6.851: Advanced Data Structures Spring 2012

Lecture 2 — 02/09, 2012

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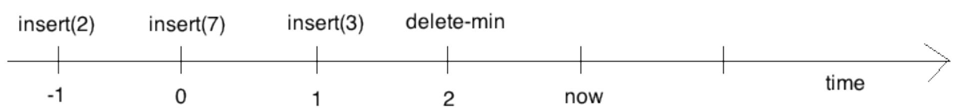
1 Overview

The main idea of persistent data structures is that when a change is made in the past, an entirely new universe is obtained. A more science-fiction approach to time travel is that you can make a change in the past and see its results not only in the current state of the data structure, but also all the changes in between the past and now.

We maintain one timeline of updates and queries for a persistent data structure:



Usually, operations are appended at the end of the timeline (present time). With retroactive data structures we can do that in the past too.

2 Retroactivity

The following operations are supported by retroactive DS:

- *Insert(t, update)* - inserts operation "update" at time t
- *Delete(t)* - deletes the operation at time t
- *Query(t, query)* - queries the DS with a "query" at time t

Uppercase Insert indicates an operation on retroactive DS, lowercase update is the operation on the actual DS.

You can think of time t as integers, but a better approach is to use an order-maintanance DS to avoid using non-integers (in case you want to insert an operation between times t and $t+1$), as mentioned in the first lecture.

There are three types of retroactivity:

- *Partial* - Query always done at $t = \infty$ (now)
- *Full* - Query at any time t (possibly in the past)
- *Nonoblivious* - Insert, Delete, Query at any time t, also if an operation modifies DS, we must say which future queries are changed.

2.1 Easy case with commutativity and inversions

Assume the following hold:

- *Commutative updates:* $x.y = y.x$ (x followed by y is the same as y followed by x); that is the updates can be reordered $\Rightarrow$ Insert(t, op) = Insert(now, op).
- *Invertible updates:* There exists an operation x^{-1}, such that $x.x^{-1} = \emptyset \Rightarrow$ Delete(t, op) = Insert(now, op^{-1})

2.1.1 Partial retroactivity

These two assumptions allow us to solve some retroactive problems easily, such as:

- *hashing*
- *array* with operation $A[i] += \Delta$ (but no direct assignment)

2.2 Full retroactivity

First, lets define the **search problem**: maintain set S of objects, subject to insert, delete, query(x, S).

Decomposable search problem [1980, 2007]: same as the search problem, with a restriction that the query must satisfy: query$(x, A \cup B) = f($query(x, A),query$(x, B))$, for some function f computed in $O(1)$ (sets A and B may overlap). Examples of problems with such a function include:

- *Dynamic nearest neighbor*
- *Successor on a line*
- *Point location*

Claim 1. *Full Retroactivity for decomposable search problems (with commutativity and inversions) can be done in $O(\lg m)$ factor overhead both in time and space (where m is the number of operations) using* **segment tree** *[1980, Bentley and Saxon (??)]*

We want to build a balanced search tree on time (leaves represent time). Every element "lives" in the data structure on the interval of time, corresponding to its insertion and deletion. Each element appears in $\lg n$ nodes.

To query on this tree at time t, we want to know what operations have been done on this tree from the beginning of time to t. Because the query is decomposable, we can look at $\lg n$ different nodes and combine the results (using the function f).

