CPSC 202 PSET 2

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A.2.1 Arithmetic

1. 0<1

   Definition: $\forall x \forall y : x < y \leftrightarrow \exists z \neq 0 : x + z = y$

   a. Assuming $0 < 1$ is true, by definition:
      - $0 < 1 \leftrightarrow \exists z \neq 0 : 0 + z = 1$

   b. Axiom 2.3 $\forall x \forall y : x + y = y + x$ applies:
      - $0 + z = z + 0$

      Therefore, $0 < 1 \leftrightarrow \exists z \neq 0 : z + 0 = 1$

   c. Axiom 2.2 states $\forall x : x + 0 = x$
      - $\exists z = 1 : 1 + 0 = 1$
      - Therefore, $1 = 1$

      0<1 holds, following from the above axioms.

2. $x + z = y + z \rightarrow x = y$.

   Proof by contradiction:

   a. Assume $x \neq y$.

   b. In our model, $\forall x : x + \infty = \infty$
      - This does not violate Axiom 2.1
      - Axiom 2.2 $x = \infty : \infty + 0 = \infty$
      - Axiom 2.3 $x = \infty : \infty + y = y + \infty$ so, $\infty = \infty$
      - Axiom 2.4 if $z = \infty$ then, $x + (y + \infty) = (x + y) + \infty$ Based on our model: $x + \infty = \infty$, then $\infty = \infty$
      - Axiom 2.5 holds because if $x = \infty : \infty + y = 0$ is false.

   c. if $z = \infty$
      - $x + \infty = y + \infty$

   d. In our model,

- $x + \infty = \infty$
- $y + \infty = \infty$
- Therefore, $\infty = \infty$

The implication fails since $x \neq y$ but in our model, $x + z = y + z$ was true.

3. $x < y \to x + z < y + z$

   a. $x + z < y + z$
      - By definition: $x + z < y + z \leftrightarrow \exists a \neq 0 : x + z + a = y + z$

   b. Axiom 2.3 states $\forall x \forall y : x + y = y + x$, therefore:
      - $z + a = a + z$, thus $x + a + z = y + z$

   c. By definition, $x < y \leftrightarrow \exists b \neq 0 : x + b = y$
      - if $b = a$
      - then $x + a = y$
      - Using the above statement, $x + a + z = y + z \equiv y + z = y + z$

   The statement holds, following from the above axioms.

4. $a < b \land c < d \to a + c < b + d$

   a. By definition: $\forall a \forall b : a < b \leftrightarrow \exists z \neq 0 : a + z = b$

   b. By definition: $\forall c \forall d : c < d \leftrightarrow \exists y \neq 0 : c + y = d$

   c. By definition $\forall a \forall b \forall c \forall d : a + c < b + d \leftrightarrow \exists x \neq 0 : a + c + x = b + d$

   d. If $x = z + y$
      - $a + c + z + y = b + d$

   e. Axiom 2.3
      - $c + z = z + c$
      - As a result, $a + z + c + y = b + d$
      - Replace $a + z$ and $c + y$

   f. $b + d = b + d$ proving that the statement holds, following from the above axioms.

A.2.2 Some distributive laws

1. For all sets A, B, C, and D: $A \subseteq C \land B \subseteq D \to A \cap B \subseteq C \cap D$.

   a. $A \subseteq C = \forall x : x \in A \to x \in C$

   b. $B \subseteq D = \forall x : x \in B \to x \in D$

   c. $A \cap B = \{x | x \in A \land x \in B\}$

   d. $x \in A \land x \in B \to x \in C \land x \in D$

   e. Therefore, $A \cap B \subseteq C \cap D$

2. For all sets A, B, C and D: $A \subseteq C \wedge B \supseteq D \rightarrow A \setminus B \subseteq C \setminus D$

   a. $A \subseteq C = \forall x : x \in A \rightarrow x \in C$

   b. $D \supseteq B = B \subseteq D = \forall x : x \in B \rightarrow x \in D$

   c. $A \setminus B = \{x | x \in A \wedge x \notin B\} \rightarrow x \in C \wedge x \notin D$

   d. $C \setminus D \ \{x | x \in C \wedge x \notin D$

   This proves $A \notin B \subseteq C \notin D$

A.2.3 Elements and subsets

1. $A \in B \in C$: D. A is not necessarily either an element or subset of C.

   a. Not necessarily an element

      - if A is $\{1\}$
      - $A \in B$ means B is $\{\{1\}\}$
      - $B \in C$ means C is $\{\{\{1\}\}\}$ therefore, $A \notin C$

   b. Not necessarily a subset

      - Using the same definitions of A, B, and C, $A \subsetneq C$

2. $A \in B \subseteq C$: A. A must be an element of C but is not necessarily a subset of C.

   a. Not necessarily a subset

      - if $A = \{1\}$, then $A \in B$ means $B = \{\{1\}\}$
      - $B \subseteq C$ means $C = \{\{1\}\}$

      Therefore, A is not necessarily a subset of C.

   b. Must be an element

      - $B \subseteq C = \forall x : x \in B \rightarrow x \in C$

      Therefore, if A is an element of B, and all elements of B are in C, then A is an element of C.

3. $A \subseteq B \in C$: D. A is not necessarily either an element or subset of C.

   a. Not necessarily an element

      - if $A = \{1\}$ and $B = \{1, 2\}$, then $A \subseteq B$
      - since $B \in C$, $C = \{\{1, 2\}\}$. $A \notin C$

   b. Not necessarily a subset

      - Using the same sets for A, B, and C above, $A \subsetneq C$ because all the elements of A are not in C.

4. $A \subseteq B \subseteq C$: B. A must be a subset of C but is not necessarily an element of C.

a. Must be a subset

- $A \subseteq B = \forall x : x \in A \rightarrow x \in B$
- $B \subseteq C = \forall x : x \in B \rightarrow x \in C$

Therefore, $\forall x : x \in A \rightarrow x \in C = A \subseteq C$

b. Not necessarily an element

- if $A = \{1\}$ and $A \subseteq B$, then $B = \{1\}$
- if $B \subseteq C$, then, $C = \{1\}$

Therefore, A is not necessarily an element in C.