ASEN 5050
SPACEFLIGHT DYNAMICS
Lecture 2: Two-Body Motion

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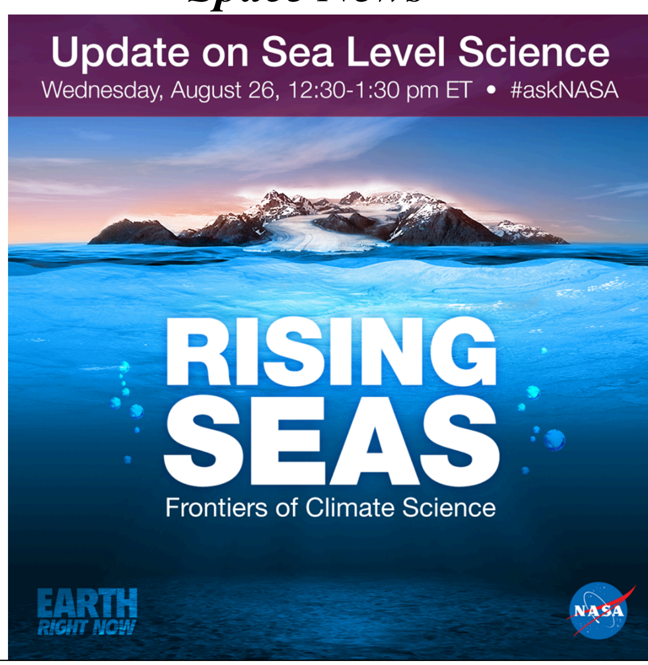
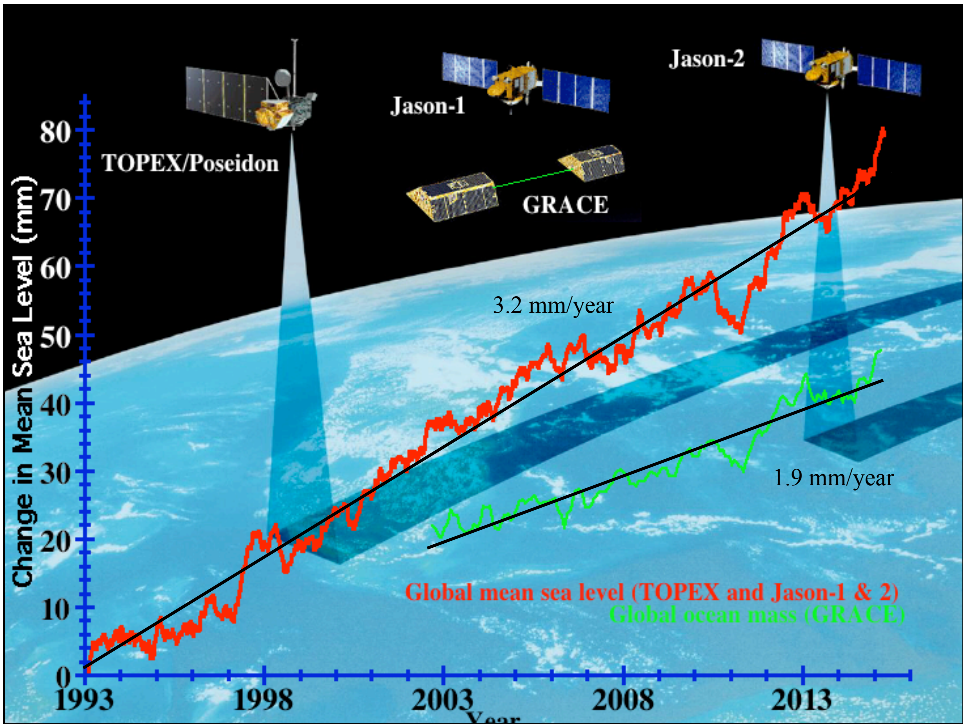
Space News





Astrodynamics

"Astrodynamics is the study of the motion of man-made objects in space, subject to both natural and artificially induced forces."

