ASEN 5050
SPACEFLIGHT DYNAMICS
Lecture 3: The Two-Body Motion

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Announcements

- Homework #1 is due now!
  - Either handed in or uploaded to D2L
  - Late policy is 10% per school day, where a "day" starts at 11:00 am.
- Homework #2 is due Thursday 9/10 at 11:00 am
- Read along in the book: Chapters 1 and 2

Space News

- SpaceX Falcon-9 return to flight?

Today's Lecture Topics

- Kepler's Laws
- Properties of conic orbits
- The Vis-Viva Equation! You will fall in love with this equation.
- Next time: Converting between the anomalies
- Then: More two-body orbital element computations

Kepler's 1st Law

"Conic Section" is the intersection of a plane with a cone. $m_{\oplus}$ is at the primary focus of the ellipse.

$$r = \frac{p}{1 + e\cos\nu} = \frac{a(1-e^2)}{1 + e\cos\nu}$$

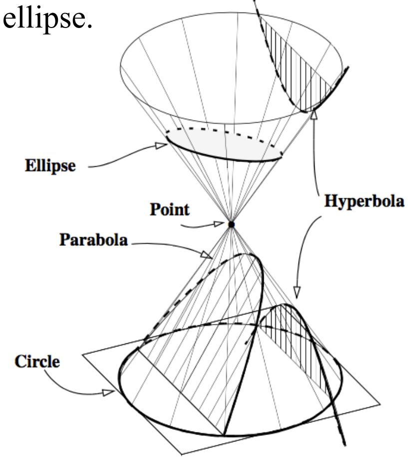
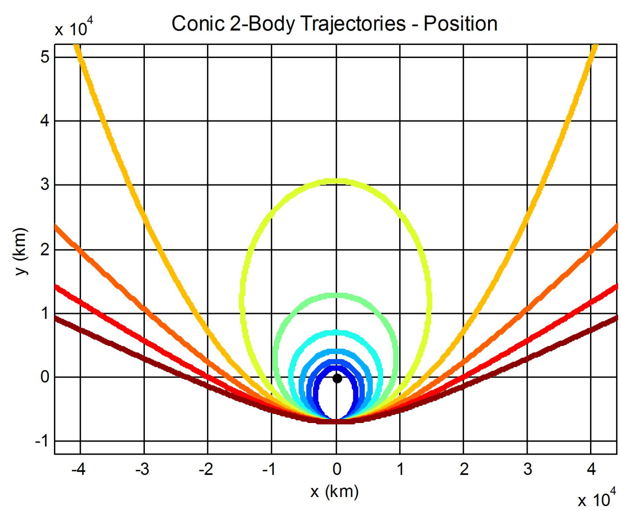


Figure 1-3. **Conic Sections.** Slicing a cone with a plane forms a conic section. When the plane is perpendicular to the axis of revolution, a circle results. Planes that are parallel to the axis of revolution yield hyperbolas (a pair as shown), and planes parallel to the outer surface yield parabolas. All other sections are ellipses, except for special cases in which the plane is *on* the surface (a line or rectilinear orbit) or only through the vertex (a point).