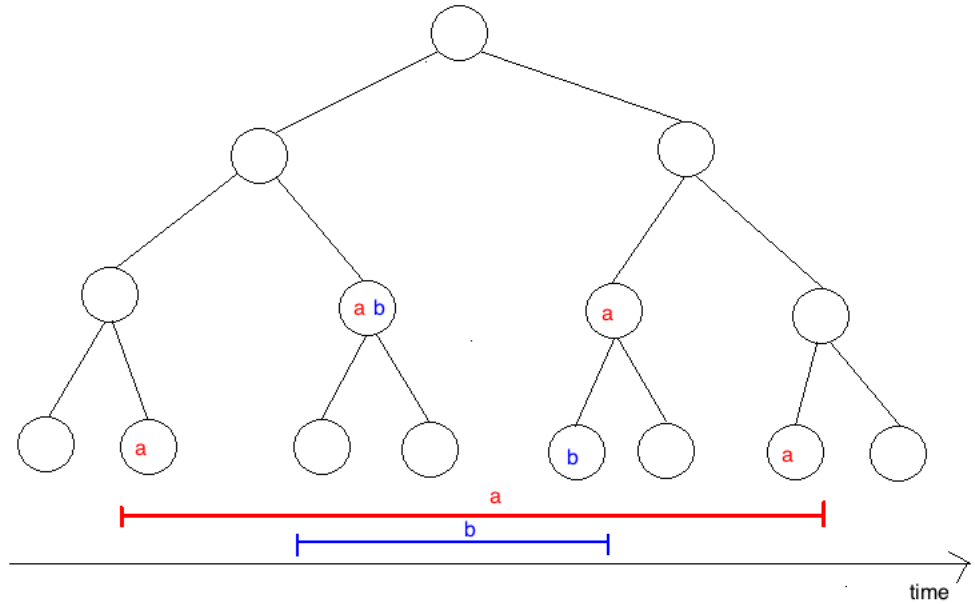


Figure 1: Segment Tree

2.3 General case of full retroactivity

Roll back method:

- write down a (linear) chain of operations and queries
- change r time units in the past with factor $O(r)$ overhead.

That's the best we can do in general.

Lower bound: $\Omega(r)$ overhead necessary.

Proof: Data Structure maintains 2 values (registers): X and Y, initially $\emptyset$. The following operations are supported: $X = x$, $Y += \Delta$, $Y = X.Y$, query 'Y?'. Perform the following operations (Cramer's rule):

$$Y += a_n, X = X.Y, Y += a_{n-1}, X = X.Y, \ldots, Y += a_0$$

which is equivalent to computing

$$Y = a_n X^n + a_{n-1} X^{n-1} + ... + a_0$$

Now, execute Insert$(t = 0, X = x)$, which changes where the polynomial is evaluated. This cannot be done faster than re-evaluating the polynomial. In history-independent algebraict decision tree, for any field, independent of pre-processing of the coefficients, need $\Omega(n)$ field operations (result from 2001), where n is the degree of the polynomial.

2.4 Cell-probe problem

How many integers (words) of memory do you need to read to solve your problem? (gives a lower bound on running time).

Claim: $\Omega(\sqrt{\frac{r}{\lg r}})$.

Open problem: $\Omega(r)$.

Proof of the lower bound claim:

- DS maintans n words (integers of $w \geq \lg n$ bits).
- Arithmetic operations are allowed.
- Query = what is the value of an integer?
- Compute FFT.
- Retroactively change the words $\Rightarrow$ Dynamic FFT.
- Changing x_i requires $\Omega(\sqrt{n})$ cell probes.

2.5 Priority Queues

Now, let us move onto some more positive results. Priority queues represent a DS where retroactive operations potentially create chain reactions but we have still obtained some nice results for. The main operations are *insert* and *delete-min* which we would like to retroactively $Insert$ and $Delete$.

Claim 2. *It is possible to implement a partially retroactive priority queue with only* $O(\lg n)$ *overhead per partially retroactive operation.*

Because of the presence of *delete-min*, the set of operations on priority queues is non-commutative. The order of updates now clearly matters, and $Insert$ing a *delete-min* retroactively has the potential to cause a chain reaction which changes everything that comes afterward. Partially retroactive priority queues are described in a paper by Demaine, Iacono, and Langerman [1].

To develop an intuition for how our DS changes given a retroactive operation, it is helpful to plot it on a two dimensional plane. The x-axis represents time and y-axis represents key value. Every $insert(t, k)$ operation creates a horizontal ray that starts at point (t, k) and shoots to the right (See Fig. 2).

Every *delete-min*() operation creates a vertical ray that starts at $(t, -\infty)$ and shoots upwards, stopping at the horizontal ray of the element it deletes. Thus, the horizontal ray becomes a line segment with end points (t, k) and (t_k, k), where t_k is the time of key k's deletion.

This combinations of inserts and deletes creates a graph of nonintersecting upside down "L" shapes, where each L corresponds to an *insert* and the *delete-min*() that deletes it. Elements which are never deleted remain rightward rays. Figure 3 demonstrates this by adding a few *delete-mins* to our previous graph of only *inserts*.

