ASEN 5050
SPACEFLIGHT DYNAMICS
Lecture 6: Orbital Elements

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Space News

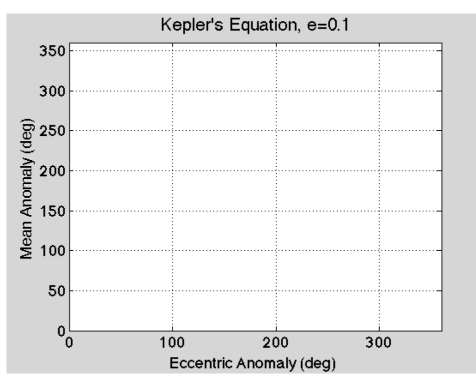
Space News

Solving Kepler's Equation

- Given M, solve for E

$$M = E - e\sin E$$

- What would a plot of this look like?
- Circular orbit?
- Elliptical?

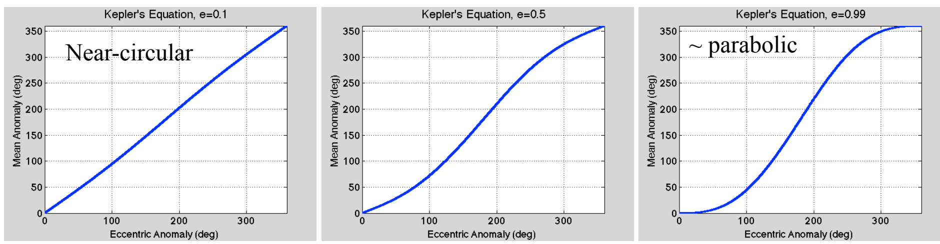


Solving Kepler's Equation

- Given M, solve for E

$$M = E - e\sin E$$

- To see what we have to do, here are a few plots of this relationship:



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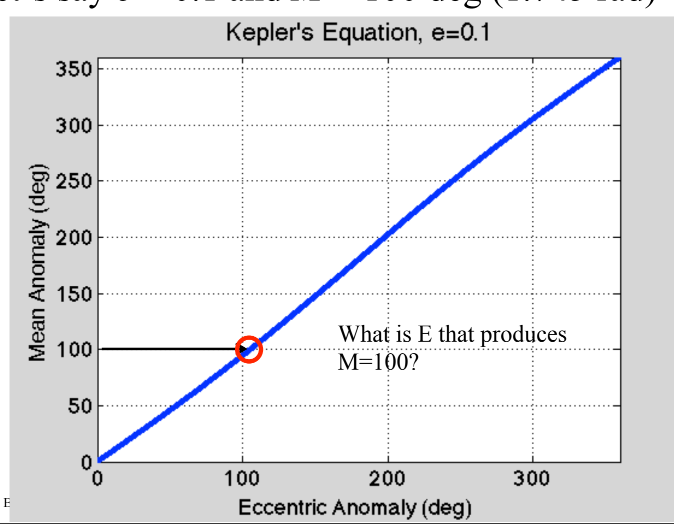
Lecture 6: The Two Body Problem

Solving Kepler's Equation

$$M = E - e\sin E$$

- Let's say $e = 0.1$ and $M = 100$ deg (1.745 rad)



Lecture 6: The Two B

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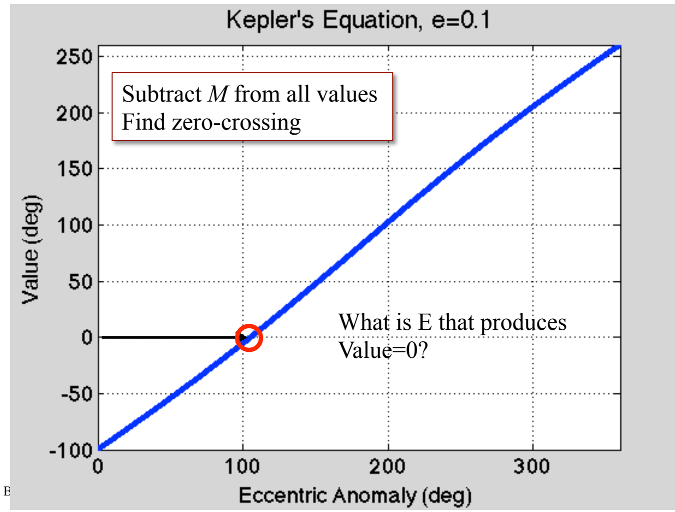
Solving Kepler's Equation

$$E - e \sin E - M_{\text{target}} = 0$$

- Let's say $e = 0.1$ and $M = 100$ deg (1.745 rad)



Solving Kepler's Equation

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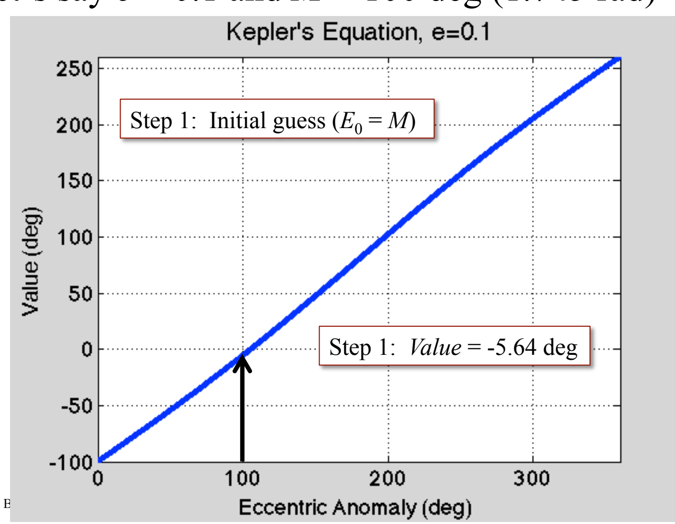

