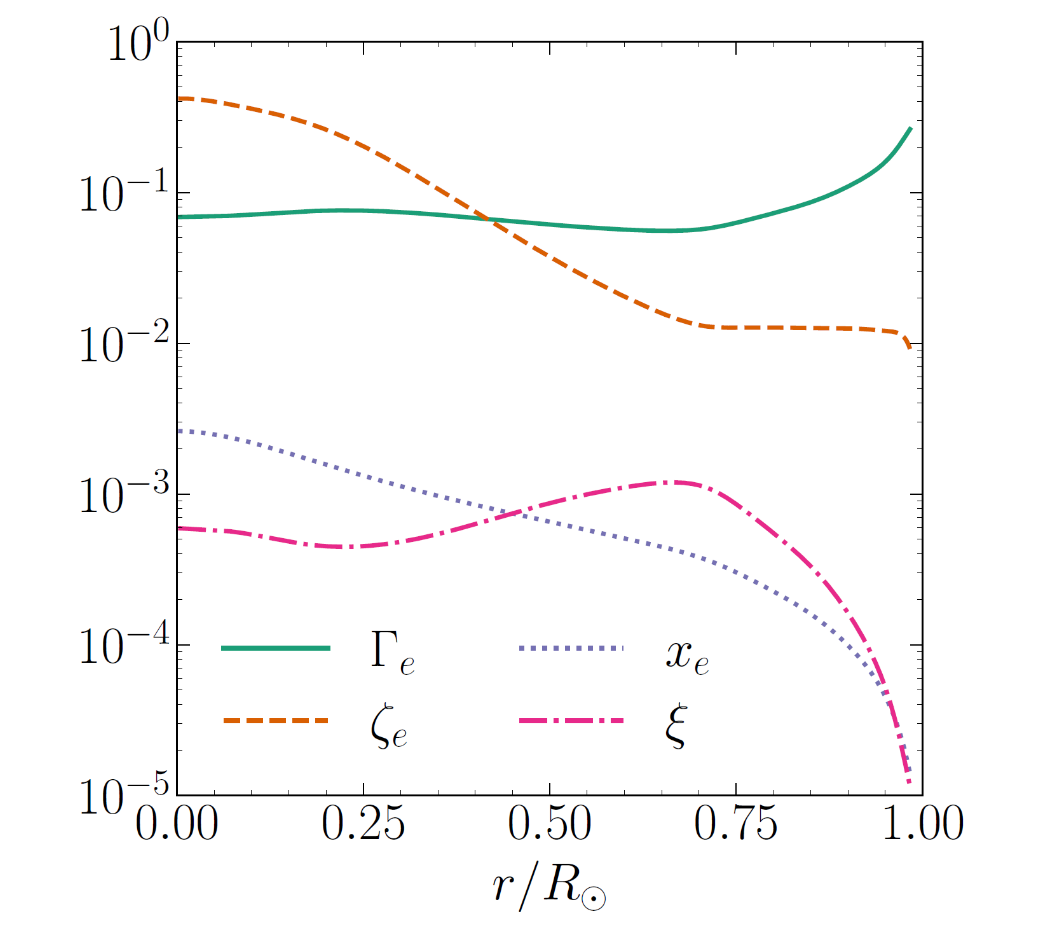
Finally, one can determine the significance of radiation pressure compared to gas pressure:

$$\xi = \frac{P_{\rm rad}}{P_{\rm gas}} = \frac{a\mu m_a T^3}{3\rho k_B} \tag{D.4}$$

where the mean molecular weight can be approximated [7] by

$$\mu = \frac{4}{3 + 5X - Z} \tag{D.5}$$

where X is the mass fraction of Hydrogen and Z is the mass fraction of metals. A plot of



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Blackbody Radiation

$$\langle E \rangle = \frac{h\nu}{\exp\left(h\nu/k_{\rm B}T\right) - 1}$$

$$P_{\rm rad} = \frac{1}{3}u = \frac{aT^4}{3} = \frac{4\sigma_{\rm SB}T^4}{3c}$$

Integrating the Planck function over all frequencies produces

$$B(T) = \int_0^\infty B_\nu(T) d\nu = \frac{ac}{4\pi}T^4 \tag{C.33}$$

as recovered in the previous section. For an isotropically emitting surface, the flux emerging from the surface is given by $F = \pi B(T)$, or

$$F = \sigma_{\rm SB}T^4 \tag{C.34}$$

Assuming the surface is a sphere of radius R, the luminosity L of the radiating blackbody is given by

$$L = 4\pi R^2 \sigma_{\rm SB} T_{\rm eff}^4 \tag{C.35}$$

where $T_{\rm eff}$ is the effective temperature of the blackbody derived from the emitted flux.

