Analysis of Stellar Structure from Newton to Einstein

Melih Kayra Kadıoğlu
Department of Physics, Koç University,
Rumelifeneri Yolu, 34450 Sariyer, Istanbul, Turkey
(Dated: January 12, 2026)

We investigate the structure of compact stars within both Newtonian gravity and general (Einsteinian) relativity. Using polytropic equations of state, we derive the Lane-Emden equation and recover the mass-radius scaling of white dwarfs, including the Chandrasekhar mass in the ultra-relativistic limit. We then solve the Tolman-Oppenheimer-Volkoff equations for a parametric neutron-star equation of state, compute mass-radius relations, baryonic and gravitational masses, binding energies, and stability criteria. Turning points and cusps in the solution sequences are shown to signal the onset of instability and determine the maximum allowed neutron-star mass. Finally, we constrain the stiffness parameter of the equation of state using the observational requirement of a $2.5\,M_\odot$ neutron star.

NEWTONIAN PHYSICS OF STELLAR STRUCTURE

We start from the Newtonian equations of hydrostatic equilibrium and mass conservation,

$$\frac{dm}{dr} = 4\pi r^2 \rho(r), \qquad \frac{dP}{dr} = -\frac{Gm(r)\rho(r)}{r^2}.$$

Since the functional form of the density is not known a priori, we follow the suggestion of the project and assume a polytropic equation of state (EOS),

$$P = K\rho^\gamma = K\rho^{1+\frac{1}{n}}.$$

We introduce the dimensionless Lane-Emden variables

$$\rho(r) = \rho_c\,\theta(\xi)^n, \qquad r = a\,\xi, \qquad P(r) = K\rho_c^{1+\frac{1}{n}}\,\theta(\xi)^{n+1},$$

where $\rho_c = \rho(0)$ and a is a constant to be determined.

Differentiating the pressure gives

$$\frac{dP}{dr} = K\rho_c^{1+\frac{1}{n}}(n+1)\theta^n \frac{d\theta}{dr}.$$

Substituting into the hydrostatic equilibrium equation yields

$$K\rho_c^{1+\frac{1}{n}}(n+1)\theta^n \frac{d\theta}{dr} = -\frac{Gm\rho_c\theta^n}{r^2}.$$

Canceling the common factor $\rho_c\theta^n$ leads to

$$K(n+1)\rho_c^{\frac{1}{n}}\frac{d\theta}{dr} = -\frac{Gm}{r^2}, \quad \Rightarrow m(r) = -\frac{K(n+1)\rho_c^{1/n}}{G}\, r^2\frac{d\theta}{dr}.$$

Differentiating and using $\frac{dm}{dr} = 4\pi r^2\rho_c\theta^n$, we obtain

$$-\frac{K(n+1)\rho_c^{1/n}}{G}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) = 4\pi r^2\rho_c\theta^n.$$

Dividing by r^2 gives

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) = -\frac{4\pi G}{K(n+1)}\,\rho_c^{1-\frac{1}{n}}\,\theta^n.$$

Changing variables to ξ via $r = a\xi$,

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) = \frac{1}{a^2}\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right).$$

Choosing

$$a^2 = \frac{(n+1)K}{4\pi G}\,\rho_c^{\frac{1}{n}-1},$$

the Lane-Emden equation follows:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \theta^n = 0.$$

Regularity at the center implies

$$\theta(0) = 1, \qquad \theta'(0) = 0.$$

To determine the series behavior near the center, we assume an even-power expansion

$$\theta(\xi) = 1 + a\xi^2 + b\xi^4 + \mathcal{O}(\xi^6).$$

Then

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\theta'\right) = 6a + 20b\xi^2 + \mathcal{O}(\xi^4),$$

and

$$\theta^n = \left(1 + a\xi^2 + b\xi^4 + \cdots\right)^n = 1 + na\xi^2 + \mathcal{O}(\xi^4).$$

Substituting into the Lane-Emden equation and matching coefficients yields

$$6a + 1 = 0 \Rightarrow a = -\frac{1}{6}, \qquad 20b + na = 0 \Rightarrow b = \frac{n}{120}.$$

Hence,

$$\theta(\xi) = 1 - \frac{1}{6}\xi^2 + \frac{n}{120}\xi^4 + \cdots,$$

which satisfies $\theta'(0) = 0$.