Solving Kepler's Equation

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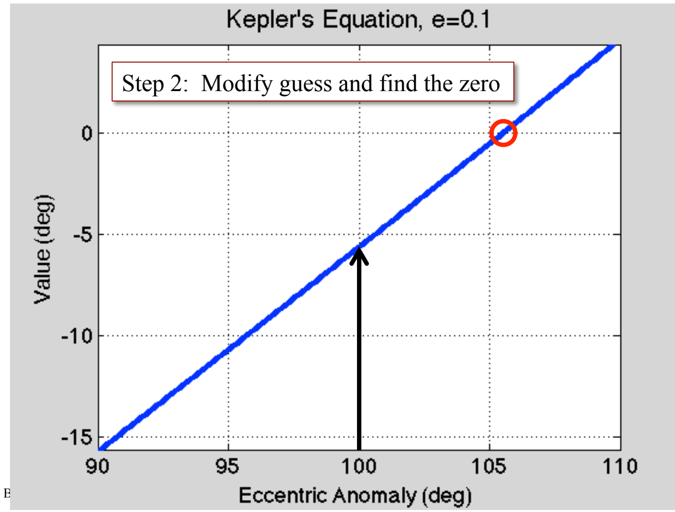

Solving Kepler's Equation

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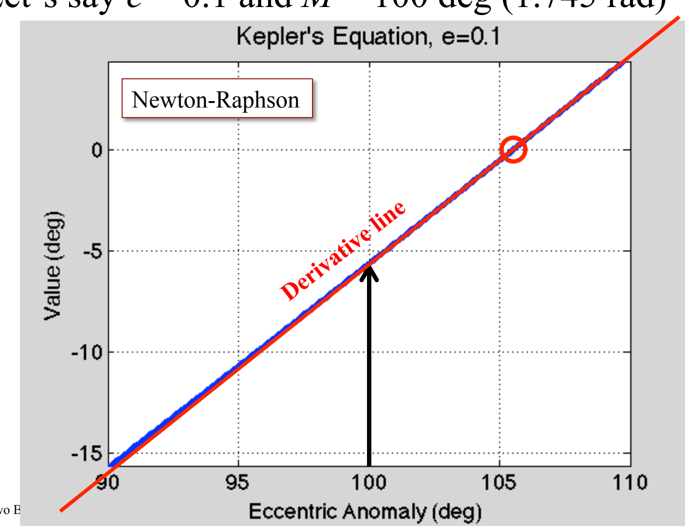

Solving Kepler's Equation

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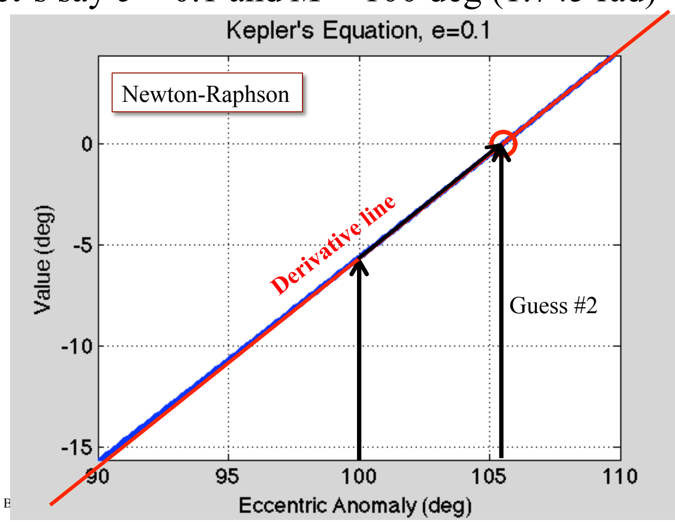

Solving Kepler's Equation

- Newton Raphson Method applied to Kepler's Equation

$$X_{n+1} = X_n + \delta_n = X_n - \frac{f(X_n)}{f'(X_n)}$$

$$E - e\sin E = M \quad \Rightarrow \quad f(E) = M - E + e\sin E = 0$$

$$f'(E) = -1 + e\cos E$$

$$E_{n+1} = E_n + \frac{M - E_n + e\sin E_n}{1 - e\cos E_n}$$

Solving Kepler's Equation

- Let's say $e = 0.1$ and $M = 100$ deg (1.745 rad)

$$E_{n+1} = E_n + \frac{M - E_n + e\sin E_n}{1 - e\cos E_n}$$

- Guess 1: $E_0 = 100$ deg
  - Error 1: -5.64 deg
- Guess 2: $E_1 = 105.546$ deg
  - Error 2: 0.0263 deg
- Guess 3: $E_2 = 105.521$ deg
  - Error 3: 0.0000 deg
- Guess 4: $E_3 = 105.521$ deg
  - Error 4: 0.0000 deg

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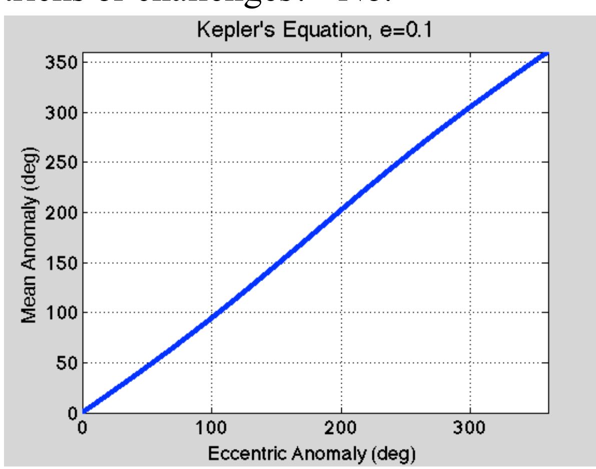
Solving Kepler's Equation

- Let's say $e = 0.1$
- Any tricks or challenges? No.



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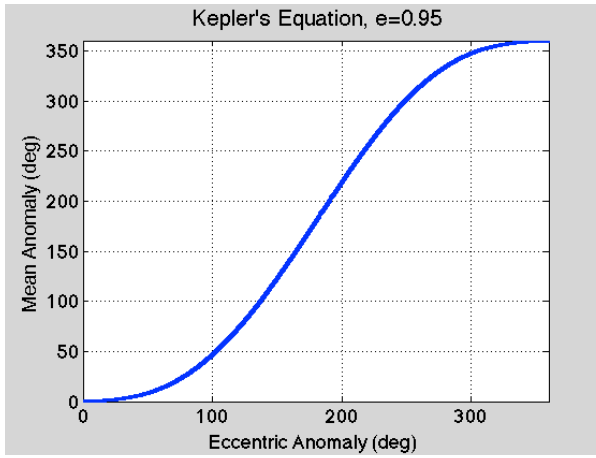
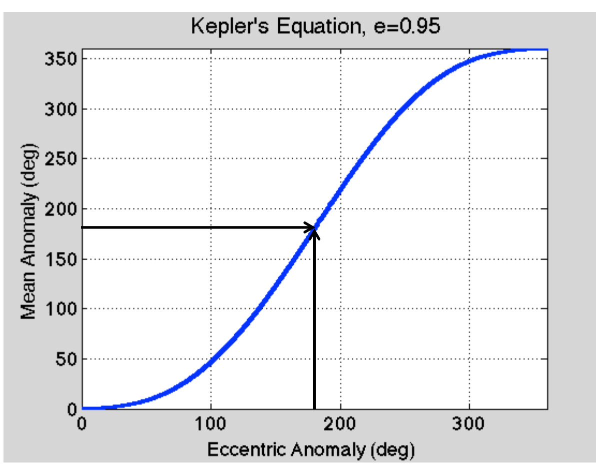
Solving Kepler's Equation

- Let's say $e = 0.95$
- Any tricks or challenges?   Not really.