The energy density per unit solid angle $u_\nu(\Omega)$ for blackbody radiation is written as the product of the density of states and the average energy:

$$u_\nu(\Omega) = \left(\frac{2\nu^2}{c^3}\right) \frac{h\nu}{\exp\left(h\nu/k_{\rm B}T\right) - 1} \tag{C.11}$$

Via the relation

$$u_\nu(\Omega) = \frac{I_\nu}{c}, \tag{C.12}$$

we can finally write Planck's law in terms of the specific intensity I_ν in units of

$$I_\nu(\nu, \Omega) = (\text{energy})(\text{time})^{-1}(\text{solid angle})^{-1}(\text{frequency})^{-1} \tag{C.13}$$

$$= \text{erg s}^{-1}\ \text{cm}^{-2}\ \text{ster}^{-1}\ \text{Hz}^{-1} \tag{C.14}$$

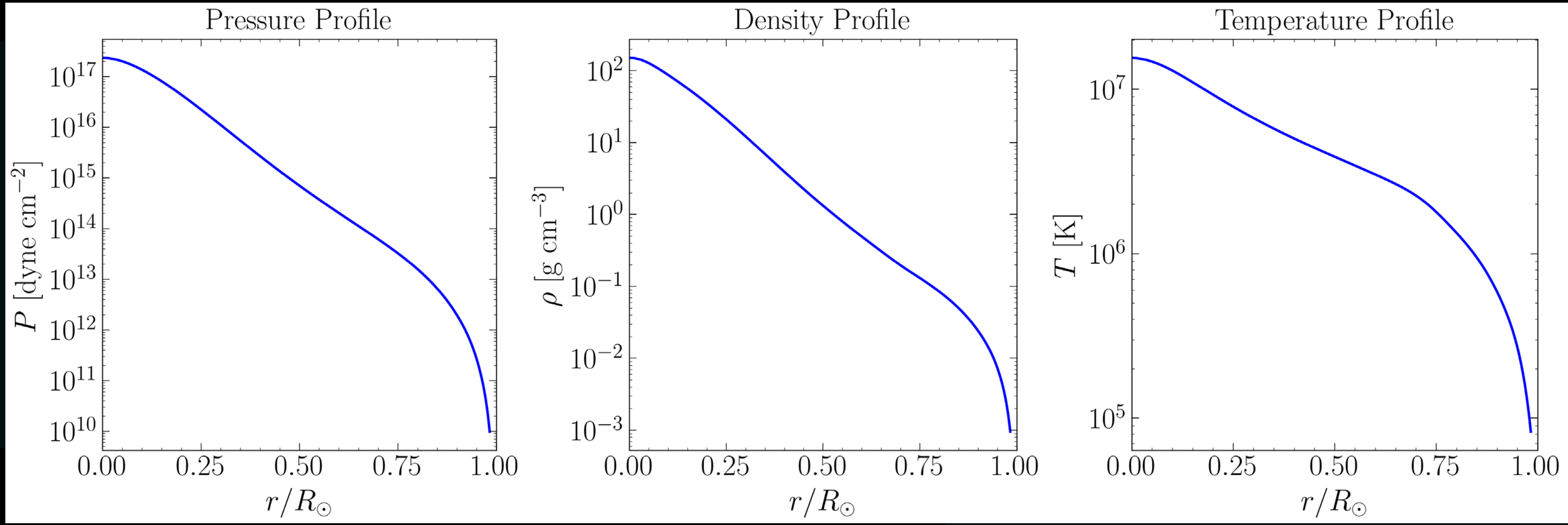
which in general, depends on spatial coordinates, direction, and frequency. Planck's law is usually labeled as $I_\nu = B_\nu$ and thus expressed as