- Tycho Brahe (1546-1601, Prague) – very precise observations
- Johann Kepler (1571-1630) – Kepler's Laws based on Brahe's observations

We'll also be considering the motion of natural bodies, subject to natural forces (and sometimes artificially induced forces).

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Prague,
Czech Republic

Kepler's Laws

1609
1. The orbit of each planet is an ellipse with the Sun at one focus.
2. The line joining the planet to the Sun sweeps out equal areas in equal time.

1619
3. The ratio of the squares of the periods of the two planets is equal to the ratio of the cubes of the semimajor axis of their orbits.

But, it was Isaac Newton who established the mathematical foundation from which Kepler's Laws can be derived.

Newton's Laws of Motion

Isaac Newton (1642-1727) established the fundamentals of celestial mechanics based on the earlier work of Tycho and Kepler. Newton formulated the basic concepts of the Laws of Motion and Law of Gravitation around 1666, but it was not until 1687 at the urging of Edmund Halley that the Principia was published. It presented 3 Laws of Motion.

Law 1. Every body continues in its state of rest, or of uniform motion in a straight line, unless it is compelled to change that state by forces impressed upon it.

This sounds trivial to us, but in the 17th century, this was new!

It was previously thought that everything tended to remain at rest or gradually achieve a state of rest.

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Law 1. Every body continues in its state of rest, or of uniform motion in a straight line, unless it is compelled to change that state by forces impressed upon it.

Law 2. The change of motion is proportional to the motive force impressed; and is made in the direction of the straight line in which that force is impressed

$$\bar{F} = \frac{d}{dt}\left(m\bar{v}\right)$$

If a body's mass doesn't change, then this simplifies to the well-known equation: $F = ma$

If a body's mass *does* change, then this may be used to derive the ideal rocket equation.

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$$\vec{F} = \frac{d}{dt}\left(m\vec{v}\right)$$

Law 3. To every action there is always opposed an equal reaction, or, the mutual actions of two bodies upon each other are always equal and directed to contrary parts.

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Universal Law of Gravitation

The Principia also includes the Universal Law of Gravitation, which states that any two point masses attract one another with a force proportional to the product of their masses and inversely proportional to the square of the distance between them.

Thus $$\left|\vec{F}_g\right| = \frac{Gm_1m_2}{r^2}$$

G = Universal Gravitational Constant
= 6.673 x 10^{-20} $km^3/kg\ s^2$

or in vector form: $$\vec{F}_g = -\frac{Gm_1m_2}{r^2}\frac{\vec{r}}{\left|\vec{r}\right|}$$

where $\vec{F}_g$ is the force on m_2 and $\vec{r}$ is a vector from m_1 to m_2. This assumes the masses are point-masses.

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Two-Body Equation

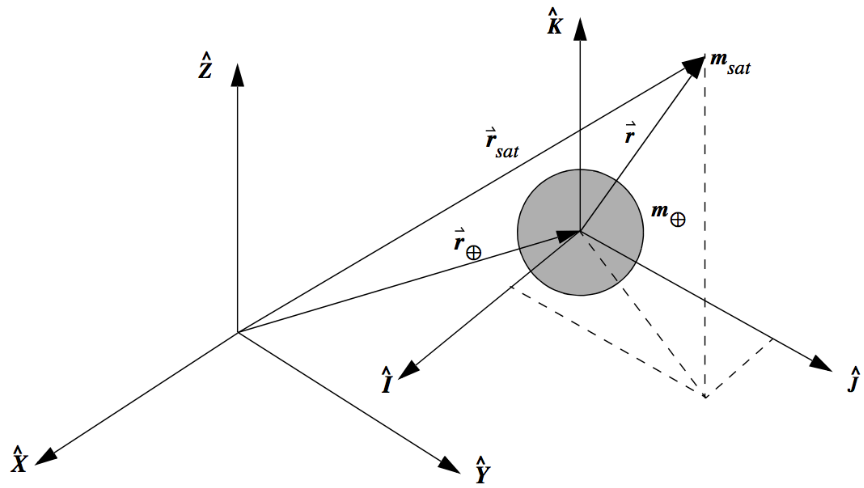
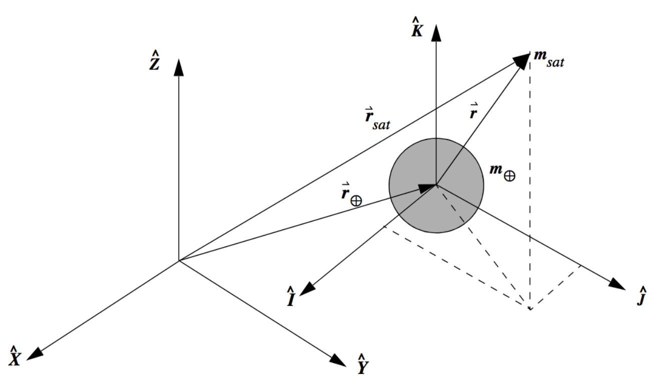


