1 Formalization

$$
\begin{array}{rrcll}
\textbf{Types} & \tau & ::= & \mathsf{nat} \mid \tau_1 \to \tau_2 & \\
\textbf{Expressions:} & e & ::= & x & \text{variable} \\
 & & \mid & z & \text{zero} \\
 & & \mid & s(e) & \text{successor} \\
 & & \mid & \mathsf{rec}(e; e_0; x.y.e_1) & \text{recursion} \\
 & & \mid & \lambda(x : \tau)\, e & \text{abstraction} \\
 & & \mid & \mathsf{ap}(e_1; e_2) & \text{application}
\end{array}
$$

Rules :

$$\frac{}{\Gamma, x : \tau \vdash x : \tau}$$

$$\frac{}{\Gamma \vdash z : \mathsf{nat}}$$

$$\frac{\Gamma \vdash e : \mathsf{nat}}{\Gamma \vdash s(e) : \mathsf{nat}}$$

$$\frac{\Gamma \vdash e : \mathsf{nat} \qquad \Gamma \vdash e_0 : \tau \qquad \Gamma, x : \mathsf{nat}, y : \tau \vdash e_1 : \tau}{\Gamma \vdash \mathsf{rec}(e; e_0; x.y.e_1) : \tau}$$

Reduction Rules :

$$(\lambda x : \tau.\, t)\, u \to\ t[x := u]$$

$$\mathsf{rec}(0;\ t_0;\ x.y.\, t_s) \to\ t_0$$

$$\mathsf{rec}(s(n);\ t_0;\ x.y.\, t_s) \to\ t_s[x := n,\ y := \mathsf{rec}(n;\ t_0;\ x, y.\, t_s)].$$

$$\frac{e\ \to\ e'}{C[e]\ \to\ C[e']} \quad \textit{where}$$

$$C\ ::= [\,] \mid z \mid s(C) \mid C\, e \mid e\, C \mid \mathsf{rec}\big(C;\ e_0;\ x.y.e_s\big)$$

2 Definitions and Theorems

Definition 1 (Normal Form)**.** *A term e is in* normal form *if there is no term e' such that*

$$e\ \to\ e'.$$

In other words, no reduction rule applies to e.

Definition 2 (Weak Normalization)**.** *A (closed) term e is* weakly normalizing*, written* $\mathsf{MN}(e)$*, if there exists a finite reduction sequence*

$$e\ \to\ e_1\ \to\ \cdots\ \to\ e_n$$

such that e_n is in normal form.

Definition 3 (Reducibility Relation)**.** *For each type τ we define the predicate $R_\tau(e)$ on closed terms e by induction on τ:*

$$R_{\mathsf{nat}}(e) \;:=\; \big(\vdash e : \mathsf{nat}\big) \;\wedge\; \mathsf{MN}(e), \tag{1}$$

$$R_{\sigma\to\tau}(e) \;:=\; \big(\vdash e : \sigma \to \tau\big) \;\wedge\; \big(\forall\, v.\; R_\sigma(v) \;\to\; R_\tau(e\,v)\big). \tag{2}$$

We often write

$$\mathcal{R}(\tau) \;=\; \{\, e \mid R_\tau(e) \,\}.$$

Theorem 1 (Confluence)**.**

$$\frac{e \;\to^*\; e_1 \qquad e \;\to^*\; e_2}{\exists e_3.\; e_1 \;\to^*\; e_3 \;\wedge\; e_2 \;\to^*\; e_3}$$

3 Proof of Weak Normalization

Lemma 1.

$$e \to e' \;\wedge\; \mathsf{MN}(e) \;\Rightarrow\; \mathsf{MN}(e').$$

Proof. Suppose $e \to e'$ and $\mathsf{MN}(e)$, so there is n with $e \to^* n$ and n in normal form. By confluence, there exists m such that

$$e' \;\to^*\; m \quad \text{and} \quad n \;\to^*\; m.$$

But n is normal, so $m = n$. Therefore $e' \to^* n$, i.e. $\mathsf{MN}(e')$. ☐

Lemma 2.

$$e \to e' \;\wedge\; \mathsf{MN}(e') \;\Rightarrow\; \mathsf{MN}(e).$$

Proof. By definition of MN, from $e \to e'$ and $e' \to^* n$ (with n normal) we get

$$e \;\to\; e' \;\to^*\; n,$$

i.e. $e \to^* n$, so $\mathsf{MN}(e)$. ☐

Lemma 3.
If $R_\tau(e)$ and $e \to e'$, then $R_\tau(e')$.

Proof. By induction on the definition of $R_\tau(e)$.

- $\tau = \mathsf{nat}$. Then

  $$R_{\mathsf{nat}}(e) = (\Gamma \vdash e : \mathsf{nat}) \;\wedge\; \mathsf{MN}(e).$$

  By lemma 1, if $e \to e'$ and $\mathsf{MN}(e)$ then also $\mathsf{MN}(e')$. Thus $R_{\mathsf{nat}}(e')$.

- $\tau = \sigma \to \rho$. We have

  $$R_{\sigma\to\rho}(e) \;=\; \Gamma \vdash e : \sigma \to \rho \;\wedge\; \mathsf{MN}(e) \;\wedge\; \forall v\,(R_\sigma(v) \implies R_\rho(e\,v)).$$

  Since if $e \to e'$, then for any $v \in R_\sigma$, $e\,v \to e'\,v$ (by reduction rule), thus for type ρ we get $R_\rho(e'\,v)$. Hence $R_{\sigma\to\rho}(e')$.

$\square$

Lemma 4.

If $e \to e'$ and $R_\tau(e')$, then $R_\tau(e)$.

Proof. Similarly, by induction on the definition of $R_\tau(e)$.

- $\tau = \mathsf{nat}$. Then
$$R_{\mathsf{nat}}(e') = (\Gamma \vdash e' : \mathsf{nat}) \;\wedge\; \mathsf{MN}(e').$$
By lemma 2, if $e \to e'$ and $\mathsf{MN}(e')$ then also $\mathsf{MN}(e)$. Thus $R_{\mathsf{nat}}(e)$.

- $\tau = \sigma \to \rho$. We have
$$R_{\sigma\to\rho}(e') \;=\; \Gamma \vdash e : \sigma \to \rho \;\wedge\; \mathsf{MN}(e') \;\wedge\; \forall v\,(R_\sigma(v) \implies R_\rho(e'\,v)).$$
Since if $e \to e'$, then for any $v \in R_\sigma$, $e\,v \to e'\,v$ (by reduction rule), thus for type ρ we get $R_\rho(e\,v)$. Hence $R_{\sigma\to\rho}(e)$.

$\square$

Lemma 5 (Fundamental Lemma)**.** *If $\Gamma = x_1 : \tau_1, \ldots, x_n : \tau_n \vdash e : \tau$ and each v_i is a closed term with $R_{\tau_i}(v_i)$, then*
$$R_\tau\big(e[x_1 := v_1, \ldots, x_n := v_n]\big).$$

Proof. $\square$

Theorem 2 (Weak Normalization)**.** *If $\vdash e : \tau$ and e is closed, then* $\mathrm{WN}(e)$.

Proof. $\square$