ASEN 5050
SPACEFLIGHT DYNAMICS
Lecture 5: Two-Body Motion

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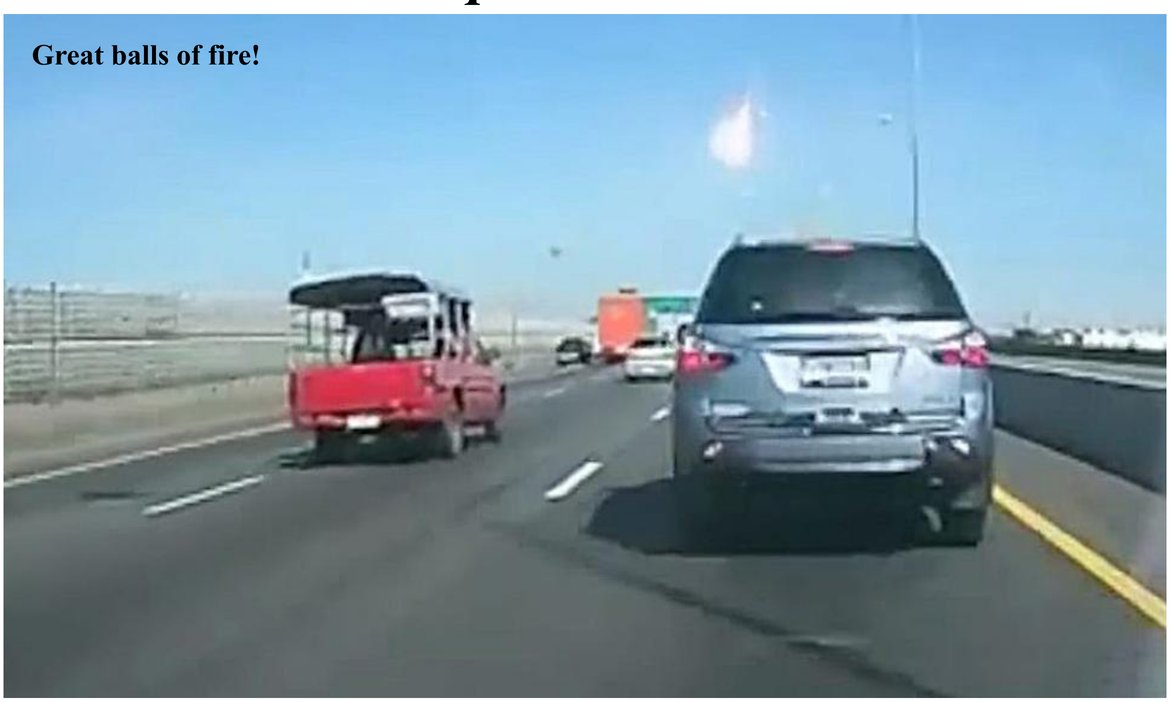
Space News



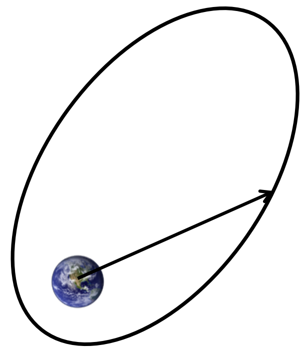
Today

- Talk about why Kepler's Equations matter and why/ how they are useful.

- For instance:

  – Say we're given a state:

  – Where will the satellite be in 10 minutes?



$$\upsilon \Rightarrow M \qquad +10 \text{ min} \qquad M_2 \Rightarrow \upsilon$$

Derivation of Kepler's Equation

If t is given:

$$M = n(t - t_p)$$

Solve $E - e\sin E = M$ for E

$$r = a(1 - e\cos E)$$

and

$$\cos\nu = \frac{a\cos E - ae}{r}$$

$$\sin\nu = \frac{b}{r}\sin E = \frac{a\sqrt{1-e^2}}{r}\sin E$$

If ν is given:

$$r = \frac{p}{1 + e\cos\nu}$$

$$\cos E = \frac{r\cos\nu + ae}{a}$$

$$\sin E = \frac{r\sin\nu}{b}$$

Solve $E - e\sin E = M$ for t (or M)

