CPSC 202 PSET 5

CPSC 202 student

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A.5.1 A recursive sequence

Show that $\forall n \in \mathbb{N} : a_n \leq 2^n$

Proof by induction on n.

a. Base cases: $n = 0$, $n = 1$, $n = 2$
   - if $n = 0, a_0 \leq 2^0$
   - $a_0 = 1$ and $2^0 = 1$, therefore $1 \leq 1$
   - if $n = 1, a_1 \leq 2^1$
   - $a_1 = 2$ and $2^1 = 1$, therefore $2 \leq 2$
   - if $n = 2, a_1 \leq 2^2$
   - $a_2 = 3$ and $2^2 = 4$, therefore $3 \leq 4$

b. Induction step: prove $a_n \leq 2^n \rightarrow a_{n+1} \leq 2^{n+1}$ for $n > 2$
   - for $n > 2, a_n = a_{n-3} + a_{n-2} + a_{n-1}$
   - if $n > 2$, then $n + 1 > 2$, and $a_{n+1} = a_{n+1-3} + a_{n+1-2} + a_{n+1-1}$.
     - Simplified: $a_{n+1} = a_{n-2} + a_{n-1} + a_n$.
     - Substitute using our definition of a_n: $a_{n+1} = a_{n-2} + a_{n-1} + a_{n-3} + a_{n-2} + a_{n-1}$
     - Simplify after substitution: $a_{n+1} = a_{n-3} + 2 * a_{n-2} + 2 * a_{n-1}$
   - Using our definition of a_{n+1}. $a_{n+1} \leq 2^{n+1}$ becomes $a_{n-3} + 2 * a_{n-2} + 2 * a_{n-1} \leq 2^n * 2$
   - Divide both sides of the inequality by 2: $a_{n-3} + a_{n-2} + a_{n-1} \leq 2^n$
   - The left hand side is equal to a_n, so $a_n \leq 2^n$, which is true, therefore, $a_{n+1} \leq 2^{n+1}$

A.5.2 Comparing products

1. Proof that $\prod_{i=1}^{n} a_i \leq \prod_{i=1}^{n} b_i$ by induction on n.

   a. Base case: n = 0
      - if n = 0, $\prod_{i=1}^{0} a_i = 1$ and $\prod_{i=1}^{0} b_i = 1$
      - $1 \leq 1$

b. Induction step: $\prod_{i=1}^{n-1} a_i \leq \prod_{i=1}^{n-1} b_i \rightarrow \prod_{i=1}^{n} a_i \leq \prod_{i=1}^{n} b_i$. We assume that n-1 is true to prove the statement holds for n.

- Take out n-1 from top: $a_n \prod_{i=1}^{n-1} a_i \leq b_n \prod_{i=1}^{n-1} b_i$
- We know $0 \leq a_n \leq b_n$ and $0 \leq \prod_{i=1}^{n-1} a_i \leq \prod_{i=1}^{n-1} b_i$
  - Let's reduce the first inequality to $0 \leq x \leq y$ and the second one to $0 \leq f \leq g$, where x, y, f, and g are just substitutions for their respective expressions with product of a sequence.
  - Because all numbers are positive, we can multiply f to all sides of the first inequality to achieve: $0 \leq xf \leq yf$ and y to both sides of the second inequality to achieve $0 \leq yf \leq yg$.
  - We can conclude that $0 \leq xf \leq yf \leq yg$ and that finally, $0 \leq xf \leq yg$
- Therefore, $a_n \prod_{i=1}^{n-1} a_i \leq b_n \prod_{i=1}^{n-1} b_i$ is true.

