Statistical physics approach to 1d and 2d Log-gases

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Optimal and Random Point Configurations
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Log-gases

N point particles $\vec{X}_N = (x_1, \ldots, x_N)$ in $\mathbb{R}^\mathrm{d}$ ($\mathrm{d} = 1, 2$)

- Logarithmic pairwise interaction $-\log|x - y|$
- Confining field/potential $V(x)$ (continuous + growth at infinity, e.g. $V(x) = |x|^2$).

Energy in the state $\vec{X}_N$

$$\boxed{\mathcal{H}_N(\vec{X}_N) := \sum_{i \neq j} -\log|x_i - x_j| + \sum_{i=1}^{N} N V(x_i)}$$

Gibbs measure

Canonical Gibbs measure at (inverse) temperature β

$$\boxed{d\mathbb{P}_{N,\beta}(\vec{X}_N) := \frac{1}{Z_{N,\beta}} \exp\left(-\frac{\beta}{2}\mathcal{H}_N(\vec{X}_N)\right) d\vec{X}_N}$$

$d\vec{X}_N =$ Lebesgue on $(\mathbb{R}^{\mathrm{d}})^N$, with $Z_{N,\beta}$ (the partition function)

$$Z_{N,\beta} := \int_{(\mathbb{R}^d)^N} \exp\left(-\frac{\beta}{2}\mathcal{H}_N(\vec{X}_N)\right) d\vec{X}_N.$$

(factor $\frac{\beta}{2}$ for reasons of convention)

Questions

Asymptotic behavior of the system ($N \to \infty$)? Fluctuations?
Dependency on β? Dependency on V ("universality")?

Motivation I - Random Matrix Theory (RMT)

Classical Gaussian Hermitian ensembles

GOE, GUE, GSE = Gaussian Orthogonal/Unitary/Symplectic Ensemble

Large $(N \times N)$ matrix with Gaussian coefficients in $\mathbb{R}, \mathbb{C}, \mathbb{H}$ (quaternions)

Coefficients independent up to symmetry (symmetric, Hermitian, self-dual).

$\rightarrow$ N real eigenvalues

A non-Hermitian ensemble: the Ginibre ensemble

Ginibre ensemble: $N \times N$ matrix with complex Gaussian i.i.d coefficients. No symmetry.

$\rightarrow$ N complex eigenvalues

RMT 2

Observation (Dyson, Ginibre)

Joint law of eigenvalues explicitly computable (thanks to Gaussian distribution). **Coincides with the canonical Gibbs measure of a log-gas.**

$$dP_{\mathrm{RMT}}(\lambda_1, \dots, \lambda_N) = \frac{1}{Z_{N,\beta}} \exp\left(-\frac{\beta}{2}\mathcal{H}_N(\lambda_1, \dots, \lambda_N)\right) d\lambda_1 \dots d\lambda_N$$

d $= 1$ for Hermitian matrices, d $= 2$ for non-Hermitian.
$\beta = 1, 2, 4$ (GOE, GUE, GSE) and $\beta = 2$ (Ginibre)
V quadratic.

In d $= 1$, for all β, existence of a tridiagonal matrix model whose eigenvalues are distributed as $\mathbb{P}_{N,\beta}$ (Dimitriu-Edelman).

Motivation II - Statistical physics

Statistical physics

- Model with singular, long-range interactions in $\mathbb{R}^d$.
- $d = 2$ 'One-component plasma', 'Coulomb gas' (log is Coulomb interaction!), 'Dyson gas', 'jellium'...
- "Real-life" implementations (vortex systems, electrostatics, Calogero-Sutherland model, Laughlin wave function)

Also approximation theory, etc.

Global behavior

Empirical measure

Encodes the global/macroscopic behavior

$$\mu_N := \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i}.$$

$$\lim_{N\to\infty}\mu_N = \mu_{\mathrm{eq},V} \text{ “equilibrium measure”}$$

In general

V confining (growth at infinity) $\Longrightarrow$ the support Σ_V of $\mu_{\mathrm{eq},V}$ is compact.

Remark: $\mu_{\mathrm{eq},V}$ depends on V, d but not on β.