Conic Sections



Geometry of Conic Sections

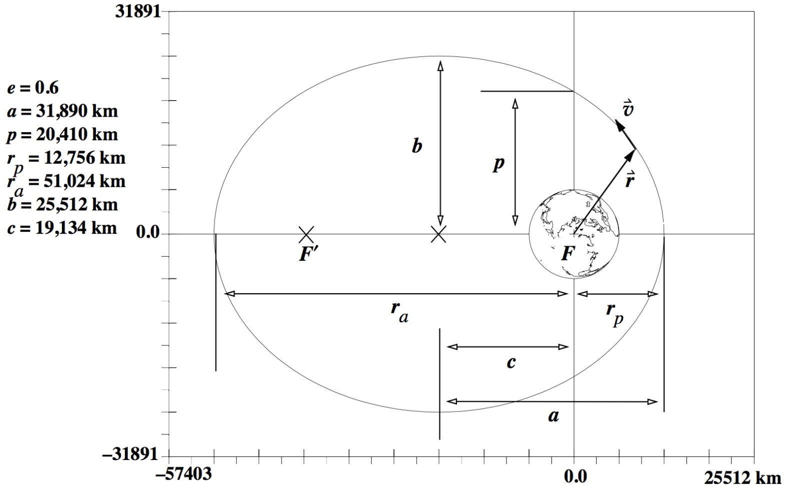


Figure 1-5. **Elliptical Orbits.** An elliptical orbit has two distinct foci with the primary focus, *F,* at the center of the Earth (or central body). The radius of apoapsis, r_a, and periapsis, r_p, denote the extreme points of the ellipse. The semimajor axis, a, and the semiminor axis, b, describe the shape of the orbit. Half the distance between the foci is c, and the semiparameter, p, locates the orbit distance normal from the semimajor axis at the focus. (Vallado, 2013)

Geometry of Conic Sections

Elliptical Orbits $0 < e < 1$

a = *semimajor axis*
b = *seminor axis*

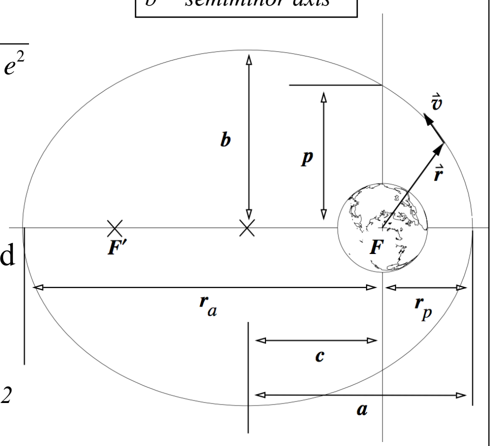
$$e = \frac{c}{a}, \qquad b = a\sqrt{1-e^2}$$

also, $$e = \frac{\sqrt{a^2 - b^2}}{a}$$

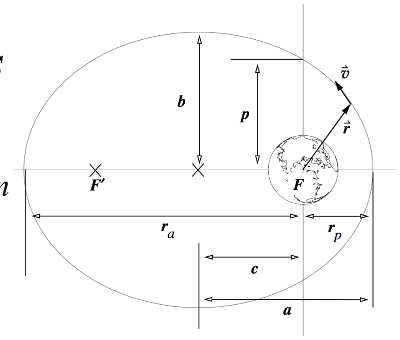
Sometimes flattening is also used

$$f = \frac{a-b}{a}$$

$$e^2 = 2f - f^2$$



Elliptic Orbits



p = *semiparameter* or *semilatus rectum*

$$p = \frac{b^2}{a} = a(1-e^2) = \frac{h^2}{\mu}$$

		Earth	*Sun*	*Moon*	
r_a	apoapsis →	apogee →	aphelion →	aposelenium →	etc.
r_p	periapsis →	perigee →	perihelion →	periselenium →	etc.

$$r_a = \frac{p}{1-e} = a(1+e)$$

$$r_p = \frac{p}{1+e} = a(1-e)$$

Geometry of Conic Sections

Elliptical Orbits $0 < e < 1$

$$r_p = a(1-e)$$
$$r_a = a(1+e)$$

Check: what's $r_a + r_p$ $\quad (a + ae) + (a - ae) = 2a$

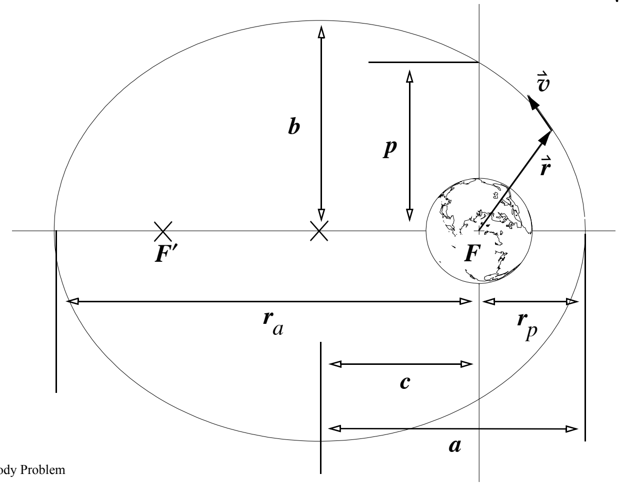
What is $r_a - r_p$ $\quad (a + ae) - (a - ae) = 2ae$

