STAT 303
Mathematical Statistics I

Term Project

Point Estimation of Gamma Distribution Parameters

January 21, 2026

1 Objective of the Project

The aim of this project is to:

- apply **point estimation methods** covered in class,
- derive and compute **Method of Moments (MoM)** and **Maximum Likelihood (MLE)** estimators,
- compare estimators using **finite-sample performance measures**,
- gain experience with **numerical likelihood optimization** and **simulation-based inference**.

This project focuses **only on point estimation** (no confidence intervals).

2 Statistical Model

Assume that the data $X_1, \ldots, X_n$ are independent and identically distributed with a **Gamma distribution**:

$$X_i \sim \text{Gamma}(k, \theta), \quad k > 0,\, \theta > 0,$$

with probability density function

$$f(x|k,\theta) = \frac{1}{\Gamma(k)\theta^k} x^{k-1} e^{-x/\theta}, \quad x > 0.$$

- k: shape parameter
- θ: scale parameter

Remark: The exponential distribution is the special case $k = 1$.

3 Data

You may use **either**:

1. a real dataset consisting of **positive observations** (e.g. waiting times, lifetimes, rainfall amounts, insurance claims), **or**
2. a simulated dataset generated from a Gamma distribution (true parameter values must be clearly stated).

Recommended sample size: $n \geq 30$.

4 Required Tasks

4.1 Descriptive Statistics

- Plot a histogram of the data.
- Compute and report the sample mean $\bar{X}$ and sample variance S^2.
- Briefly comment on skewness and suitability of the Gamma model.

4.2 Point Estimation

(A) Method of Moments (MoM)
For the Gamma distribution:

$$E(X) = k\theta, \quad \mathrm{Var}(X) = k\theta^2.$$

Using sample moments, derive the MoM estimators:

$$\hat{k}_{MM} = \frac{\bar{X}^2}{S^2}, \quad \hat{\theta}_{MM} = \frac{S^2}{\bar{X}}.$$

You must:

- show the derivation,
- compute $\hat{k}_{MM}$ and $\hat{\theta}_{MM}$ for your dataset.

(B) Maximum Likelihood Estimation (MLE)
The log-likelihood function is:

$$\ell(k, \theta) = (k-1)\sum_{i=1}^{n} \ln x_i - \frac{1}{\theta}\sum_{i=1}^{n} x_i - nk\ln\theta - n\ln\Gamma(k).$$

Show that:

$$\hat{\theta}_{MLE} = \frac{\bar{X}}{\hat{k}_{MLE}},$$

where $\hat{k}_{MLE}$ satisfies

$$\ln(\hat{k}) - \psi(\hat{k}) = \ln(\bar{X}) - \frac{1}{n}\sum_{i=1}^{n} \ln x_i,$$

and $\psi(\cdot)$ denotes the **digamma function**.
You must:

- explain why $\hat{k}_{MLE}$ has no closed-form solution,
- compute $\hat{k}_{MLE}$ numerically,
- compute $\hat{\theta}_{MLE}$.

4.3 Comparison on Observed Data

- Compare MoM and MLE numerically.
- Overlay fitted Gamma densities (MoM and MLE) on the histogram.
- Briefly interpret differences.

5 Simulation Study (Main Component)

Simulation Design
Choose **two parameter settings**, for example:

- Scenario 1 (high skewness): $k = 1, \theta = 2$
- Scenario 2 (moderate skewness): $k = 5, \theta = 1$

Sample sizes: $n \in \{20, 50, 100\}$.
Number of replications: $R = 2000$.

Performance Measures
For each estimator and each sample size, compute:

- **Bias**: $\mathrm{Bias}(\hat{\theta}) = E(\hat{\theta}) - \theta$,
- **Variance**,
- **Mean Squared Error (MSE)**.

Do this for both k and θ, and for both MoM and MLE.

Output

- Tables summarizing Bias, Variance, and MSE.
- At least one plot showing **MSE vs sample size**.

6 Discussion and Conclusions

Discuss:

- Which estimator performs better and why?
- Effect of skewness (small k).
- Effect of sample size.
- Practical recommendations.

7 Report Format

- **Length:** 8–10 pages (excluding code appendix)
- **Include:**
  - clear mathematical notation
  - well-labeled tables and figures
  - reproducible code in an appendix (R or Python).

8 Submission Instructions

Please strictly follow the submission guidelines below:

- The report must be prepared using **LaTeX** and submitted in **PDF format**.
- Please note that all reports will be uploaded to **Turnitin for originality and AI check.**
- The PDF report should be uploaded to the **"Term Project Reports"** on ODTÜClass.
- All codes (R, Python, etc.) in the appendix must be submitted as a separate file (in your preferred format) to the **"Term Project Codes"** on ODTÜClass.

9 Timeline

This project is designed to be completed in **6–7 days**. Due date is January 21, 2026 at 23:59.

Academic Integrity
All submitted work must be original. Use of software is encouraged, but **interpretation and explanation must be your own.**