Figure 1-12. Geometry for Two Bodies in an Inertial Reference Frame. XYZ is assumed to be an inertial coordinate system. IJK is displaced from XYZ, but does not rotate or accelerate with respect to XYZ.

Two-Body Equation

Using this, we can derive the <u>two-body equation</u> of the relative motion of one body WRT another. In the inertial frame we have:

$$\vec{F}_{gsat} = m_{sat}\ddot{\vec{r}}_{sat} = -\frac{Gm_{\oplus}m_{sat}}{r^2}\frac{\vec{r}}{r} \qquad \vec{F}_{g\oplus} = m_{\oplus}\ddot{\vec{r}}_{\oplus} = \frac{Gm_{\oplus}m_{sat}}{r^2}\left(\frac{\vec{r}}{r}\right)$$



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$$\ddot{\vec{r}} = \ddot{\vec{r}}_{sat} - \ddot{\vec{r}}_{\oplus} \quad \Longrightarrow \quad \ddot{\vec{r}} = -\frac{G(m_{\oplus} + m_{sat})}{r^2}\frac{\vec{r}}{r}$$

which is a differential equation describing the motion of $\boldsymbol{m_{sat}}$ WRT $\boldsymbol{m_{\oplus}}$. Often, we ignore the mass $\boldsymbol{m_{sat}}$ and replace $\boldsymbol{Gm_{\oplus}}$ with $\boldsymbol{\mu}$:

$$\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r} \qquad \textit{(basic two-body equation)}$$

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$$\boxed{\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r} \qquad \textit{(basic two-body equation)}}$$

A Dynamical System

- Let's consider the dynamical equation:

$$\ddot{x}(t) = a_0$$ where a_0 is a known value

- How many degrees of freedom does $x(t)$ have?

- It has 2!

$$\dot{x}(t) = a_0 t + v_0 \qquad x(t) = \frac{1}{2} a_0 t^2 + v_0 t + x_0$$

- We have to specify both v_0 and x_0 to determine $x(t)$.

A Second Dynamical System

- Now let's consider the dynamical equation:

$$\ddot{\vec{r}}(t) = \vec{a}_0$$ where a_0 is a known vector

- How many degrees of freedom does $r(t)$ have?

- It has 6: 2 for each axis (x, y, z)

- Integrals of motion: $x_0, \dot{x}_0, \quad y_0, \dot{y}_0, \quad z_0, \dot{z}_0$

The Two-Body Dynamical System

- Two bodies in motion have six degrees of freedom each, relative to an inertial reference; 12 total.

3 coordinates x 2nd order x 2 bodies = 12 DOF

- In order for the two-body motion to have a closed-form solution, we must be able to find 12 integrals of motion that define the orbits.