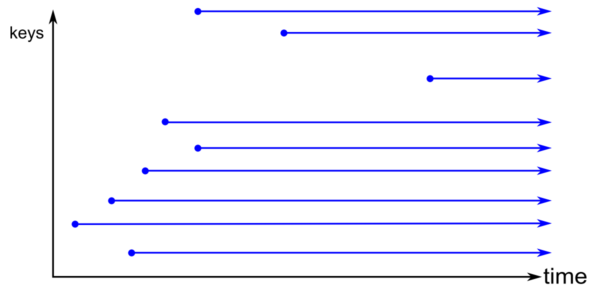


Figure 2: Graph of priority queue featuring only a set of *inserts*

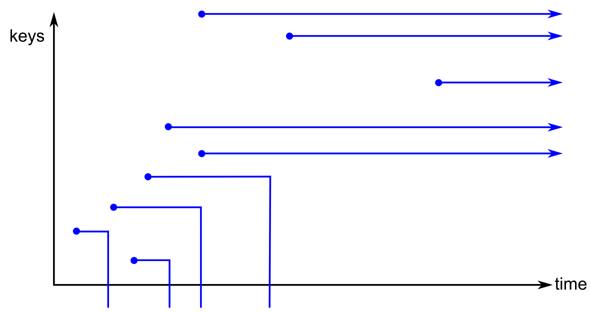


Figure 3: Adding *del-min*() operations leads to these upside down "L" shapes.

The rest of the discussion will focus on $Insert(t,\text{“}insert(k)\text{”})$. It should be easy enough to convince yourself that $Delete(t,\text{“}delete\text{-}min\text{”})$ has equivalent analysis if you think about Delete as inserting the deleted element at the time of deletion.

Consider the priority queue represented by figure 4. It's similar to the previous ones seen but with more inserts and deletes to better illustrate the chain-reactions of retroactive operations. Figure 5 shows what happens when elements are retroactively inserted. The retroactive operations and their chain reactions are shown in red. The cascading changes add up fast.

However, since we're only requiring partial retroactivity we only need to determine what element $Insert(t,\text{“}insert(k)\text{”})$ inserts into Q_{now} where Q_{now} is the priority queue at the present time. Naively, it is easy to see that the element inserted at Q_{now} is: $\max\{k, k' \mid k' \text{ deleted at time} \geq t\}$. That is, the element that makes it to the "end" is the biggest element that was previously deleted (ie. the end of the chain-reaction shown in Figure 3) or simply k if it is bigger than those (ie. the insert caused no chain reactions).

Problem: Maintaining "deleted" elements is hard. It requires us to maintain the various chain-reactions which isn't efficient. Instead, we would like to simply keep track of inserts. Such a transformation is possible as long as we define the new concept of "bridges".

Definition 3. *We define time t to be a bridge if $Q_t \subseteq Q_{now}$.*

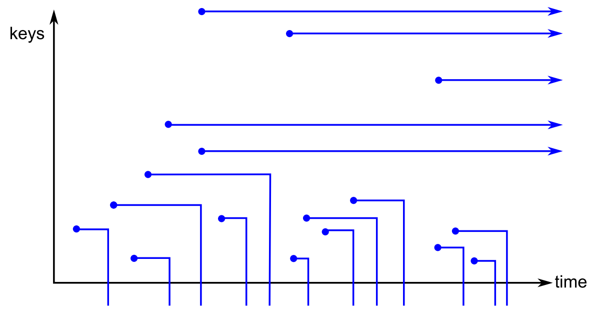


Figure 4: An L view representation of a priority queue with a more complicated set of updates

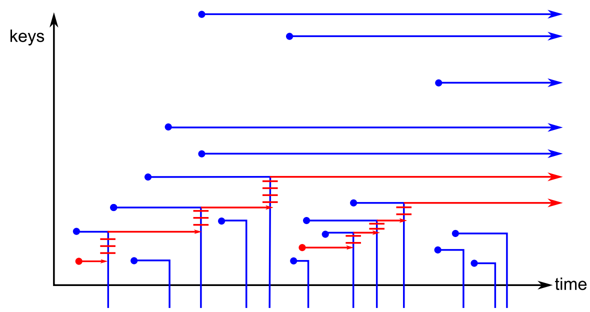


Figure 5: Retroactive inserts start at red dots and cause subsequent *delete-min* operations to effect different elements as shown.

This simply means that all of the elements at a bridge t' are also present at t_{now}. You can think of bridges as separating the chaotic chain-reactions that happen during retroactive $Insert$s as seen in figure 6.

If t' is the bridge preceding time t, then
$\max\{k' \mid k' \text{ deleted at time} \geq t\} = \max\{k' \notin Q_{now} \mid k' \text{ inserted at time} \geq t'\}$

With that transformation, we only need to maintain three data structures which will allow us to perform partially retroactive operations with only $O(\lg n)$ overhead.