2. Show $\forall k \in \mathbb{N} : \exists n_k : \forall n \in \mathbb{N} : n \geq n_k : k^n \leq n!$. Proof by induction on n:

a. Base case: $n = n_k = 2k^2$

- $k^{2k^2} \leq (2k^2)!$
- $(2k^2)! = \prod_{i=1}^{2k^2} i = (\prod_{i=1}^{k^2-1} i)(\prod_{i=k^2}^{2k^2} i) \geq \prod_{i=k^2}^{2k^2} i \geq \prod_{i=k^2}^{2k^2} k^2$ since $i \geq k^2$ and we proved it in 5.2.1.
- Simplifying: $\prod_{i=k^2}^{2k^2} k^2 = (k^2)^{2k^2-k^2+1} = (k^2)^{k^2+1} = k^{2k^2+2} = k^{2k^2}k^2 \geq k^{2k^2}$
- Therefore, $(2k^2)! \geq k^{2k^2}$

b. Induction step: $n! \geq k^n \rightarrow (n+1)! \geq k^{n+1}$

- $(n+1)! = (n+1)n!$ and $k^{n+1} = k^n k$, so $(n+1)n! \geq k^n k$
- We know $n! \geq k^n$ and need to prove $n+1 \geq k$
  - Since $n \geq 2k^2$, then $n+1 \geq 2k^2+1$
  - $n+1 \geq 2k^2+1 \geq 2k^2 \geq k$
  - Therefore, $n+1 \geq k$
- Therefore, $(n+1)! \geq k^{n+1}$

A.5.3 Rubble removal

The minimum and maximum number of days are the same. Proof by induction:

a. The number of days it takes to get rid of all the rocks = $d = \sum_{i=1}^{n}(2n_i - 1)$

b. Base case: d = 1

- $\sum_{i=1}^{1}(2n_i - 1) = 2n_i - 1 = 2(1) - 1 = 1$

c. Induction step, where d+1 represents the day before: $d \rightarrow d+1$

- Case 1: Throw away a rock
  - $d+1 = n_1, n_2, ..., n_k$ where each element represents rock piles and n_k is the pile with one rock in it (which we throw away), where $d+1 = \sum_{i=1}^{k}(2n_i - 1)$

- $d = n_1, n_2, ..., n_{k-1}$, where $d = \sum_{i=1}^{k-1}(2n_i - 1)$
- Take out k for d+1: $d + 1 = (\sum_{i=1}^{k-1}(2n_i - 1) + 2_{n_k} - 1)$
  $= d + 2_{n_k} - 1 = d + 2 - 1 = d + 1$
- Therefore, for this case, the induction hypothesis holds.

- Case 2: Split the pile
  - $d + 1 = n'_1, n'_2, ..., n'_{k-1}$
  - For the next day, n'_{k-1} is split into two piles, n_{k-1} and n_k,
    where $d = n'_1, n'_2, ..., n_{k-1}, n_k$ and $n'_{k-1} = n_{k-1}n_k$
  - Assuming d is true : $d = \sum_{i=1}^{k-2}(2n'_i - 1) + \sum_{i=k-1}^{k}(2n_i - 1)$, we need to prove $d + 1 = \sum_{i=1}^{k-1}(2n'_i - 1)$
    * $\sum_{i=1}^{k-1}(2n'_i - 1) = \sum_{i=1}^{k-2}(2n'_i - 1) + 2n'_{k-1} - 1$
    * $2n'_{k-1} - 1 = 2(n_{k-1} + n_k) - 1 = 2n_{k-1} - 1 + 2n_k - 1 + 1$, which is the expanded version of the second sum that defines d: $\sum_{i=k-1}^{k}(2n_i - 1)$
    * So, $\sum_{i=1}^{k-2}(2n'_i - 1) + 2n'_{k-1} - 1 = \sum_{i=1}^{k-2}(2n'_i - 1) + \sum_{i=k-1}^{k}(2n_i - 1) + 1$ which is equal to d + 1.

Therefore, because in our base case and both cases, we achieved the same number of days, there minimum and maximum number of days to get rid of all the rocks is the same number d for a given number of rocks whether you achieve the base case, or implement case 1 or 2 in the induction step first.