

CPSC 202 PSET 8

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A.8.1 Minimal and maximal elements

1. Proof: There exists a nonempty $R \subseteq \mathcal{P}(\mathbb{N})$ with no maximal elements.
 - Let's say A_{max} is our maximal element in our infinite set, where max is any number which designates the maximal power set.
 - However, if we add 1 to max, we achieve a new maximal. Because we can continuously add one, there is no maximal.
2. Proof: There exists a nonempty $S \subseteq \mathcal{P}(\mathbb{N})$ with no minimal elements.
 - Similar to our first proof, we can assign A_{min} to be our minimal element. Then we can subtract 1 from A_{min} for A_{min-1} . A_{min} is no longer the minimal in the case. Because we can keep subtracting one, there is no minimal element.
3. Proof: There exists a nonempty $T \subseteq \mathcal{P}(\mathbb{N})$ with no minimal nor maximal elements.

If we consider the set of even natural numbers, we can take the union of that set with the odd natural numbers. There is no maximal.

Considering the set of even natural numbers, we take the set difference of $A_{min} = 0$, do this continuously for each next $A_{min} = 2, 4, \dots$, proving that there is no minimum as we can remove these infinitely.

A.8.2 No trailing zeros

$$x \sim y \iff x = 2^k y \vee y = 2^k x \quad k \in \mathbb{N}$$

1. Show that \sim is an equivalence relation:
 - a. It's reflexive: $x \sim x \iff x = 2^k x \vee x = 2^k x, k \in \mathbb{N}$

If $k = 0$, then, $x = 2^0 x = x = 1 * x$. Therefore, $x = x$ and \sim is reflexive.
 - b. It's symmetric: $x \sim y \rightarrow y \sim x$

$x \sim y \iff x = 2^k y \vee y = 2^k x, k \in \mathbb{N}$
 $y \sim x \iff y = 2^k x \vee x = 2^k y, k \in \mathbb{N}$ Assume $x \sim y$:

- Case 1: $x = 2^k y$
Plug x into $y \sim x$: $y = 2^k 2^k y \vee 2^k y = 2^k y$
 $2^k y = 2^k y$, therefore, $y \sim x$ is true.
 - Case 2: $y = 2^k x$
Plug y into $y \sim x$: $2^k x = 2^k x \vee x = 2^k 2^k x$
 $2^k x = 2^k x$, therefore, $y \sim x$ is true.
- c. It's transitive: $x \sim y \wedge y \sim z \rightarrow x \sim z$
 $x = 2^k y \vee y = 2^k x, k \in \mathbb{N} \equiv x = 2^m y, m \in \mathbb{Z}$
 $y \sim z : y = 2^{m'} z$
 $x = 2^m 2^{m'} z = 2^{m+m'} z, m + m' \in \mathbb{Z}$
Therefore, $x \sim z$
2. Show that there is a bijection $f : \mathbb{N} \rightarrow \mathbb{N} / \sim$

$$f(x) = \begin{cases} [0]_{\sim} & x = 0 \\ [2x - 1]_{\sim} & x > 0 \end{cases}$$

- a. Prove injective: $\forall x, y, f(x) = f(y) \rightarrow x = y$
Case 1: $x = 0$
 $f(x) = [0]_{\sim} = f(y) = 0 \rightarrow x = y$
Case 2: $x > 0, y > 0$
 $[2x - 1]_{\sim} = [2y - 1]_{\sim}$
 $2x - 1 = 2^k(2y - 1)$
Assume w/o loss of generality that $k \geq 0$
If $k > 0$, then the equality does not hold. Therefore, k must equal 0.
 $2x - 1 = 2y - 1 \rightarrow x = y$

A.8.3 Domination

$$f \preceq g \iff \forall x \in \mathbb{R} : f(x) \leq g(x)$$

1. Prove that \preceq is a partial order:
 - a. Reflexive:
 - $f \preceq f : f(x) = f(x)$
 - $f(x) \leq f(x)$
 - Therefore, $f \preceq f$
 - b. Antisymmetric: $f \preceq g \wedge g \preceq f \rightarrow f = g$
 - $f(x) \leq g(x) \wedge g(x) \leq f(x) \rightarrow f(x) = g(x)$
 - c. Transitive: $f \preceq g \wedge g \preceq h \rightarrow f \preceq h$
 - $f(x) \leq g(x) \wedge g(x) \leq h(x) \rightarrow f(x) \leq h(x) \rightarrow f \preceq h$
2. Prove or disprove: \preceq is a total order. Disproof:
 - $f(x) = x, g(x) = 5$

- For $x < 5$, $f(x) \leq g(x)$, but for $x > 5$, $f(x) \geq g(x)$. Therefore, \preceq is not a total order, because $f(x) \leq g(x) \wedge f(x) \geq g(x)$
3. Prove or disprove \preceq is a lattice: Each pair of elements has a greatest lower bound and a least upper bound. Proof:
- $f \wedge g$ (greatest lower bound) and $f \vee g$ (least upper bound)
 - $f \wedge g(x) = \min(f(x), g(x))$
 - $f \vee g(x) = \max(f(x), g(x))$