# CS 4495 Computer Vision

## *N-Views (2) – Essential and Fundamental Matrices*

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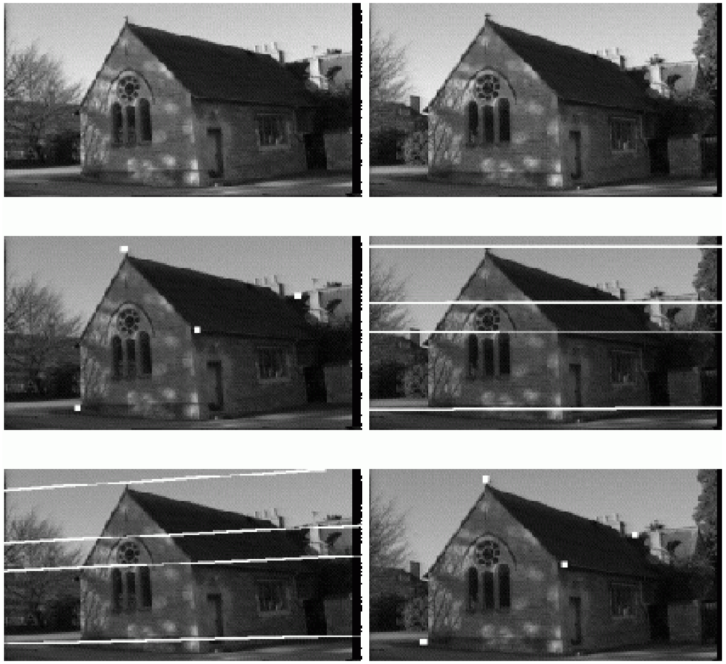


# Administrivia

- Today: Second half of N-Views (n = 2)
- PS 3: Will hopefully be out by Thursday
  - Will be due October 6th.
  - Will be based upon last week and today's material
  - We may revisit the logistics – suggestions?

