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Aufgabe 1:

Seien A und B die folgenden Matrizen über $\mathbb{R}$:

$$A = \begin{pmatrix} 2 & \sqrt{3} \\ 0 & \frac{3}{2} \end{pmatrix}, B = \begin{pmatrix} 1 & \sqrt{3} \\ 2 & \frac{1}{2} \end{pmatrix}$$

$$A + B = \begin{pmatrix} 2+1 & \sqrt{3}+\sqrt{3} \\ 0+2 & \frac{3}{2}+\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 3 & 2\sqrt{3} \\ 2 & 2 \end{pmatrix}$$

$$A - B = \begin{pmatrix} 2-1 & \sqrt{3}-\sqrt{3} \\ 0-2 & \frac{3}{2}-\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

$$(A+B)(A-B) = \begin{pmatrix} 3 & 2\sqrt{3} \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 3 \times 1 + 2\sqrt{3} \times (-2) & 3 \times 0 + 2\sqrt{3} \times 1 \\ 2 \times 1 + 2 \times (-2) & 2 \times 0 + 2 \times 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 - 4\sqrt{3} & 2\sqrt{3} \\ -2 & 2 \end{pmatrix}$$

Aufgabe 2:

$$A_1A_2 = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 3 \\ 1 & 2 & -3 \end{pmatrix} = n.d.$$

$$A_1A_3 = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \times 2 + 1 \times 3 + 3 \times (-1) \\ 1 \times 2 + 2 \times 3 + 1 \times (-1) \\ 1 \times 2 + 1 \times 3 + 1 \times (-1) \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \\ 4 \end{pmatrix}$$

$$A_1A_4 = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 13 & 23 \end{pmatrix} = n.d.$$

$$A_1A_5 = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -3 & 2 \\ 4 & -3 \\ 0 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \times (-3) + 1 \times 4 + 3 \times 0 & 2 \times 2 + 1 \times (-3) + 3 \times (-3) \\ 1 \times (-3) + 2 \times 4 + 1 \times 0 & 1 \times 2 + 2 \times (-3) + 1 \times (-3) \\ 1 \times (-3) + 1 \times 4 + 1 \times 0 & 1 \times 2 + 1 \times (-3) + 1 \times (-3) \end{pmatrix} = \begin{pmatrix} -2 & -8 \\ 5 & -7 \\ 1 & -4 \end{pmatrix}$$

$$A_1A_6 = \begin{pmatrix}2 & 1 & 3\\1 & 2 & 1\\1 & 1 & 1\end{pmatrix}\begin{pmatrix}2 & 1\\1 & 2\end{pmatrix} = n.d.$$

$$A_2A_1 = \begin{pmatrix}2 & -1 & 3\\1 & 2 & -3\end{pmatrix}\begin{pmatrix}2 & 1 & 3\\1 & 2 & 1\\1 & 1 & 1\end{pmatrix}$$

$$= \begin{pmatrix}2\times 2+(-1)\times 1+3\times 1 & 2\times 1+(-1)\times 2+3\times 1 & 2\times 3+(-1)\times 1+3\times 1\\1\times 2+2\times 1+(-3)\times 1 & 1\times 1+2\times 2+(-3)\times 1 & 1\times 3+2\times 1+(-3)\times 1\end{pmatrix}$$

$$= \begin{pmatrix}6 & 3 & 8\\1 & 2 & 2\end{pmatrix}$$

$$A_2A_3 = \begin{pmatrix}2 & -1 & 3\\1 & 2 & -3\end{pmatrix}\begin{pmatrix}2\\3\\-1\end{pmatrix} = \begin{pmatrix}2\times 2+(-1)\times 3+3\times(-1)\\1\times 2+2\times 3+(-3)\times(-1)\end{pmatrix} = \begin{pmatrix}-2\\11\end{pmatrix}$$

$$A_2A_4 = \begin{pmatrix}2 & -1 & 3\\1 & 2 & -3\end{pmatrix}\begin{pmatrix}13 & 23\end{pmatrix} = n.d.$$

$$A_2A_5 = \begin{pmatrix}2 & -1 & 3\\1 & 2 & -3\end{pmatrix}\begin{pmatrix}-3 & 2\\4 & -3\\0 & -3\end{pmatrix}$$

$$= \begin{pmatrix}2\times(-3)+(-1)\times 4+3\times 0 & 2\times 2+(-1)\times(-3)+3\times(-3)\\1\times(-3)+2\times 4+(-3)\times 0 & 1\times 2+2\times(-3)+(-3)\times(-3)\end{pmatrix} = \begin{pmatrix}-10 & -2\\5 & 5\end{pmatrix}$$