Hmmmm, so what is $\frac{r_a - r_p}{r_a + r_p}$ $\quad e = \frac{r_a - r_p}{r_a + r_p}$

Elliptic Orbits

- What is the velocity of a satellite at each point along an elliptic orbit?

$$\varepsilon = \frac{v^2}{2} - \frac{\mu}{r} \quad \Rightarrow \quad v = \sqrt{2\varepsilon + \frac{2\mu}{r}}$$



Elliptic Orbits

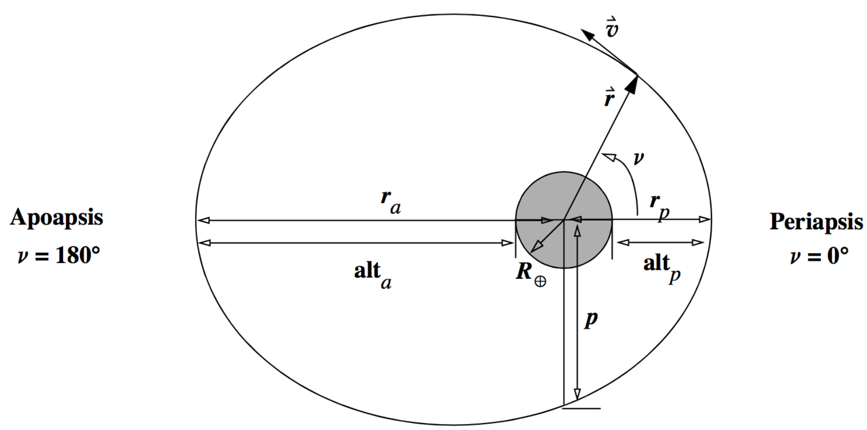


Figure 1-9. Radius of Apoapsis and Periapsis, as well as True Anomaly. Apoapsis occurs farthest from the planet; periapsis is closest. Notice the difference between altitudes and radius values. True anomaly, ν, locates the satellite's current position. We measure it in the direction of satellite motion from periapsis.

Geometry of Conic Sections

Parabolic Orbit

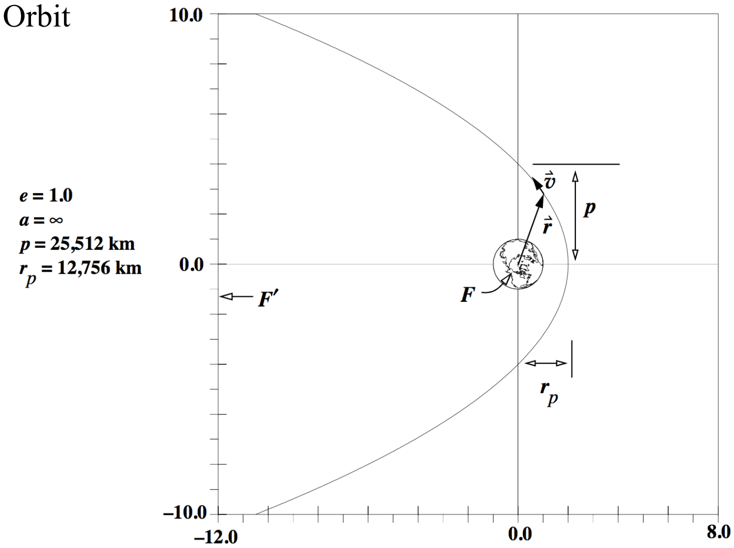


Figure 1-7. Parabolic Orbits. Parabolic orbits are considered open because they don't repeat. The semimajor axis is infinite. The second focus is at infinity.

