Problem 8.1. Show that $\mathsf{P}[\mathcal{A} \cap \mathcal{B}]\mathsf{P}[\mathcal{A} \cup \mathcal{B}] \leq \mathsf{P}[\mathcal{A}]P[\mathcal{B}]$

Solution: Note that $\mathsf{P}[\mathcal{A}] = \mathsf{P}[\mathcal{A} \setminus \mathcal{B}] + \mathsf{P}[\mathcal{A} \cap \mathcal{B}]$ and $\mathsf{P}[\mathcal{B}] = \mathsf{P}[\mathcal{B} \setminus \mathcal{A}] + \mathsf{P}[\mathcal{A} \cap \mathcal{B}]$. We then have

$$\begin{aligned}
\mathsf{P}[\mathcal{A}]\mathsf{P}[\mathcal{B}] &= (\mathsf{P}[\mathcal{A} \setminus \mathcal{B}] + \mathsf{P}[\mathcal{A} \cap \mathcal{B}])(\mathsf{P}[\mathcal{B} \setminus \mathcal{A}] + \mathsf{P}[\mathcal{A} \cap \mathcal{B}]) \\
&= \mathsf{P}[\mathcal{A} \setminus \mathcal{B}]\mathsf{P}[\mathcal{B} \setminus \mathcal{A}] + \mathsf{P}[A \cap \mathcal{B}](\mathsf{P}\left[\mathcal{A} \setminus \mathcal{B} + \mathsf{P}\left[[]\, \mathcal{A} \cap \mathcal{B}\right] + \mathsf{P}\left[\mathcal{B} \setminus \mathcal{A}\right]\right) \\
&= \mathsf{P}[\mathcal{A} \setminus \mathcal{B}]\mathsf{P}[\mathcal{B} \setminus \mathcal{A}] + \mathsf{P}[A \cap \mathcal{B}]\mathsf{P}\left[\mathcal{A} \cup \mathcal{B}\right] \\
&\leq \mathsf{P}[A \cap \mathcal{B}]\mathsf{P}\left[\mathcal{A} \cup \mathcal{B}\right],
\end{aligned}$$

as desired. □

Problem 8.2. Suppose $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are events such that $\mathcal{A} \cap \overline{\mathcal{C}} = \mathcal{B} \cap \overline{\mathcal{C}}$. Show that $|\mathsf{P}\left[\mathcal{A}\right] - \mathsf{P}\left[\mathcal{B}\right]| \leq \mathsf{P}\left[\mathcal{C}\right]$.

Solution: Assume WLOG that $\mathsf{P}\left[\mathcal{A} \cap \mathcal{C}\right] \geq \mathsf{P}\left[\mathcal{B} \cap \mathcal{C}\right]$. We then have

$$\begin{aligned}
|\mathsf{P}\left[\mathcal{A}\right] - \mathsf{P}\left[\mathcal{B}\right]| &= |(\mathsf{P}\left[\mathcal{A} \setminus C\right] + \mathsf{P}\left[\mathcal{A} \cap \mathcal{C}\right]) - (\mathsf{P}\left[\mathcal{B} \setminus C\right] + \mathsf{P}\left[\mathcal{B} \cap \mathcal{C}\right])| \\
&= |\mathsf{P}\left[\mathcal{A} \cap \mathcal{C}\right] - \mathsf{P}\left[\mathcal{B} \cap \mathcal{C}\right]| \\
&= \mathsf{P}\left[\mathcal{A} \cap \mathcal{C}\right] - \mathsf{P}\left[\mathcal{B} \cap \mathcal{C}\right] \\
&\leq \mathsf{P}\left[\mathcal{A} \cap \mathcal{C}\right] \\
&\leq \mathsf{P}\left[\mathcal{C}\right].
\end{aligned}$$

□

Problem 8.3. Let m be a positive integer, and let $\alpha(m)$ be the probability that a number chosen at random from $\{1, \ldots, m\}$ is divisible by either $4, 5$, or 6. Write down an exact formula for $\alpha(m)$, and also show that $\alpha(m) = 14/30 + O(1/m)$.

Solution: Let $\mathcal{D}_k$ denote the set of all integers $1 \leq j \leq m$ such that $k|j$. We then have

$$\begin{aligned}
\mathsf{P}\left[\mathcal{D}_4 \cup \mathcal{D}_5 \cup \mathcal{D}_6\right] &= \mathsf{P}\left[D_4\right] + \mathsf{P}\left[D_5\right] + \mathsf{P}\left[D_6\right] - \mathsf{P}\left[\mathcal{D}_4 \cap \mathcal{D}_5\right] - \mathsf{P}\left[\mathcal{D}_4 \cap \mathcal{D}_6\right] - \mathsf{P}\left[\mathcal{D}_5 \cap \mathcal{D}_6\right] + \mathsf{P}\left[\mathcal{D}_4 \cap \mathcal{D}_5 \cap \mathcal{D}_6\right] \\
&= \mathsf{P}\left[D_4\right] + \mathsf{P}\left[D_5\right] + \mathsf{P}\left[D_6\right] - \mathsf{P}\left[\mathcal{D}_{20}\right] - \mathsf{P}\left[\mathcal{D}_{12}\right] - \mathsf{P}\left[\mathcal{D}_{30}\right] + \mathsf{P}\left[\mathcal{D}_{60}\right] \\
&= \frac{1}{m}\left(\left\lfloor\frac{m}{4}\right\rfloor + \left\lfloor\frac{m}{5}\right\rfloor + \left\lfloor\frac{m}{6}\right\rfloor - \left\lfloor\frac{m}{20}\right\rfloor - \left\lfloor\frac{m}{12}\right\rfloor - \left\lfloor\frac{m}{30}\right\rfloor + \left\lfloor\frac{m}{60}\right\rfloor\right) \\
&= \frac{1}{m}\left(\frac{m}{4} + \frac{m}{5} + \frac{m}{6} - \frac{m}{20} - \frac{m}{12} - \frac{m}{30} + \frac{m}{60} + O(1)\right) \\
&= \frac{1}{m}\left(\frac{28m}{60} + O(1)\right) \\
&= \frac{1}{m}\left(\frac{28m}{60} + O(1)\right) \\
&= 14/30 + O(1)/m \\
&= 14/30 + O(1/m).
\end{aligned}$$