Examples

- d $= 1$, V quadratic $\to$ Wigner's semicircle law

$$d\mu(x) = \frac{1}{2\pi}\sqrt{4 - x^2}dx$$

  $=$ eigenvalues of symmetric/Hermitian Gaussian random matrices (in fact, *universality*).

- d $= 2$, V quadratic $\to$ Circular law

$$d\mu(x) = \frac{1}{\pi}\mathbf{1}_{|x|\leq 1}dx$$

  $=$ eigenvalues of Ginibre ensemble (in fact, *universality*).

Variational characterization of macroscopic behavior

Variational property of $\mu_{\mathrm{eq},V}$

$\mu_{\mathrm{eq},V}$ = minimizer on $\mathcal{P}(\mathbb{R}^d)$ of *weighted logarithmic energy*

$$\boxed{I_V(\mu) := \iint -\log|x-y|\,d\mu(x)d\mu(y) + \int V(x)d\mu(x).}$$

Large deviations at speed N^2 (Guionnet-Zeitouni, Ben Arous-Zeitouni, Hiai-Petz, etc.)

Remark

$I_V(\mu)$ can be seen as "continuum" or "mean-field" limit of finite-N energy

$$\mathcal{H}_N(\vec{X}_N) := \sum_{i\neq j} -\log|x_i - x_j| + \sum_{i=1}^{N} NV(x_i) \approx N^2 I_V(\mu_N)$$

Further questions

Microscopic behavior

Zoom into the system by $N^{1/\mathrm{d}} \to$ finite (N-) point configuration. What does it look like?

Fluctuations

In which sense does $\mu_N \approx \mu_{\mathrm{eq},V}$ (empirical measure $\approx$ equilibrium measure)?

- At small scales ($O(1) \to O(N^{-\frac{1}{d}+\varepsilon})$)?
- Deviations bounds?
- Central limit theorem?

Possible approaches

- Integrable probability. For $\beta = 1, 2, 4$, algebraic structure. E.g. $\beta = 2$, for d $= 1, 2$, *determinantal point process*.
- Orthogonal polynomial (Riemann-Hilbert problems).
- Dynamical approach (Dyson Brownian Motion).
- Loop equations.

"Statistical physics" approach

- β, V general, d $= 1, 2$ (and maybe other interactions, dimensions?)
- Energy $\mathcal{H}_N(\vec{X}_N)$, volume term $d\vec{X}_N$.
- More robust, less precise?

Splitting formula

Sandier-Serfaty

$\mathcal{H}_N(\vec{X}_N) = \sum_{i \neq j} -\log|x_i - x_j| + N\sum_{i=1}^{N} V(x_i)$ becomes

$$\boxed{\mathcal{H}_N(\vec{X}_N) = N^2 I_V(\mu_{\mathrm{eq},V}) - \frac{N\log N}{\mathrm{d}} + 2N\zeta_N(\vec{X}_N) + F_N^{\mu_{\mathrm{eq},V}}(\vec{X}_N)}$$

- First-order energy
- Constant term (due to scaling)
- Confining term (say, 0 on support Σ_V and $+\infty$ outside)
- Second-order interaction energy

Second-order interaction I

Old term

$$\sum_{i \neq j} -\log|x_i - x_j| = N^2 \iint_{(\mathbb{R}^d \times \mathbb{R}^d) \setminus \triangle} -\log|x-y| \, d\mu_N^{\otimes 2}(x,y)$$

Old system: N point charges.

New term

$$\boxed{F_N^{\mu_{\mathrm{eq},V}}(\vec{X}_N) = \iint_{(\mathbb{R}^d \times \mathbb{R}^d) \setminus \triangle} -\log|x-y| (d\nu'_N - d\mu'_{\mathrm{eq},V})^{\otimes 2}(x,y)}$$

New system (zoomed!):

- N point charges $\nu'_N = \sum_{i=1}^N \delta_{N^{1/d} x_i}$
- Negative background of density $\mu'_{\mathrm{eq},V}(x) = \mu_{\mathrm{eq},V}(N^{-1/d} x)$