For the special case $n = 1$, the Lane–Emden equation reduces to

$$\theta'' + \frac{2}{\xi}\theta' + \theta = 0,$$

whose regular solution is

$$\theta(\xi) = \frac{\sin\xi}{\xi}.$$

The first zero occurs at $\xi_1 = \pi$, so the stellar surface is at $\theta(\pi) = 0$.

Using the Lane-Emden substitutions,

$$\rho(r) = \rho_c\,\theta(\xi)^n, \qquad r = a\,\xi, \qquad R = a\,\xi_n, \qquad \theta(\xi_n) = 0,$$

the enclosed mass becomes

$$m(\xi) = 4\pi\rho_c a^3 \int_0^{\xi} \xi'^2 \theta(\xi')^n \, d\xi',$$

and the total mass is

$$M = 4\pi\rho_c a^3 \int_0^{\xi_n} \xi^2 \theta(\xi)^n \, d\xi.$$

From the Lane-Emden equation,

$$\frac{d}{d\xi}\left(\xi^2\theta'\right) = -\xi^2\theta^n,$$

so integration gives

$$\int_0^{\xi_n} \xi^2\theta^n \, d\xi = -\xi_n^2\theta'(\xi_n).$$

Therefore,

$$M = 4\pi\rho_c a^3\left[-\xi_n^2\theta'(\xi_n)\right] = 4\pi\rho_c R^3\left(-\frac{\theta'(\xi_n)}{\xi_n}\right).$$

Defining the Lane-Emden constant

$$\omega_n \equiv -\xi_n^2\theta'(\xi_n) > 0,$$

and eliminating ρ_c, one finds for $n \neq 1$ the mass–radius scaling

$$M \propto R^{\frac{3-n}{1-n}}.$$

An explicit expression is

$$M = 4\pi\,\omega_n\,\xi_n^{-\frac{3-n}{1-n}}\left(\frac{(n+1)K}{4\pi G}\right)^{\frac{n}{n-1}} R^{\frac{3-n}{1-n}}.$$

To relate the theoretical polytropic model to observational white-dwarf data, we use the surface gravity

$$g = \frac{GM}{R^2},$$

from which the radius follows as

$$R = \sqrt{\frac{GM}{g}}.$$

Since the data provide $\log_{10} g$ in CGS units,

$$g = 10^{\log g}\ [\mathrm{cm\,s^{-2}}].$$

Writing the mass in solar units,

$$M = \left(\frac{M}{M_\odot}\right) M_\odot,$$

we obtain in CGS

$$R = \sqrt{\frac{G_{\mathrm{cgs}}(M/M_\odot)M_\odot}{10^{\log g}}}.$$

Converting to Earth radii gives

$$\frac{R}{R_\oplus} = \frac{1}{R_\oplus}\sqrt{\frac{G_{\mathrm{cgs}}M_\odot}{10^{\log g}}\,M_{\mathrm{WD}}},$$

where M_{WD} is the mass in solar units. The constants are

$$G_{\mathrm{cgs}} = 6.67430\times10^{-8}\ \mathrm{cm^3\,g^{-1}\,s^{-2}}, \qquad M_\odot = 1.98847\times10^{33}\ \mathrm{g},$$

$$R_\oplus = 6.371\times10^{8}\ \mathrm{cm}.$$

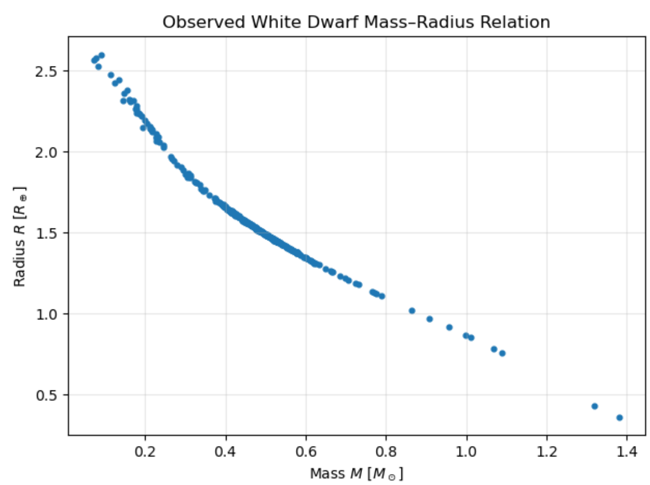

FIG. 1.

