ASEN 5050
SPACEFLIGHT DYNAMICS
Two-Body Motion

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Announcements

- Homework #2 is due a week from today!
  - Either handed in or uploaded to D2L
  - Late policy is 10% per school day, where a "day" starts at 9:00 am.

Space News

Russian extends travel time to International Space Station

Posted August 26

MOSCOW — The Russian Federal Space Agency says the next manned trip to the International Space Station will be extended from the usual six hours to two days.

Roscosmos said in a statement released on Wednesday that the decision was made due to security concerns after the space lab had to adjust its orbit last month, dodging space junk.

The roll-out of a new Soyuz rocket in March 2013 allowed Russia to cut travel time to the orbiting lab from two days travel in cramped quarters to just six hours.

The next manned mission to the ISS is due to blast off from the Russia-leased launch pad in Kazakhstan on September 2, carrying a Russian, a Kazakh and a Dane.

Homework #1

- Darn orbital dynamics. The ISS is not cooperative at the moment. Here are the next passes (viewed from Boulder):

ISS - Visible Passes

Search period start: 03 September 2015 00:00
Search period end: 13 September 2015 00:00
Orbit: 400 x 403 km, 51.6° (Epoch: 01 September)

Passes to include: visible only all

Click on the date to get a star chart and other pass details.

Date	Brightness (mag)	Start Time	Start Alt.	Start Az.	Highest point Time	Highest point Alt.	Highest point Az.	End Time	End Alt.	End Az.	Pass type
06 Sep	-0.4	05:49:25	10°	SSE	05:50:34	12°	SE	05:51:50	10°	ESE	visible
08 Sep	-1.5	05:37:22	10°	SSW	05:40:10	27°	SE	05:43:00	10°	ENE	visible
09 Sep	-0.7	04:46:09	12°	SSE	04:47:03	14°	SE	04:48:51	10°	E	visible
10 Sep	-3.1	05:28:00	23°	SW	05:29:53	67°	SE	05:33:07	10°	ENE	visible
11 Sep	-1.7	04:37:08	30°	ESE	04:37:08	30°	ESE	04:39:39	10°	ENE	visible
11 Sep	-2.1	06:10:10	10°	W	06:13:06	30°	NNW	06:16:02	10°	NE	visible
12 Sep	-3.0	05:18:48	37°	W	05:19:42	51°	NW	05:22:55	10°	NE	visible

Homework #1

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		Time	Alt.	Az.	Time	Alt.	Az.	Time	Alt.	Az.	
03 Oct	-2.1	19:23:19	10°	SSW	19:26:03	25°	SE	19:28:03	15°	E	visible
03 Oct	-0.7	20:59:21	10°	W	21:00:41	20°	W	21:00:41	20°	W	visible
04 Oct	-3.0	20:05:48	10°	WSW	20:09:01	64°	NW	20:10:01	39°	NNE	visible
05 Oct	-3.3	19:12:33	10°	SW	19:15:46	63°	SE	19:19:00	10°	ENE	visible
05 Oct	-0.8	20:49:52	10°	WNW	20:51:50	20°	NW	20:51:50	20°	NW	visible
06 Oct	-1.7	19:56:02	10°	W	19:58:59	31°	NNW	20:00:54	17°	NNE	visible
07 Oct	-2.7	19:02:25	10°	WSW	19:05:37	54°	NNW	19:08:50	10°	NE	visible
07 Oct	-0.6	20:40:40	10°	NW	20:42:28	14°	NNW	20:42:28	14°	NNW	visible
08 Oct	-1.0	19:46:37	10°	WNW	19:49:04	19°	NNW	19:51:22	11°	NNE	visible
09 Oct	-1.6	18:52:43	10°	W	18:55:36	28°	NNW	18:58:28	10°	NE	visible
09 Oct	-0.4	20:31:35	10°	NNW	20:32:47	12°	N	20:32:47	12°	N	visible
10 Oct	-0.7	19:37:27	10°	NW	19:39:15	14°	NNW	19:41:01	10°	NNE	visible
11 Oct	-0.5	20:22:12	10°	NNW	20:22:56	11°	N	20:22:56	11°	N	visible
12 Oct	-0.6	19:28:22	10°	NNW	19:29:31	11°	N	19:30:39	10°	NNE	visible

Today's Lecture Topics

- Converting between the anomalies
- Then: More two-body orbital element computations

Our toolbox:
- Newton's law of gravitation
- Specific Energy
- Vis-Viva Equation

Most everything else comes straight from these.