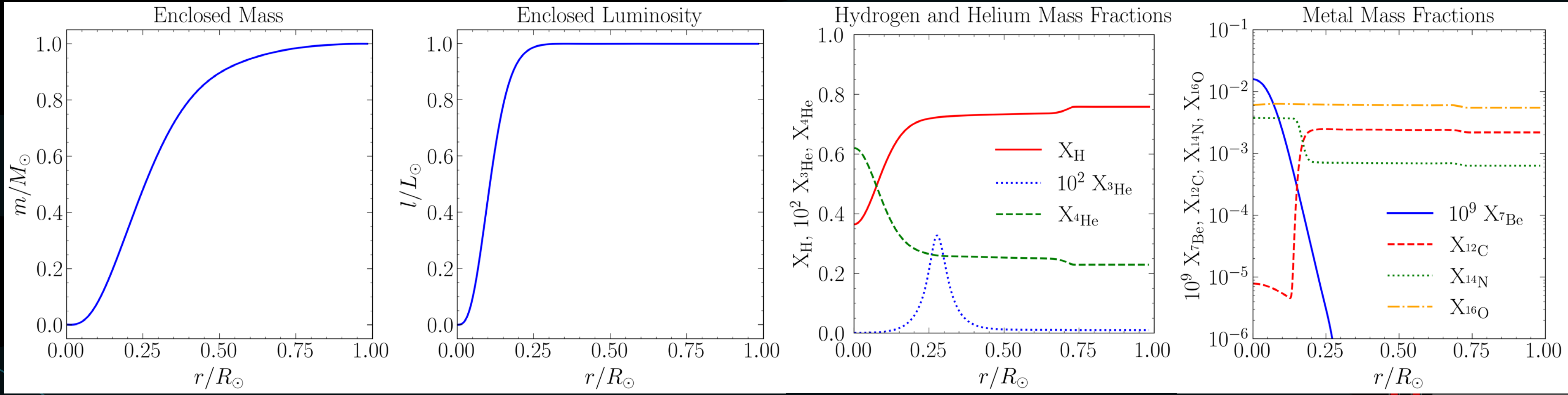
$$B_\nu(T) = \frac{2h\nu^3/c^2}{\exp\left(h\nu/k_{\rm B}T\right) - 1} \tag{C.15}$$

in terms of frequency. If expressed in terms of per unit wavelength,

$$B_\lambda(T) = \frac{2hc^2/\lambda^5}{\exp\left(hc/\lambda k_{\rm B}T\right) - 1} \tag{C.16}$$

Solar Variable Profiles





Quantum Tunneling

Consider a 1-dimensional potential barrier described by

$$U(x) = \begin{cases} 0 & x < 0 \\ U_0 & 0 \leq x \leq L \\ 0 & x \leq L \end{cases}$$

where U_0 and L are finite and positive. The time-independent Schrödinger equation is given by

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x) \tag{10}$$

We will consider the case where the energy E of the incoming particle is smaller than the height of the potential barrier. We can split up the problem into three distinct regions.

Region I: $-\infty < x < 0$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_I(x)}{dx^2} = E\psi_I(x) \tag{11}$$

Region II: $0 < x < L$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_{II}(x)}{dx^2} + U_0\psi(x) = E\psi_{II}(x) \tag{12}$$

Region II: $L < x < \infty$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_{III}(x)}{dx^2} = E\psi_{III}(x) \tag{13}$$

Solutions must be continuous and differentiable at the boundaries

$$\psi_I(0) = \psi_{II}(0) \tag{14}$$
$$\psi_{II}(L) = \psi_{III}(L) \tag{15}$$
$$\left.\frac{d\psi_I(x)}{dx}\right|_{x=0} = \left.\frac{d\psi_{II}(x)}{dx}\right|_{x=0} \tag{16}$$
$$\left.\frac{d\psi_{II}(x)}{dx}\right|_{x=L} = \left.\frac{d\psi_{III}(x)}{dx}\right|_{x=L} \tag{17}$$

The solutions in Regions I and III take the form of

$$\psi_I(x) = Ae^{-ikx} + Be^{-ikx} \tag{18}$$
$$\psi_{III}(x) = Fe^{ikx} \tag{19}$$
$$\tag{20}$$

for $k = \sqrt{2mE}/h$. In Region II,

$$\frac{d^2\psi_{II}(x)}{dx^2} = \beta^2\psi_{II}(x) \tag{21}$$

for $\beta^2 = \frac{2m}{\hbar^2}(U_0 - E) > 0$ with wave function

$$\psi_{II}(x) = Ce^{-\beta x} + De^{\beta x}. \tag{22}$$

Use the boundary conditions to determine the constants. The tunneling probability is given as

$$T(L,E) = \left|\frac{F}{A}\right|^2 \tag{23}$$

so while it may seem that we have four equations and five constants, we divide by A to obtain the tunneling probability, killing one constant. The expression that leads to F/A is lengthy and gives

$$T(L,E) = \frac{1}{\cosh^2(\beta L) + (\gamma/2)^2\sinh^2(\beta L)} \tag{24}$$

with

$$\left(\frac{\gamma}{2}\right)^2 = \frac{1}{4}\left(\frac{1 - E/U_0}{E/U_0} + \frac{E/U_0}{1 - E/U_0} - 2\right) \tag{25}$$

For a wide and high barrier, which the Coulomb barrier can be thought of to be,

$$T(L,E) \approx 16\frac{E}{U_0}\left(1 - \frac{E}{U_0}\right)e^{-2\beta L} \tag{26}$$

where the product βL can be related to the ratio of the Gamow energy and the particle energy for fusion.