We now obtain the polytropic form of the cold white-dwarf equation of state by expanding the degenerate-electron EOS at small x. Assume the standard parametric form

$$\rho = D\,x^q \qquad \Rightarrow \qquad x = \left(\frac{\rho}{D}\right)^{1/q},$$

and

$$P(x) = C\,F(x).$$

For $x \ll 1$ the leading term is

$$F(x) = \frac{8}{5}x^5 + \mathcal{O}(x^7),$$

so that

$$P \simeq \frac{8C}{5}x^5 = \frac{8C}{5D^{5/q}}\rho^{5/q}.$$

Comparing with the polytropic form

$$P = K_*\,\rho^{1+\frac{1}{n_*}},$$

we identify

$$1 + \frac{1}{n_*} = \frac{5}{q}, \qquad n_* = \frac{q}{5-q}, \qquad K_* = \frac{8C}{5D^{5/q}}.$$

For a polytrope $P = K\rho^{1+1/n}$, the Lane–Emden solution gives

$$R = a\,\xi_1, \qquad M = 4\pi a^3 \rho_c\,\omega_n, \qquad a^2 = \frac{(n+1)K}{4\pi G}\rho_c^{\frac{1}{n}-1},$$

with $\omega_n = -\xi_1^2\theta'(\xi_1)$. Eliminating ρ_c yields the mass–radius relation

$$R = \mathcal{A}(n)\,K^{\frac{n}{3-n}}\,M^{\frac{1-n}{3-n}},$$

where

$$\mathcal{A}(n) = \xi_1(4\pi\omega_n)^{-\frac{1-n}{3-n}}\left(\frac{n+1}{4\pi G}\right)^{\frac{n}{3-n}}.$$

The corresponding central density is

$$\rho_c = \left[\frac{M}{4\pi\omega_n}\left(\frac{4\pi G}{(n+1)K}\right)^{3/2}\right]^{\frac{2n}{3-n}}.$$

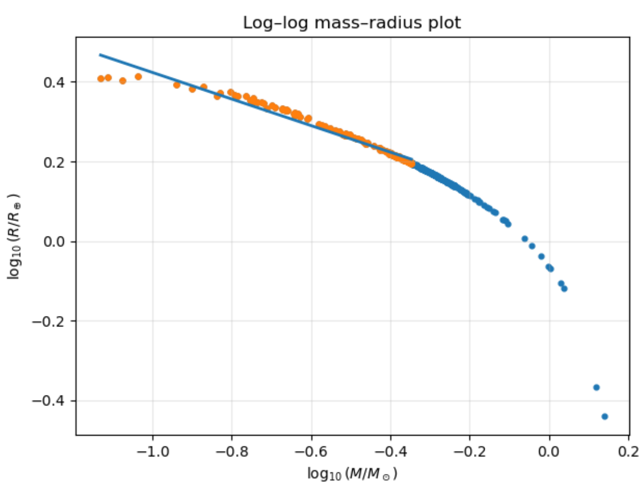

FIG. 2.

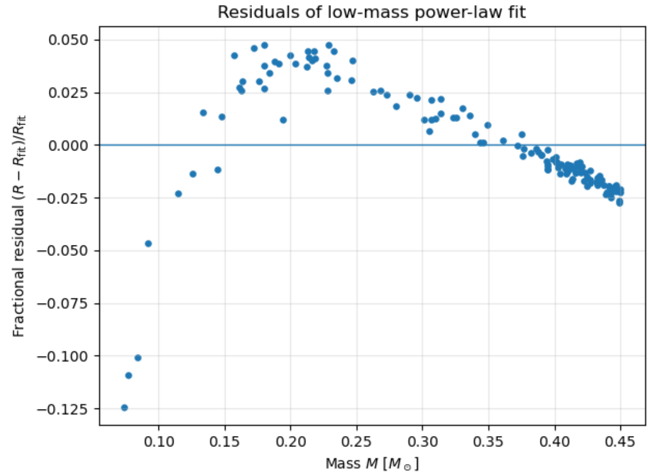

FIG. 3.