Properties of Conic Sections

	v	ε	e	a
Ellipses	$< \sqrt{\frac{2\mu}{r}}$	< 0	$0 \le e < 1$	$a > 0$
Parabolas	$= \sqrt{\frac{2\mu}{r}}$	$= 0$	$e = 1$	$a = \infty$
Hyperbolas	$> \sqrt{\frac{2\mu}{r}}$	> 0	$e > 1$	$a < 0$

Since $\frac{h^2}{\mu} = a(1-e^2)$ is positive

Orbital Period

- Last time we proved Kepler's 2nd and 3rd laws and arrived at an expression for the orbital period of an ellipse:

$$P = 2\pi\sqrt{\frac{a^3}{\mu}} \qquad \text{units} = \sqrt{\frac{\mathrm{km}^3}{\mathrm{km}^3/\mathrm{s}^2}} = \text{seconds}$$

Mean Motion

- We also arrived at an expression for the *mean motion*:

$$n = \frac{2\pi \text{ radians}}{P \text{ sec}}$$

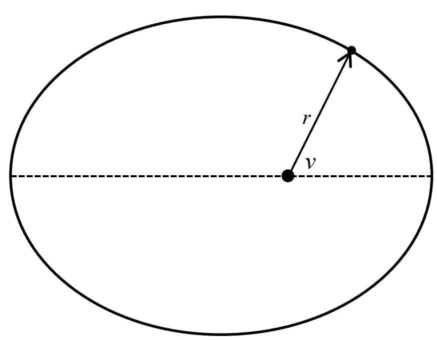
$$P = 2\pi\sqrt{\frac{a^3}{\mu}}$$

$$n = \frac{2\pi}{P} = \frac{2\pi}{2\pi\sqrt{\frac{a^3}{\mu}}}$$

$$= \sqrt{\frac{\mu}{a^3}} \text{ rad/s}$$

Where are we in an orbit?

- We have:
  - Our position in an orbit relative to Earth
  - The time
  - The true anomaly, ν
- We want to know:
  - How long it will take to get somewhere
  - The time profile of ν(t)
- We pose the answer by determining how much area is swept out in some amount of time.

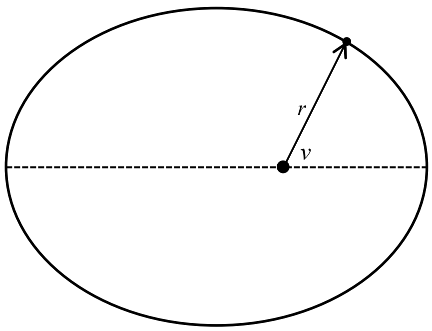


The Anomalies

- The true anomaly, ν
  - The actual, measured angle
  - Notice that this does not advance at a constant rate in an elliptical orbit
- Mean anomaly, *M*
  - An angle that *does* advance at a constant rate in an elliptical orbit.

$$M = n(t - t_p)$$

- Eccentric anomaly, *E*
  - An angle that helps translate from the true anomaly to the mean anomaly