Nuclear Cross Section

$$\mathrm{p(tunnel)} \approx \exp\left(-\sqrt{\frac{E_G}{E}}\right) \tag{G.12}$$

where $E_G = 2m_r c^2(\pi\alpha Z_a Z_b)$ is the Gamow energy in the CGS units ($\alpha = e^2/\hbar c$). In the literature, the tunneling probability is usually written as

$$\exp\left(-\sqrt{\frac{E_G}{E}}\right) = \exp\left(-2\pi\eta\right), \tag{G.13}$$

where $\eta = Z_a Z_b \frac{e^2}{\hbar}\sqrt{\frac{m_r}{2E}}$ is the Sommerfield parameter, and the right-hand-side of Equation G.13 is called the Gamow factor.

Below the Coulomb barrier, the cross section for charged-particle induced nuclear reactions drops rapidly [20] as

$$\sigma(E) \propto \exp\left(-2\pi\eta\right). \tag{G.14}$$

In addition, cross sections also depend on the energetics outside of the realm of nuclear physics. One can write an energy-dependent cross section in terms of the de Broglie wavelength

$$\lambda = \frac{\hbar}{\sqrt{2m_a E_a}} \tag{G.15}$$

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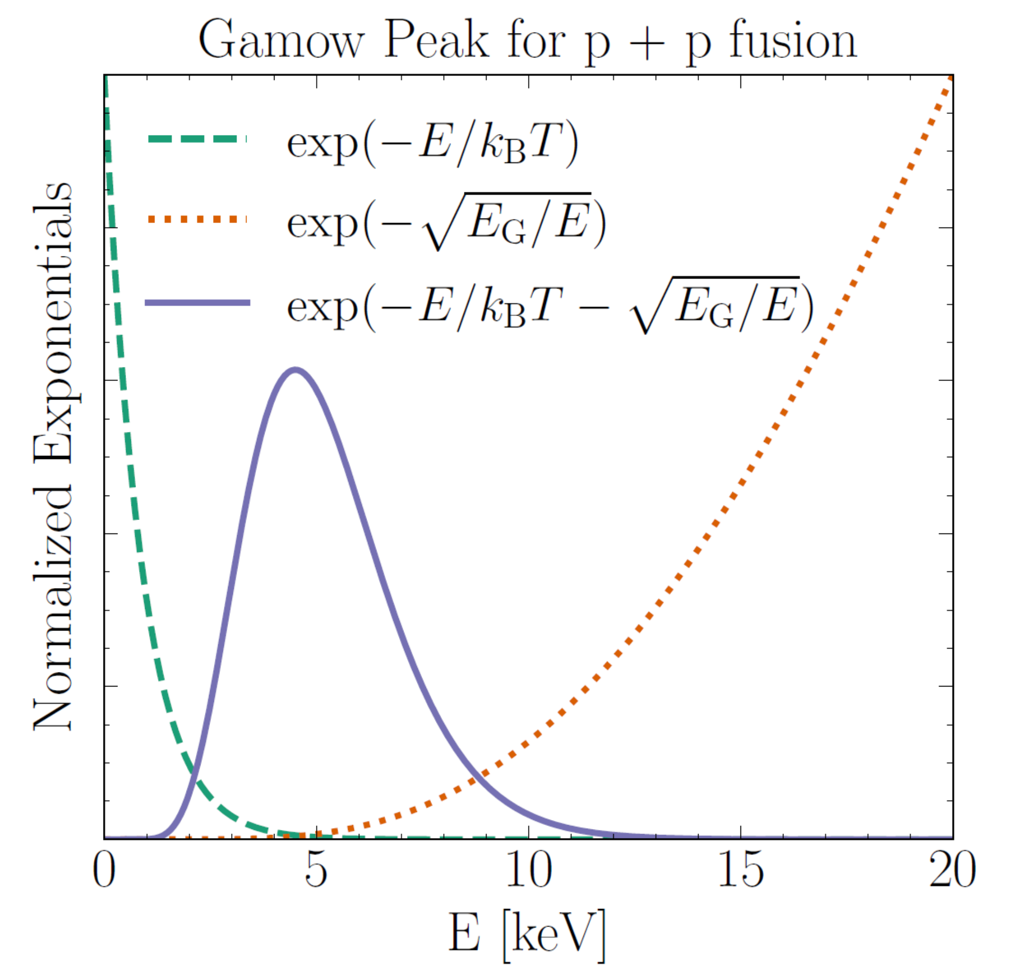
of the projectile particle as

$$\sigma(E) \propto \pi\lambda^2 \propto \frac{1}{E}. \tag{G.16}$$

The remaining nuclear physics to be measured via experiment is packaged into the astrophysical S factor $S(E)$, arriving at

$$\sigma(E) = \frac{S(E)}{E}\exp\left(-2\pi\eta\right). \tag{G.17}$$



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Nuclear Reaction Rates

$$\langle \sigma v \rangle = \int_0^\infty f(v_{\rm rel})\sigma(v_{\rm rel})v_{\rm rel}dv_{\rm rel} \tag{E.12}$$