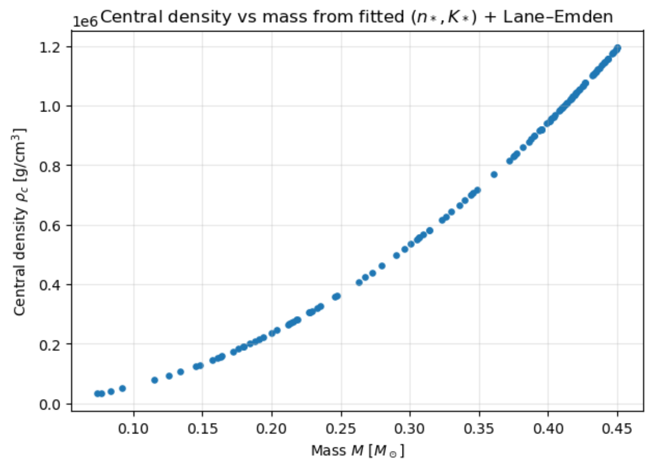

FIG. 4.

In Fig. 2, the low-mass data lie approximately on a straight line in log–log scale, confirming the polytropic power-law behavior $R \propto M^\beta$. Deviations at higher masses signal the breakdown of the small-x approximation. The residuals in Fig. 3 are small in the fitted range, and the central density in Fig. 4 increases monotonically with mass, as expected for gravitational compression.

Using the fitted q and K_*, the EOS constants satisfy

$$K_* = \frac{8C}{5D^{5/q}} \qquad \Rightarrow \qquad C = \frac{5K_*}{8}D^{5/q}.$$

For a trial value of D, the constant C is fixed and the full Newtonian structure equations

$$\frac{dm}{dr} = 4\pi r^2\rho, \qquad \frac{dP}{dr} = -\frac{Gm\rho}{r^2}, \qquad P = P(\rho; C, D)$$

can be solved and compared to the observed mass–radius data.

Theoretical values for $\mu_e = 2$ are

$$C_{\rm th} = \frac{m_e^4 c^5}{24\pi^2\hbar^3}, \qquad D_{\rm th} = \frac{m_u m_e^3 c^3 \mu_e}{3\pi^2\hbar^3}.$$

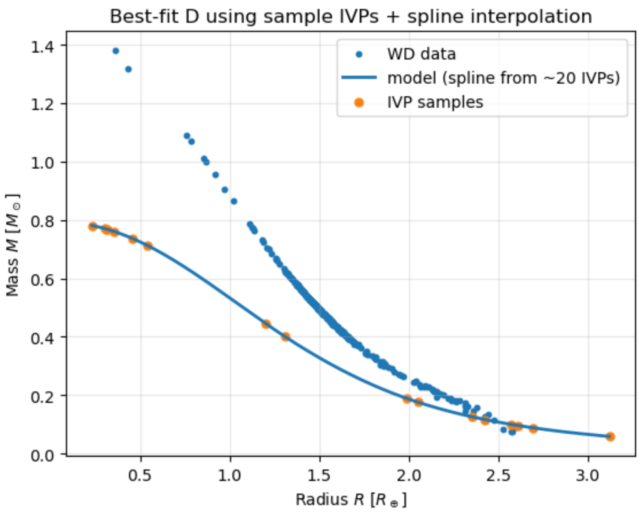

FIG. 5.