Solving Kepler's Equation

- Let's say $e = 0.95$, $M = 180$ deg

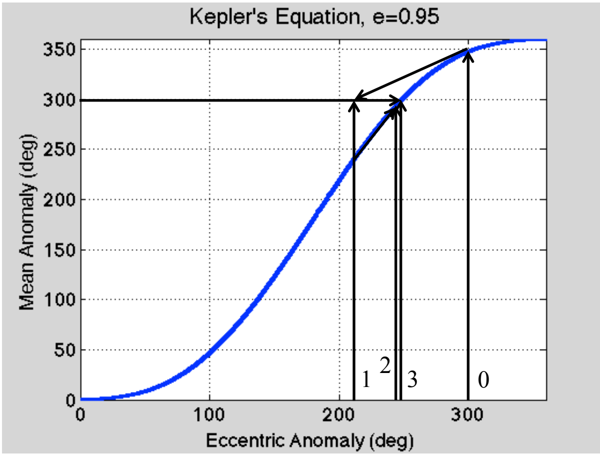


<u>Iteration History:</u>

E0 = 180.0000 deg
   Error = 0.0000 deg

Solving Kepler's Equation

- Let's say $e = 0.95$, $M = 300$ deg



Iteration History:

E0 = 300.0000 deg
Value = 47.1386 deg

E1 = 210.2122 deg
Error = -62.3980 deg

E2 = 244.4787 deg
Error = -6.4014 deg

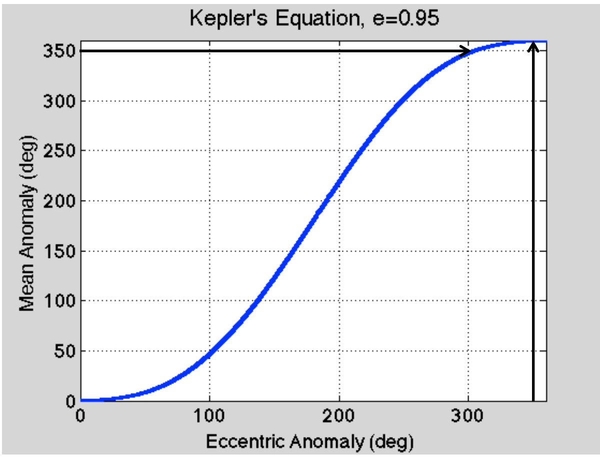
E3 = 249.0209 deg
Error = -0.1562 deg

E4 = 249.1375 deg
Error = -0.0001 deg

E5 = 249.1376 deg
Error = -0.0000 deg

Solving Kepler's Equation

- Let's say $e = 0.95$, $M = 350$ deg



Iteration History:

E0 = 350.0000 deg
Value = 9.4518 deg

E1 = 203.3066 deg
Error = -125.1577 deg

E2 = 270.1472 deg
Error = -25.4220 deg

E3 = 295.6314 deg
Error = -5.2939 deg

E4 = 304.6185 deg
Error = -0.5873 deg

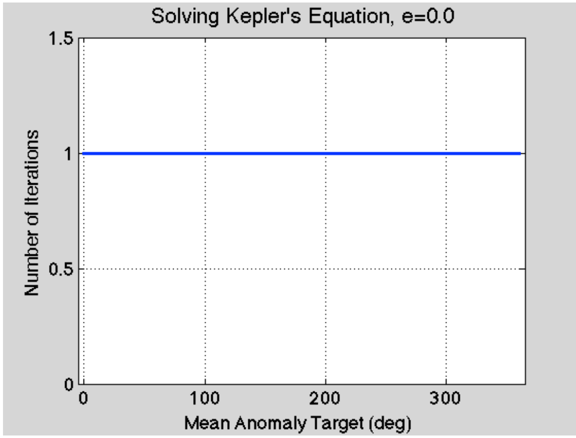
E5 = 305.8945 deg
Error = -0.0111 deg

E6 = 305.9195 deg
Error = -0.0000 deg

Solving Kepler's Equation

- How many iterations does it take?

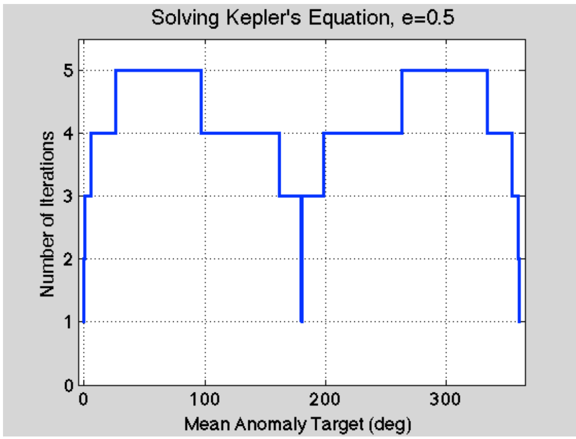
- $e = 0$



Solving Kepler's Equation

- How many iterations does it take?

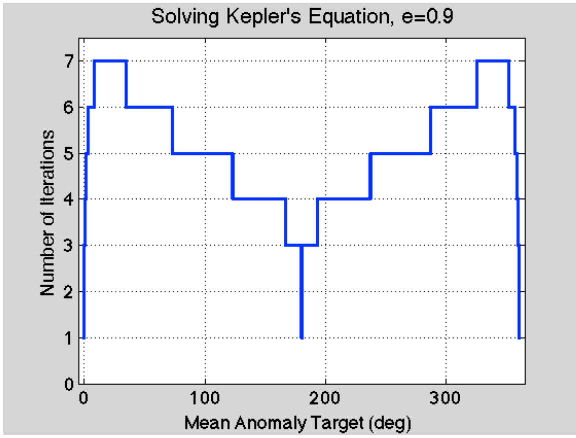
- $e = 0.5$



Solving Kepler's Equation

- How many iterations does it take?