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12 Integrals of Motion

- Body 1: $x_0^1, \dot{x}_0^1, \quad y_0^1, \dot{y}_0^1, \quad z_0^1, \dot{z}_0^1$
- Body 2: $x_0^2, \dot{x}_0^2, \quad y_0^2, \dot{y}_0^2, \quad z_0^2, \dot{z}_0^2$

- These are 12 IoMs, but they're not very helpful. They don't tell us much about the orbit of either body.
- Can we identify other IoMs that are more helpful in describing an analytical solution to an orbit?
- We're lucky, we already know some: total system momentum, total system angular momentum, total system energy – these are Integrals of Motion. Let's take a look.

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Integrals of Motion

- Let's make a few assumptions:

  1) the mass of the satellite is negligible,
  2) an inertial coordinate system,
  3) the bodies are spherically symmetric with uniform density,
  4) no other forces act except for the gravitational attraction of the two bodies.

- i.e., we're going to use just these equations and physics:

$$\ddot{\vec{r}}_{sat} = -\frac{Gm_{\oplus}}{r^2}\frac{\vec{r}}{r} \qquad \ddot{\vec{r}}_{\oplus} = \frac{Gm_{sat}}{r^2}\left(\frac{\vec{r}}{r}\right)$$

Center-of-Mass Integral

COM relationship: $$\vec{r}_{CM} = \frac{m_{\oplus}\vec{r}_{\oplus} + m_{sat}\vec{r}_{sat}}{m_{\oplus} + m_{sat}}$$

Differentiate twice $$\ddot{\vec{r}}_{CM} = \frac{m_{\oplus}\ddot{\vec{r}}_{\oplus} + m_{sat}\ddot{\vec{r}}_{sat}}{m_{\oplus} + m_{sat}}$$

Recall these:

$$\vec{F}_{g\oplus} = m_{\oplus}\ddot{\vec{r}}_{\oplus} = \frac{Gm_{\oplus}m_{sat}}{r^2}\left(\frac{\vec{r}}{r}\right) \qquad \vec{F}_{gsat} = m_{sat}\ddot{\vec{r}}_{sat} = -\frac{Gm_{\oplus}m_{sat}}{r^2}\frac{\vec{r}}{r}$$

Center-of-Mass Integral

COM relationship: $\vec{r}_{CM} = \dfrac{m_\oplus \vec{r}_\oplus + m_{sat}\vec{r}_{sat}}{m_\oplus + m_{sat}}$

Differentiate twice $\ddot{\vec{r}}_{CM} = \dfrac{m_\oplus \ddot{\vec{r}}_\oplus + m_{sat}\ddot{\vec{r}}_{sat}}{m_\oplus + m_{sat}} = 0$

Integrating twice $\vec{r}_{CM} = \vec{c}_1 t + \vec{c}_2$ (straight line)

Since these are vectors, there are 3 "c_1"s and 3 "c_2"s for a total of 6 Integrals of Motion!

Conservation of Linear Momentum

Angular Momentum Integral

Want to show that the angular momentum is constant.

$$\vec{h} = \vec{r} \times \dot{\vec{r}} = \text{constant} \quad \Rightarrow \quad \frac{d}{dt}(\vec{r} \times \dot{\vec{r}}) = 0$$

Work this one yourself! Tools you may need:

$$\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r}$$

$$\frac{d}{dt}(a \times b) = \dot{a} \times b + a \times \dot{b}$$

Angular Momentum Integral

Want to show that the angular momentum is constant.

$$\vec{h} = \vec{r} \times \dot{\vec{r}} = \text{constant} \quad \Rightarrow \quad \frac{d}{dt}(\vec{r} \times \dot{\vec{r}}) = 0$$

$$\frac{d}{dt}(\vec{r} \times \dot{\vec{r}}) = \cancelto{0}{\dot{\vec{r}} \times \dot{\vec{r}}} + \vec{r} \times \ddot{\vec{r}}$$

$$\frac{d}{dt}(\vec{r} \times \dot{\vec{r}}) = \vec{r} \times \ddot{\vec{r}}$$

Plug in our EOM: $\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r}$

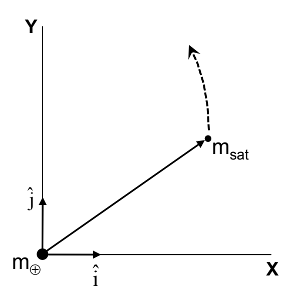
$$\frac{d}{dt}(\vec{r} \times \dot{\vec{r}}) = \vec{r} \times \left(-\frac{\mu}{r^3}\vec{r}\right)$$

$$= -\frac{\mu}{r^3}\left(\vec{r} \times \vec{r}\right) = 0$$

Thus, $\vec{h}$ is perpendicular to the orbital plane and constant in direction, yielding 3 more Integrals.