Another Useful Relation:

$$\boxed{\tan\frac{\nu}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}}$$

Example Using Mean/Eccentric Anomalies

For a satellite in an Earth orbit with h_A=3000 km and h_P=300 km, how long does it take to go from an altitude of 1000 km to one of 2000 km?

$$r = a(1 - e\cos E)$$

$$r_\oplus = 6378 \text{ km} \quad \Rightarrow \quad r_P = 6678 \text{ km}, \;\; r_A = 9378 \text{ km}$$

$$a = \frac{r_A + r_P}{2} = 8028 \text{ km}$$

$$e = \frac{r_A - r_P}{r_A + r_P} \qquad \Rightarrow e = \frac{2700}{16056} = 0.1682$$

Example Using Mean/Eccentric Anomalies

For a satellite in an Earth orbit with h_A=3000 km and h_P=300 km, how long does it take to go from an altitude of 1000 km to one of 2000 km?

$$r = a(1 - e\cos E)$$

$$r = (8028 \text{ km})(1 - (0.1682)\cos E)$$

Plug in r = (6378 + 1000) km and (6378 + 2000) km to determine the Eccentric Anomaly at those critical moments.

Compute the time between them.

Example Using Mean/Eccentric Anomalies

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Between r_A and r_P

$$r_1 = 7378km \;\Rightarrow\; r_1 = a(1 - e\cos E_1) \;\Rightarrow\; \cos E_1 = 0.4815, \quad E_1 = 61.22^\circ \text{ or } \cancel{298.78^\circ}$$

$$r_2 = 8378km \;\Rightarrow\; r_2 = a(1 - e\cos E_2) \;\Rightarrow\; \cos E_2 = -0.2593, \quad E_2 = 105.03^\circ \text{ or } \cancel{254.97^\circ}$$

Find the time of flight between $\boldsymbol{E_1=61.22°}$ to $\boldsymbol{E_2=105.03°}$

Example Using Mean/Eccentric Anomalies

For a satellite in an Earth orbit with h_A=3000 km and h_P=300 km, how long does it take to go from an altitude of 1000 km to one of 2000 km?

Find the time of flight between $\boldsymbol{E_1=61.22°}$ to $\boldsymbol{E_2=105.03°}$

$$M_1 = E_1 - e\sin E_1 \;\rightarrow\; M_1 = 52.78^\circ$$

$$M_2 = E_2 - e\sin E_2 \;\rightarrow\; M_2 = 95.72^\circ$$

$$n = \sqrt{\frac{\mu}{a^3}} = 181^\circ/\text{hr}$$

$$\Delta M = n\Delta t \;\rightarrow\; \Delta t = 14.2 \text{ minutes}$$

Example Using Mean/Eccentric Anomalies

What is the **altitude** of this same satellite 10 minutes past apogee?

At apoapse, M_1=180°.

10 minutes past apoapse:

$$M_2 = M_1 + 181^\circ / hr\left(\frac{10}{60}hr\right) = 210.16^\circ$$

$$M_2 = 210.16^\circ = E_2 - e\sin E_2 \qquad e = 0.1682$$

Solve for E_2

$$r = a(1 - e\cos E_2) \qquad a = 8028\ km$$
$$h = r - 6378km$$

Example Problems

1) An Earth satellite is observed to have a perigee height of 100 km and an apogee height of 600 km. Find the period (and e).

$$a = \frac{r_A + r_P}{2} = \frac{100 + 6378 + 600 + 6378}{2} = 6728\ km$$
$$P = 2\pi\sqrt{a^3/\mu} = 5492.11\ sec = 1.5256\ hours$$
$$e = \frac{r_A - r_P}{r_A + r_P} = \frac{6978 - 6478}{6978 + 6478} = 0.0372$$

Can also find v_A, v_P, p, μ (if P given).