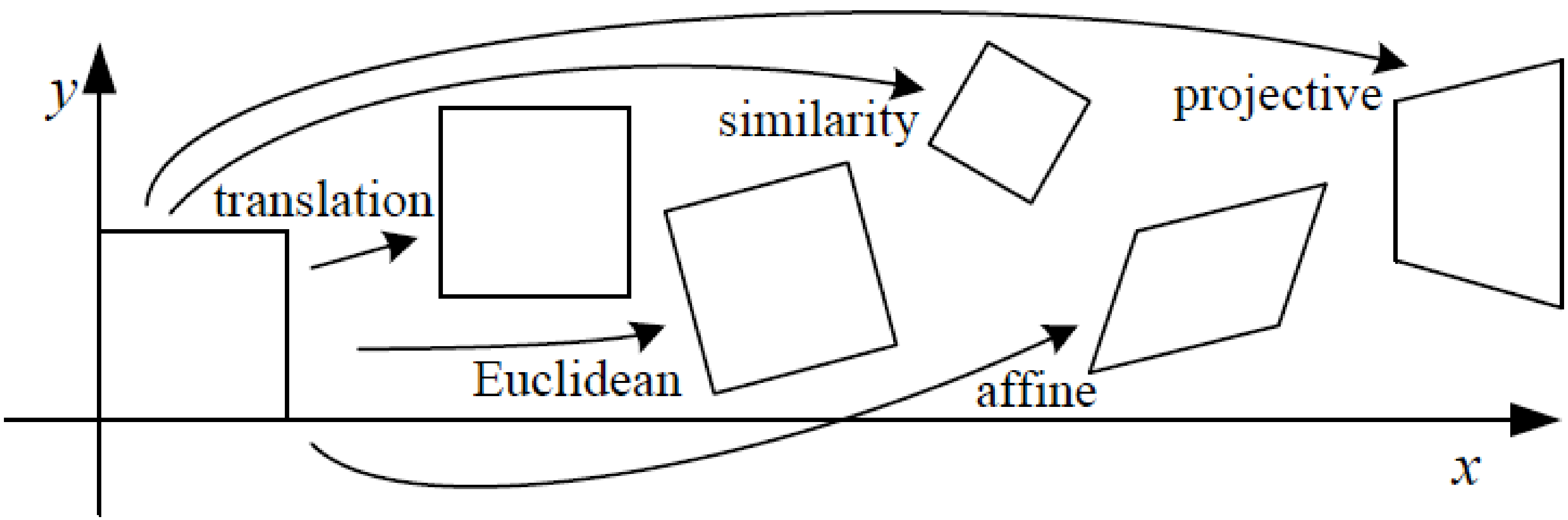
# Two views…and two lectures

- Projective transforms from image to image
- Some more projective geometry
  - Points and lines and planes
- Two arbitrary views of the same scene
  - Calibrated – “Essential Matrix”
  - Two uncalibrated cameras “Fundamental Matrix”
    - Gives epipolar lines

# Last time

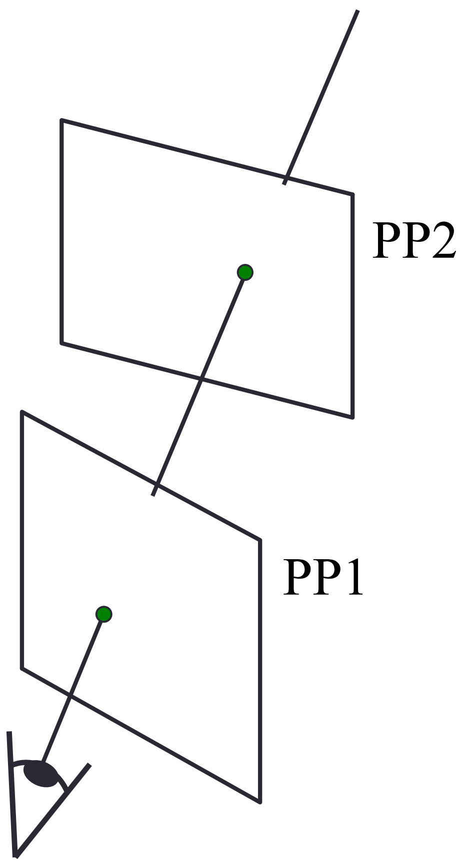
- Projective Transforms: Matrices that provide transformations including translations, rotations, similarity, affine and finally general (or perspective) projection.

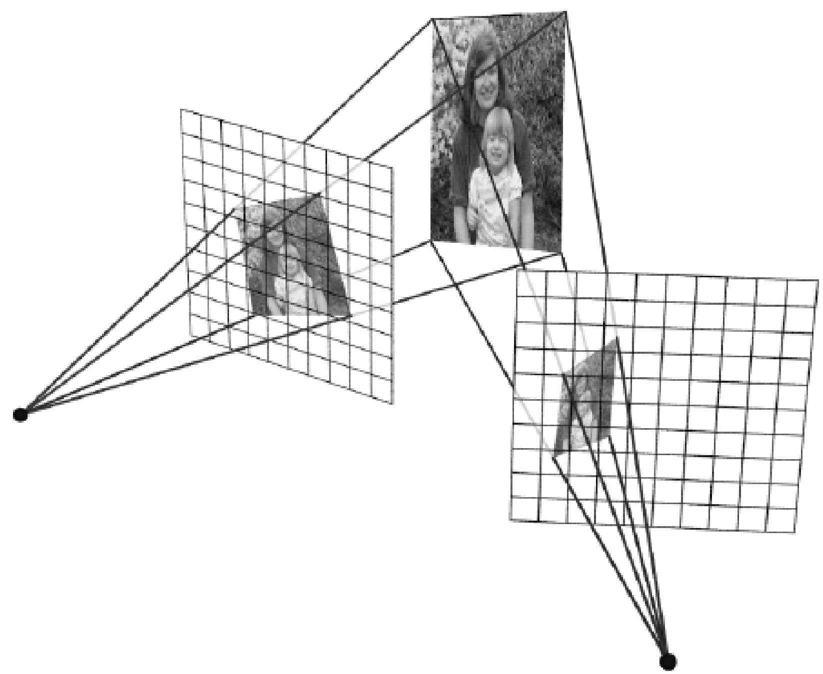
- When 2D matrices are 3x3; for 3D they are 4x4.