- We will store Q_{now} as a balanced BST. It will be changed once per update.
- We will store a balanced BST where the leaves equal insertions, ordered by time, and augmented with $\forall$ node $x : \max\{k' \notin Q_{now} \mid k' \text{ inserted in } x\text{'s subtree}\}$.
- Finally, we will store a balanced BST where the leaves store all updates, ordered by time, and augmented by the following:

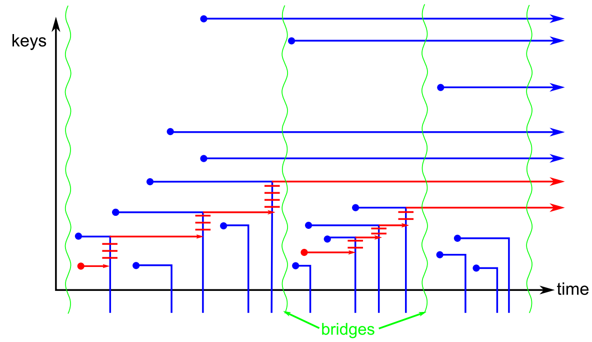


Figure 6: Bridges have been inserted as green wavy lines. Notice how they only cross elements present to the end of visible time.

$$\begin{cases} 0 & \text{for inserts with } k \in Q_{now} \\ +1 & \text{for other inserts, } k \notin Q_{now} \\ \text{-1} & \text{for } \textit{delete-mins} \end{cases}$$

as well as subtree sums.

Now, we have to uses these data structures to execute our *Insert*. This can be done in $O(\lg n)$ with the following steps.

- First, for an *Insert* at time t we must find the preceding bridge at time t'. Using our BBST of all updates, we know that a bridge is a time(leaf) at which has a prefix of updates summing to 0 (with respect to the augmentation). This is possible in $O(\lg n)$ time due to the stored subtree sums. In brief, traverse the tree to find t, calculating the prefix sum as you descend. If t has a prefix sum of 0 then we're done. Otherwise, walk back up the tree to find the preceding update with a prefix sum of 0.
- Next, in our BBST of insertions we descend to t' and work back up the tree looking for the maximum node not already in Q_{now}. Because of the augmentation we store, this can be done in $O(\lg n)$ time.
- Finally, there are a collection of steps related to updating the data structures once we make our update, but given our data structures it should be easy to convince yourself that it can be done in $O(\lg n)$ time.

2.5.1 Other Structures

- *queue:* $O(1)$ partial, $O(\lg m)$ full
- *deque:* $O(\lg m)$ full
- *union-find (incremental connectivity):* $O(\lg m)$ full

- *priority-queue:* $O(\sqrt{m}\lg m)$ full. This comes from the fact that any partially retroactive DS can be made fully retroactive with a $O(\sqrt{m})$ factor overhead. It's an open problem whether or not we can do better.
- *successor:* This was actually the motivating problem for retroactivity. $O(\lg m)$ partial because it's a search problem. $O(\lg^2 m)$ full because it's also decomposable. However, Giora and Kaplan gave us a better solution of $O(\lg m)$ [2]! This new algorithm uses many data structures we'll learn about later; including fractional cascading (L3) and vam Emde Boas (L11).

2.6 Blah blah blah

Here is a subsection.

2.6.1 Blah blah blah

Here is a subsubsection. You can use these as well.

2.7 Using Boldface

Make sure to use lots of boldface.

Question: How would you use boldface?

Example: This is an example showing how to use boldface to help organize your lectures.

Some Formatting. Here is some formatting that you can use in your notes:

- *Item One* – This is the first item.
- *Item Two* – This is the second item.
- . . . and here are other items.

If you need to number things, you can use this style:

1. *Item One* – Again, this is the first item.
2. *Item Two* – Again, this is the second item.
3. . . . and here are other items.

Bibliography. Please give real bibliographical citations for the papers that we mention in class. See below for how to include a bibliography section. If you use BibTeX, integrate the .bbl file into your .tex source. You should reference papers like this: "The FKS dictionary originates in a paper by Fredman, Komlós and Szemerédi [**?**]." In general, the name of the authors should appear in text at most once (for the first citation); further citations look like: "Our proof follows that of [**?**]".

Take a look at previous lectures (TeX files are available) to see the details. A excellent source for bibliographical citations is DBLP. Just Google DBLP and an author's name.

References

[1] Erik D. Demaine, John Iacono, Stefan Langerman: *Retroactive data structures.* SODA 2004: 281-290

[2] Giora and Kaplan: