Name/NetID:

I - Logic.

1. Let A, B, C be three propositions (or Boolean variables). Prove that the following is a *tautology*, i.e. that it is true for every possible Boolean value (true/false) of A, B, C.

$$(A \implies B) \implies ((B \implies C) \implies (A \implies C)).$$

2. Write the formal negation of the following statement (the connector $\neg$ should not appear in your answer).

$$\forall x, \forall y, \forall z, \forall n, ((x^n + y^n = z^n) \implies ((n \leq 2) \vee (xyz = 0))).$$

II - Sets Let A be the set $A := \{a, \{a\}\}$.

1. Write down the power set of A, denoted by 2^A.

2. Write down the power set of the power set of A, denoted by 2^{2^A}. *Be patient.*

3. Compute the symmetric difference $2^A \triangle 2^{2^A}$.

III - Relations & Functions Let S be some finite set. We define two relations on 2^S as follows: let $A \in 2^S$ and $B \in 2^S$:

- We say that A is *equipotent* to B and we write $A \approx B$ if there exists a bijective map from A to B.
- We say that A is *subpotent* to B and we write $A \preceq B$ if there exists an injective map from A to B.

1. Prove that "is equipotent to" is an equivalence relation on 2^S.

2. Prove that "is subpotent to" is reflexive and transitive.

3. Show that, in general, "is subpotent to" is not antisymmetric.

IV - Functions & modulo As usual, $\mathbb{N}$ denotes the set of natural integers $\{0, 1, 2, \dots\}$.

$$\boxed{\text{Let } f : \mathbb{N} \to \mathbb{N} \text{ be a map satisfying } \forall n \in \mathbb{N}, f \circ f(n) = n + 2017.}$$

1. Prove that f is one-to-one.

2. Prove that $\forall n \in \mathbb{N}, f(n + 2017) = f(n) + 2017$.

3. Deduce from the previous question that $\forall k \geq 0, \forall n \in \mathbb{N}, f(n + 2017k) = f(n) + 2017k$. *You may treat $k = 0$ separately and then argue by induction on $k \geq 1$.*

We let π be the map $\pi : \mathbb{N} \to \{0, \ldots, 2016\}$ defined as follows: for any n in $\mathbb{N}$, $\pi(n)$ is the unique integer in $\{0, \ldots, 2016\}$ which is equal to n mod 2017.

4. Explain why for any n in $\mathbb{N}$, there exists $k \geq 0$ such that $n = \pi(n) + 2017k$.

5. Conversely, prove that for any x, y in $\mathbb{N}$, if there exists $k \geq 0$ such that $x = y + 2017k$ then $\pi(x) = \pi(y)$.

6. Deduce from previous questions that for any n in $\mathbb{N}$, there exists $k \geq 0$ such that

$$f(n) = f(\pi(n)) + 2017k.$$

7. Deduce from previous questions that $\pi \circ f = \pi \circ f \circ \pi$.

V - Number theory

1. Prove by induction that for all $n \geq 1$, $3^{2n} + 7$ is divisible by 4

2. Give all solutions to the following system of two equations: $x = 4 \mod 5$ and $x = 6 \mod 13$.

3. Find the inverse of 167 mod 253.

Bonus questions

- Exercise 2: How many elements are there in $A \cup \left(2^A \times (2^{2^A} \cup 2^{A \times 2^A})\right)$?
- Exercise 3: Prove that "is subpotent to" can be extended in a meaningful way as an order relation on the quotient set. $2^S / \approx$
- Exercise 4: Show that $\pi \circ f$ is an involution from $\{0, \ldots, 2016\}$ and that it does not have a fixed point. Deduce that f cannot exist.
- Exercise 5 What is the last digit of $7^{7^{7^{7^{7^{7^{7}}}}}}$?

Scratch paper.

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