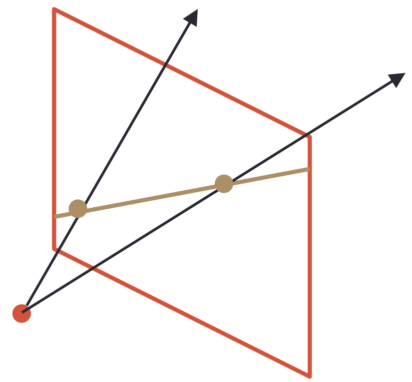
# Last time: Homographies

- Provide mapping between images (image planes) taken from same center of projection; also mapping between any images of a planar surface.



# Last time: Projective geometry



- A line is a *plane* of rays through origin
  - all rays (x,y,z) satisfying: ax + by + cz = 0

$$\text{in vector notation}: \quad 0 = \underbrace{\begin{bmatrix} a & b & c \end{bmatrix}}_{\mathbf{l}} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\mathbf{p}}$$

- A line is also represented as a homogeneous 3-vector **l**

# Projective Geometry: lines and points

## 2D Lines:

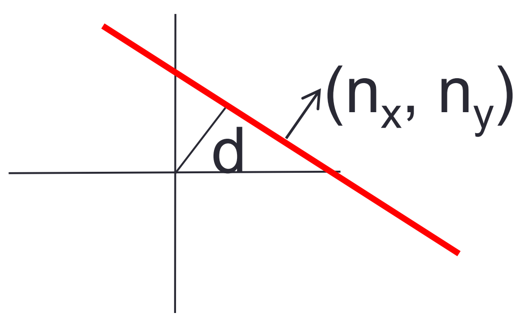
$$a x + b y + c = 0$$

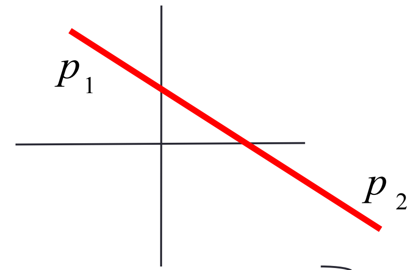
$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

Eq of line

$$\mathbf{l}^T \mathbf{x} = 0$$

$$l = \begin{bmatrix} a & b & c \end{bmatrix} \Rightarrow \begin{bmatrix} n_x & n_y & d \end{bmatrix}$$