$$A_2A_6 = \begin{pmatrix}2 & -1 & 3\\1 & 2 & -3\end{pmatrix}\begin{pmatrix}2 & 1\\1 & 2\end{pmatrix} = n.d.$$

$$A_3A_1 = \begin{pmatrix}2\\3\\-1\end{pmatrix}\begin{pmatrix}2 & 1 & 3\\1 & 2 & 1\\1 & 1 & 1\end{pmatrix} = n.d.$$

$$A_3A_2 = \begin{pmatrix}2\\3\\-1\end{pmatrix}\begin{pmatrix}2 & -1 & 3\\1 & 2 & -3\end{pmatrix} = n.d.$$

$$A_3A_4 = \begin{pmatrix}2\\3\\-1\end{pmatrix}\begin{pmatrix}13 & 23\end{pmatrix} = \begin{pmatrix}2\times 13 & 2\times 23\\3\times 13 & 3\times 23\\(-1)\times 13 & (-1)\times 23\end{pmatrix} = \begin{pmatrix}26 & 46\\39 & 69\\-13 & -23\end{pmatrix}$$

$$A_3A_5 = \begin{pmatrix}2\\3\\-1\end{pmatrix}\begin{pmatrix}-3 & 2\\4 & -3\\0 & -3\end{pmatrix} = n.d.$$

$$A_3A_6 = \begin{pmatrix}2\\3\\-1\end{pmatrix}\begin{pmatrix}2 & 1\\1 & 2\end{pmatrix} = n.d.$$

$$A_4A_1 = \begin{pmatrix}13 & 23\end{pmatrix}\begin{pmatrix}2 & 1 & 3\\ 1 & 2 & 1\\ 1 & 1 & 1\end{pmatrix} = n.d.$$

$$A_4A_2 = \begin{pmatrix}13 & 23\end{pmatrix}\begin{pmatrix}2 & -1 & 3\\ 1 & 2 & -3\end{pmatrix}$$

$$= \begin{pmatrix}13\times 2 + 23\times 1 & 13\times(-1) + 23\times 2 & 13\times 3 + 23\times(-3)\end{pmatrix} = \begin{pmatrix}49 & 33 & -30\end{pmatrix}$$

$$A_4A_3 = \begin{pmatrix}13 & 23\end{pmatrix}\begin{pmatrix}2\\ 3\\ -1\end{pmatrix} = n.d.$$

$$A_4A_5 = \begin{pmatrix}13 & 23\end{pmatrix}\begin{pmatrix}-3 & 2\\ 4 & -3\\ 0 & -3\end{pmatrix} = n.d.$$

$$A_4A_6 = \begin{pmatrix}13 & 23\end{pmatrix}\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix} = \begin{pmatrix}13\times 2 + 23\times 1 & 13\times 1 + 23\times 2\end{pmatrix} = \begin{pmatrix}49 & 59\end{pmatrix}$$

$$A_5A_1 = \begin{pmatrix}-3 & 2\\ 4 & -3\\ 0 & -3\end{pmatrix}\begin{pmatrix}2 & 1 & 3\\ 1 & 2 & 1\\ 1 & 1 & 1\end{pmatrix} = n.d.$$

$$A_5A_2 = \begin{pmatrix}-3 & 2\\ 4 & -3\\ 0 & -3\end{pmatrix}\begin{pmatrix}2 & -1 & 3\\ 1 & 2 & -3\end{pmatrix}$$

$$= \begin{pmatrix}(-3)\times 2 + 2\times 1 & (-3)\times(-1) + 2\times 2 & (-3)\times 3 + 2\times(-3)\\ 4\times 2 + (-3)\times 1 & 4\times(-1) + (-3)\times 2 & 4\times 3 + (-3)\times(-3)\\ 0\times 2 + (-3)\times 1 & 0\times(-1) + (-3)\times 2 & 0\times 3 + (-3)\times(-3)\end{pmatrix} = \begin{pmatrix}-4 & 7 & -15\\ 5 & -10 & 21\\ -3 & -6 & 9\end{pmatrix}$$

$$A_5A_3 = \begin{pmatrix}-3 & 2\\ 4 & -3\\ 0 & -3\end{pmatrix}\begin{pmatrix}2\\ 3\\ -1\end{pmatrix} = n.d.$$

$$A_5A_4 = \begin{pmatrix}-3 & 2\\ 4 & -3\\ 0 & -3\end{pmatrix}\begin{pmatrix}13 & 23\end{pmatrix} = n.d.$$