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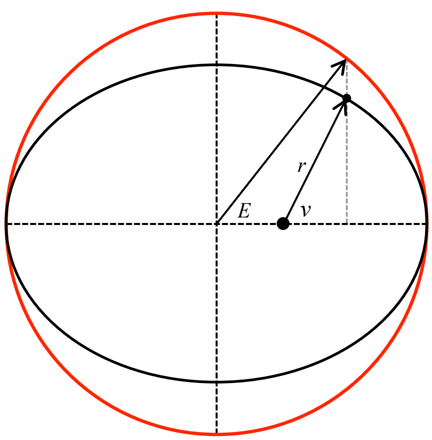
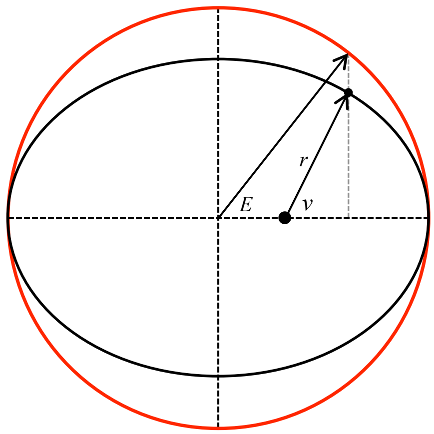
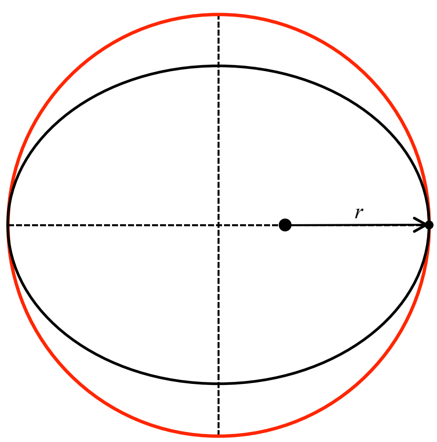


The Anomalies

- Some quick mental exercises:



