APPENDIX J

Root Locus Rules: Derivations

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By

Norman S. Nise

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APPENDIX J

Root Locus Rules: Derivations

J.1 Derivation of the Behavior of the Root Locus at Infinity (Kuo, 1987)

Let the open-loop transfer function be represented as follows:

$$KG(s)H(s) = \frac{K(s^m + a_1 s^{m-1} + \cdots + a_m)}{(s^{m+n} + b_1 s^{m+n-1} + \cdots + b_{m+n})} \tag{J.1}$$

or

$$KG(s)H(s) = \frac{K}{\left(\dfrac{s^{m+n} + b_1 s^{m+n-1} + \cdots + b_{m+n}}{s^m + a_1 s^{m-1} + \cdots + a_m}\right)} \tag{J.2}$$

Performing the indicated division in the denominator, we obtain

$$KG(s)H(s) = \frac{K}{s^n + (b_1 - a_1)s^{n-1} + \cdots} \tag{J.3}$$

In order for a pole of the closed-loop transfer function to exist,

$$KG(s)H(s) = -1 \tag{J.4}$$

Assuming large values of s that would exist as the locus moves toward infinity, Eq. (J.3) becomes

$$s^n + (b_1 - a_1)s^{n-1} = -K \tag{J.5}$$

Factoring out s^n, Eq. (J.5) becomes

$$s^n\left(1 + \frac{b_1 - a_1}{s}\right) = -K \tag{J.6}$$

Taking the nth root of both sides, we have

$$s\left(1 + \frac{b_1 - a_1}{s}\right)^{1/n} = -K^{1/n} \tag{J.7}$$

If the term

$$\left(1 + \frac{b_1 - a_1}{s}\right)^{1/n} \tag{J.8}$$

is expanded into an infinite series where only the first two terms are significant,[1] we obtain

$$s\left(1 + \frac{b_1 - a_1}{ns}\right) = (-K)^{1/n} \tag{J.9}$$

Distributing the factor s on the left-hand side yields

$$s + \frac{b_1 - a_1}{n} = (-K)^{1/n} \tag{J.10}$$

Now, letting $s = \sigma + j\omega$ and $(-K)^{1/n} = |K^{1/n}|e^{j(2k+1)\pi/n}$, where

$$(-1)^{1/n} = e^{j(2k+1)\pi/n} = \cos\left(\frac{(2k+1)\pi}{n}\right) + j\sin\left(\frac{(2k+1)\pi}{n}\right) \tag{J.11}$$

Eq. (J.10) becomes

$$\sigma + j\omega + \frac{b_1 - a_1}{n} = \left|K^{1/n}\right|\left[\cos\frac{(2k+1)\pi}{n} + j\sin\frac{(2k+1)\pi}{n}\right] \tag{J.12}$$

where $k = 0, \pm 1, \pm 2, \pm 3, \ldots$ Setting the real and imaginary parts of both sides equal to each other, we obtain

$$\sigma + \frac{b_1 - a_1}{n} = \left|K^{1/n}\right|\cos\frac{(2k+1)\pi}{n} \tag{J.13a}$$

$$\omega = \left|K^{1/n}\right|\sin\frac{(2k+1)\pi}{n} \tag{J.13b}$$

Dividing the two equations to eliminate $\left|K^{1/n}\right|$, we obtain

$$\frac{\sigma + \dfrac{b_1 - a_1}{n}}{\omega} = \frac{\cos\dfrac{(2k+1)\pi}{n}}{\sin\dfrac{(2k+1)\pi}{n}} \tag{J.14}$$

Finally, solving for ω, we find

$$\omega = \left[\tan\frac{(2k+1)\pi}{n}\right]\left[\sigma + \frac{b_1 - a_1}{n}\right] \tag{J.15}$$

The form of this equation is that of a straight line,

$$\omega = M(\sigma - \sigma_0) \tag{J.16}$$

where the slope of the line, M, is

$$M = \tan\frac{(2k+1)\pi}{n} \tag{J.17}$$

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[1] This is a good approximation since s is approaching infinity for the region applicable to the derivation.