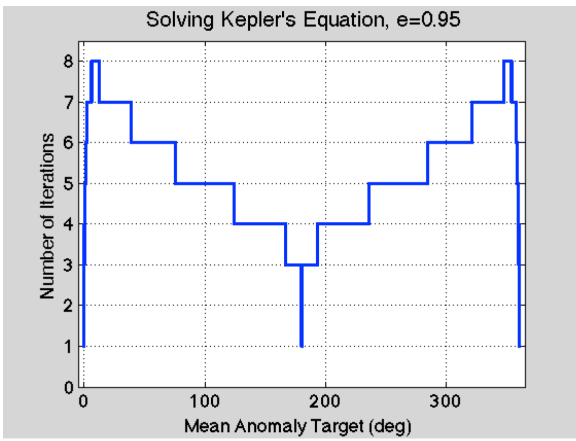
- $e = 0.9$



Solving Kepler's Equation

- How many iterations does it take?

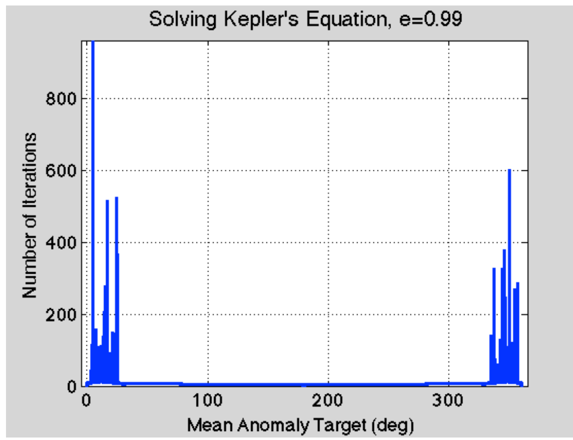
- $e = 0.95$



Solving Kepler's Equation

- How many iterations does it take?

- $e = 0.99$



Not good!

Kepler's Equation

- Algorithm 2 in Vallado:

ALGORITHM 2: ***KepEqtnE*** $(M, e \Rightarrow E)$

IF $-\pi < M < 0$ or $M > \pi$

let $E = M - e$

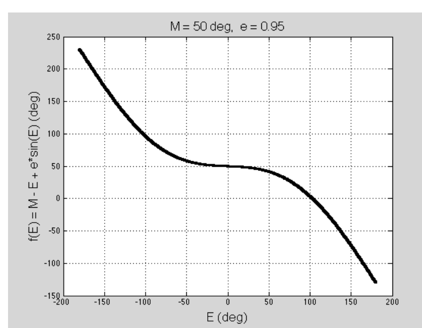
ELSE

let $E = M + e$

LOOP

$$E_{n+1} = E_n + \frac{M - E_n + e\,\mathrm{SIN}(E_n)}{1 - e\,\mathrm{COS}(E_n)}$$

UNTIL $|E_{n+1} - E_n| <$ tolerance



Orbital Elements

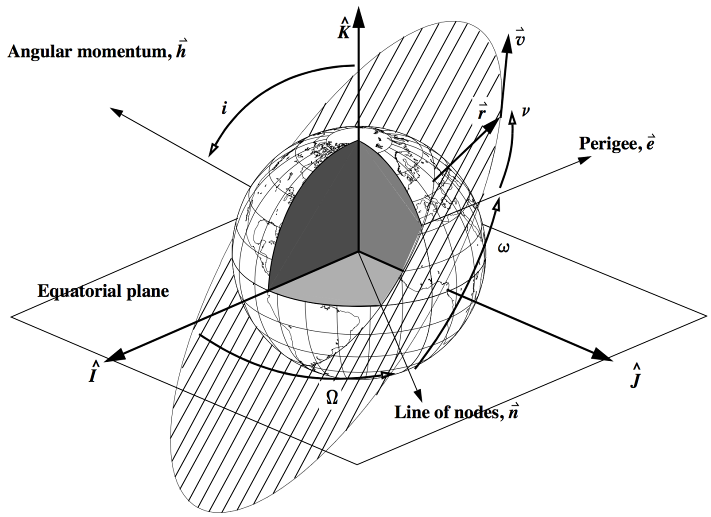


Figure 2-16. Classical Orbital Elements. The six classical orbital elements are the *semimajor axis, a*; *eccentricity*, *e*; *inclination*, *i*; *right ascension of the ascending node*, Ω—often referred to as simply the *node*; *argument of perigee*, ω; and *true anomaly*, ν. I haven't shown scale and shape elements (a and e) because I've introduced them in Chap. 1.

Two Line Element Sets

Card #	Satellite Number	Class	International Designator (Year / Lch# / Piece)	Epoch Yr	Epoch Day of Year (plus fraction)	Mean motion derivative (rev/day /2)	Mean motion second derivative (rev/day2 /6)	Bstar (/ER)	Eph	Elem num	Chk Sum
			Year / Lch# / Piece			S	S .	S . S E			
1	16609	U	86 / 017 / A	93	352.53502934	.00007889	00000 0	10529 -3	0	34	2

Card #	Satellite Number	Inclination (deg)	Right Ascension of the Node (deg)	Eccentricity	Arg of Perigee (deg)	Mean Anomaly (deg)	Mean Motion (rev/day)	Epoch Rev	Chk
2	16609	51.6190	13.3340	0005770	102.5680	257.5950	15.59114070	44786	9

Figure 2-19. Transmission Format for the Two-Line Element Set. This example TLE set uses data from the previous example in the text. Note the use of implied decimal points. S is the sign of the values, and E is the exponent.

Available at http://www.celestrak.com

$$\bar{n} = \sqrt{\mu / \bar{a}^3} \qquad e \qquad i \qquad \Omega \qquad \omega \qquad M$$

$$\frac{\dot{n}}{2} \qquad \frac{\ddot{n}}{6} \qquad B^* = \frac{1}{2}\frac{C_D A}{m}\rho_0 \qquad UTC$$

$\bar{a}$, $\bar{n}$ are "Kozai" means. B* is a drag parameter.