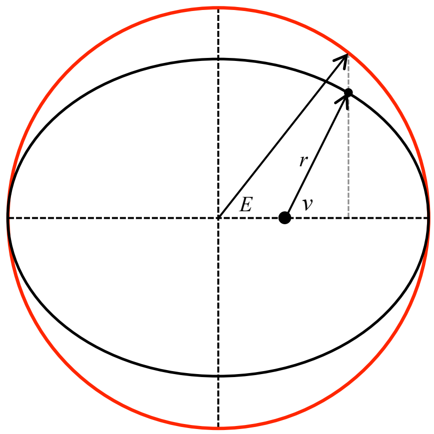
The Anomalies



Satellite is at periapse

True Anomaly	=	**0 deg**
Mean Anomaly	=	**0 deg**
Eccentric Anomaly	=	**0 deg**

The Anomalies



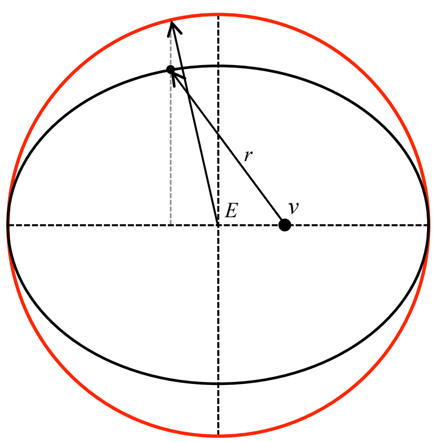
Satellite is at ν = 80 deg e = 0.3

True Anomaly = **80 deg**

Mean Anomaly = **48 deg**

Eccentric Anomaly = **63 deg**

The Anomalies



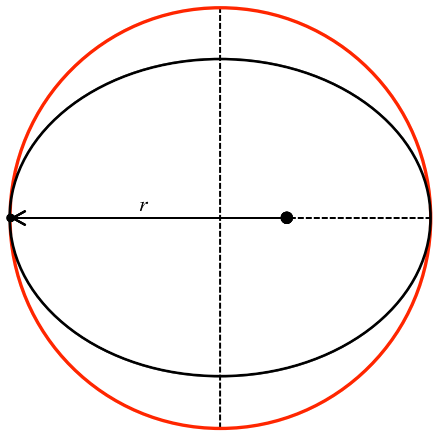
Satellite is at ν = 120 deg e = 0.3

True Anomaly = **120 deg**

Mean Anomaly = **87 deg**

Eccentric Anomaly = **104 deg**

The Anomalies



Satellite is at apoapse

True Anomaly = **180 deg**

Mean Anomaly = **180 deg**

Eccentric Anomaly = **180 deg**

The Anomalies

- The true anomaly advances quickly away from periapse
  - The mean anomaly advances steadily
  - The eccentric anomaly is in between
- At apoapse, they all catch up.
- Beyond apoapse, it's all in reverse (i.e., symmetric)
  - True anomaly advances the slowest away from apoapse
  - Mean anomaly advances steadily
  - Eccentric anomaly is in between

The Anomalies

Let's see some math!

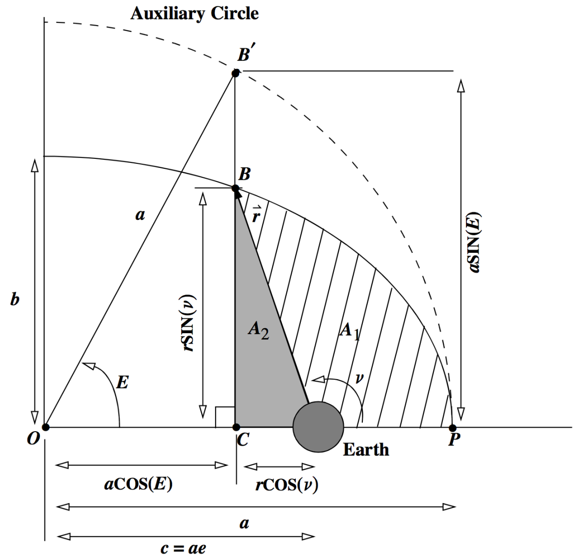


Figure 2-2. Geometry of Kepler's Equation. The eccentric anomaly uses an auxiliary circle as shown. The ultimate goal is to determine the area, A_1, which allows us to calculate the time. (Vallado, 2013)

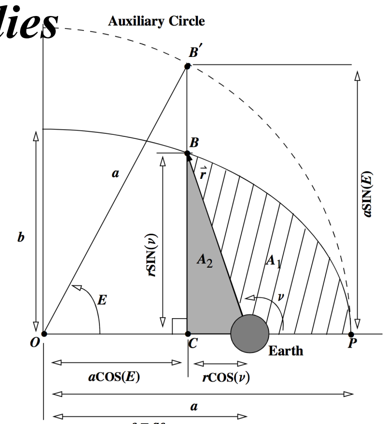
The Anomalies

True Anomaly

Mean Anomaly

Eccentric Anomaly

Note: $c = a - r_p$
$= a - a(1-e)$
$= ae$



For a coordinate system centered on Earth, write the location of the satellite in terms of E

$$X_{SAT} = a\cos E - ae$$

$$\text{Eq. of Ellipse} \quad \frac{X_{SAT}^2}{a^2} + \frac{Y_{SAT}^2}{b^2} = 1 \Rightarrow Y_{SAT} = b \sin E$$

Derivation of Kepler's Equation

Now,

$$
\begin{aligned}
r^2 &= X_{SAT}^2 + Y_{SAT}^2 \\
&= (a\cos E - ae)^2 + (\overset{a\sqrt{1-e^2}}{b}\sin E)^2 \\
&= a^2\left[\cos^2 E - 2e\cos E + e^2 + (1-e^2)\sin^2 E\right] \\
&= a^2\left[1 - 2e\cos E + e^2(1-\sin^2 E)\right] \\
&= a^2\left[1 - 2e\cos E + e^2\cos^2 E\right] \\
&= a^2\left(1 - e\cos E\right)^2 \quad \Rightarrow \quad \boxed{r = a(1 - e\cos E)}
\end{aligned}
$$