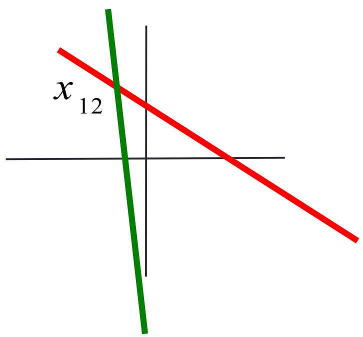



$$p_1 = \begin{bmatrix} x_1 & y_1 & 1 \end{bmatrix}$$

$$p_2 = \begin{bmatrix} x_2 & y_2 & 1 \end{bmatrix}$$

$$l = p_1 \times p_2$$



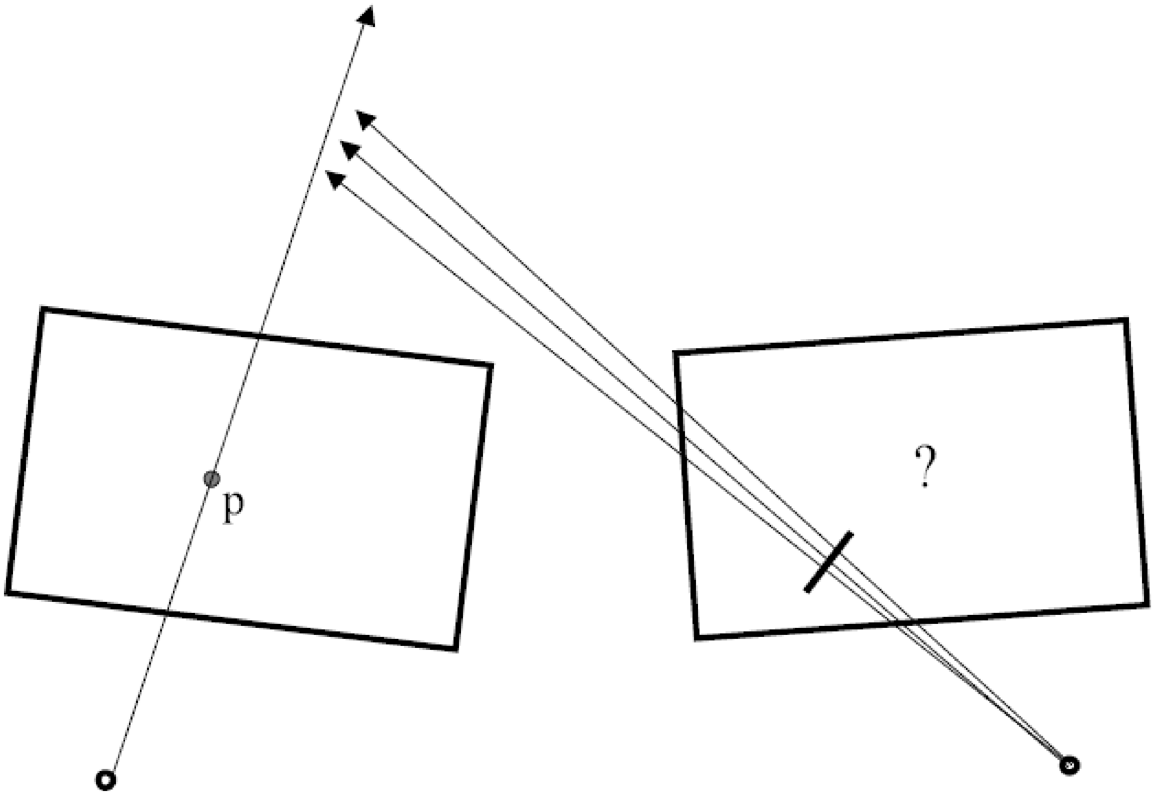

$$l_1 = \begin{bmatrix} a_1 & b_1 & c_1 \end{bmatrix}$$

$$l_2 = \begin{bmatrix} a_2 & b_2 & c_2 \end{bmatrix}$$

$$x_{12} = l_1 \times l_2$$