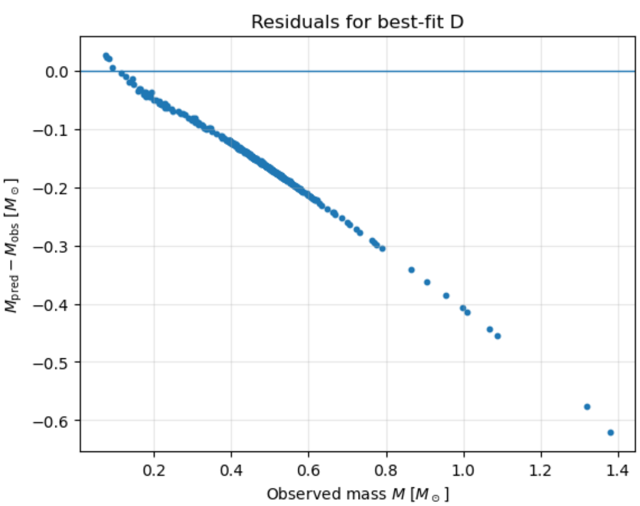

FIG. 6.

In the ultra-relativistic limit $x \gg 1$ the pressure function is

$$F(x) = x(2x^2 - 3)\sqrt{1+x^2} + 3\,\mathrm{asinh}(x),$$

whose asymptotic expansion is

$$F(x) = 2x^4 - 2x^2 + 3\ln(2x) + \cdots.$$

Thus

$$P \simeq 2C\,x^4, \qquad \rho = Dx^3,$$

so that

$$P \simeq \frac{2C}{D^{4/3}}\rho^{4/3}.$$

This corresponds to a polytrope with $n = 3$ and

$$K_{\rm rel} = \frac{2C}{D^{4/3}}.$$

For $n = 3$, the mass becomes independent of the central density:

$$M_{\rm Ch} = 4\pi\omega_3 \left(\frac{K_{\rm rel}}{\pi G}\right)^{3/2} = \frac{4\omega_3}{\sqrt{\pi}} \left(\frac{K_{\rm rel}}{G}\right)^{3/2}.$$

Using the exact constants

$$C = \frac{m_e^4 c^5}{24\pi^2\hbar^3}, \qquad D = \frac{m_u m_e^3 c^3 \mu_e}{3\pi^2\hbar^3},$$

one obtains the Chandrasekhar mass

$$M_{\rm Ch} = \frac{\omega_3}{2}\sqrt{3\pi}\,\frac{(\hbar c/G)^{3/2}}{m_u^2\mu_e^2}.$$

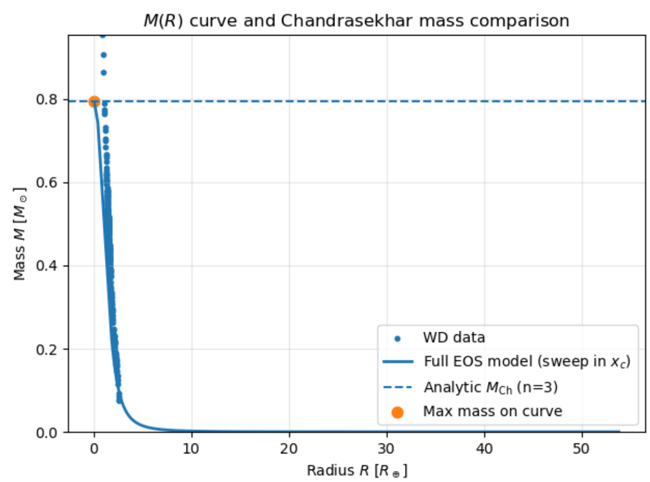

FIG. 7.

EINSTEINIAN PHYSICS OF STELLAR STRUCTURE

We use the parametric polytropic equation of state (EOS)

$$p = K\,\rho_r^\Gamma,$$
$$\rho = \rho_r + \frac{K}{\Gamma - 1}\,\rho_r^\Gamma,$$

where ρ_r is the rest-mass (baryon) density and ρ is the total energy density. The project specifies $\Gamma = 1.3569$ and fixes K by giving a reference pressure at a reference value of ρ_r.