Example Problems

2) How many days each year is the Earth farther from the Sun than 1AU? *(1 AU=149,597,870 km)*

$$\Delta t = \frac{M_2 - M_1}{n} \qquad n = \frac{360}{365.26} = 0.9856 \text{ deg}/day$$

$$M_1 = E_1 - e\sin E_1$$
$$M_2 = E_2 - e\sin E_2$$
$$r = a(1 - e\cos E)$$

We need $r > a$ $\qquad e = 0.0167$

$$or\ r = a(1 - ecosE) > a$$
$$1 - ecosE > 1$$
$$ecosE < 0$$
$$cosE < 0$$
$$90^\circ < E < 270^\circ$$

$$E_1 = 90^\circ \quad M_1 = E_1 - e\sin E_1 = 89.04^\circ; \qquad E_2 = 270^\circ \quad M_2 = E_2 - e\sin E_2 = 270.96^\circ$$

$$\Delta t = \frac{M_2 - M_1}{n}$$

$$\boxed{\Delta t = 184.6 \quad days}$$

Example Problems

3) Neglecting the eccentricity of Neptune's orbit, how many years in each Pluto orbit is Pluto closer to the Sun than Neptune?

$$r_P = a_P(1 - e_P cosE) \qquad a_P = 39.544\ AU, \quad e_P = 0.249$$
$$r_N = a_N \quad (e_N = 0.009) \qquad a_N = 30.11\ AU$$

Need $r_P = a_P(1 - e_P \cos E) < a_N$

$$\Rightarrow \cos E > \frac{1}{e_P}(1 - \frac{a_N}{a_P}) = \frac{1}{0.249}(1 - \frac{30.11}{39.544}) = 0.956077$$

$$\Rightarrow |E| < 17.0445^\circ$$

$$\Rightarrow |M| < 12.9097^\circ \qquad (M = E - e_P \sin E)$$

$$\text{Duration} = \frac{2 \times 12.9097^\circ}{n_P} = 17.7 \text{ years}$$

$$n_P = {}^{360^\circ}\!\!\diagup_{246.74 \text{ years}}$$

Canonical Units

1. Reduce size of numbers
2. More mathematically stable
3. Speed up algorithms
4. Allow different orgs to use standard values
5. Reduce maintenance programming

Define distance unit to be one Earth radius: ***1 ER = $R_{\oplus}$ = 6378.137 km***

We want $\mu_{\oplus}$=1, so define $\mu_{\oplus} = \dfrac{ER^3}{TU^2}$

Thus our time unit (TU) is: $TU = \sqrt{\dfrac{R_{\oplus}^{3}}{\mu_{\oplus}}}$

TU is time for satellite to cover 1 radian in a circular orbit of radius $R_{\oplus}$.

$$TU = \sqrt{\frac{6378.137^3}{398600.44}} = 806.8 \text{ seconds}$$

Vallado uses canonical units in examples throughout the book.

Canonical Units: Example 1

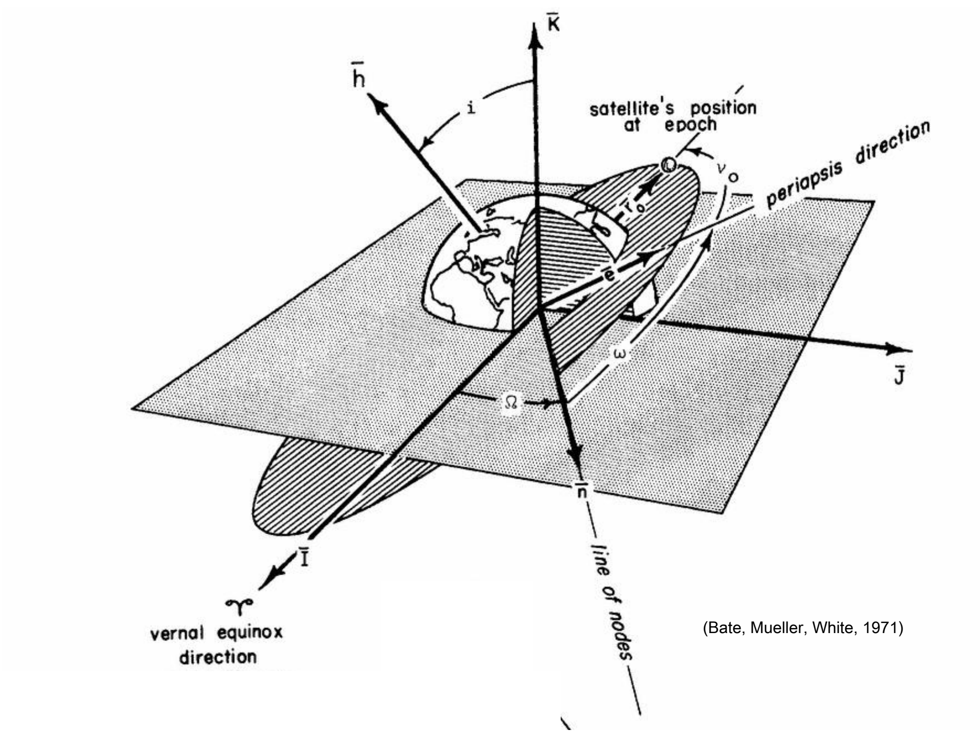
Given: A geosynchronous orbit
Find: The semi-major axis (a)