Parabolic Orbit

$$e = 1, \qquad \varepsilon = 0, \qquad r = \frac{p}{1+\cos\nu}$$

$$e = \sqrt{1 + \frac{2\varepsilon h^2}{\mu^2}} = 1$$

$$\varepsilon = \frac{\mathrm{v}^2}{2} - \frac{\mu}{r} = 0 \qquad \Rightarrow \qquad \mathrm{v} = \sqrt{\frac{2\mu}{r}}$$

Note: As $\nu \to 180°$

$\mathrm{r} \to \infty$

$\mathrm{v} \to 0$

A parabolic orbit is a borderline case between an open hyperbolic orbit and a closed elliptic orbit

Geometry of Conic Sections

Hyperbolic Orbit

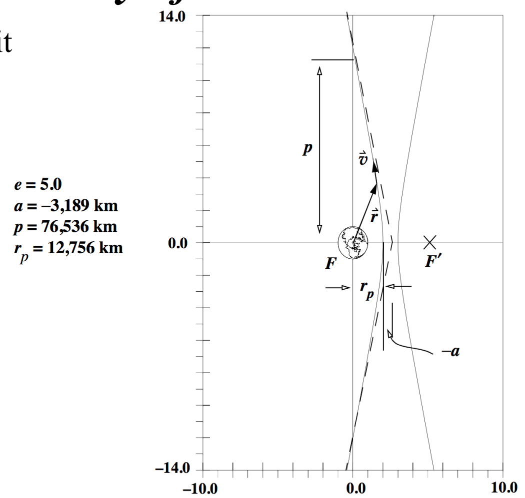


Figure 1-8. **Hyperbolic Orbits.** For a hyperbolic orbit, the semimajor axis is negative, and the eccentricity is larger than unity. Notice the rapid departure from the Earth with a *modest* eccentricity. The orbit approaches the dashed-line asymptotes. Hyperbolas occur in pairs called branches. The second branch is a mirror image of the first branch about the conjugate axis, shown as the dashed vertical line.

(Vallado, 2013)

Hyperbolic Orbit

$e > 1, \qquad \varepsilon > 0$

$$\varepsilon = \frac{\mathrm{v}^2}{2} - \frac{\mu}{r} \;, \qquad \mathrm{v}^2 = 2\varepsilon + \frac{2\mu}{r}$$

as $r \to \infty$, $\mathrm{v}^2 \to \; 2\varepsilon \; = \; \mathrm{v}_0^2 - \dfrac{2\mu}{r_0}$ for any other point on the hyperbola

or $\mathrm{v}_\infty^2 = 2\varepsilon$ Excess Velocity at $r = \infty$

and we will show that $\varepsilon = -\dfrac{\mu}{2a} \;\Rightarrow\; \mathrm{v}_\infty^2 = -\dfrac{\mu}{a} =$ hyperbolic excess velocity

Hyperbolic Orbit

- Interplanetary transfers use hyperbolic orbits everywhere
  - Launch
  - Gravity assists
  - Arrivals
  - Probes

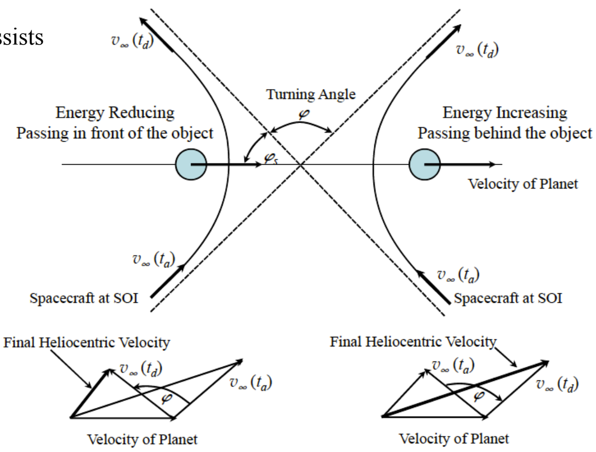


Figure 12-11. Gravity Assist Trajectories. An illustration of two gravity assists: one that increases a spacecraft's energy relative to the Sun (right) and one that decreases it (left). The spacecraft velocities are the same relative to the planet at both the arrival and departure from the SOI of the planet. The turning angle (φ) shows the change in direction after the flyby, relative to the assisting body.