In general relativity, hydrostatic equilibrium is governed by the Tolman–Oppenheimer–Volkoff (TOV) equations:

$$\frac{dm}{dr} = 4\pi r^2 \rho,$$
$$\frac{d\nu}{dr} = 2\,\frac{m + 4\pi r^3 p}{r(r-2m)},$$
$$\frac{dp}{dr} = -(\rho + p)\,\frac{m + 4\pi r^3 p}{r(r-2m)} = -\frac{1}{2}(\rho + p)\frac{d\nu}{dr}.$$

We integrate from the center $r = 0$ to the stellar surface $r = R$ defined by

$$p(R) = 0 \quad \Longleftrightarrow \quad \rho(R) = 0,$$

since the EOS implies $p = 0 \Rightarrow \rho_r = 0 \Rightarrow \rho = 0$.

The regular central boundary conditions are

$$m(0) = 0, \qquad p(0) = p_c.$$

The metric potential ν is defined only up to an additive constant, so we may choose

$$\nu(0) = 0$$

without loss of generality, and later shift it so that $\nu(\infty) = 0$.

We adopt the project's geometric-unit scaling,

$$M_\odot \text{ as mass unit}, \qquad r_0 = \frac{GM_\odot}{c^2} \text{ as length unit},$$

so that

$$M = m\,M_\odot, \qquad R = r\,r_0.$$

When plotting, we convert R to kilometers via $R_{\rm km} = r\,r_0/10^3$ and keep M in units of $M_\odot$.

To obtain the mass–radius relation, we sweep over central pressures (or central $\rho_{r,c}$) and integrate the TOV system outward until the event $p(r) = 0$ is reached. Each integration yields

$$M(p_c) = m(R), \qquad R(p_c) = R.$$

Collecting these points produces the neutron-star M–R curve shown in Fig. 8.

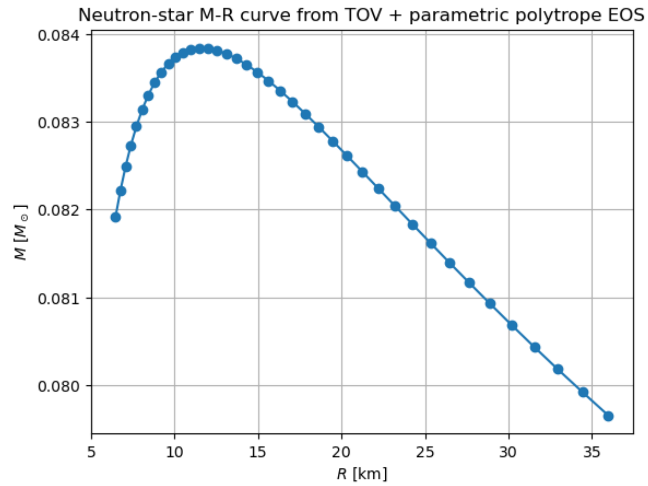

FIG. 8.

The gravitational mass $m(r)$ is not equal to the sum of the rest masses of the particles, because in general relativity the total mass-energy includes internal energy and (negative) gravitational binding energy. The total rest mass of the particles defines the *baryonic mass* $m_P(r)$, which satisfies

$$\frac{dm_P}{dr} = 4\pi\left(1 - \frac{2m}{r}\right)^{-1/2} r^2 \rho_r.$$

This equation is integrated together with the TOV system, with central condition

$$m_P(0) = 0,$$

and the same surface condition $p(R) = 0$. The total gravitational and baryonic masses are

$$M = m(R), \qquad M_P = m_P(R).$$

The fractional binding energy is defined by

$$\Delta \equiv \frac{M_P - M}{M}.$$

From the resulting sequences we plot $\Delta(R)$ and $M(M_P)$. A cusp in these curves signals an extremum along the one-parameter family labeled by p_c.

A cusp in the M–R or M–M_P relation corresponds to a turning point, where

$$\frac{dM}{dp_c} = 0, \qquad \frac{dM_P}{dp_c} = 0.$$

Beyond this point, both masses decrease as p_c increases, indicating the onset of instability. Thus the cusp marks the maximum gravitational mass $M_{\rm max}$ and the corresponding maximum baryonic mass $M_{P,{\rm max}}$.