Polar Coordinates

- Next, we need to re-frame the problem in polar coordinates to tackle more integrals of motion.



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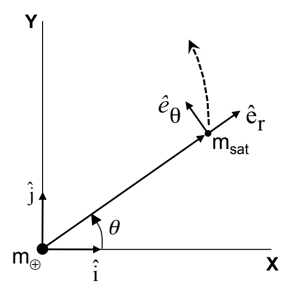
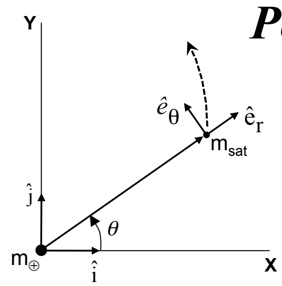


Polar Coordinates



$$\vec{r} = r\cos\theta\hat{i} + r\sin\theta\hat{j} = r\hat{e}_r$$

$$\dot{\vec{r}} = (\dot{r}\cos\theta - r\sin\theta\dot{\theta})\hat{i} + (\dot{r}\sin\theta + r\cos\theta\dot{\theta})\hat{j}$$

$$= \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$

$$\ddot{\vec{r}} = \ddot{r}\hat{e}_r + \dot{r}\dot{\hat{e}}_r + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta + r\dot{\theta}\dot{\hat{e}}_\theta$$

$$= (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$$

$$\hat{e}_\theta = -\sin\theta\hat{i} + \cos\theta\hat{j}$$

$$\hat{e}_r = \cos\theta\hat{i} + \sin\theta\hat{j}$$

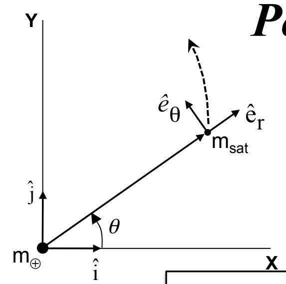
$$\dot{\hat{e}}_r = (-\sin\theta\hat{i} + \cos\theta\hat{j})\dot{\theta}$$

$$= \dot{\theta}\hat{e}_\theta$$

$$\dot{\hat{e}}_\theta = -\cos\theta\dot{\theta}\hat{i} - \sin\theta\dot{\theta}\hat{j}$$

$$= -\dot{\theta}\hat{e}_r$$

Polar Coordinates



$$\hat{e}_\theta = -\sin\theta\hat{i} + \cos\theta\hat{j}$$

$$\hat{e}_r = \cos\theta\hat{i} + \sin\theta\hat{j}$$

$$\dot{\hat{e}}_r = (-\sin\theta\hat{i} + \cos\theta\hat{j})\dot{\theta}$$

$$= \dot{\theta}\hat{e}$$

$\vec{r} = r\cos\theta\hat{i} +$ … $\sin\theta\dot{\theta}\hat{j}$

$\dot{\vec{r}} = (\dot{r}\cos\theta -$

$= \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}$

$\ddot{\vec{r}} = \ddot{r}\hat{e}_r + \dot{r}\dot{\hat{e}}_r$

$= (\ddot{r} - r\dot{\theta}^2)\hat{e}$

$$\dot{\hat{e}}_r = \dot{\theta}\hat{e}_\theta$$

$$\dot{\hat{e}}_\theta = -\dot{\theta}\hat{e}_r$$

$$\vec{r} = r\hat{e}_r$$

$$\dot{\vec{r}} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$

$$\ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$$

The Energy Integral

Dot $\dot{\vec{r}}$ into the two-body EOM

$$\ddot{\vec{r}} + \frac{\mu}{r^3}\vec{r} = 0$$

$$\dot{\vec{r}}\cdot\ddot{\vec{r}} + \dot{\vec{r}}\cdot\frac{\mu}{r^3}\vec{r} = 0$$