(Vallado, 2013)

Properties of Conic Sections

	v	**ε**	**e**	**a**
Ellipses	$< \sqrt{\frac{2\mu}{r}}$	< 0	$0 \le e < 1$	$a > 0$
Parabolas	$= \sqrt{\frac{2\mu}{r}}$	$= 0$	$e = 1$	$a = \infty$
Hyperbolas	$> \sqrt{\frac{2\mu}{r}}$	> 0	$e > 1$	$a < 0$

Since $\frac{h^2}{\mu} = a(1-e^2)$ is positive

Flight Path Angle

This is also a good time to define the flight path angle, ϕ_{fpa}, as the angle from the local horizontal to the velocity vector.

- \+ from periapsis to apoapsis
- \- from apoapsis to periapsis
- 0 at periapsis and apoapsis

Only elliptic orbits ($h = r_a v_a = r_p v_p$)

Always 0 for circular orbits

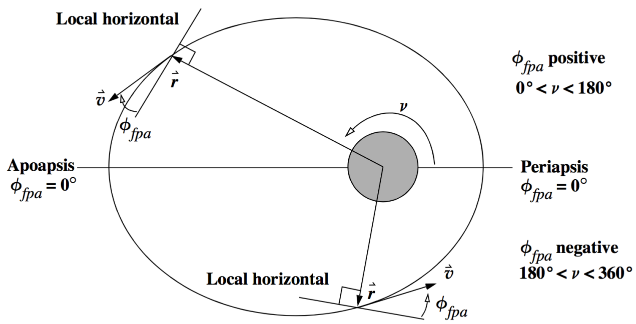


Figure 1-10. Geometry for the Flight-path Angle. The flight-path angle is always measured from the local horizontal to the velocity vector. It's always positive while the satellite travels from periapsis to apoapsis and negative for travel from apoapsis to periapsis. I've exaggerated the diagram for clarity.

Flight Path Angle

Another useful relationship is:

$$h = r\mathrm{v}\cos\phi_{fpa} \qquad \left(\text{recall: } \vec{h} = \vec{r} \times \vec{v}\right)$$

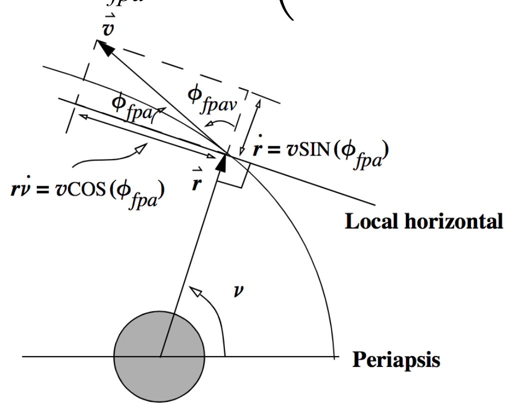


Figure 1-13. Geometry of the Flight-Path Angle. The flight-path angle, ϕ_{fpa}, is measured from the local horizontal *to* the velocity vector, which we can break into radial and transverse components. The vertical angle ϕ_{fpav} is sometimes used in derivations.

Specific Energy

Recall the energy equation: $\varepsilon = \dfrac{v^2}{2} - \dfrac{\mu}{r}$

Note at periapse $h = r_p v_p$ $r_p = a(1-e)$

$$p = \frac{h^2}{\mu} = a(1-e^2) \quad \Rightarrow \quad h = \sqrt{\mu a(1-e^2)}$$

$$\text{So, at periapse: } v_p^2 = \frac{h^2}{r_p^2} = \frac{\mu a(1-e^2)}{a^2(1-e)^2} = \frac{\mu(1+e)}{a(1-e)}$$

$$\text{Thus} \qquad \varepsilon = \frac{1}{2}\left[\frac{\mu(1+e)}{a(1-e)}\right] - \frac{\mu}{a(1-e)} = \frac{-\frac{1}{2}\mu(1-e)}{a(1-e)} = -\frac{\mu}{2a}$$

$$\boxed{\varepsilon = -\frac{\mu}{2a}}$$

Vis-Viva Equation

The energy equation: $\varepsilon = \dfrac{v^2}{2} - \dfrac{\mu}{r} = -\dfrac{\mu}{2a}$