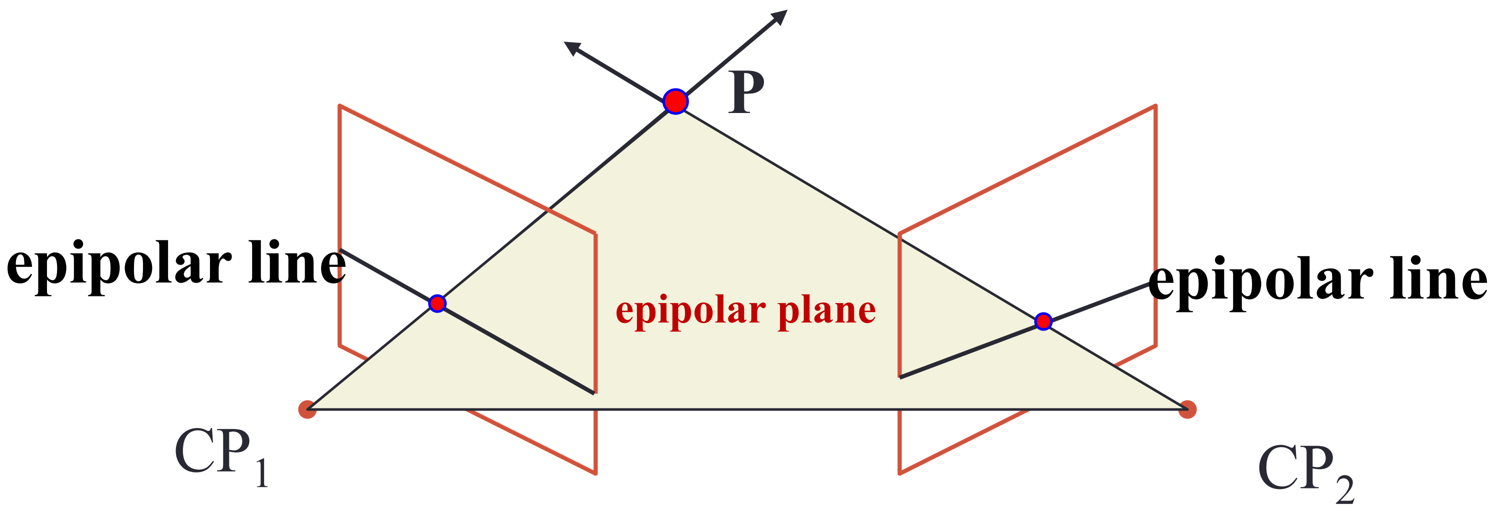
# Motivating the problem: stereo

- Given two views of a scene (the two cameras not necessarily having optical axes) what is the relationship between the location of a scene point in one image and its location in the other?



# Stereo correspondence

- Determine Pixel Correspondence
  - Pairs of points that correspond to same scene point