P = 24 sidereal hours = 86164/806.8 = 106.795869 TU

$$a = \left(\mu\left(\frac{P}{2\pi}\right)^2\right)^{1/3} = \left(1\left(\frac{106.7958697}{2\pi}\right)^2\right)^{1/3} = 6.610734645 \text{ ER}$$

$$a = 6.610734645(6378.1363) = 42164.17124 \text{ km}$$

Orbital Elements



Orbital Elements

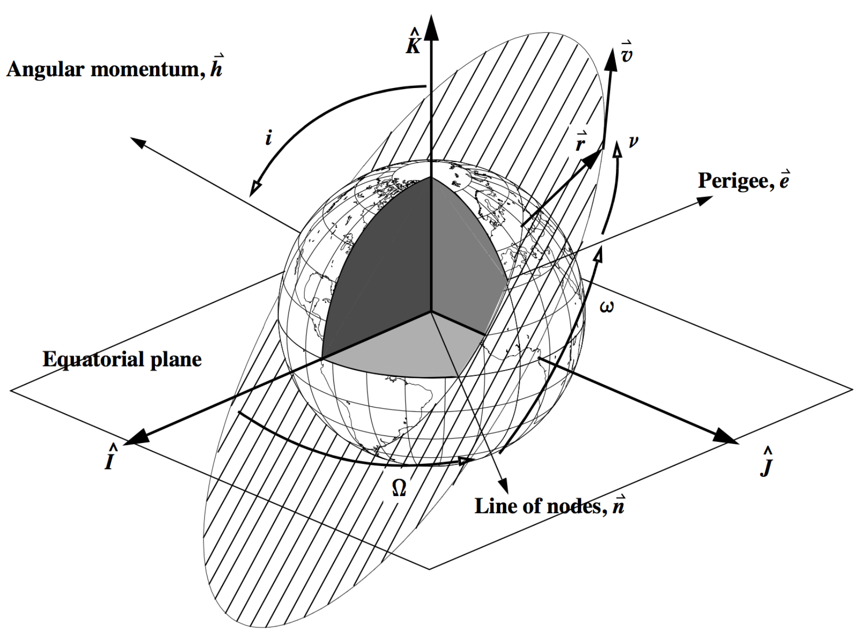


Figure 2-16. Classical Orbital Elements. The six classical orbital elements are the *semimajor axis, a*; *eccentricity*, *e*; *inclination*, *i*; *right ascension of the ascending node*, Ω—often referred to as simply the *node*; *argument of perigee*, ω; and *true anomaly*, ν. I haven't shown scale and shape elements (a and e) because I've introduced them in Chap. 1.

Orbital Elements

Law of Cosines $\vec{A} \cdot \vec{B} = AB\cos\alpha$

- useful for many angular relationships

We'll start with deriving the eccentricity vector
Then inclination, and the other angles

Orbital Elements

From our two-body derivation, we have:

$$\dot{\vec{r}} \times \vec{h} = \mu \frac{\vec{r}}{r} + \vec{B} \quad \text{(Eq 1-23 from Vallado)}$$