Dropping subscripts and inserting the Maxwell-Boltzmann distribution,

$$\langle \sigma v \rangle = 4\pi \left(\frac{m_r}{2\pi k_{\rm B}T}\right)^{3/2} \int_0^\infty v^3\sigma(v)\exp\left(-\frac{m_r v^2}{2k_{\rm B}T}\right) dv. \tag{E.13}$$

In terms of the energy $E = \frac{1}{2}m_r v^2$,

$$\boxed{\langle \sigma v \rangle = \left(\frac{8}{\pi m_r}\right)^{1/2} \frac{1}{(k_{\rm B}T)^{3/2}} \int_0^\infty \sigma(E)E\exp\left(-\frac{E}{k_{\rm B}T}\right) dE} \tag{E.14}$$

$$\boxed{\Gamma_{ab} = \frac{n_a n_b}{1+\delta_{ab}} \left(\frac{8}{\pi m_r}\right)^{1/2} \frac{1}{(k_{\rm B}T)^{3/2}} \int_0^\infty S(E)\exp\left(-\frac{E}{k_{\rm B}T} - 2\pi\eta\right) dE}$$

$$\langle \sigma v \rangle = 1.3005 \times 10^{-15} \left[\frac{Z_a Z_b}{A T_6^2}\right]^{1/3} f_0 S_{\rm eff} \exp(-\tau)\ {\rm cm}^3\ {\rm s}^{-1}$$

Particle Diffusion

$$\Phi_N(r) = -\frac{1}{3}\ell\bar{v}\frac{dn}{dr}, \tag{H.8}$$

which is known as Fick's first law of diffusion with a diffusion coefficient of $D = -\frac{1}{3}\ell\bar{v}$. The resulting flux of energy is obtained by swapping density for energy density u.

$$\Phi_E(r) = -\frac{1}{3}\ell\bar{v}\frac{du}{dr} \tag{H.9}$$

The form of u depends on the particles which transport energy.

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Radiative Transport

$$\kappa_\nu = \frac{\sigma_\nu}{\mu m_a} = \frac{1}{n\ell_\nu \mu m_a} = \frac{1}{\rho \ell_\nu} = \frac{a_\nu}{\rho}$$

$$\tau_\nu = \frac{s}{\ell_\nu} = s\rho\kappa_\nu = sn\sigma_\nu = sa_\nu$$

$$\boxed{\frac{1}{\kappa} = \frac{\pi}{\sigma_{\mathrm{SB}} c T^3} \int_0^\infty \frac{1}{\kappa_\nu} \frac{\partial B_\nu}{\partial T} d\nu \equiv \frac{1}{\bar{\kappa}}}$$

$\partial B_\nu / \partial T = (\alpha_1 \alpha_2 \nu^4/T^2) \cdot \exp(\alpha_2 \nu/T)/[\exp(\alpha_2 \nu/T) - 1]^2$

$\alpha_1 = 2hc^2 = 1.191066 \cdot 10^{-5}$ for radiance in units of $\mathrm{mW \cdot m^{-2} \cdot steradian^{-1}/cm^{-1}}$.
$\alpha_2 = hc/k = 1.438833$ K cm

- At low temperatures and partial ionization, b-b and b-f absorption dominates from the bound electrons.
- As ionization occurs at higher temperatures, free-free opacity takes over, but as T continues to increase, $\kappa_{\mathrm{f-f}}$ decreases and scattering from free electrons dominates.