Second-order interaction II

Some intuition

Recall:

$$\nu_N' = \sum_{i=1}^{N} \delta_{N^{1/d} x_i}, \quad \mu_{\mathrm{eq},V}(x) = \mu_{\mathrm{eq},V}(N^{-1/d} x)$$

- $F_N^{\mu_{\mathrm{eq},V}}(\vec{X}_N)$ measures the difference between finite-N point configuration ν_N' and continuous background $\mu_{\mathrm{eq},V}'$.
- Fact: $F_N^{\mu_{\mathrm{eq},V}}$ is typically of order N.
- The limit $N \to \infty$ of ν_N' is an infinite point configuration.
- The limit $N \to \infty$ of $\mu_{\mathrm{eq},V}'$ is a constant density.

Microscopic behavior I

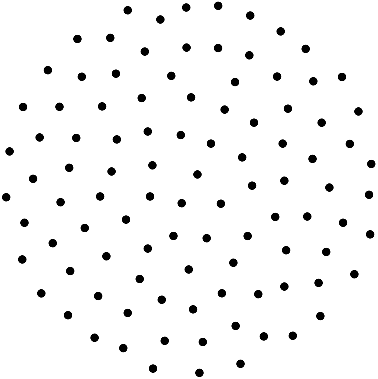


Figure: $\beta = 400$

Microscopic behavior I

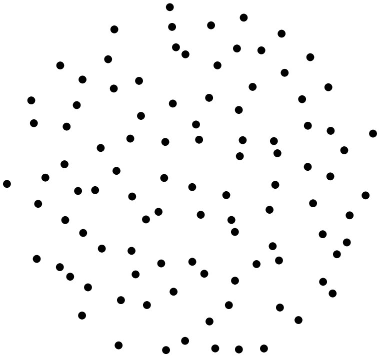


Figure: $\beta = 5$

Microscopic behavior II

Non-averaged point process

Let $z \in \mathring{\Sigma}_V$ be fixed. Zoom around z at scale $N^{1/\mathsf{d}}$.

$$\boxed{\mathcal{C}_{N,z} : \vec{X}_N \mapsto \sum_{i=1}^{N} \delta_{N^{1/\mathsf{d}}(x_i - z)}.}$$

Values in $\mathcal{X}$, the space of point configurations.

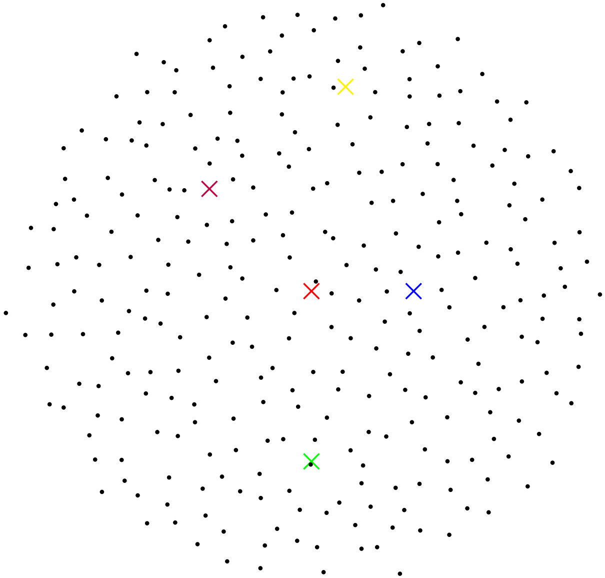
Empirical field

Let $\Omega \subset \Sigma_V$ be fixed. Randomly pick z in Ω and zoom around it.

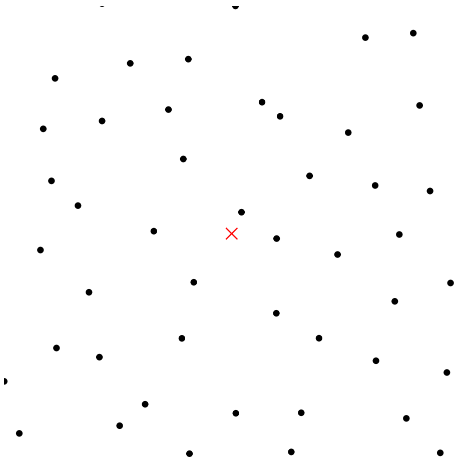
$$\boxed{\overline{\mathcal{C}}_{N,\Omega} : \vec{X}_N \mapsto \frac{1}{|\Omega|} \int_{\Omega} \delta_{\mathcal{C}_{N,z}} dz}$$

Average of $\mathcal{C}_{N,z}$'s. Values in $\mathcal{P}(\mathcal{X})$.