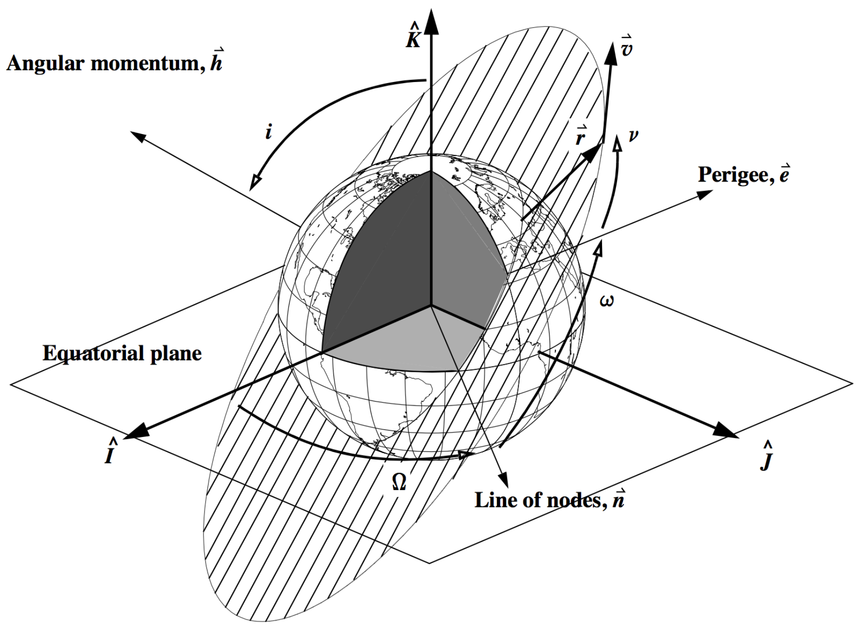
Since $e = \dfrac{B}{\mu}$,

then $\vec{e} = \dfrac{\vec{B}}{\mu}$, the <u>**eccentricity vector**</u>, and

$$\vec{e} = \frac{\vec{\mathrm{v}} \times \vec{h}}{\mu} - \frac{\vec{r}}{r}$$

where $|\vec{e}| = e$. $\vec{e}$ points towards periapse.

Orbital Elements



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Orbital Elements

Now, let's define our other orbital elements.

The ***inclination***, i, refers to the tilt of the orbit plane. It is the angle between $\hat{K}$ and $\vec{h}$, and varies from 0-180°.

$0^\circ < i < 90^\circ$	Prograde orbit (w/Earth's rotation)
$90^\circ < i < 180^\circ$	Retrograde orbit (against Earth rotation)
$i = 90^\circ$	Polar Orbit

$$\cos i = \frac{\hat{K} \cdot \vec{h}}{|\hat{K}||\vec{h}|}$$

Orbital Elements

The ***right ascension of the ascending node***, Ω, is the angle in the equatorial plane from $\hat{I}$ to the ascending node. The ***ascending node*** is the point on the equator where the satellite passes from South to North (opposite for the descending node).

The ***line of nodes*** connects the ascending and descending nodes. The node vector, $\vec{n}$, points towards the ascending node and is denoted:

$$\vec{n} = \hat{K} \times \vec{h}$$

The ***node*** lies between 0° and 360°.

$$\cos\Omega = \frac{\hat{I} \cdot \vec{n}}{|\hat{I}||\vec{n}|} = \frac{n_x}{n} \qquad \text{If } n_y < 0, \text{ then } \Omega = 360^\circ - \Omega$$

$$\sin\Omega = \frac{h_x}{h \sin i} \qquad \cos\Omega = \frac{-h_y}{h \sin i}$$

Orbital Elements

The ***argument of periapse***, ω, measured from the ascending node, locates the closest point of the orbit (periapse) and is the angle between $\vec{n}$ and $\vec{e}$.

$$\cos\omega = \frac{\vec{n} \cdot \vec{e}}{|\vec{n}||\vec{e}|} \qquad If\ (e_z < 0),\ \omega = 360^\circ - \omega$$

Orbital Elements

The ***true anomaly***, ν, is the angle between periapse and the satellite position; thus:

$$\cos\nu = \frac{\vec{e} \cdot \vec{r}}{|\vec{e}||\vec{r}|} \qquad \textit{If } (\vec{r} \cdot \vec{\mathrm{v}}) < 0,\ \nu = 360^\circ - \nu$$

($\vec{r} \cdot \vec{\mathrm{v}}$ is positive going away from periapse, negative coming towards periapse.)