Solving for v yields the Vis-Viva Equation!

$$\boxed{v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right) \qquad \textit{Vis-Viva Equation}}$$

or

$$\boxed{v = \sqrt{\frac{2\mu}{r} - \frac{\mu}{a}}}$$

	v	ε	e	a
Ellipses	$< \sqrt{\frac{2\mu}{r}}$	< 0	$0 \le e < 1$	$a > 0$
Parabolas	$= \sqrt{\frac{2\mu}{r}}$	$= 0$	$e = 1$	$a = \infty$
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Vis-Viva Equation

The energy equation: $\varepsilon = \frac{\mathrm{v}^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$

Solving for v yields the Vis-Viva Equation!

$$\boxed{\mathrm{v}^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right) \qquad \textit{Vis-Viva Equation}}$$

or $$\boxed{\mathrm{v} = \sqrt{\frac{2\mu}{r} - \frac{\mu}{a}}}$$

Additional derivables

$$\boxed{\mathrm{v}_c = \sqrt{\frac{\mu}{r}}}$$

$$\boxed{\mathrm{v} = \sqrt{\frac{2\mu}{r}} = \mathrm{v}_{escape}}$$

$$\boxed{\begin{array}{l}\mathrm{v}_p = \sqrt{\dfrac{\mu\,(1+e)}{a\,(1-e)}} \\ \mathrm{v}_a = \sqrt{\dfrac{\mu\,(1-e)}{a\,(1+e)}}\end{array}}$$

And any number of other things.

Proving Kepler's 2nd and 3rd Laws

$$dA = \int_0^r r dv dr = \frac{1}{2} r^2 dv$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{dv}{dt} = \frac{h}{2} \qquad (h = r^2 \dot{\theta})$$

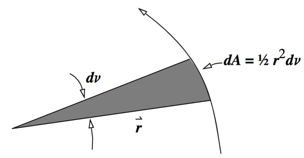

Figure 1-14. Area Swept Out by a Satellite. According to Kepler's second law, satellites sweep out equal areas in equal time intervals.

$$2\int dA = h \int_{t_1}^{t_2} dt$$

Proves 2nd Law

area of ellipse $= \pi ab$

$P = t_2 - t_1 =$ period

thus $P = \dfrac{2\pi ab}{h}$ $\qquad b^2 = a^2(1-e^2) = ap \;\Rightarrow\; b = \sqrt{ap}$

$$h = \sqrt{\mu a(1-e^2)}$$

$$P = \frac{2\pi a^{3/2} \cancel{p^{1/2}}}{\mu^{1/2} \cancel{\sqrt{a(1-e^2)}}} = \boxed{2\pi \sqrt{\frac{a^3}{\mu}} = P}$$

Expression of Kepler's 3rd Law $\left[\dfrac{P_1^2}{P_2^2} = \dfrac{a_1^3}{a_2^3} \right]$

Proving Kepler's 2nd and 3rd Laws

If we define the mean motion, $\boldsymbol{n}$, to be:

$$n = \frac{2\pi}{P} \quad \text{then} \quad n = \sqrt{\frac{\mu}{a^3}} \quad \text{or,}$$

$$\boxed{n^2 a^3 = \mu}$$

Mean angular rate of change of the object in orbit

Shuttle (300km)	*90 min*
Earth Obs (800 km)	*101 min*
GPS (20,000 km)	*~12 hrs*
GEO (36,000 km)	*~24 hrs*

	v	$\boldsymbol{\varepsilon}$	$\boldsymbol{e}$	$\boldsymbol{a}$
Ellipses	$< \sqrt{\frac{2\mu}{r}}$	< 0	$0 \le e < 1$	$a > 0$
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