$$A_5A_6 = \begin{pmatrix}-3 & 2\\ 4 & -3\\ 0 & -3\end{pmatrix}\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix} = \begin{pmatrix}(-3)\times 2 + 2\times 1 & (-3)\times 1 + 2\times 2\\ 4\times 2 + (-3)\times 1 & 4\times 1 + (-3)\times 2\\ 0\times 2 + (-3)\times 1 & 0\times 1 + (-3)\times 2\end{pmatrix} = \begin{pmatrix}-4 & 1\\ 5 & -2\\ -3 & -6\end{pmatrix}$$

$$A_6A_1 = \begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}\begin{pmatrix}2 & 1 & 3\\ 1 & 2 & 1\\ 1 & 1 & 1\end{pmatrix} = n.d.$$

$$A_6A_2 = \begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}\begin{pmatrix}2 & -1 & 3\\ 1 & 2 & -3\end{pmatrix}$$

$$= \begin{pmatrix} 2\times 2 + 1\times 1 & 2\times(-1) + 1\times 2 & 2\times 3 + 1\times(-3) \\ 1\times 2 + 2\times 1 & 1\times(-1) + 2\times 2 & 1\times 3 + 2\times(-3) \end{pmatrix} = \begin{pmatrix} 5 & 0 & 3 \\ 4 & 3 & -3 \end{pmatrix}$$

$$A_6A_3 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = n.d.$$

$$A_6A_4 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 13 & 23 \end{pmatrix} = n.d.$$

$$A_6A_5 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -3 & 2 \\ 4 & -3 \\ 0 & -3 \end{pmatrix} = n.d.$$

Aufgabe 3:

Sei A die folgende Matrix über $\mathbb{R}$:

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1\times 1 + 2\times 0 & 1\times 2 + 2\times 1 \\ 0\times 1 + 1\times 0 & 0\times 2 + 1\times 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1\times 1 + 4\times 0 & 1\times 2 + 4\times 1 \\ 0\times 1 + 1\times 0 & 0\times 2 + 1\times 1 \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1\times 1 + 6\times 0 & 1\times 2 + 6\times 1 \\ 0\times 1 + 1\times 0 & 0\times 2 + 1\times 1 \end{pmatrix} = \begin{pmatrix} 1 & 8 \\ 0 & 1 \end{pmatrix}$$

Lemma. *Für $\forall n \in \mathbb{N}$ gilt:*

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}$$

Beweis. Beweis durch vollständige Induktion:
IA: $n = 1$.

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

IV: Die Aussage gelte für ein beliebiges $n \in \mathbb{N}$, d.h.

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}$$

Induktionsschluss: Zu zeigen ist, dass die Aussage für $n + 1$ gilt, also dass

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^{n+1} = \begin{pmatrix} 1 & 2(n+1) \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}^{n+1} = \begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}^{n}\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix} = \begin{pmatrix}1 & 2n\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}1\times 1 + 2n\times 0 & 1\times 2 + 2n\times 1\\ 0\times 1 + 1\times 0 & 0\times 2 + 1\times 1\end{pmatrix} = \begin{pmatrix}1 & 2(n+1)\\0 & 1\end{pmatrix}$$

$$A^n = \begin{pmatrix}1 & 2n\\0 & 1\end{pmatrix} \text{ für } \forall n \in \mathbb{N}$$

Aufgabe 4:

(a) Beweis:

$$(A(B+B'))_{ij} = \sum_{k=1}^{m} A_{ik}(B+B')_{kj} = \sum_{k=1}^{m} A_{ik}(B_{kj}+B'_{kj}) = \sum_{k=1}^{m} A_{ik}B_{kj} + A_{ik}B'_{kj}$$
$$= (AB + AB')_{ij}$$

$$((A+A')B)_{ij} = \sum_{k=1}^{m}(A+A')_{ik}B_{kj} = \sum_{k=1}^{m}(A_{ik}+A'_{ik})B_{kj} = \sum_{k=1}^{m} A_{ik}B_{kj} + A'_{ik}B_{kj}$$
$$= (AB + AB')_{ij}$$

(b) Beweis:

$$((AB)C)_{ij} = \sum_{k=1}^{n}(AB)_{ik}C_{kj} = \sum_{k=1}^{n}\sum_{q=1}^{m}(A_{iq}B_{qk})C_{kj} = \sum_{q=1}^{m}\sum_{k=1}^{n} A_{iq}(B_{qk}C_{kj})$$
$$= \sum_{q=1}^{m} A_{iq}\sum_{k=1}^{n}(B_{qk}C_{kj}) = \sum_{q=1}^{m} A_{iq}(BC)_{qj} = (A(BC))_{ij}$$