Special Cases

Elliptical Equatorial Orbits – Ω is undefined, so we use true longitude of periapse, $\tilde{\omega}_{true}$,

$$\cos\tilde{\omega}_{true} = \frac{\hat{I} \cdot \vec{e}}{|\hat{I}||\vec{e}|} \qquad \text{If } (e_y < 0),\ \text{then } \tilde{\omega}_{true} = 360^\circ - \tilde{\omega}_{true}$$

For $i = 0°$, equivalent to astronomers' longitude of periapse, $\tilde{\omega}$, where $\tilde{\omega} = \Omega + \omega$

Special Cases

Circular Orbits – ω is undefined, use <u>argument of latitude</u>, *u*, where:

$$\cos u = \frac{\vec{n} \cdot \vec{r}}{|\vec{n}||\vec{r}|} \qquad \text{If } (r_z < 0), \text{ then } u = 360^\circ - u$$

Special Cases

Circular Equatorial – ω and Ω undefined

$$\cos(\lambda_{true}) = \frac{\hat{I} \cdot \vec{r}}{|\hat{I}||\vec{r}|} \qquad If\ (r_j < 0),\ \lambda_{true} = 360^\circ - \lambda_{true}$$

Two Line Element Sets

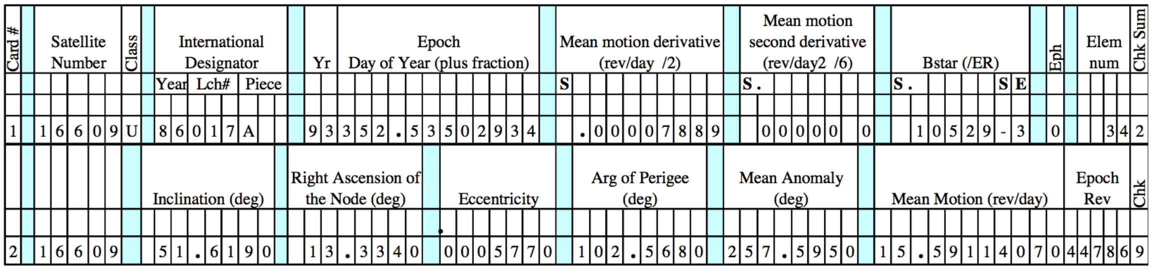

Figure 2-19. Transmission Format for the Two-Line Element Set. This example TLE set uses data from the previous example in the text. Note the use of implied decimal points. S is the sign of the values, and E is the exponent.

Available at http://www.celestrak.com

$$\bar{n} = \sqrt{\mu / \bar{a}^3} \qquad e \qquad i \qquad \Omega \qquad \omega \qquad M$$

$$\frac{\dot{n}}{2} \qquad \frac{\ddot{n}}{6} \qquad B^* = \frac{1}{2}\frac{C_D A}{m}\rho_0 \qquad UTC$$

$\bar{a}$, $\bar{n}$ are "Kozai" means. B* is a drag parameter.

Two Line Element Sets

Example

1 16609U 86017A 93352.53502934 .00007889 00000 0 10529-3 34
2 16609 51.6190 13.3340 0005770 102.5680 257.5950 15.59114070 44786

Epoch: Dec 18, 1993 12h 50min 26.5350 sec UTC

$$\bar{n} = 15.59114070 \;\; rev/day \Rightarrow \bar{a} = 1.06118087 \;\; ER = 6768.357 \;\; km$$

$$\frac{\dot{n}}{2} = 7.889 \times 10^{-5} \;\; rev/day^2 \qquad \frac{\ddot{n}}{6} = 0.0 \;\; rev/day^3$$

$$B^* = 1.0529 \times 10^{-4} \qquad e = 0.0005770 \qquad i = 51.6190^\circ$$

$$\Omega = 13.3340^\circ \qquad \omega = 102.5680^\circ \qquad M = 257.5950^\circ$$

Errors can be as large as a km or more.

Orbital Elements from $\vec{r}$ and $\vec{v}$ (and t)

Algorithm 9 in the book p. 112 - 116

First compute the following vectors

$$\vec{h} = \vec{r} \times \vec{v} \qquad \vec{n} = \hat{K} \times \vec{h} \qquad \vec{e} = \frac{\vec{v} \times \vec{h}}{\mu} - \frac{\vec{r}}{r}$$

$$h = |\vec{h}| \qquad n = |\vec{n}| \qquad e = |\vec{e}|$$

Compute the energy: $\frac{v^2}{2} - \frac{\mu}{r} = \varepsilon$

$$a = -\frac{\mu}{2\varepsilon} \quad \left(or\ p = h^2/\mu \Rightarrow a = p/(1-e^2)\right)$$