Thus, the angle of the line in radians with respect to the positive extension of the real axis is

$$\theta = \frac{(2k+1)\pi}{n} \tag{J.18}$$

and the σ intercept is

$$\sigma_0 = -\left[\frac{b_1 - a_1}{n}\right] \tag{J.19}$$

From the theory of equations,[2]

$$b_1 = -\sum \text{finite poles} \tag{J.20a}$$
$$a_1 = -\sum \text{finite zeros} \tag{J.20b}$$

Also, from Eq. (J.1) ,

$$\begin{aligned} n &= \text{number of finite poles} - \text{number of finite zeros} \\ &= \#\text{finite poles} - \#\text{finite zeros} \end{aligned} \tag{J.21}$$

By examining Eq. (J.16) , we conclude that the root locus approaches a straight line as the locus approaches infinity. Further, this straight line intersects the σ axis at

$$\sigma_0 = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#\text{finite poles} - \#\text{finite zeros}} \tag{J.22}$$

which is obtained by substituting Eqs. (J.20) and (J.21) into Eq. (J.19).

Let us summarize the results: *The root locus approaches straight lines as asymptotes as the locus approaches infinity. Further, the equation of the asymptotes is given by the real-axis intercept and the angle with respect to the real axis as follows:*

$$\sigma_0 = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#\text{finite poles} - \#\text{finite zeros}} \tag{J.23}$$

$$\theta = \frac{(2k+1)\pi}{\#\text{finite poles} - \#\text{finite zeros}} \tag{J.24}$$

where $k = 0, \pm 1, \pm 2, \pm 3, \ldots$ Notice that the running index, k, in Eq. (J.24) yields a multiplicity of lines that account for the many branches of a root locus that approach infinity.

J.2 Derivation of Transition Method for Breakaway and Break-in Points

The *transition* method for finding real-breakaway and break-in points without differentiating can be derived by showing that the natural log of $1/[G(\sigma)H(\sigma)]$ has a zero derivative at the same value of σ as $1/[G(\sigma)H(\sigma)]$ (Franklin, 1991).

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[2]Given an nth-order polynomial of the form $s^n + a_{n-1}s^{n-1} + \cdots$, the coefficient, a_{n-1}, is the negative sum of the roots of the polynomial.

We now show that if we work with the natural log we can eliminate the step of differentiation.

First find the derivative of the natural log of $1/[G(\sigma)H(\sigma)]$ and set it equal to zero. Thus,

$$\frac{d}{d\sigma}\ln\left[\frac{1}{G(\sigma)H(\sigma)}\right] = G(\sigma)H(\sigma)\frac{d}{d\sigma}\left[\frac{1}{G(\sigma)H(\sigma)}\right] = 0 \tag{J.25}$$

Since $G(\sigma)H(\sigma)$ is not zero at the breakaway or break-in points, letting

$$\frac{d}{d\sigma}\ln\left[\frac{1}{G(\sigma)H(\sigma)}\right] = 0 \tag{J.26}$$

will thus yield the same value of σ as letting

$$\frac{d}{d\sigma}\left[\frac{1}{G(\sigma)H(\sigma)}\right] = 0 \tag{J.27}$$

Hence,

$$\begin{aligned}\frac{d}{d\sigma}\ln\left[\frac{1}{G(\sigma)H(\sigma)}\right] &= \frac{d}{d\sigma}\ln\left[\frac{(\sigma+p_1)(\sigma+p_2)\cdots(\sigma+p_n)}{(\sigma+z_1)(\sigma+z_2)\cdots(\sigma+z_m)}\right] \\ &= \frac{d}{d\sigma}[\ln(\sigma+p_1)+\ln(\sigma+p_2)\cdots\ln(\sigma+p_n)-\ln(\sigma+z_1) \\ &\qquad -\ln(\sigma+z_2)\cdots-\ln(\sigma+z_m)] \\ &= \frac{1}{\sigma+p_1}+\frac{1}{\sigma+p_2}\cdots+\frac{1}{\sigma+p_n}-\frac{1}{\sigma+z_1} \\ &\qquad -\frac{1}{\sigma+z_2}\cdots-\frac{1}{\sigma+z_m}=0\end{aligned} \tag{J.28}$$

or

$$\sum_{i=1}^{n}\frac{1}{\sigma+p_i} = \sum_{i=1}^{m}\frac{1}{\sigma+z_i} \tag{J.29}$$

where z_i and p_i are the negatives of the zero and pole values of $G(s)H(s)$, respectively. Equation (J.29) can be solved for σ, the real axis values that minimize or maximize K, yielding the breakaway and break-in points without differentiating.

Bibliography

Franklin, G. F.; Powell, J. D.; and Emami-Naeini, A. *Feedback Control of Dynamic Systems,* 2d ed. Addison-Wesley, Reading, Mass., 1991.

Kuo, B. *Automatic Control Systems,* 5th ed. Prentice-Hall, Englewood Cliffs, N.J., 1987.