CPSC 202 PSET 10

CPSC 202 student

11/29/17 5pm

A.10.1: Too many injections

a. $f : A \to S$, $g : B \to S$, and $h : C \to S$ are all injections, therefore, f(A)=g(B)=h(C) means the ranges are the same.

b. k represents the elements in the domain, therefore, the range is equal to k. If n represents the elements in S, the range is represented by $\binom{n}{k}$.

c. Given that their ranges are the same, and k represents the size of the domain, there are k! different ways to write each function, or $k!^3$ injective functions.

d. Therefore, there are $\binom{n}{k}$ ways to choose a range and $k!^3$ injective functions, and $\binom{n}{k} \cdot k!^3$ different triples of functions f, g, and h, or $n_k \cdot k!^2$.

A.10.2 Binomial coefficients

Show that, for any $k, m, n \in \mathbb{N}$ s.t. $0 \leq k \leq m \leq n$, $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$.

a. Expanding binomial coefficients:
   - By definition, $\binom{n}{m} = \frac{n!}{m! \cdot (n-m)!}$
     $\binom{m}{k} = \frac{m!}{k! \cdot (m-k)!}$
     $\binom{n-k}{m-k} = \frac{(n-k)!}{(m-k)! \cdot ((n-k)-(m-k))!} = \frac{(n-k)!}{(m-k)! \cdot (n-m)!}$
   - Therefore, $\binom{n}{m}\binom{m}{k} = \frac{n! \cdot m!}{m! \cdot k! \cdot (n-m)! \cdot (m-k)!} = \frac{n!}{k! \cdot (n-m)!(m-k)!}$
     and $\binom{n}{k}\binom{n-k}{m-k} = \frac{n! \cdot (n-k)!}{k! \cdot (n-k)! \cdot (m-k)! \cdot (n-m)!} = \frac{n!}{k! \cdot (m-k)! \cdot (n-m)!} = \frac{n!}{k! \cdot (n-m)! \cdot (m-k)!}$

The two expressions are equal therefore, the statement is true.

A.10.3 Variable names

Give a closed form expression for the number of legal variable names of length n.

a. We can get 0 or more letters from a,b,c followed by 0 or more letters from 0,1,2,3.

b. We know for a sequence a_0, a_1, a_2, a_3, there are $\sum_{n=0}^{\infty} a_n \cdot z^n = F(z)$, where a_n represents the number of ways to write a string, n is the length of the sequence, z is a placeholder to make the sum converge, and F(z) is the generating function in closed form.

c. For the nth coefficient series: $\sum_{n=0}^{\infty} a_n \cdot z^n = \sum_{n=0}^{\infty} (az)^n = \frac{1}{1-az}$
Therefore, the generating function for a,b,c is $\frac{1}{1-3z}$ and the generating function for 0,1,2,3 is $\frac{1}{1-4z}$

d. $\frac{1}{1-4z} \cdot \frac{1}{1-3z} = \frac{A}{1-3z} + \frac{B}{1-4z}$
$\frac{A}{1-3z} + \frac{B}{1-4z} = \frac{A(1-4z)+B(1-3z)}{(1-4z)(1-3z)}$
$A(1-4z) + B(1-3z) = 1 + 0z$
$A - A(4z) + B - B(3z) = 1 + 0z$
A+B = 1
$-A(4z) - B(3z) = 0z$
A=-3, B=4

e. $\frac{-3}{1-3z} + \frac{4}{1-4z} = -3\frac{1}{1-3z} + 4\frac{1}{1-4z}$
$= -3\sum_{n=0}^{\infty}(3-z)^n + 4\sum_{n=0}^{\infty}(4z)^n$
$= \sum_{n=0}^{infty} -(3)^{n+1} \cdot z^n + \sum_{n=0}^{\infty} 4^{n+1} \cdot z^n$
$= \sum_{n=0}^{\infty}(4^{n+1} - 3^{n+1}) \cdot z^n$

Therefore, for a sequence of length n, there are $4^{n+1} - 3^{n+1}$ to write a variable name.