Microscopic behavior of eigenvalues and charged particles

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Courant Instructor Day

Random Matrix Theory - I

$(M_{i,j})_{1\leq i,j\leq N}$ random variables $\longrightarrow M$, a $N\times N$ random matrix

Quantum physics

- Interactions $\leftrightarrow$ Hermitian operator.
- Energy levels $\leftrightarrow$ eigenvalues.
- Complicated system $\leftrightarrow$ "Random interaction".

Random Matrix Theory - II

Dyson, 1962

(. . .) a mathematical idealization of the notion of "all physical systems with equal probability"

Universality?

Some properties may be **universal** i.e. fairly independent of the law of the matrix.

Statistical physics

(Classical, at equilibrium, in Euclidean space, ...)

N particles $x_1, \ldots, x_N$ in $\mathbb{R}^d$.

- State of the system: $\vec{X}_N = (x_1, \ldots, x_N)$.
- Energy in the state $\vec{X}_N$ given by $\mathcal{H}_N(\vec{X}_N)$.

Complicated system, $N \approx 10^{23}$ (Avogadro) very large $\longrightarrow$ "random state"

- Microcanonical: the energy E is fixed and $\{\vec{X}_N, \mathcal{H}_N(\vec{X}_N) = E\}$ are equiprobable
- Canonical: the temperature $T = \frac{1}{\beta}$ is fixed and density

$$dP(\vec{X}_N) = \frac{1}{Z_{N,\beta}} \exp\left(-\beta \mathcal{H}_N(\vec{X}_N)\right) d\vec{X}_N$$

$Z_{N,\beta}$ = normalizing constant

Connection - I

$N \times N$ coefficients for the random matrix = independent Gaussian random variables.

1. + symmetry condition (Hermitian matrix, real eigenvalues)
2. no symmetry condition (non-Hermitian matrix, complex eigenvalues)

$\vec{X}_N = (x_1, \ldots, x_N)$ the eigenvalues. Explicit computation:

$$dP(\vec{X}_N) = \frac{1}{Z_N} \exp\left(- \left(c_1 \sum_{i \neq j} - \log |x_i - x_j| + c_2 \sum_{i=1}^{N} N |x_i|^2 \right) \right) d\vec{X}_N$$

$d\vec{X}_N$ = Lebesgue measure on $\mathbb{R}$ or $\mathcal{C}$.

Connection - II

Morality

Eigenvalues $\subset$ (charged) particles with

1. Logarithmic repulsion $\leftrightarrow \sum_{i \neq j} - \log |x_i - x_j|$ (singular + long-range)
2. External potential $\leftrightarrow N|x|^2$ (confining)
3. Some values of the (inverse) temperature β

Conversely?

"True" in the Hermitian case (β-ensembles).

Macroscopic behavior - I

$\vec{X}_N = (x_1, \ldots, x_N) \longrightarrow$ Density/Empirical measure $\mu_N := \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$

$$\int f(x) d\mu_N(x) := \frac{1}{N}\sum_{i=1}^N f(x_i) \quad \text{(definition)}$$

Fact: Equilibrium measure μ_{eq} such that $\lim_{N\to\infty} \mu_N = \mu_{\mathrm{eq}}$ almost surely.

$$\int f(x) d\mu_N(x) \approx \int f(x) d\mu_{\mathrm{eq}}(x)$$

except for very small probability (order $\exp(-N^2)$)

Macroscopic behavior - II

$$d\mathbb{P}_{N,\beta}(\vec{X}_N) = \frac{1}{Z_{N,\beta}} \exp\left(-\beta\left(\sum_{i\neq j} -\log|x_i - x_j| + \sum_{i=1}^{N} NV(x)\right)\right) d\vec{X}_N$$

- Macroscopic behavior does not depend on β, depends on V (non-universal).
- Obtained by minimizing an energy functional from classical potential theory.
- Examples: semi-circle law $\frac{1}{2\pi}\sqrt{4-x^2}\mathbf{1}_{[-2,2]}$, circular law $\frac{1}{\pi}\mathbf{1}_{|x|\leq 1}$.

β large

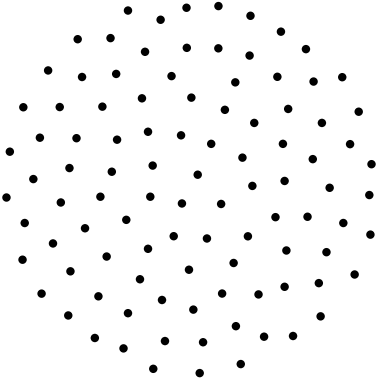

Figure: $\beta = 400$

β small

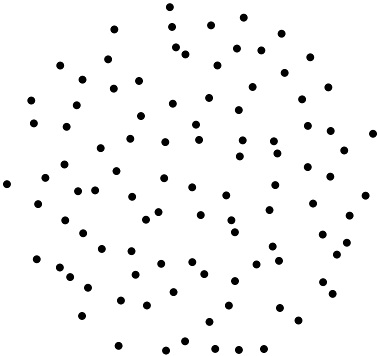

Figure: $\beta = 5$

Microscopic behavior

Observable "empirical field" P_N (law of a random point configuration).

- "Typically a square lattice"
- "1/2 triangular lattice, 1/2 square lattice"
- "1/3 lattice, 2/3 disorder"

Theorem

The empirical field behaves like minimizers of a certain "free energy" functional

$$\beta \mathit{Energy} + \mathit{Entropy}.$$

In particular, point processes arising in RMT must minimize the functional.

Competition: Energy favors "order" (crystal?) and entropy favors disorder (Poisson).
Microscopic behavior depends on β, not on V ('universal').

Perspectives

- Other models (random polynomials, unitary matrices..).
- Understand minimizers (phase transition?).

Thank you for your attention!