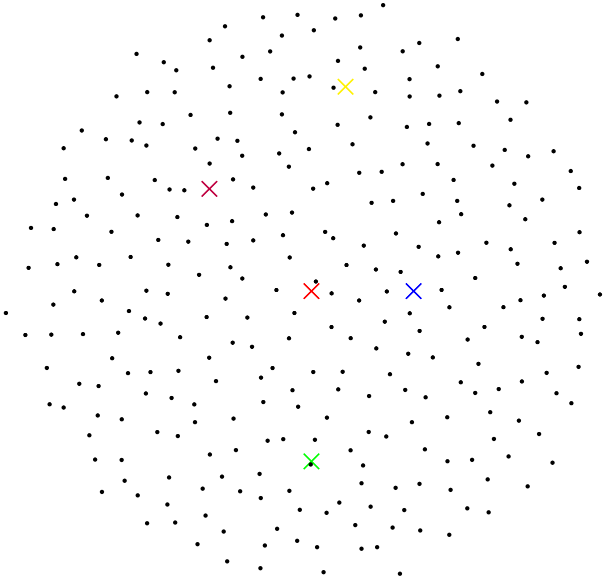
Empirical field II



Empirical field II



Empirical field II



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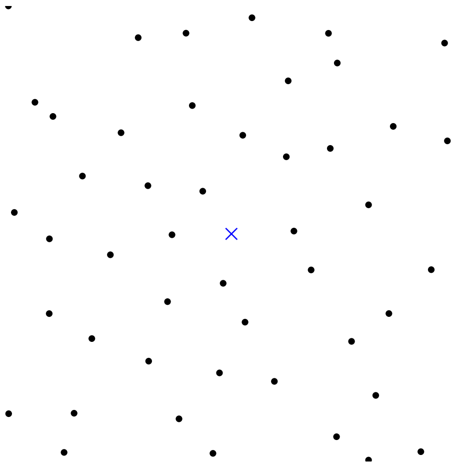

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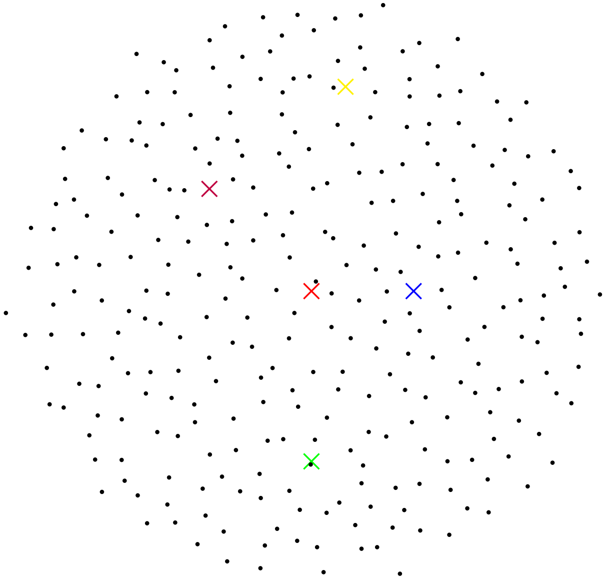

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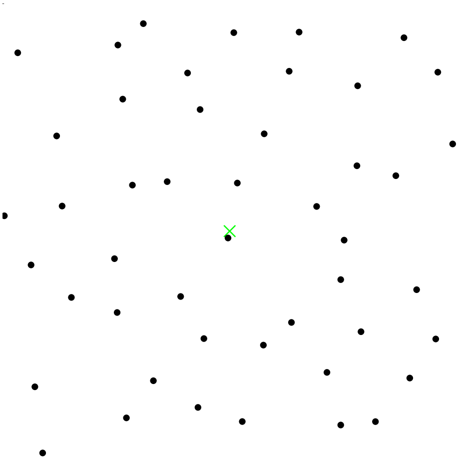