Convective Transport

Useful to the discussion of energy transport is the adiabatic temperature gradient given by

$$\nabla_{\rm ad} = \frac{P\delta}{T\rho c_P}. \tag{F.52}$$

By plugging in

$$P = \frac{\rho}{\mu m_a} k_{\rm B} T, \tag{F.53}$$

$$\delta = 1 \text{ (for an ideal gas), and} \tag{F.54}$$

$$c_P = \frac{\gamma}{\gamma - 1} \frac{k_{\rm B}}{\mu m_a}, \tag{F.55}$$

one obtains

$$\nabla_{\rm ad} = \frac{\gamma - 1}{\gamma}. \tag{F.56}$$

For a monatomic ideal gas with $\gamma = \frac{5}{3}$,

$$\boxed{\nabla_{\rm ad} = \frac{2}{5}} \tag{F.57}$$

Convective Stability 1

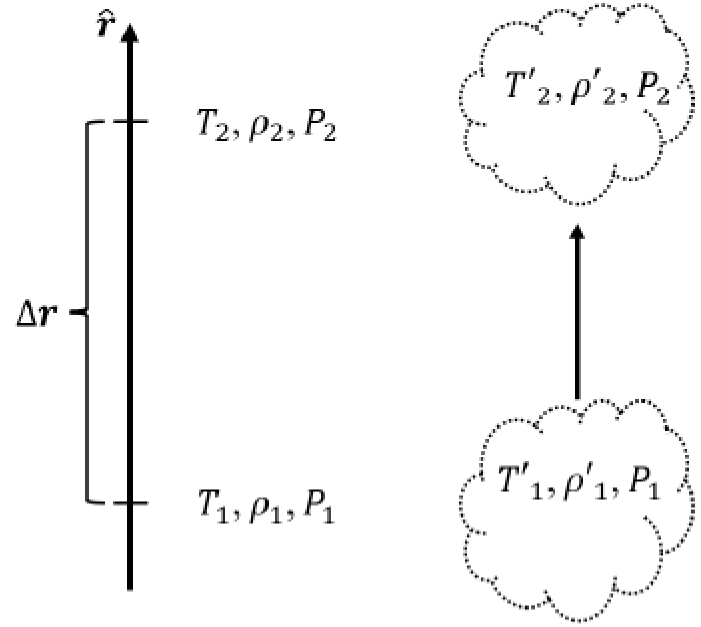


Figure 21: Two points of interest are considered within a plane parallel region of a star. The gas in the surroundings of the star has temperature T, density ρ, and pressure P, while quantities within the gas parcel are primed. Note that $P' = P$ since the gas parcel is assumed to have the same pressure as its surroundings. During convective motion, the gas parcel rise from position 1 to position 2 adiabatically, then dissolves. The distance between position 1 and position 2, Δr, is of order the mixing length.

Convective Stability 2

$$\lambda_P = -\frac{P}{dP/dr} = -\left(\frac{\partial \ln P}{\partial r}\right)^{-1} = \frac{P}{g\rho}$$

Suppose that the temperature within the gas parcel is greater than that of the surroundings: $T' > T$. Since pressures are balanced, $\rho' < \rho$. Archimedes' principle states that the parcel experiences a net upwards force

$$F_{\text{buoyancy}} = \rho V g - \rho' V g \tag{I.1}$$

where V is the parcel's volume and g is the local gravitational acceleration. Ideally, the

appreciably over the mixing length. As such, for a parcel of gas to cease experiencing a net buoyant force upwards, the following condition must be satisfied:

$$\left(\frac{\partial \rho}{\partial r}\right)_{\text{parcel}} > \left(\frac{\partial \rho}{\partial r}\right)_{\text{star}} \tag{I.3}$$

Note that $\partial\rho/\partial r$ is a negative quantity. The goal is to express the above condition in terms of local properties. I begin from the differential density, which has dependencies on temperature and pressure via the ideal gas equation of state:

$$d\rho = \left(\frac{\partial \rho}{\partial P}\right) dP + \left(\frac{\partial \rho}{\partial T}\right) dT \tag{I.4}$$

In order to rewrite the various terms in the above equation in terms of thermodynamic quantities, one can divide both sides by ρ and multiply each term by the corresponding quantity:

$$\frac{d\rho}{\rho} = \frac{1}{\rho}\frac{P}{P}\frac{\partial \rho}{\partial P} dP + \frac{1}{\rho}\frac{T}{T}\frac{\partial \rho}{\partial T} dT \tag{I.5}$$

$$= \frac{P}{\rho}\frac{\partial \rho}{\partial P}\frac{dP}{P} + \frac{T}{P}\frac{\partial \rho}{\partial T}\frac{dT}{T} \tag{I.6}$$

$$= \alpha\frac{dP}{P} - \delta\frac{dT}{T} \tag{I.7}$$

As such, the density gradient can be obtained by dividing by the differential dr:

$$\frac{1}{\rho}\frac{\partial \rho}{\partial r} = \frac{\alpha}{P}\frac{\partial P}{\partial r} - \frac{\delta}{T}\frac{\partial T}{\partial r} \tag{I.8}$$

