CPSC 202 PSET 6

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A.6.1 An oscillating sum

Give a closed-form expression for $f(n)$ and prove that it is correct.

My closed form expression for f(n):

$$g(n) = \begin{cases} \frac{n}{2} & \text{if n is even} \\ \frac{n+1}{2} & \text{if n is odd} \end{cases}$$

Proven by induction on n:

a. Base case: $n = 0, f(n) = g(n)$
   - $f(0) = (-1)^n \cdot \sum_{k=0}^{0} (-1)^0 \cdot 0 = 0$
   - $g(0) = \frac{0}{2} = 0$

b. Induction case: $f(n) = g(n) \rightarrow f(n+1) = g(n+1)$
   - $f(n+1) = (-1)^{n+1} \cdot \sum_{k=0}^{n+1} (-1)^k \cdot k = (-1)^{n+1} \cdot [\sum_{k=0}^{n} (-1)^k \cdot k + (-1)^{n+1} \cdot (n+1)] = (-1)[(-1)^n \cdot \sum_{k=0}^{n} (-1)^k \cdot k + (-1)^{2n-1} \cdot (n+1)]$
     where $(-1)^{2n-1} = (-1)^{2n}(-1) = 1(-1) = -1$
     Therefore, $f(n+1) = (-1)(g(n) - (n+1))$

   C1. n is even: $f(n+1) = (-1)(\frac{n}{2} - (n+1)) = (-1)(-\frac{n+2}{2}) = \frac{n+2}{2} = \frac{n+1+1}{2} = \frac{n+1}{2} + \frac{1}{2} = \lceil \frac{n+1}{2} \rceil$

   C2. n is odd: $f(n+1) = (-1)(\frac{n+1}{2} - (n+1)) = (-1)(-\frac{n+1}{2}) = \frac{n+1}{2} = \lceil \frac{n+1}{2} \rceil$

   Therefore, $f(n) = g(n) \rightarrow f(n+1) = g(n+1)$

A.6.2 An approximate sum

Show that $\sum_{k=1}^{n} k^2 \cdot 2^k = \Theta(n^2 \cdot 2^n)$ Θ means that the function must be O and Ω. Find $c \geq 0$ s.t. $N = 1$

a. $O(f(n)) : \exists c, N$ s.t. $g(n) \leq c(f(n)), n \geq N$
   - $\sum_{k=1}^{n} 2^k \cdot k^2 = 2^1 \cdot 1^2 + 2^2 \cdot 2^2 + ... + 2^n \cdot n^2 \geq 2^n \cdot n^2 \cdot 1$
   - c = 1

b. $\Omega(f(n)) : \exists c$ s.t. $g(n) \geq c(f(n)), n \geq N$

- $\sum_{k=1}^{n} k^2 \cdot 2^k \leq \sum_{k=1}^{n} n^2 \cdot 2^k = n^2 \cdot \sum_{k=1}^{n} 2^k = n^2 \cdot \frac{2-2^{n+1}}{1-2} = n(2^{n+1}-2) \leq n^2 \cdot 2^{n+1} = n^2 \cdot 2^n \cdot 2$
- c = 2

A.6.3 A stretched function

Prove $f(g(n))$ is in $O(n)$

- $f(n) = O(n) \rightarrow \exists c1, N1$ where for $n \geq N1, f(n) \leq c1 \cdot n$
- $g(n) = O(n) \rightarrow \exists c2, N2$ where for $n \geq N2, g(n) \leq c2 \cdot n$
- $f(g(n)) = O(n) \rightarrow \exists c3, N3$ where for $n \geq N3, f(g(n)) \leq c3 \cdot n$
- Case 1: If $g(n) > N1$ and $n > N2$
  - Then, $(f(g(n)) \leq c1 \cdot g(n) \leq c1 \cdot c2 \cdot n(c1 \cdot c2 = c3)$ and $N3 \geq N2$
- Case 2: If $|g(n) \leq N1|$
  - $-N1 \leq |g(n)| \leq N1$
  - If $g(n) \leq N1$
    - $n \geq N3 \geq \frac{m}{c1 \cdot c2}$ for $n < N3$
    - $f(g(n)) \leq m = \frac{m}{c1 \cdot c2} \cdot c1 \cdot c2 \leq N3 \cdot c1 \cdot c2 \leq c1 \cdot c2 \cdot n$
  - $f(g(n))$ is bounded, $f(g(n)) < m$
- $n \geq \frac{f(g(n)}{c3} \cdot \frac{f(g(n)}{c3} \leq n \cdot c3$ so $n \geq \frac{m}{c3}$