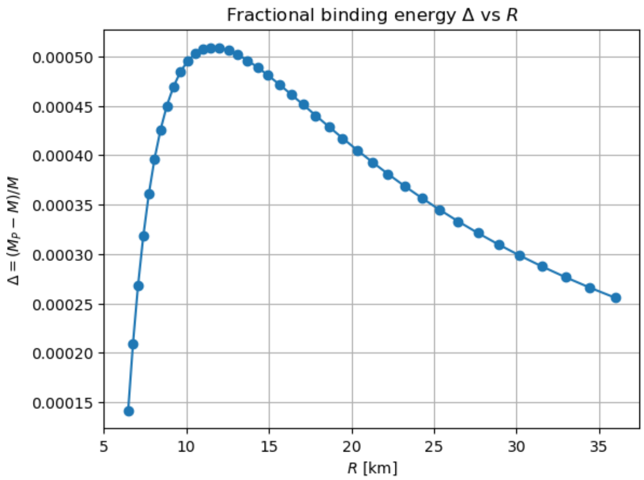

FIG. 9.

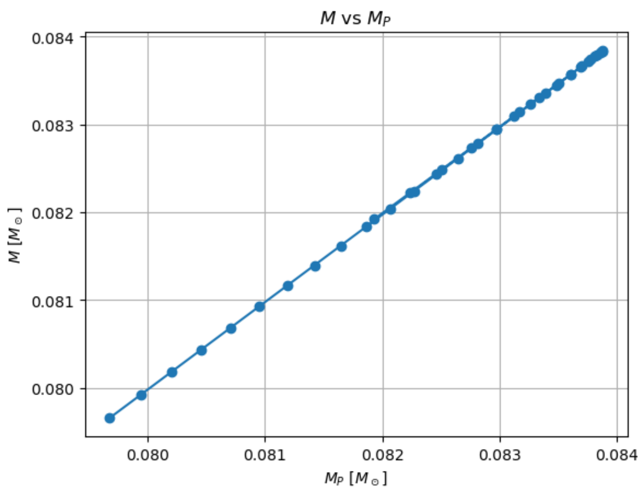

FIG. 10.

Defining the central total density $\rho_c = \rho(0)$, a simple turning-point stability criterion is

$$\frac{dM}{d\rho_c} > 0 \;\Rightarrow\; \text{stable}, \qquad \frac{dM}{d\rho_c} < 0 \;\Rightarrow\; \text{unstable}.$$

The maximum allowed mass occurs at

$$\frac{dM}{d\rho_c} = 0,$$

which coincides with the cusp of the $M(\rho_c)$ curve and separates the stable and unstable branches.

From the numerical sequence we find

$$M_{\max} \simeq 0.08384\, M_\odot,$$

at

$$\rho_c \simeq 1.11 \times 10^{18}\ \mathrm{kg\,m^{-3}}, \qquad R \simeq 11.49\ \mathrm{km}.$$

Using M_P instead of M yields the same qualitative behavior, because both are derived from the same one-parameter family of equilibrium solutions; the cusp in the M–M_P curve shows that their extrema occur at the same turning point.

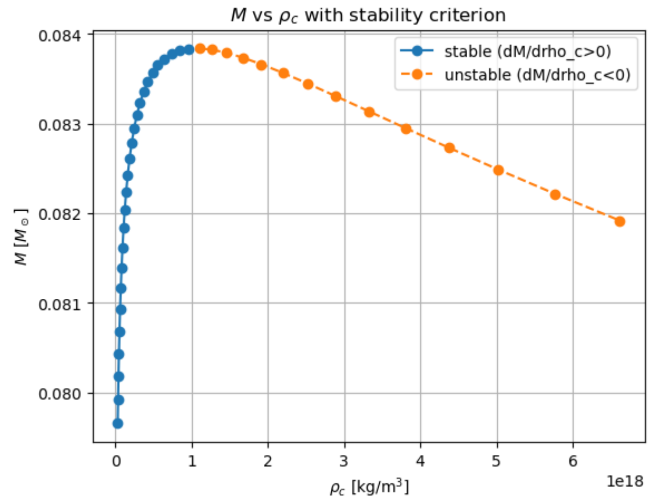

FIG. 11.