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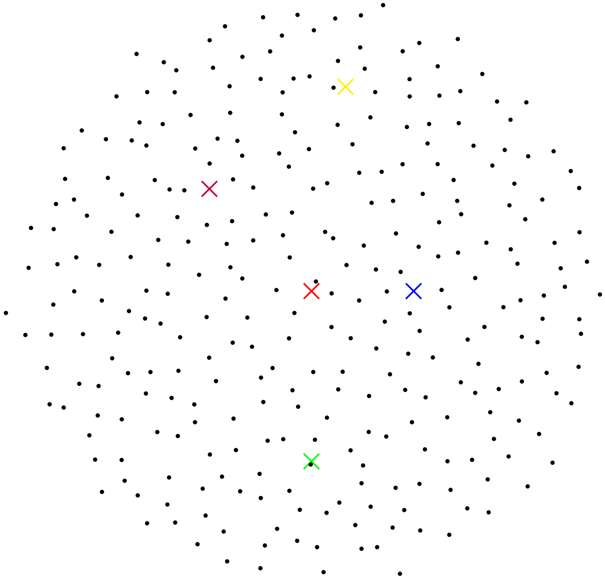

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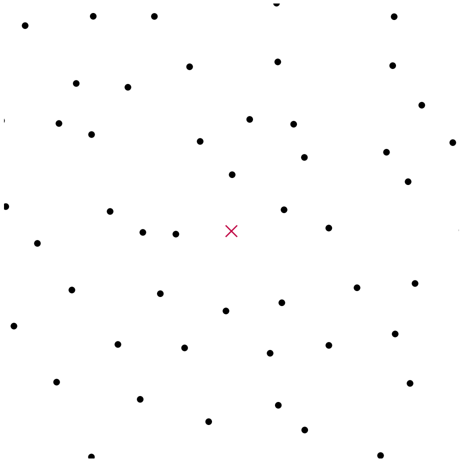

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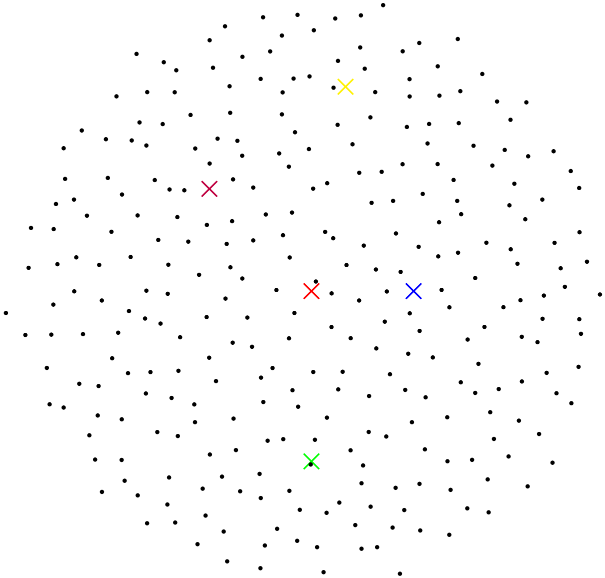
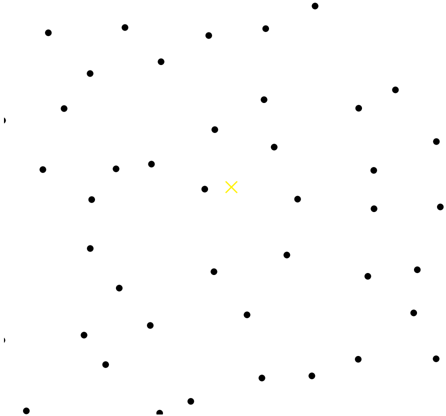
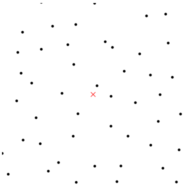


Empirical field II



Empirical field II

Empirical field = random variable



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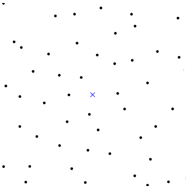