Now we will derive Kepler's Equation

Derivation of Kepler's Equation

Remember $r = \dfrac{p}{1 + e\cos\nu}$

$$\dot{r} = \frac{-p\left(-e\sin\nu\right)\ \dot{\nu}}{\left(1+e\cos\nu\right)^2} = \frac{p\left(e\sin\nu\right)\ \dot{\nu}r^2}{p^2} = \frac{he\sin\nu}{p}$$

But also $r = a\left(1 - e\cos E\right)$

$$\dot{r} = ae\sin E\dot{E}$$

So, $ae\dot{E}\sin E = \dfrac{he\sin\nu}{p}$

$$a\cancel{e}\dot{E}\cancel{\sin E} = \frac{h\cancel{e}}{p}\left(\frac{b}{r}\cancel{\sin E}\right)$$

Note $b\,sin\,E = r\,sin\nu$

$$sin\nu = \frac{b}{r}sin\,E$$

Derivation of Kepler's Equation

Thus, $$r\dot{E} = \frac{hb}{pa} = \frac{\sqrt{\mu a\left(1-e^2\right)}\,a\sqrt{1-e^2}}{a\left(1-e^2\right)a} \qquad \frac{h^2}{\mu} = p$$

$$r\dot{E} = \frac{\sqrt{\mu a}\,a\left(1-e^2\right)}{a^2\left(1-e^2\right)} = \sqrt{\mu}\,\frac{a^{3/2}}{a^{4/2}} = \sqrt{\frac{\mu}{a}}$$

Thus, $$a\left(1-e\cos E\right)\dot{E} = \sqrt{\frac{\mu}{a}}$$

$$\dot{E} - e\cos E\dot{E} = \frac{\mu^{1/2}}{a^{3/2}} = n$$

Integrating, $E - e\sin E = nt + \text{constant}$

Derivation of Kepler's Equation

If reference time is perigee, E = 0,

$$\cancelto{0}{E_{t_p}} - e\sin\cancelto{0}{E_{t_p}} = nt_p + \text{constant} \quad \Rightarrow \quad \text{constant} = -nt_p$$

$$E - e\sin E = \underbrace{n\left(t - t_p\right)}_{\substack{\textit{Angle swept out at the}\\ \textit{mean angular velocity,}\\ \textit{n, the mean motion}}} \equiv M \qquad \text{Kepler's Equation}$$

M = Mean Anomaly

$$M = n(t - t_p) = E - e\sin E$$

Kepler's Equation relates the Mean Anomaly to the Eccentric Anomaly

Derivation of Kepler's Equation

Can we relate either the Mean or Eccentric Anomaly to the True Anomaly?

Yes!

$$r = a(1 - e\cos E)$$

$$r = \frac{p}{1 + e\cos\nu}$$

Useful Formula:

$$r = a(1 - e\cos E) = \frac{p}{1 + e\cos\nu} = \frac{a(1 - e^2)}{1 + e\cos\nu}$$

Conversions

- Any issues with these relationships?

$$M = n(t - t_p) = E - e\sin E$$

$$r = a(1 - e\cos E) = \frac{p}{1 + e\cos\nu} = \frac{a(1 - e^2)}{1 + e\cos\nu}$$

- Quadrants! Two satellites with ν = 10 deg and ν = -10 deg will be impossible to distinguish.

Quadrant Checks

- True Anomaly → Eccentric Anomaly (Eq 2-9)

$$\mathrm{SIN}(E) = \frac{\mathrm{SIN}(\nu)\sqrt{1-e^2}}{1+e\,\mathrm{COS}(\nu)} \qquad \mathrm{COS}(E) = \frac{e+\mathrm{COS}(\nu)}{1+e\,\mathrm{COS}(\nu)}$$

- Eccentric Anomaly → True Anomaly

$$\mathrm{SIN}(\nu) = \frac{\mathrm{SIN}(E)\sqrt{1-e^2}}{1-e\,\mathrm{COS}(E)} \qquad \mathrm{COS}(\nu) = \frac{\mathrm{COS}(E)-e}{1-e\,\mathrm{COS}(E)}$$

(both ν and *E* are either 0 – 180 deg or 180 – 360 deg)