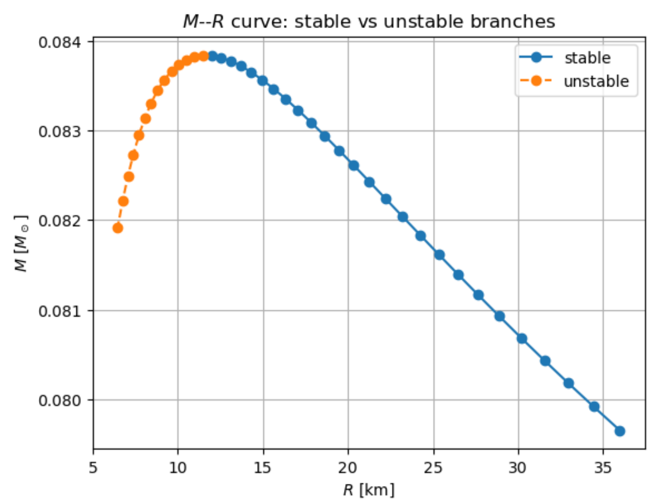

FIG. 12.

Keeping Γ fixed, varying K changes the stiffness of the EOS. Larger K implies higher pressure at a given density and therefore a larger maximum supported mass. For each K we compute

$$M_{\max}(K) = \max_{\rho_c} M(\rho_c),$$

and plot $M_{\max}$ as a function of K.

The observation of a neutron star with $M \simeq 2.5\, M_\odot$ requires

$$M_{\max}(K) \geq 2.5\, M_\odot.$$

From the numerical curve we obtain the lower bound

$$K \geq K_{\rm min} \simeq 3.85 \times 10^9 \text{ (SI)}, \qquad \frac{K}{K_0} \geq 12.59.$$

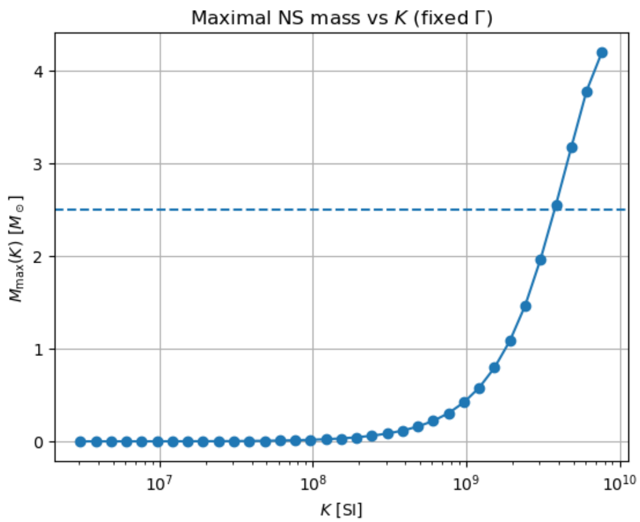

FIG. 13.

Outside the star ($r > R$), where $p = 0$ and $m(r) = M$ is constant, the metric equation reduces to

$$\frac{d\nu}{dr} = \frac{2M}{r(r-2M)}.$$

Integrating from R to r gives

$$\begin{aligned}\nu(r) - \nu(R) &= \int_R^r \frac{2M}{\tilde{r}(\tilde{r}-2M)}\, d\tilde{r} \\ &= \ln\left(\frac{r-2M}{r}\right) - \ln\left(\frac{R-2M}{R}\right).\end{aligned}$$

Since $\ln[(r-2M)/r] = \ln(1-2M/r)$, the exterior solution is

$$\nu(r > R) = \ln\left(1 - \frac{2M}{r}\right) - \ln\left(1 - \frac{2M}{R}\right) + \nu(R).$$

CONCLUSION

Starting from Newtonian polytropes and extending to full general relativity, we have shown how the equation of state determines the global structure, stability, and maximum mass of compact stars. The appearance of turning points and cusps in the equilibrium sequences provides a clear and universal criterion for the onset of instability and for the existence of a maximal stellar mass, linking microscopic physics of dense matter to macroscopic observables.

Citations and References

[1] S. Chandrasekhar, *An Introduction to the Study of Stellar Structure* (The University of Chicago Press, Chicago, 1939).