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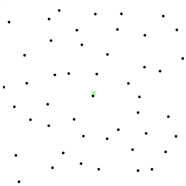

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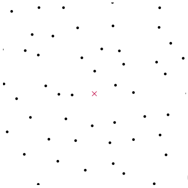

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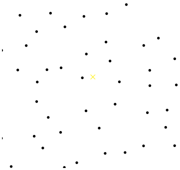

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Microscopic behavior - III

Rate function

For $m > 0$, define $\mathcal{F}_\beta^m$ on the space $\mathcal{P}(\mathcal{X})$

$$\mathcal{F}_\beta^m(P) := \beta \mathbb{W}_m^{\mathrm{elec}}(P) + \mathbf{ent}[P|\Pi^m]$$

$\mathbb{W}_m^{\mathrm{elec}}(P)$ is an energy functional, $\mathbf{ent}[P|\Pi^m]$ is a relative entropy functional, Π^m = Poisson point process of intensity m.

Theorem (L. - Serfaty)

Zoom around a point z for which $\mu_{\mathrm{eq},V}(z) = m$ and do a small averaging.

The law of the empirical field **concentrates on minimizers of** $\mathcal{F}_\beta^m$ *except for events of probability* $\exp(-N)$.

(Large Deviation Principle at speed N with rate function built out of $\mathcal{F}_\beta^m$)

Variational property of microscopic behavior

- Microscopic behavior $\iff$ minimizers of free energy $\beta \mathbb{W}_m^{\mathrm{elec}}(P) + \mathbf{ent}[P|\Pi^m]$
- Depend on m only through scaling, hence some kind of universality.

Terms

- Specific relative entropy $\mathbf{ent}[P|\Pi^1]$ minimized for $P = \Pi^1$.
- Energy $\mathbb{W}^{\mathrm{elec}}(P)$ expected to favorize ordered configurations (true for $\mathrm{d} = 1$).

Statistical physics approach?

Basic idea

$$\mathbb{P}_{N,\beta}\left(\text{Empirical field} \approx P\right)$$
$$\propto \int_{(\Sigma_V)^N} \mathbf{1}_{\text{Empirical field}\approx P} \exp\left(-\beta F_N^{\mu_{\text{eq}},V}(\vec{X}_N)\right) d\vec{X}_N$$

Separate between

- Energy contribution

$$\text{Empirical field} \approx P \Longrightarrow F_N^{\mu_{\text{eq}},V}(\vec{X}_N) \approx N\mathbb{W}^{\text{elec}}(P).$$

- Volume contribution

$$\int_{(\Sigma_V)^N} \mathbf{1}_{\text{Empirical field}\approx P} d\vec{X}_N \approx \exp\left(-N\mathbf{ent}[P|\Pi]\right).$$

+ throw away bad configurations (high energy/small volume)

What is known about microscopic behavior?

- For d $= 1$ and $\beta > 0$, existence of a limit point process (Valkó-Virág, also Killip-Stoiciu) named "Sine_β" point process (for V quadratic). The description involves counting explosions of a certain diffusion on the hyperbolic disk. Existence + properties of the limit (Holcomb-Valkó, Allez-Dumaz).
- Universality of the microscopic behavior in d $= 1$ with respect to V (Bourgade-Erdös-Yau-Lin, Bekerman-Figalli-Guionnet)
- For d $= 2$, $\beta = 2$, the finite-N Ginibre ensemble is determinantal, converges to a well-understood limit point process (Ginibre). Some universality for $\beta = 2$ (Ameur-Hedenmalm-Makarov).

What can we say about our rate function?

$$\mathcal{F}_\beta(P) := \beta \mathbb{W}^{\mathrm{elec}}(P) + \mathbf{ent}[P|\Pi^1]$$

- The Sine_β point processes are minimizers of $\mathcal{F}_\beta$ for $\beta > 0$ in the $d = 1$ case. Expected to be unique.
- The Ginibre point process minimizes $\mathcal{F}_\beta$ for $\beta = 2$ in the $d = 2$ case.
- Minimizers of $\mathcal{F}_\beta$ tend (in entropy sense) to a Poisson point process as $\beta \to 0$. (Known for Sine_β by studying the diffusion in $\mathrm{d} = 1$, Allez-Dumaz)
- In dimension 1 minimizers of $\mathcal{F}_\beta$ converge to $P_\mathbb{Z}$ ("clock process") as $\beta \to \infty$. $P_\mathbb{Z}$ is the unique stationary minimizer of the energy.