$$e = |\vec{e}| \quad \text{or} \quad e = \left[1 + 2\varepsilon h^2/\mu^2\right]^{1/2}$$

$$i = \cos^{-1}(h_z/h)$$

$$\Omega = \cos^{-1}\left(\frac{n_x}{n}\right) \qquad \text{If } (n_y < 0),\ \Omega = 360° - \Omega$$

$$\omega = \cos^{-1}\left(\frac{\vec{n}\cdot\vec{e}}{ne}\right) \qquad \text{If } (e_z < 0),\ \omega = 360° - \omega$$

$$\nu = \cos^{-1}\left(\frac{\vec{e}\cdot\vec{r}}{er}\right) \qquad \text{If } (\vec{r}\cdot\vec{v} < 0),\ \nu = 360° - \nu$$

Orbital Elements from $\vec{r}$ and $\vec{v}$ (and t)

Test using Example 2-5 in book

Also,

$$\tan\left(\frac{\nu}{2}\right) = \sqrt{\frac{1+e}{1-e}}\tan\left(\frac{E}{2}\right)$$

$$M = E - e\sin E$$

$$t_p = t - \frac{(E - e\sin E)}{n}$$

$$n = \sqrt{\frac{\mu}{a^3}}$$

Review of Two-Body Problem

$$m_{sat}\ddot{\vec{r}}_{sat} = -\frac{Gm_{\oplus}m_{sat}}{r^2}\frac{\vec{r}}{r} \qquad m_{\oplus}\ddot{\vec{r}}_{\oplus} = \frac{Gm_{\oplus}m_{sat}}{r^2}\frac{\vec{r}}{r}$$

$$\ddot{\vec{r}} = -\frac{G(m_{\oplus}+m_{sat})}{r^2}\frac{\vec{r}}{r} \cong -\frac{\mu}{r^3}\vec{r} \qquad \mu = Gm_{\oplus}$$

$$\vec{h} = \vec{r}\times\vec{\mathrm{v}} \qquad \vec{r}_{CM} = \vec{a}t+\vec{b} \qquad h = r^2\dot{\theta} = r^2\dot{\nu}$$

$$\varepsilon = \underbrace{\frac{\mathrm{v}^2}{2}}_{\text{KE}} - \underbrace{\frac{\mu}{r}}_{\text{PE}}$$ Energy Integral

both relative to one of the bodies

$$r = \frac{p}{1+e\cos\nu}$$

Equation of conic section

$\nu = \theta - \omega$ true anomaly

$p = h^2/\mu$ semilatus rectum

$e = Ah^2/\mu$

Review of Two-Body Problem

Elliptical Orbits $0 \le e < 1$

a = semimajor axis

b = semiminor axis

$$b = a\sqrt{1-e^2} \qquad e = \frac{\sqrt{a^2-b^2}}{a}$$

$$p = \frac{b^2}{a} = a(1-e^2) = \frac{h^2}{\mu}$$

$$r_P = \frac{p}{1+e} = a(1-e) \qquad r_A = \frac{p}{1-e} = a(1+e)$$

Flight Path Angle $h = r\mathrm{v}\cos\phi_{\mathrm{fpa}}$

$$\varepsilon = -\frac{\mu}{2a} \Rightarrow \mathrm{v}^2 = \mu\left(\frac{2}{r}-\frac{1}{a}\right)$$ *Vis - Viva Equation*

Review of Two-Body Problem

$$\Rightarrow \mathrm{v}_P = \sqrt{\frac{\mu}{a}\frac{(1+e)}{(1-e)}} \qquad \mathrm{v}_A = \sqrt{\frac{\mu}{a}\frac{(1-e)}{(1+e)}}$$

$$\mathrm{v}_C = \sqrt{\frac{\mu}{r}} \qquad \mathrm{v}_{ESC} = \sqrt{\frac{2\mu}{r}} \quad (a = \infty)$$

Also $e = \sqrt{1 + \dfrac{2\varepsilon h^2}{\mu^2}}$

$$P = 2\pi\sqrt{a^3/\mu} \qquad n = \sqrt{\frac{\mu}{a^3}} \qquad n^2 a^3 = \mu$$

$$r = a(1 - e\cos E)$$

$$E - e\sin E = n(t - t_p) = M$$