□

Problem 8.4. Let $\{\mathcal{A}_i\}_{i\in I}$ be a finite family of events, where $n := |I|$. For $m = \{0, \ldots, n\}$, define

$$\alpha_m := \sum_{k=1}^{m} (-1)^{k-1} \sum_{\substack{J\subseteq I \\ |J|=k}} \mathsf{P}\left[\bigcap_{j\in J} \mathcal{A}_j\right].$$

Also, define

$$\alpha := \mathsf{P}\left[\bigcup_{i\in I} \mathcal{A}_i\right].$$

Show that $\alpha \leq \alpha_m$ if m is odd and $\alpha \geq \alpha_m$ if m is even.

Solution: Before embarking on the proof, we will prove two lemmas.

Lemma 1. *If $n \in \mathbb{N}$, then*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

Proof. By the binomial theorem, we have

$$\begin{aligned} 0 &= (1-1)^n \\ &= \sum_{k=0}^{n} (-1)^k (1)^{n-1} \binom{n}{k} \\ &= \sum_{k=0}^{n} (-1)^k \binom{n}{k}. \end{aligned}$$

□

Lemma 2. *Let $n, m \in \mathbb{N}$, and let*

$$\beta_m = \sum_{k=0}^{m} (-1)^k \binom{n}{k}.$$

If m is even, then $\beta_m \geq 0$. If m is odd, then $\beta_m \leq 0$.

Proof. We will proceed by induction. If $n = 0$, then the result is trivial.

Otherwise, suppose the result holds for some $n \in \mathbb{N}$, and consider β_m. We have

$$\begin{aligned}
\beta_m &= \sum_{k=0}^{m}(-1)^k\binom{n+1}{k} \\
&= \binom{n+1}{0} + \sum_{k=1}^{m}(-1)^k\left[\binom{n}{k} + \binom{n}{k-1}\right] \\
&= 1 + \sum_{k=1}^{m}(-1)^k\binom{n}{k} + \sum_{k=1}^{m}(-1)^k\binom{n}{k-1} \\
&= 1 + \sum_{k=1}^{m}(-1)^k\binom{n}{k} + \sum_{k=1}^{m}(-1)^k\binom{n}{k-1} \\
&= \sum_{k=0}^{m}(-1)^k\binom{n}{k} + \sum_{k=0}^{m-1}(-1)^{k+1}\binom{n}{k} \\
&= \sum_{k=0}^{m}(-1)^k\binom{n}{k} - \sum_{k=0}^{m-1}(-1)^k\binom{n}{k}
\end{aligned}$$

If m is even, we have by the induction hypothesis that the first term is nonnegative and that the second term is nonpositive (since $m-1$) is odd. Thus, in total, $\beta_m \geq 0$ when m is odd.

On the other hand, if m is odd, we have by the induction hypothesis that the first term is nonpositive and that the second term is nonnegative (since $m-1$) is even. Thus, in total, $\beta_m \geq 0$ when m is odd.

Therefore, we have the result for all $n \in \mathbb{N}$. $\square$

We now begin the proof of Bonferroni's inequalities.

Our approach is to count how many times and with what coefficients an element $\omega \in \Omega := \bigcup_{i \in I} \mathcal{A}_i$ appears in α and in α_m. Thus, for each $\omega \in \Omega$, let $I_\omega \subseteq I$ denote the set of all $i \in I$ such that $\omega \in \mathcal{A}_i$, and let $r_\omega = |I_\omega|$.

First, observe that in the definition of α_m, the probability $\mathsf{P}[\omega]$ appears in the inner sum if and only if $J \subseteq I_\omega$. For $k \leq r_\omega$, there are exactly $\binom{r_\omega}{k}$ k-element subsets of I_ω. We thus have

$$\alpha_m = \sum_{\omega \in \Omega}\sum_{k=1}^{m}(-1)^{k-1}\binom{r_\omega}{k}\mathsf{P}[\omega].$$

Note, then, that

$$\begin{aligned}\beta_m &:= \sum_{k=1}^{m} (-1)^{k-1} \binom{r_\omega}{k} \\ &= 1 - \sum_{k=0}^{m} (-1)^{k} \binom{r_\omega}{k},\end{aligned}$$

If m is even, we have by Lemma 2 that $\beta_m \leq 1$. Multiplying my $\mathsf{P}[\omega]$ and summing over all $\omega \in \Omega$, we find that

$$\sum_{\omega\in\Omega} \sum_{k=1}^{m} (-1)^{k-1} \binom{r_\omega}{k} \mathsf{P}[\omega] \leq \sum_{\omega\in\Omega} \mathsf{P}[\omega];$$

that is, $\alpha_m \leq \alpha$ when m is even. Similarly, we have $\alpha_m \geq \alpha$ when m is odd, giving us the result. □