Physical description of minimizers

For finite N the law of the (finite) point process

$$dP_N(\vec{X}_N) = \frac{1}{Z_N} \exp(-\beta \mathsf{Energy}_N) d\vec{X}_N$$

minimizes

$$\beta \mathbf{E}_{\cdot} [\mathsf{Energy}_N] + \mathrm{Ent}[\cdot].$$

Infinite volume, is $\exp\left(-\beta \mathsf{Energy}_\infty\right) d\Pi^1$ the minimizer of

$$\mathcal{F}_\beta(P) := \beta \mathbf{E}_P[\mathsf{Energy}_\infty] + \mathbf{ent}[P|\Pi^1]$$

(where Π = Poisson)? Answer: no.

DLR equations

Dobrushin-Lanford-Ruelle (DLR) formalism: condition on the exterior.

DLR description for canonical Gibbs

On a fixed box Λ, the law of P is given by

$$\mathbf{E}_P[F] = \int dP(\gamma) \int F(\eta) \frac{\exp\left(-\beta \mathrm{Energy}(\eta, \gamma_{\Lambda^c})\right)}{Z^{\gamma}_{\beta,\Lambda}} d\mathbf{B}_{\Lambda, \#\gamma\cap\Lambda}(\eta)$$

where

- F is a local test function on the space of point configurations
- γ, η are point configurations
- γ_{Λ^c} is the exterior
- η is sampled in Λ instead of γ_Λ, same number of points, according to a Gibbs-like measure

$$\frac{1}{Z^{\gamma}_{\beta,\Lambda}} \exp\left(-\beta \mathrm{Energy}(\eta, \gamma_{\Lambda^c})\right) d\mathbf{B}_{\Lambda, \#\gamma\cap\Lambda}(\eta)$$

DLR for Sine_β

Theorem (Dereudre - Hardy - L. - Maïda (in progress))

For $\beta > 0$, Sine_β *satisfies a canonical Gibbs DLR description.*

Already known for "integrable case" ($\beta = 2$) Bufetov, Kuijlaars-Miña-Díaz.

Applications?

Fluctuations

CLT for fluctuations

The "statistical physics" approach can be used to study fluctuations of log-gases, and to obtain CLT's

$$\int \varphi \left(\sum_{i=1}^{N} \delta_{x_i} - N\mu_{\mathrm{eq},V} \right) \to \text{Gaussian}.$$

L.-Serfaty for d $= 2$ (see also Bauerschmidt-Bourgade-Nikula-Yau), Bekerman-L.-Serfaty d $= 1$ (well-known since Johansson, Shcherbina, Borot-Guionnet, see also Lambert-Ledoux-Webb). Some similarities with "loop/Dyson-Schwinger equations" (cf. Paul's talk), with crucial use of a consequence of LDP: expansion of $Z_{N,\beta}$ to order N.

Questions

"Optimal and random point configurations"

Two big questions for $d = 2$

1. Minimizers of the energy? [Optimal]
2. Phase transition? [Random]

For 1. the conjectured answer is triangular/hexagonal/Abrikosov lattice. "Crystallization conjecture". Proof???

What about the phase transition?

Origin

Alastuey-Jancovici(1981) “On the classical two-dimensional one-component Coulomb plasma”

> *It is an intriguing question to know whether the exact model has a phase transition (or perhaps several ones).*

Order parameter? Probably two-point correlation function $\rho_2(x, y)$.
For $\beta = 2$ (high temperature, Ginibre ensemble), known exactly

$$\rho_2(x, y) \approx 1 - \exp(-|x - y|^2).$$

Numerical evidence - I

Caillol-Levesque-Weis-Hansen (1982) "A Monte Carlo Study of the Classical Two-Dimensional One-Component Plasma"

We have located the fluid-solid phase transition by **comparing the free energies of both phases** *[liquid and solid].*

