1 The next-order energy

We recall that the next-order energy $F_N^\mu(\vec{X}_N)$, where μ is a probability measure and $\vec{X}_N$ a N-tuple of points, was defined (in the $\mathsf{Log1}, \mathsf{Log2}$ cases) as the following quantity

$$F_N^\mu(\vec{X}_N) := \iint_{\triangle^c} -\log|x-y| \left(\sum_{i=1}^N \delta_{x_i} - N d\mu\right)(x) \left(\sum_{i=1}^N \delta_{x_i} - N d\mu\right)(y)$$

1. Compute the the next-order energy $F_N^\mu(\vec{X}_N)$, when μ is the uniform measure on the unit **circle** and $\vec{X}_N$ are the N-th roots of unity. *Hint: first, compute the following logarithmic potential*

$$x \mapsto \int -\log|x-y| \left(\sum_{i=1}^N \delta_{x_i} - N d\mu\right)(y)$$

2. Compute the average of the next-order energy $F_N^\mu(\vec{X}_N)$ when μ is the uniform measure on the unit **disk** and $\vec{X}_N$ are N points chosen uniformly and independently in the disk.

3. Let μ be the uniform measure on the unit **circle** and let $\vec{X}_N$ be N points on the circle, with angles $\theta_1, \dots, \theta_N$ in $[0, 2\pi)$.
   (a) Express the next-order energy $F_N^\mu(\vec{X}_N)$ in terms of the θ_i's and of the function $g = -\log|\sin(\cdot)|$.
   (b) (*) Using convexity, show that the minimum of $F_N^\mu(\vec{X}_N)$ is obtained when the $\vec{X}_N$ are equally spaced on the circle.

2 Discrepancy estimates

A priori estimates on the fluctuations, as mentioned in the lecture control the distance between the empirical measure $\frac{1}{N}\sum_{i=1}^N \delta_{x_i}$ and its asymptotic (almost sure) value μ only against smooth (at least Lipschitz) test functions. In particular, we cannot use them directly to bound *discrepancies*, namely the difference between the number of points in a domain Ω and the mass $N\mu(\Omega)$ given by μ to this domain. Let us work in the $\mathsf{Log2}$ case, and for $R > 1$ we denote by D_R (“D” as “discrepancy”) the following quantity

$$D_R := \int_{B_R} \left(\sum_{i=1}^N \delta_{x_i} - N d\mu\right),$$

where B_R denotes the disk of center 0 and radius R.

1. Using a version of Green's formula, prove that, up to some multiplicative constant, we have

$$D_T = \int_{\partial B_T} \nabla H_N^\mu \cdot \vec{n},$$

   where $\vec{n}$ denotes the outer normal unit vector to the circle ∂B_T.

2. First, let us assume that $D_R > 0$, i.e. there are “more points than expected”. Show that for any $T > 0$ we have

$$D_{R+T} \geq D_R - CN\|\mu\|_\infty TR,$$

   for some constant C, and deduce (using a mean value argument) that

$$\frac{D_R^2}{2C\|\mu\|_\infty} \leq RN \int_{B_{2R}} |\nabla H_N^\mu|$$

3. Show that a similar inequality holds if $D_R < 0$.

4. Observe that, when replacing ∇H_N^μ by its truncation at distance $\frac{1}{\sqrt{N}}$, we have

$$\int_{B_{2R}} |\nabla H_N^\mu| \leq \int_{B_{2R}} |\nabla H^\mu_{N,\frac{1}{\sqrt{N}}}| + CN,$$

for some constant C. Using Cauchy-Schwarz's inequality and the fact that the electric energy is typically of order N, deduce that D_R is typically of order at most $N^{3/4}$. This can be improved in various ways, but it gives a quantitative bound on the convergence of the empirical measure to μ.

3 A priori bounds on the fluctuations

We recall the result of Proposition 3.4 in the Log2 case: if φ is a 1-Lipschitz function in $\mathbb{R}^2$ supported on the unit disk and μ is a probability measure, then

$$\left| \int_{\mathbb{R}^2} \varphi(x) \left(\sum_{i=1}^N \delta_{x_i} - N d\mu \right)(x) \right| \leq C \left(\|\nabla H^\mu_{N,\vec{\eta}}\|_{L^2} + \sqrt{N} \right), \tag{1}$$

where $\nabla H^\mu_{N,\vec{\eta}}$ is the electric field generated by $\sum_{i=1}^N \delta_{x_i} - N d\mu$, truncated at distance η near each point charge - *the details of the truncation procedure are not important here.*

1. For $0 < \alpha \leq 1$ we denote by $C^{0,\alpha}$ the space of α-Hölder functions φ satisfying

$$|\varphi|_{C^{0,\alpha}} := \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{|x-y|^\alpha} < +\infty,$$

in particular $C^{0,1}$ is the space of Lipschitz functions and (1) controls the fluctuations of a $C^{0,1}$ function in terms of the electric energy. Give an a priori bound for the fluctuations of a $C^{0,\alpha}$ test function *Hint: use interpolation or regularise by hand.*

4 Exponential moments of the fluctuations

The a priori bounds on the fluctuations, together with the exponential moments on the energy, yield a concentration bound for the fluctuations of any Lipschitz function φ in the Log2 case of the form

$$\mathbb{P}_{N,\beta}\left[\left| \sum_{i=1}^N \varphi(x_i) - N \int \varphi d\mu \right| \geq M\sqrt{N} \right] \leq \exp(-N(M^2 - C)), \tag{2}$$

for some constant C.

1. Estimate the exponential moments of the fluctuations in the non-interacting case, i.e. give the leading order of

$$\mathbf{E}_{\mu^{\otimes N}}\left(\exp\left(\left(\int \varphi d\mathrm{fluct}_N^\mu \right)^2 \right) \right),$$

where $\mu^{\otimes N}$ denotes the law of N independent points chosen according to μ.

2. Compare (2) with the case of non-interacting particles where $x_1, \ldots, x_N$ are sampled independently according to the measure μ.