The condition for convective stability now reads

$$\left(\frac{\alpha}{P}\frac{\partial P}{\partial r} - \frac{\delta}{T}\frac{\partial T}{\partial r}\right)_{\text{parcel}} > \left(\frac{\alpha}{P}\frac{\partial P}{\partial r} - \frac{\delta}{T}\frac{\partial T}{\partial r}\right)_{\text{star}} \tag{I.9}$$

since pressure in the gas parcel is in balance with the surrounding gas, the first term on each side drops out.

$$\left(\frac{\delta}{T}\frac{\partial T}{\partial r}\right)_{\text{parcel}} < \left(\frac{\delta}{T}\frac{\partial T}{\partial r}\right)_{\text{star}} \tag{I.10}$$

Rewrite this condition in terms of logarithmic derivatives, we can multiply by the pressure scale height and cancel δ:

$$\left(-\frac{P}{dP/dr}\frac{1}{T}\frac{\partial T}{\partial r}\right)_{\text{parcel}} > \left(-\frac{P}{dP/dr}\frac{1}{T}\frac{\partial T}{\partial r}\right)_{\text{star}} \tag{I.11}$$

$$\left(\frac{\partial \ln T}{\partial \ln P}\right)_{\text{parcel}} > \left(\frac{\partial \ln T}{\partial \ln P}\right)_{\text{star}} \tag{I.12}$$

The left-hand-side is merely the adiabatic temperature gradient ∇_{ad}, while the right-hand-side requires explanation. This quantity is, in theory, also a temperature gradient. In the absence of convection, radiative diffusion is the mechanism via which energy is transported. Since we seek a condition wherein convection sets in, we assume that radiative transport is happening by default, and label the right-hand-side as ∇_{rad}. Finally, we arrive at the Schwarzschild stability criterion:

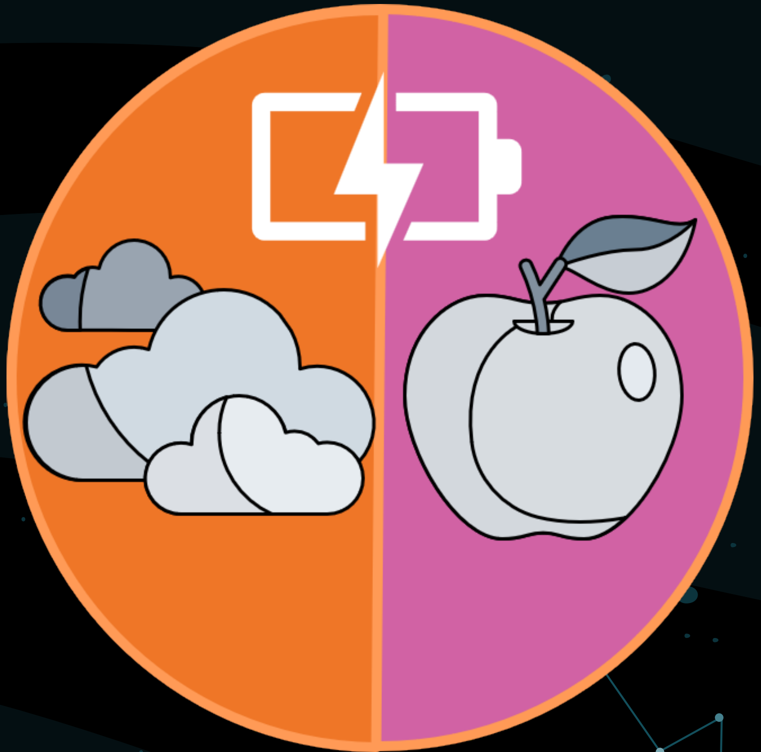
$$\boxed{\nabla_{\text{rad}} < \nabla_{\text{ad}}} \tag{I.13}$$

Virial Theorem

- Describes the relationship between the internal energy of gas particles and the star's gravitational binding energy via an equation of state parameter

- Encodes how energy is allocated within stars

$$\zeta E_i = E_g$$



Virial Theorem 2

$$\frac{dP}{dr} = -\frac{Gm(r)}{r^2}\rho(r)$$

$$\frac{\partial}{\partial m} = \frac{1}{4\pi r^2 \rho}\frac{\partial}{\partial r}$$

in Equation B.18, we can suggestively multiply the left-hand-side by $4\pi r^3$ and integrate by parts over mass elements dm from 0 to M, the mass of the star:

$$\int_0^M dm 4\pi r^3 \frac{\partial P}{\partial m} = [4\pi r^3 P]|_0^M - \int_0^M 12\pi dm r^2 \frac{\partial r}{\partial m} P \tag{B.56}$$

The surface term vanishes, since no mass is contained at the center of the star and pressure has been defined to vanish at the stellar surface. Using Equation B.12, the integrand of the

