Calculus 3 - 123.003 - Midterm 1

Thursday, October 13th 2016
Documents, calculators, cell phones, computers etc. are **not** allowed.

- Please use a pen, write intelligibly, and emphasize the main points of your reasoning, the crucial steps of your calculations and especially your results.
- Use a **draft** if you need one. Striking out some mistakes are OK, confusing your draft and your answer sheet is not.
- Write your answers as sentences, and prefer words over symbols whenever possible.
- Double check long computations, and make sure that your final answer is in the simplest form possible (i.e. all the cancellations have been taken into account).

  For example $1 + (x+1)^2 - 2x - \ln(\sin^2 + \cos^2 x)$ is not a final answer, $x^2 + 2$ is.
- The seven exercises are completely independent, and you can treat them in the order of your choice (but please make it clear). Inside the exercises, some questions are independent, feel free to skip one (but please make it clear). Do not get stuck on one exercise, move on after a few minutes if you really have no idea, and come to it later.
- Exercise 7 is a **bonus** question, which requires to take initiatives. It might remind you of an exercise treated in class.
- Show your work. Any beginning of an honest answer will be taken into account. On the other hand, do not try to fool me (or yourself) by writing nonsense.
- Trust yourself. Good luck!

The scale of points for each exercise will be determined according to how well you will have collectively done. A first approximation would be that all "atomic" questions are approximately worth the same.

1. Let us define three vectors in $\mathbb{R}^3$ as follows

$$\vec{u} = \langle 3, -5, 2 \rangle, \quad \vec{v} = \langle 4, 1, -1 \rangle, \quad \vec{w} = \langle 1, -1, 1 \rangle.$$

   (a) Compute the dot product $\vec{u} \cdot \vec{v}$
   (b) Compute the cross product $\vec{u} \times \vec{w}$
   (c) Compute $(\vec{w} \times \vec{u}) \cdot \vec{w}$
   (d) Compute the area of the parallelogram generated by $\vec{u}$ and $\vec{v}$.

2. Let f be the function defined on $\mathbb{R}^2$ by

$$f : (x, y) \mapsto \exp(\cos(x)\sin(y)).$$

   (a) Compute the first derivatives $\partial_x f$ and $\partial_y f$.
   (b) Compute the second derivatives of f.

3. Let $\mathcal{P}$ be the plane passing through the points $(1, 2, 3)$, $(-1, 2, -3)$ and $(-4, 5, 0)$ in $\mathbb{R}^3$.

   (a) Compute a unit normal vector to $\mathcal{P}$.
   (b) Give an equation of $\mathcal{P}$.

4. Study the existence of the following limits

$$\lim_{(x,y)\to(0,0)} \frac{x^4 - y^3}{x^2 + y^6}, \quad \lim_{(x,y)\to(0,0)} \frac{x^2 y^3}{x^4 + y^2}, \quad \lim_{(x,y)\to(0,0)} \frac{\cos(x^2) + e^{3y}}{1 + \sin^2(xy)}.$$

5. Let r be a map from $[0, 1]$ to $\mathbb{R}^3$ defined by

$$r(t) := \left(t, \frac{t^2}{2}, \frac{2\sqrt{2}}{3} t^{3/2} \right).$$

   It is a space curve parametrized by $t \in [0, 1]$.

   (a) Is r continuous on $[0, 1]$? Is it differentiable on $[0, 1]$? (Give a short, but precise explanation).
   (b) For any $t \in [0, 1]$, compute $r'(t)$.
   (c) For any $t \in [0, 1]$, compute $\int_0^t r(s)ds$.
   (d) For any $t \in [0, 1]$, we denote by $\mathbf{Arc}(t)$ the arclength of the curve between the point of parameter 0 and the point of parameter t. Prove that

$$\mathbf{Arc}(t) = \frac{t^2}{2} + t.$$

   (e) Show that **Arc** is a bijection from $[0, 1]$ to $[0, \frac{3}{2}]$ and compute its inverse bijection $\mathbf{Arc}^{-1}$.
   (f) Reparametrize the curve with respect to arclength.

6. We consider two surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ in $\mathbb{R}^3$, given by the equations

$$\mathcal{S}_1 := \left\{ 4x^2 + 4y^2 + z^2 = 4 \right\}, \quad \mathcal{S}_2 := \left\{ x^2 + 2y^2 - z^2 = 1 \right\}.$$

   (a) Identify these two quadric surfaces.
   (b) Describe the intersection $\mathcal{S}_1 \cap \mathcal{S}_2$.

7. (**Bonus.**) Let $r : t \mapsto r(t) \in \mathbb{R}^3$ be a space curve defined on $\mathbb{R}$, whose second derivative exists everywhere on $\mathbb{R}$, and which is such that for any t in $\mathbb{R}$, the acceleration $r''(t)$ is colinear to $r(t)$. Prove that there exists a plane $\mathcal{P}$ such that $r(t)$ belongs to $\mathcal{P}$ for all t.