Two Line Element Sets

Example

1 16609U 86017A 93352.53502934 .00007889 00000 0 10529-3 34
2 16609 51.6190 13.3340 0005770 102.5680 257.5950 15.59114070 44786

Epoch: Dec 18, 1993 12h 50min 26.5350 sec UTC

$\bar{n} = 15.59114070 \ rev/day \Rightarrow \ \bar{a} = 1.06118087 \ ER = 6768.357 \ km$

$\frac{\dot{n}}{2} = 7.889 \times 10^{-5} \ rev/day^2 \qquad \frac{\ddot{n}}{6} = 0.0 \ rev/day^3$

$B^* = 1.0529 \times 10^{-4} \qquad e = 0.0005770 \qquad i = 51.6190^\circ$

$\Omega = 13.3340^\circ \qquad \omega = 102.5680^\circ \qquad M = 257.5950^\circ$

Errors can be as large as a km or more.

Orbital Elements from $\vec{r}$ and $\vec{v}$ (and t)

Algorithm 9 in the book p. 112 - 116

First compute the following vectors

$$\vec{h} = \vec{r} \times \vec{v} \qquad \vec{n} = \hat{K} \times \vec{h} \qquad \vec{e} = \frac{\vec{v} \times \vec{h}}{\mu} - \frac{\vec{r}}{r}$$

$$h = |\vec{h}| \qquad n = |\vec{n}| \qquad e = |\vec{e}|$$

Compute the energy: $\frac{v^2}{2} - \frac{\mu}{r} = \varepsilon$

$$a = -\frac{\mu}{2\varepsilon} \quad \left(or \ p = h^2/\mu \Rightarrow a = p/(1-e^2) \right) \qquad \Omega = \cos^{-1}\left(\frac{n_x}{n}\right) \quad \text{If } (n_y < 0),\ \Omega = 360^\circ - \Omega$$

$$e = |\vec{e}| \quad \text{or} \quad e = \left[1 + 2\varepsilon h^2/\mu^2\right]^{1/2} \qquad \omega = \cos^{-1}\left(\frac{\vec{n}\cdot\vec{e}}{ne}\right) \quad \text{If } (e_z < 0),\ \omega = 360^\circ - \omega$$

$$i = \cos^{-1}(h_z/h) \qquad \nu = \cos^{-1}\left(\frac{\vec{e}\cdot\vec{r}}{er}\right) \quad \text{If } (\vec{r}\cdot\vec{v} < 0),\ \nu = 360^\circ - \nu$$

Orbital Elements from $\vec{r}$ and $\vec{v}$ (and t)

Test using Example 2-5 in book

Also,

$$\tan\left(\frac{\nu}{2}\right) = \sqrt{\frac{1+e}{1-e}}\tan\left(\frac{E}{2}\right)$$

$$M = E - e\sin E$$

$$t_p = t - \frac{(E - e\sin E)}{n}$$

$$n = \sqrt{\frac{\mu}{a^3}}$$

Example 2-5

Example 2-5. Finding the Orbital Elements with *RV2COE*.

GIVEN: $\vec{r}_{IJK} = 6524.834\,\hat{I} + 6862.875\,\hat{J} + 6448.296\,\hat{K}$ km

$\vec{v}_{IJK} = 4.901\,327\,\hat{I} + 5.533\,756\,\hat{J} - 1.976\,341\,\hat{K}$ km/s

FIND: Classical orbital elements

The magnitudes are $r = 11456.57$ km and $v = 7.651\,888$ km/s

Begin by finding the specific angular momentum:

$$\vec{h} = \vec{r} \times \vec{v} = \begin{bmatrix} 6524.834 & 6862.875 & 6448.296 \\ 4.901\,327 & 5.533\,756 & -1.976\,341 \end{bmatrix} \text{km}^2/\text{s}$$

$$= -49246.7\hat{I} + 44500.5\hat{J} + 2469.6\hat{K}$$

The magnitude of angular momentum is 66420.10 km^2/s.

Example 2-5

Find the node vector (not the mean motion) using a cross product:

$$\vec{n} = \hat{K} \times \vec{h} = -44500.5\hat{I} - 49246.7\hat{J} \qquad |\vec{n}| = 66374.17 \text{ km}^2/\text{s}$$

$$\vec{e} = \frac{1}{\mu}\left[\left(v^2 - \frac{\mu}{r}\right)\vec{r} - (\vec{r} \cdot \vec{v})\vec{v}\right] = -0.3146\hat{I} - 0.385\,23\hat{J} + 0.668\,04\hat{K}$$

$$e = 0.832\,853$$

$$\xi = \frac{v^2}{2} - \frac{\mu}{r} = \frac{7.651\,888^2}{2} - \frac{398600.4418}{11456.67} = -5.516\,604 \text{ km}^2/\text{s}^2$$

Because the orbit isn't parabolic (the eccentricity is not 1.0), find the semimajor axis by $a = -\frac{\mu}{2\xi}$:

$a = 36{,}127.343$ km

Then, find the semiparameter using the specific angular momentum:

$$p = \frac{h^2}{\mu} = \frac{66420.10^2}{398600.4418} = 11{,}067.790 \text{ km}$$

Example 2-5

The next task is to determine the angles. Find the inclination with a cosine expression. No quadrant check is necessary.

$$\cos(i) = \frac{h_K}{|\vec{h}|} = \frac{2469.6}{66420.10}$$

$$i = 87.870°$$

Find the right ascension of the ascending node. Notice the quadrant check here affects the final value.

$$\cos(\Omega) = \frac{n_I}{|\vec{n}|} = \frac{-44500.5}{66374.17} \qquad \text{Initially, } \Omega \text{ is } 132.102°$$

$$\text{IF}(n_J < 0)\text{THEN } \Omega = 360° - \Omega \qquad \Omega = 227.898°$$

Example 2-5

Find the argument of perigee similarly but without modification for quadrants:

$$\cos(\omega) = \frac{\vec{n} \cdot \vec{e}}{|\vec{n}||\vec{e}|} = \frac{32970.953mu}{(66374.17)(0.832\,853)} \qquad \omega = 53.38°$$

$$\text{IF}(e_K < 0)\text{THEN } \omega = 360° - \omega$$

Finally, find the true anomaly:

$$\cos(\nu) = \frac{\vec{e} \cdot \vec{r}}{|\vec{e}||\vec{r}|} = \frac{-388.773}{(0.832\,853)(11456.67)} \qquad \nu = 92.335°$$

$$\text{IF}(\vec{r} \cdot \vec{v} < 0)\text{THEN } \nu = 360° - \nu$$

Evaluate the special angles recalling the limitations imposed for planes in each calculation. Determine the elements relating to the location of perigee. Notice the difference in the values of the true longitude of perigee and the longitude of perigee!