When plotted as functions of β, **the two free-energy curves intersect at** $\beta = 140$*; the fluid phase has the lower free energy and is hence the stable phase, below* $\beta = 140$*, while above that value of the coupling constant the solid is the stable phase.*

Note that a simple comparison of the free energies of the fluid and solid phases **does not rule out the possible existence of an intermediate "hexatic" phase***, the first-order transition being replaced by a succession of two "continuous" (second-order) transitions.*

Two-point correlation

Caillol-Levesque-Weis-Hansen

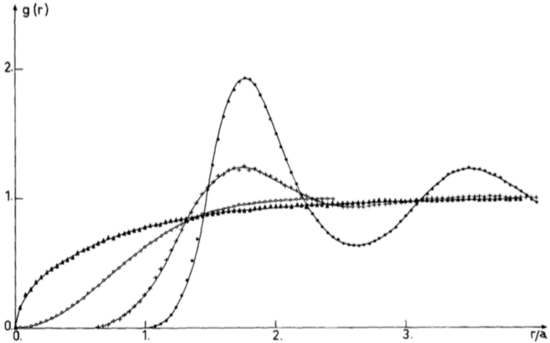


Fig. 1. Pair distribution function $g(r)$ versus $x = r/a$, from the Monte Carlo simulations on a sphere at $\Gamma = 0.5$ (triangles), $\Gamma = 2$ (circles), $\Gamma = 10$ (crosses), and $\Gamma = 40$ (dots).

Two-point correlation

Caillol-Levesque-Weis-Hansen

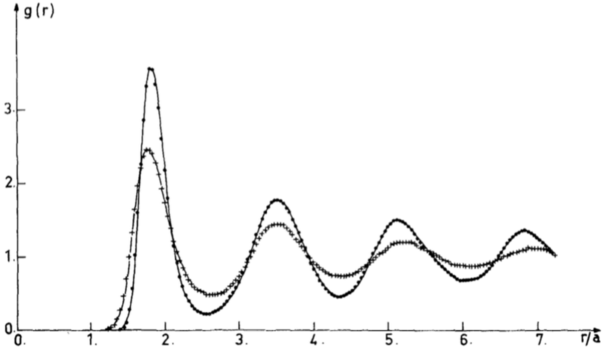
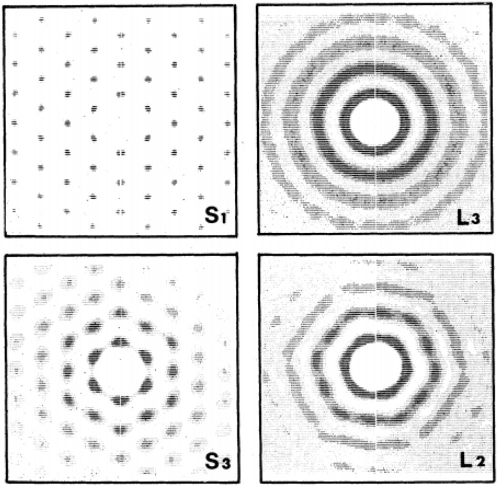


Fig. 2. Same as Fig. 1 for $\Gamma = 80$ (crosses) and $\Gamma = 200$ (dots).

Numerical evidence - II

Choquard, Clerouin "Cooperative Phenomena below Melting of the One-Component Two-Dimensional Plasma" (1983)



Compare S_3 ($\beta = 140$) and L_2 ($\beta = 137$)!!

Numerical evidences - III

Krauth et al. (2010's)

> *Generic two-dimensional particle systems* **cannot crystallize at finite temperature** *because of the importance of fluctuations, yet they may form solids.*
>
> *A crystal thus possesses long-range orientational order. The* **positional correlations of a two-dimensional solid decay to zero as a power law** *at large distances [but] the lattice distortions* **preserve long-range orientational order**.

Questions

- Can we replicate the numerics?
- Is there a phase transition? Of which nature? Non-uniqueness of minimisers (up to rotations)?
- Take $\beta = 2$. The positional decay of correlations is $\exp(-|x-y|^2)$. Can it really be specific to $\beta = 2$? Is there some sort of KT transition?
- Where does $\beta = 140$ come from?

Thank you for your attention!