$$\dot{\vec{r}}\cdot\ddot{\vec{r}} + \frac{\mu\dot{r}}{r^2} = 0$$

$\vec{r} = r\hat{e}_r$

$\dot{\vec{r}} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$

So, $\vec{r}\cdot\dot{\vec{r}} = r\dot{r}$

Note that $\frac{d}{dt}\left(\frac{\dot{\vec{r}}\cdot\dot{\vec{r}}}{2}\right) = \dot{\vec{r}}\cdot\ddot{\vec{r}}$

$$\frac{d}{dt}\left(\frac{\mu}{r}\right) = \frac{-\mu\dot{r}}{r^2}$$

Thus $\frac{d}{dt}\left(\frac{\dot{\vec{r}}\cdot\dot{\vec{r}}}{2}\right) - \frac{d}{dt}\left(\frac{\mu}{r}\right) = 0 \Longrightarrow \frac{d}{dt}\left(\frac{v^2}{2} - \frac{\mu}{r}\right) = 0$

The Energy Integral

Integrating this: $$\frac{d}{dt}\left(\frac{v^2}{2} - \frac{\mu}{r}\right) = 0$$

Yields: $$\frac{v^2}{2} - \frac{\mu}{r} = c$$

c is a constant, and is equal to the specific energy of the satellite relative to the central body. We'll call it ε (for "energy")

The Energy Integral

$$\varepsilon = \frac{v^2}{2} - \frac{\mu}{r}$$

$\frac{v^2}{2}$ Relative kinetic energy per unit mass

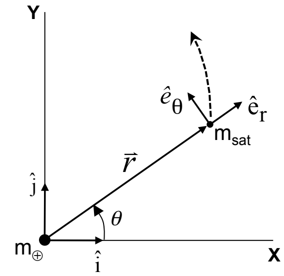
$-\frac{\mu}{r}$ Relative potential energy per unit mass

Known as the Energy Integral

So, we have 10 constants of motion: $\bar{c}_1,\ \bar{c}_2,\ \bar{h},\ \varepsilon$

Kepler's First Law

Since the orbit lies in a plane, let's express our relative two-body equations in polar coordinates



$$\dot{\hat{e}}_r = \dot{\theta}\hat{e}_\theta$$

$$\dot{\hat{e}}_\theta = -\dot{\theta}\hat{e}_r$$

$$\vec{r} = r\hat{e}_r$$

$$\dot{\vec{r}} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$

$$\ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$$

Kepler's First Law

For our problem, $\ddot{\vec{r}} = -\dfrac{\mu}{r^3}\vec{r}$

$$\ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$$

$$-\frac{\mu}{r^3}\vec{r} = -\frac{\mu}{r^3}r\hat{e}_r = -\frac{\mu}{r^2}\hat{e}_r$$

Hence: $(\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta = -\dfrac{\mu}{r^2}\hat{e}_r$

Thus, in polar coordinates:

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$$

Kepler's First Law

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$$

The 2nd equation is equivalent to: $\frac{d}{dt}(r^2\dot{\theta}) = 0$

$$h = r^2\dot{\theta}$$

which is our angular momentum integral

Kepler's First Law

Must solve $\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2}$

Change independent variable ***t*** to ***θ.***

Change dependent variable ***r*** to ***1/u***.