Example 2-5

$$\cos(\tilde{\omega}_{true}) = \frac{e_I}{|\vec{e}|} = \frac{-0.3146}{0.832\,853}$$, and initially, $\tilde{\omega}_{true}$ is 112.194°

$$\text{IF}(e_J < 0)\text{THEN}\tilde{\omega}_{true} = 360° - \tilde{\omega}_{true} \quad , \quad \tilde{\omega}_{true} = 247.806°$$

$$\tilde{\omega} = \Omega + \omega = 227.89° + 53.38° = 281.27°$$

Find the argument of latitude:

$$\cos(u) = \frac{\vec{n} \cdot \vec{r}}{|\vec{n}||\vec{r}|} \qquad \text{IF}(r_K < 0)\text{THEN } u = 360° - u$$

$$u = \cos^{-1}\left(\frac{(-44500.5)(6524.834) + (-49246.7)(6862.875)}{(66374.17)(11456.67)}\right) = 145.605\,49°$$

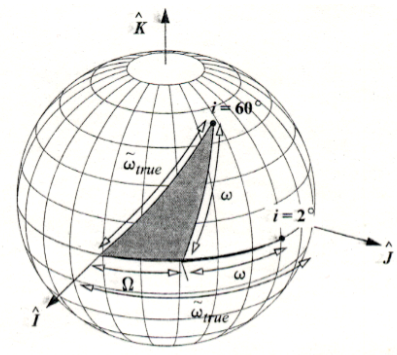
Finally, determine the true longitude:

$$\cos(\lambda_{true}) = \frac{r_I}{|\vec{r}|} = \frac{6524.834}{11456.67} \qquad \lambda_{true} = 55.282\,587°$$

$$\text{IF}(r_J < 0)\text{THEN } \lambda_{true} = 360° - \lambda_{true}$$

▲

Longitude of Periapsis



$i = 2°$, $e = 0.000\,01$	$i = 60°$, $e = 0.000\,01$
$\Omega = 30°$, $\omega = 40°$	$\Omega = 30°$, $\omega = 40°$
$\tilde{\omega}_{true} = 69.988\,27°$	$\tilde{\omega}_{true} = 59.820\,08°$

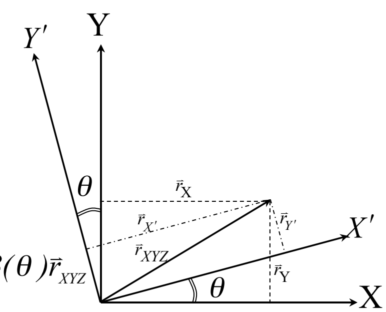
Figure 2-17. Representation of Longitude of Periapsis. Notice the effect inclination has on the values for the longitude of periapsis. The values of $(\Omega + \omega)$ and $\tilde{\omega}_{true}$ are close when the inclination is small but very different as the inclination increases.

Principal Axis Rotations

$$r_{X'} = r_X \cos\theta + r_Y \sin\theta$$

$$r_{Y'} = -r_X \sin\theta + r_Y \cos\theta$$

$$r_{Z'} = 0 = r_Z$$



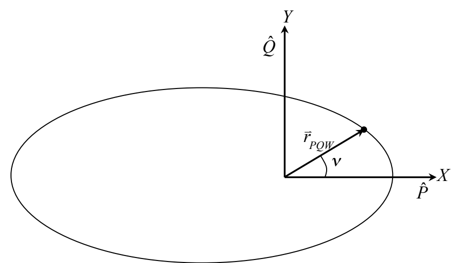
$$\vec{r}_{X'Y'Z'} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \vec{r}_{XYZ} = ROT3(\theta)\vec{r}_{XYZ}$$

$$ROT2(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \qquad ROT1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

$\vec{r}$ and $\vec{v}$ From Orbital Elements

Express the position and velocity in the perifocal system ($\hat{x}$ goes through periapse, $\hat{z}$ in the direction of $\bar{h}$, $\bar{y}$ perpendicular to $\hat{x}$ and $\hat{z}$ in the orbit plane.)

$$\vec{r}_{PQW} = r\cos\nu\,\hat{P} + r\sin\nu\,\hat{Q} = \begin{bmatrix} \dfrac{p\cos\nu}{1+e\cos\nu} \\ \dfrac{p\sin\nu}{1+e\cos\nu} \\ 0 \end{bmatrix}$$



$$\vec{v}_{PQW} = \begin{bmatrix} \dot{r}\cos\nu - r\dot{\nu}\sin\nu \\ \dot{r}\sin\nu + r\dot{\nu}\cos\nu \\ 0 \end{bmatrix}$$

$$r\dot{\nu} = \frac{h}{r} = \frac{\sqrt{\mu p}(1+e\cos\nu)}{p} = \sqrt{\frac{\mu}{p}}(1+e\cos\nu)$$

$$\dot{r} = \frac{r\dot{\nu}e\sin\nu}{1+e\cos\nu} = \sqrt{\frac{\mu}{p}}(e\sin\nu), \quad \text{thus}$$

$$\vec{v}_{PQW} = \begin{bmatrix} -\sqrt{\mu/p}\,\sin\nu \\ \sqrt{\mu/p}\,(e+\cos\nu) \\ 0 \end{bmatrix}$$

$\vec{r}$ and $\vec{v}$ From Orbital Elements

Now we simply need to rotate into the geocentric equatorial system. The order of the rotations <u>does</u> matter

$$\vec{r}_{IJK} = ROT3(-\Omega)ROT1(-i)ROT3(-\omega)\vec{r}_{PQW} = \left[\frac{IJK}{PQW}\right]\vec{r}_{PQW}$$

$$\vec{v}_{IJK} = ROT3(-\Omega)ROT1(-i)ROT3(-\omega)\vec{v}_{PQW} = \left[\frac{IJK}{PQW}\right]\vec{v}_{PQW}$$

$$\left[\frac{IJK}{PQW}\right] = \begin{bmatrix} \cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i & -\cos\Omega\sin\omega - \sin\Omega\cos\omega\cos i & \sin\Omega\sin i \\ \sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i & -\sin\Omega\sin\omega + \cos\Omega\cos\omega\cos i & -\cos\Omega\sin i \\ \sin\omega\sin i & \cos\omega\sin i & \cos i \end{bmatrix}$$

Algorithm 10 in book.