More Quadrant Checks

- True Anomaly → Eccentric Anomaly (Eq 2-14)

$$\mathrm{TAN}\left(\frac{E}{2}\right) = \sqrt{\frac{1-e}{1+e}}\,\mathrm{TAN}\left(\frac{\nu}{2}\right)$$

- Eccentric Anomaly → True Anomaly (Eq 2-13)

$$\mathrm{TAN}\left(\frac{\nu}{2}\right) = \sqrt{\frac{1+e}{1-e}}\,\mathrm{TAN}\left(\frac{E}{2}\right)$$

Derivation of Kepler's Equation

So, $$\cos\nu = \frac{\cos E - e}{1 + e\cos E} = \frac{a(\cos E - e)}{r}$$

$$\sin\nu = \sqrt{1-\cos^2\nu} = \frac{\sqrt{1-e^2}\,\sin E}{1+e\cos E} = \frac{a\sqrt{1-e^2}\,\sin E}{r}$$

$$\tan^2\frac{\nu}{2} = \frac{1-\cos\nu}{1+\cos\nu} = \frac{1-\dfrac{\cos E - e}{1-e\cos E}}{1+\dfrac{\cos E - e}{1-e\cos E}} = \frac{1-e\cos E-\cos E+e}{1-e\cos E+\cos E-e}$$

$$= \frac{(1+e)(1-\cos E)}{(1-e)(1+\cos E)} = \frac{1+e}{1-e}\tan^2\frac{E}{2}$$

$$\boxed{\tan\frac{\nu}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}}$$

If we have ν and need more:

- If ν is given:

$$r = \frac{p}{1+e\cos\nu}$$

$$\cos E = \frac{r\cos\nu + ae}{a}$$

$$\sin E = \frac{r\sin\nu}{b}$$

Solve $E\text{-}e\sin E = M$ for t (or M)

If we have t and need more:

- If t is given: $M = n(t - t_p)$

Solve $E - e\sin E = M$ for E

$$r = a(1 - e\cos E)$$

and $$\cos\nu = \frac{a\cos E - ae}{r}$$

$$\sin\nu = \frac{b}{r}\sin E$$

$$= \frac{a\sqrt{1-e^2}}{r}\sin E$$

Derivation of Kepler's Equation

If t is given:

$M = n(t - t_p)$

Solve $E - e\sin E = M$ for E

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Solve $E - e\sin E = M$ for t (or M)

Another Useful Relation: $$\boxed{\tan\frac{\nu}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}}$$

Solving Kepler's Equation

Want to use Newton-Raphson Iteration.

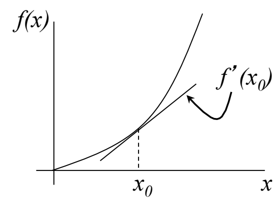
Assume we want to solve ***f(y) = 0*** for ***y***.

Assume $\boldsymbol{y = x + \delta}$, where $\boldsymbol{x}$ is an approximate guess, and $\boldsymbol{\delta}$ is a small correction. Expanding in a Taylor Series:

$$0 = f(y) = f(x+\delta) \approx f(x) + f'(x)\delta + \frac{f''(x)}{2!}\delta^2 + \ldots$$

Neglecting 2[nd] order terms and higher, we can solve for $\boldsymbol{\delta}$:

$$\delta = -\frac{f(x)}{f'(x)}$$



Solving Kepler's Equation

We need to iterate this since we have neglected higher order terms:

$$X_{n+1} = X_n + \delta_n = X_n - \frac{f(X_n)}{f'(X_n)}$$

Iterate until δ_n is acceptably small ($\delta = y-x$)

We want to solve $E - e\sin E = M \quad \Rightarrow \quad f(E) = M - E + e\sin E = 0$

$$f'(E) = -1 + e\cos E$$

So

$$E_{n+1} = E_n + \frac{M - E_n + e\sin E_n}{1 - e\cos E_n}$$

Careful with ***e = 1***, i.e., don't use for high ***e***.