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} \quad \text{or in general} \quad \frac{d}{dt} = \dot{\theta}\frac{d}{d\theta}$$

but $r^2\dot{\theta} = h$ $\qquad \frac{d}{dt} = \frac{h}{r^2}\frac{d}{d\theta} = hu^2\frac{d}{d\theta}$

Kepler's First Law

$$\dot{r} = \frac{dr}{dt} = hu^2 \frac{dr}{d\theta} = hu^2 \frac{d}{d\theta}\left(\frac{1}{u}\right) = -\frac{hu^2}{u^2}\frac{du}{d\theta} = -h\frac{du}{d\theta}$$

$$\ddot{r} = \frac{d}{dt}\left(\frac{dr}{dt}\right) = hu^2 \frac{d}{d\theta}\left(\frac{dr}{dt}\right) = hu^2 \frac{d}{d\theta}\left(-h\frac{du}{d\theta}\right) = -h^2u^2\frac{d^2u}{d^2\theta}$$

also $r\dot{\theta}^2 = r\left(\frac{h}{r^2}\right)^2 = \frac{h^2}{r^3} = h^2u^3$

So, $\boxed{\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2}} \Rightarrow -h^2u^2\frac{d^2u}{d\theta^2} - h^2u^3 = -\mu u^2$

or $\frac{d^2u}{d^2\theta} + u = \frac{\mu}{h^2} =$ constant

Kepler's First Law

$$\frac{d^2u}{d^2\theta} + u = \frac{\mu}{h^2} = constant$$

Which has the form of an undamped spring-mass (harmonic oscillator) system with a constant forcing term.

The general solution to this equation is:

$$u = \frac{\mu}{h^2} + A\cos(\theta - \omega)$$

where ***A*** and ***ω*** are two more constants of motion.

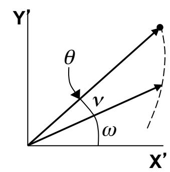
Kepler's First Law

Substituting $u = 1/r$ $\quad r = \dfrac{1}{\dfrac{\mu}{h^2} + A\cos(\theta - \omega)} = \dfrac{h^2/\mu}{1 + \dfrac{Ah^2}{\mu}\cos(\theta - \omega)}$

Let $\quad \nu = \theta - \omega$ *(true anomaly)* $\quad \omega =$ *argument of periapse*

$p = h^2/\mu$ *(semilatus rectum or semiparameter)*

$e = Ah^2/\mu$ *(eccentricity)*



Thus $\quad \boxed{r = \dfrac{p}{1 + e\cos\nu} = \dfrac{a(1 - e^2)}{1 + e\cos\nu}}$

This is exactly the equation of a conic section. This is sometimes called the "trajectory equation", and extends Kepler's First Law from ellipses to parabolas and hyperbolas.

Geometry of Conic Sections

"Conic Section" is the intersection of a plane with a cone. $\boldsymbol{m_\oplus}$ is at the primary focus of the ellipse.

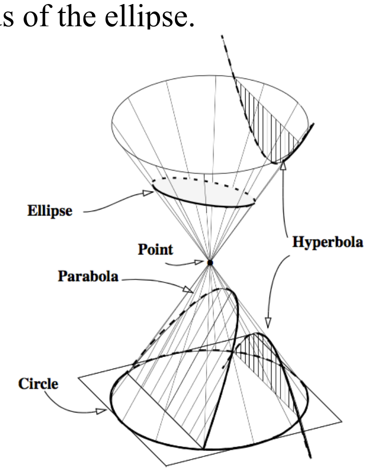


Figure 1-3. **Conic Sections.** Slicing a cone with a plane forms a conic section. When the plane is perpendicular to the axis of revolution, a circle results. Planes that are parallel to the axis of revolution yield hyperbolas (a pair as shown), and planes parallel to the outer surface yield parabolas. All other sections are ellipses, except for special cases in which the plane is *on* the surface (a line or rectilinear orbit) or only through the vertex (a point).

Summary and where we're going

- We showed that we have 12 constants of motion, or *integrals of motion* in the 2-body problem, using:
  - Conservation of linear momentum: 6 constants
  - Conservation of angular momentum: 3 constants
  - Conservation of energy: 1 constant
  - Kepler's equations: 2 constants
- Next lecture we'll show how we convert these into the six Keplerian orbital elements
  - Semi-major axis, eccentricity, inclination, right ascension of ascending node, argument of periapse, and true anomaly