Ex. 2-6

Example 2-6

▼ **Example 2-6. Finding Position and Velocity Vectors (*COE2RV* Test Case).**

GIVEN: $p = 11{,}067.790$ km, $e = 0.832\,85$, $i = 87.87°$, $\Omega = 227.89°$, $\omega = 53.38°$, $\nu = 92.335°$

FIND: $\vec{r}_{IJK}\ \vec{v}_{IJK}$

We have to change the rotation angles if we're using special orbits (equatorial or circular), but this orbit doesn't have special cases. From the given information, form the PQW position and velocity vectors:

$$\vec{r}_{PQW} = \begin{bmatrix} \dfrac{p\cos(\nu)}{1+e\cos(\nu)} \\ \dfrac{p\sin(\nu)}{1+e\cos(\nu)} \\ 0 \end{bmatrix} = \begin{bmatrix} \dfrac{11067.790\cos(92.335)°}{1+0.832\,85\cos(92.335)°} \\ \dfrac{11067.790\sin(92.335)}{1+0.832\,85\cos(92.335)°} \\ 0 \end{bmatrix} = \begin{bmatrix} -466.7639 \\ 11447.0219 \\ 0 \end{bmatrix} \text{km}$$

$$\vec{v}_{PQW} = \begin{bmatrix} -\sqrt{\dfrac{\mu}{p}}\sin(\nu) \\ \sqrt{\dfrac{\mu}{p}}(e+\cos(\nu)) \\ 0 \end{bmatrix} = \begin{bmatrix} -\sqrt{\dfrac{398600.4418}{11067.790}}\sin(92.335) \\ \sqrt{\dfrac{398600.4418}{11067.790}}(0.832\,85+\cos(92.335)) \\ 0 \end{bmatrix} = \begin{bmatrix} -5.996\,222 \\ 4.753\,601 \\ 0 \end{bmatrix} \frac{\text{km}}{\text{s}}$$

Example 2-6

Rotate these vectors to the geocentric equatorial system using the following rotation matrices:

$$\vec{r}_{IJK} = [\text{ROT3}(-\Omega)][\text{ROT1}(-i)][\text{ROT3}(-\omega)]\vec{r}_{PQW}$$
$$\vec{v}_{IJK} = [\text{ROT3}(-\Omega)][\text{ROT1}(-i)][\text{ROT3}(-\omega)]\vec{v}_{PQW}$$

Or, use the expanded matrix with a computer to do the many trigonometric operations, which result in the transformation matrix

$$\left[\frac{IJK}{PQW}\right] = \begin{bmatrix} -0.377\,860\,07 & 0.554\,641\,79 & -0.741\,346\,25 \\ -0.462\,525\,60 & 0.580\,556\,38 & 0.670\,092\,80 \\ 0.802\,054\,76 & 0.596\,092\,93 & 0.037\,166\,95 \end{bmatrix}$$

Finally, multiply each vector to apply the transformation:

Example 2-6

$$\vec{r}_{IJK} = \left[\frac{IJK}{PQW}\right]\vec{r}_{PQW} = \begin{bmatrix} -0.377\,860\,07 & 0.554\,641\,79 & -0.741\,346\,25 \\ -0.462\,525\,60 & 0.580\,556\,38 & 0.670\,092\,80 \\ 0.802\,054\,76 & 0.596\,092\,93 & 0.037\,166\,95 \end{bmatrix} \begin{bmatrix} -466.7639 \\ 11447.0219 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 6525.344 \\ 6861.535 \\ 6449.125 \end{bmatrix} \text{km}$$

$$\vec{v}_{IJK} = \left[\frac{IJK}{PQW}\right]\vec{v}_{PQW} = \begin{bmatrix} -0.377\,860\,07 & 0.554\,641\,79 & -0.741\,346\,25 \\ -0.462\,525\,60 & 0.580\,556\,38 & 0.670\,092\,80 \\ 0.802\,054\,76 & 0.596\,092\,93 & 0.037\,166\,95 \end{bmatrix} \begin{bmatrix} -5.996\,222 \\ 4.753\,601 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4.902\,276 \\ 5.533\,124 \\ -1.975\,709 \end{bmatrix} \text{km/s}$$

f and *g* Series

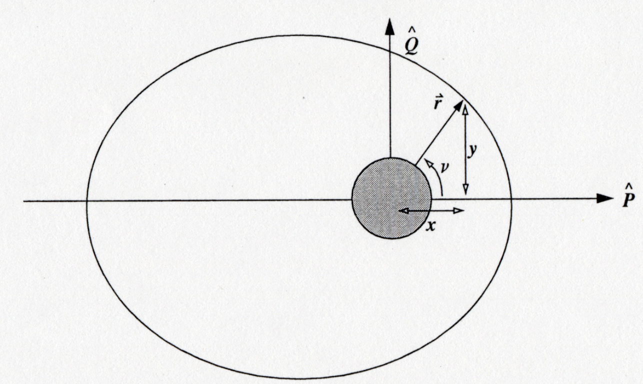


Figure 2-14. Geometry for the Perifocal Coordinate System. We can break a satellite's position into two components (x, y) in the orbital plane.

$$\vec{r} = x\hat{P} + y\hat{Q} \qquad \vec{r} = f\vec{r}_o + g\vec{v}_o$$

$$\vec{v} = \dot{x}\hat{P} + \dot{y}\hat{Q} \qquad \vec{v} = \dot{f}\vec{r}_o + \dot{g}\vec{v}_o$$

f and g Series

Start by crossing the position vector into the initial velocity vector:

$$\vec{r} \times \vec{v}_o = (f\vec{r}_o + g\vec{v}_o) \times \vec{v}_o$$
$$= f(\vec{r}_o \times \vec{v}_o) + g(\vec{v}_o \times \vec{v}_o)$$

The second term is zero, and the other terms are normal to the plane:

$$x\dot{y}_o - \dot{x}_o y = fh$$
$$f = \frac{x\dot{y}_o - \dot{x}_o y}{h}$$

Differentiating this last equation:

$$\dot{f} = \frac{\dot{x}\dot{y}_o - \dot{x}_o \dot{y}}{h}$$

f and g Series

Now cross the initial position vector into the position vector:

$$\vec{r}_o \times \vec{r} = \vec{r}_o \times (f\vec{r}_o + g\vec{v}_o)$$
$$= f(\vec{r}_o \times \vec{r}_o) + g(\vec{r}_o \times \vec{v}_o)$$