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho}$$

second term in Equation B.56 reduces to $3P/\rho$. Now moving our attention to the right-hand-side of Equation B.18, repeating the above integration scheme produces the gravitational binding energy of the star E_g given by

$$E_g \equiv -\int_0^M \frac{Gm}{r} dm \tag{B.57}$$

Finally, equating both integrated sides of B.18, we obtain one possible form of the virial theorem:

$$\boxed{3\int_0^M dm \frac{P}{\rho} = -E_g} \tag{B.58}$$

One immediate consequence of Equation B.58 is that the gravitational binding energy the star varies if the internal structure of the star changes, i.e., if there is expansion and contraction. Of course, any radial motion must be slow enough to maintain hydrostatic equilibrium. We can gain further insight into the consequences of the virial theorem by involving previously explored thermodynamic identities.

Using Equation B.49, we can continue to develop the virial theorem by subsisting P/ρ to involve the internal energy of gas particles, assuming that the equation of state does not change throughout the star:

$$3(\gamma - 1)\int_0^M dm u = E_g \tag{B.59}$$

Defining the integral in Equation B.59 as the total internal energy of the star E_i, and $\zeta = 3(\gamma - 1)$, we obtain another more general form of the virial theorem:

$$\boxed{\zeta E_i = E_g} \tag{B.60}$$

If we define the total energy of the star W as the sum of internal and gravitational binding energy,

$$W = E_i + E_g < 0 \text{ for a gravitationally bound star,} \tag{B.61}$$

along with Equation B.60, we see that total energy, internal energy, and gravitational binding energy are all coupled:

$$W = (1-\zeta)E_i = \frac{\zeta - 1}{\zeta} E_g \tag{B.62}$$

As the total energy of a star changes, its internal density/pressure profile changes. Since any gas of finite temperature radiates, we can start to consider the thermodynamics of the luminosity L of the star and how that relates to its total energy. Consider a simple example wherein the entire star contracts (slowly). By conservation of energy, we must have

$$\frac{dW}{dt} + L = 0. \tag{B.63}$$

Combining this with Equation B.62, essentially differentiating in time,

$$L = (\zeta - 1)\frac{dE_i}{dt} = -\frac{\zeta - 1}{\zeta}\frac{dE_g}{dt} \tag{B.64}$$

As the star produces energy and eventually radiates it away, a portion of this energy is used to heat the star. This may be interpreted as stars having negative specific heats.

We can also define a characteristic timescale describing the loss of internal energy to radiation. Barring any pathological choices for the adiabatic index γ, the luminosity of a contracting star is of order the rate of change of gravitational binding energy and internal energy. From simple dimensional analysis, the Kelvin-Helmholtz timescale [5] can be constructed as

$$\tau_{KH} \equiv \frac{|E_g|}{L} \approx \frac{E_i}{L} \tag{B.65}$$

For a sun-like star, one can estimate [27] the gravitational binding energy as

$$E_g = \frac{3GM^2}{4R}. \tag{B.66}$$

where again, M and R are the mass and radius of the entire star. Using values characteristic of the Sun,

$$\tau_{KH} \approx 2 \times 10^7 \text{ years} \left(\frac{M}{M_\odot}\right)^2 \left(\frac{L}{L_\odot}\right)^{-1} \left(\frac{R}{R_\odot}\right)^{-1}. \tag{B.67}$$

Maxwell's Equations

Maxwell's Equations in differential form are given by

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{47}$$
$$\nabla \cdot \vec{B} = 0 \tag{48}$$
$$\nabla \times \vec{E} = -\frac{\partial B}{\partial t} \tag{49}$$
$$\nabla \times \vec{B} = \mu_0 \left(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \tag{50}$$

When in vacuum, there are no charges $\rho = 0$ and no currents $\vec{J}$:

$$\nabla \cdot \vec{E} = 0 \tag{51}$$
$$\nabla \cdot \vec{B} = 0 \tag{52}$$
$$\nabla \times \vec{E} + \frac{\partial B}{\partial t} = 0 \tag{53}$$
$$\nabla \times \vec{B} - \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} = 0 \tag{54}$$

The curl of the curl identity is

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} \tag{55}$$

Taking the curl of the curl equations gives us

$$\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} - \nabla^2 \vec{E} = 0 \tag{56}$$
$$\mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} - \nabla^2 \vec{B} = 0 \tag{57}$$

Defining the speed of electromagnetic waves as $c = (\mu_0 \epsilon_0)^{-1/2}$,

$$\frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = \nabla^2 \vec{E} \tag{58}$$
$$\frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} = \nabla^2 \vec{B} \tag{59}$$

which is a wave equation, thus proving the existence of electromagnetic waves.