The first term is zero, and the other terms are normal to the plane:

$$x_o y - x y_o = gh$$
$$g = \frac{x_o y - x y_o}{h}$$

Differentiating this last equation:

$$\dot{g} = \frac{x_o \dot{y} - \dot{x} y_o}{h}$$

f and g Series

Look at the cross-product:

$$\vec{h} = \vec{r} \times \vec{v} = (f\vec{r}_o + g\vec{v}_o) \times (\dot{f}\vec{r}_o + \dot{g}\vec{v}_o)$$

$$= f\dot{f}(\vec{r}_o \times \vec{r}_o) + f\dot{g}(\vec{r}_o \times \vec{v}_o) + \dot{f}g(\vec{v}_o \times \vec{r}_o) + g\dot{g}(\vec{v}_o \times \vec{v}_o)$$

$$= f\dot{g}\vec{h} - \dot{f}g\vec{h}$$

Which can only be true if: $1 = f\dot{g} - \dot{f}g$

$$x = r\cos\nu \qquad y = r\sin\nu$$

$$\dot{x} = -\sqrt{\frac{\mu}{p}}\sin\nu \qquad \dot{y} = \sqrt{\frac{\mu}{p}}(e + \cos\nu)$$

f and g Series

Which gives:

$$f = 1 - \left(\frac{r}{p}\right)\left(1 - \cos\Delta\nu\right)$$

$$g = \frac{r r_o \sin\Delta\nu}{\sqrt{\mu p}}$$

$$\dot{f} = \sqrt{\frac{\mu}{p}}\tan\left(\frac{\Delta\nu}{2}\right)\left(\frac{1 - \cos\Delta\nu}{p} - \frac{1}{r} - \frac{1}{r_o}\right)$$

$$\dot{g} = 1 - \left(\frac{r_o}{p}\right)\left(1 - \cos\Delta\nu\right)$$

f and g Series

So, to summarize, given an initial position and velocity, we can calculate a future postion and velocity given the change in the true anomaly Δν:

$$\vec{r} = f\vec{r}_o + g\vec{v}_o$$

$$\vec{v} = \dot{f}\vec{r}_o + \dot{g}\vec{v}_o$$

$$\vec{r} \cdot \vec{r}_o = rr_o \cos\Delta\nu$$

$$f = 1 - \left(\frac{r}{p}\right)(1 - \cos\Delta\nu)$$

$$g = \frac{rr_o \sin\Delta\nu}{\sqrt{\mu p}}$$

$$\dot{f} = \sqrt{\frac{\mu}{p}}\tan\left(\frac{\Delta\nu}{2}\right)\left(\frac{1-\cos\Delta\nu}{p} - \frac{1}{r} - \frac{1}{r_o}\right)$$

$$\dot{g} = 1 - \left(\frac{r_o}{p}\right)(1 - \cos\Delta\nu)$$

Which you can test using Example 2-4 in the textbook.

f and g Series: State Transition Matrix

We can reexpress our f and g series representation:

$$\vec{r} = f\vec{r}_o + g\vec{v}_o$$

$$\vec{v} = \dot{f}\vec{r}_o + \dot{g}\vec{v}_o$$

in terms of a state-variable relationship:

$$\vec{X} = \begin{bmatrix} \vec{r} \\ \vec{v} \end{bmatrix} \qquad \vec{X}_o = \begin{bmatrix} \vec{r}_o \\ \vec{v}_o \end{bmatrix} \qquad \vec{X} = \Phi\vec{X}_o = \begin{bmatrix} F & G \\ \dot{F} & \dot{G} \end{bmatrix}\vec{X}_o$$

$$\text{where } F = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & f \end{bmatrix}, \text{ etc.,}$$

and Φ is the State Transition Matrix

f and g Series

- What uses do these functions have?
  - Given two states, find the time of flight between them.
  - Given two states, find an orbit that connects them.
    - Big fan of this application.
  - Using an iterative technique, such as Newton Raphson, can determine a future state given a current state and a transfer time or transfer angle.

ASEN 5050
SPACEFLIGHT DYNAMICS
Coordinate and Time Systems

Dr. Steve Nerem
University of Colorado - Boulder

Coordinate Systems

Celestial Sphere

- Celestial poles intersect Earth's rotation axis.
- Celestial equator extends Earth equator.
- Direction of objects measured with right ascension (α) and declination (δ).

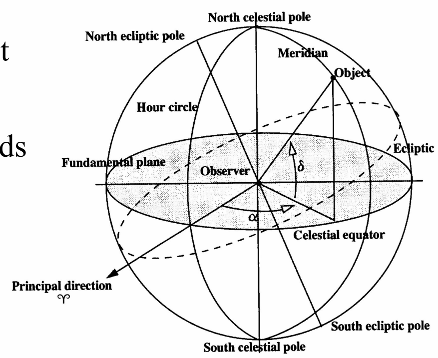


Figure 1-15. Geometry of the Celestial Sphere. The celestial sphere is based on an observer's perceived view of objects in space. A meridian, or hour circle, is any circle that passes through the observer at the center. I've shown the principal direction to help us find right ascension, α, and declination, δ.

(Vallado, 1997)

Coordinate Systems

The **Vernal Equinox** defines the reference direction.

The **ecliptic** is defined as the mean plane of the Earth's orbit about the Sun.

The angle between the Earth's mean equator and the ecliptic is called the **obliquity of the ecliptic**, ε~23.5°.

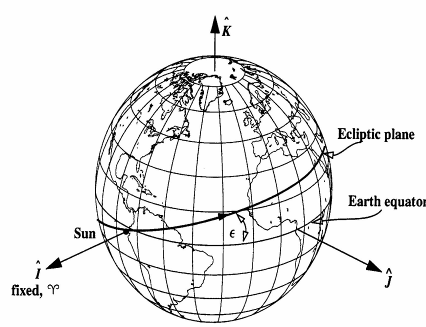


Figure 1-16. Geometry of the Vernal Equinox. The Earth's mean orbit around the Sun forms the ecliptic plane. The Earth's equatorial plane is inclined about 23.5° to the ecliptic. When the Sun is at the intersection of the two planes (has zero declination) and is at the ascending node, as viewed from Earth, it's the first day of spring.

(Vallado, 1997)