Epipolar Constraint

- Reduces correspondence problem to 1D search along *conjugate epipolar lines*

# Example: converging cameras

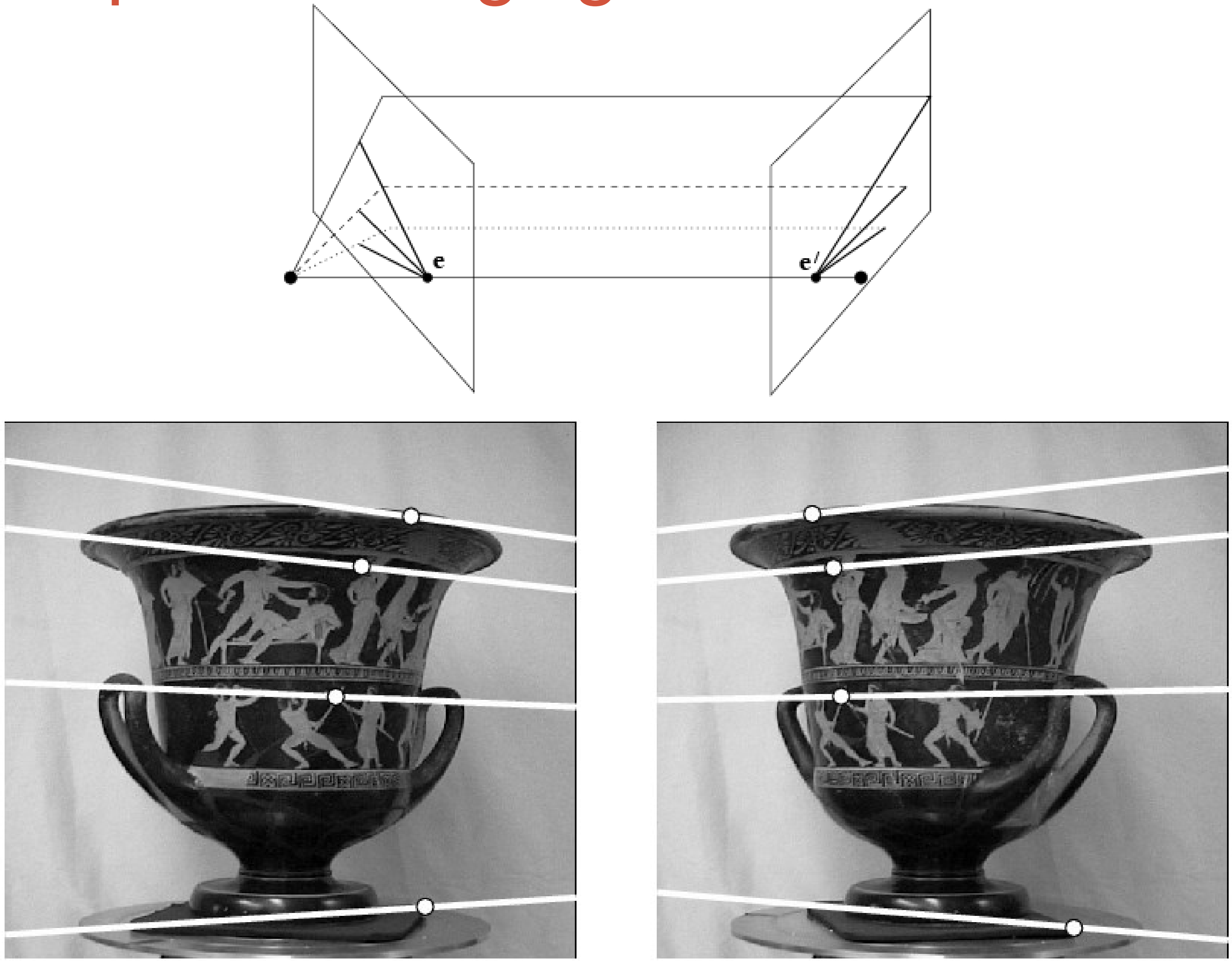
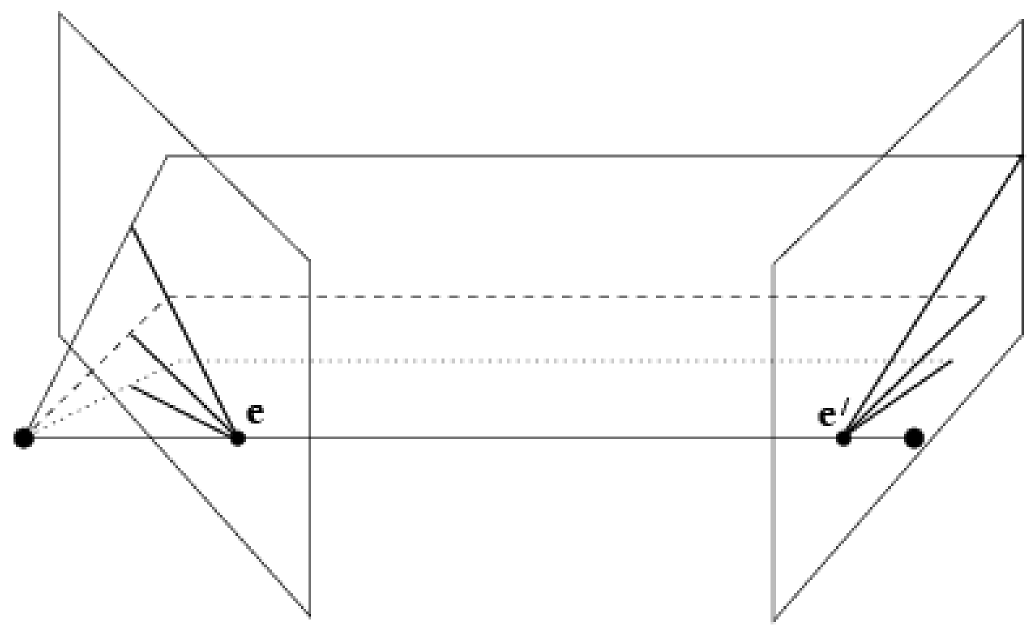



Figure from Hartley & Zisserman

# Epipolar geometry: terms

- Baseline: line joining the camera centers
- Epipole: point of intersection of baseline with image plane
- Epipolar plane: plane containing baseline and world point
- Epipolar line: intersection of epipolar plane with the image plane

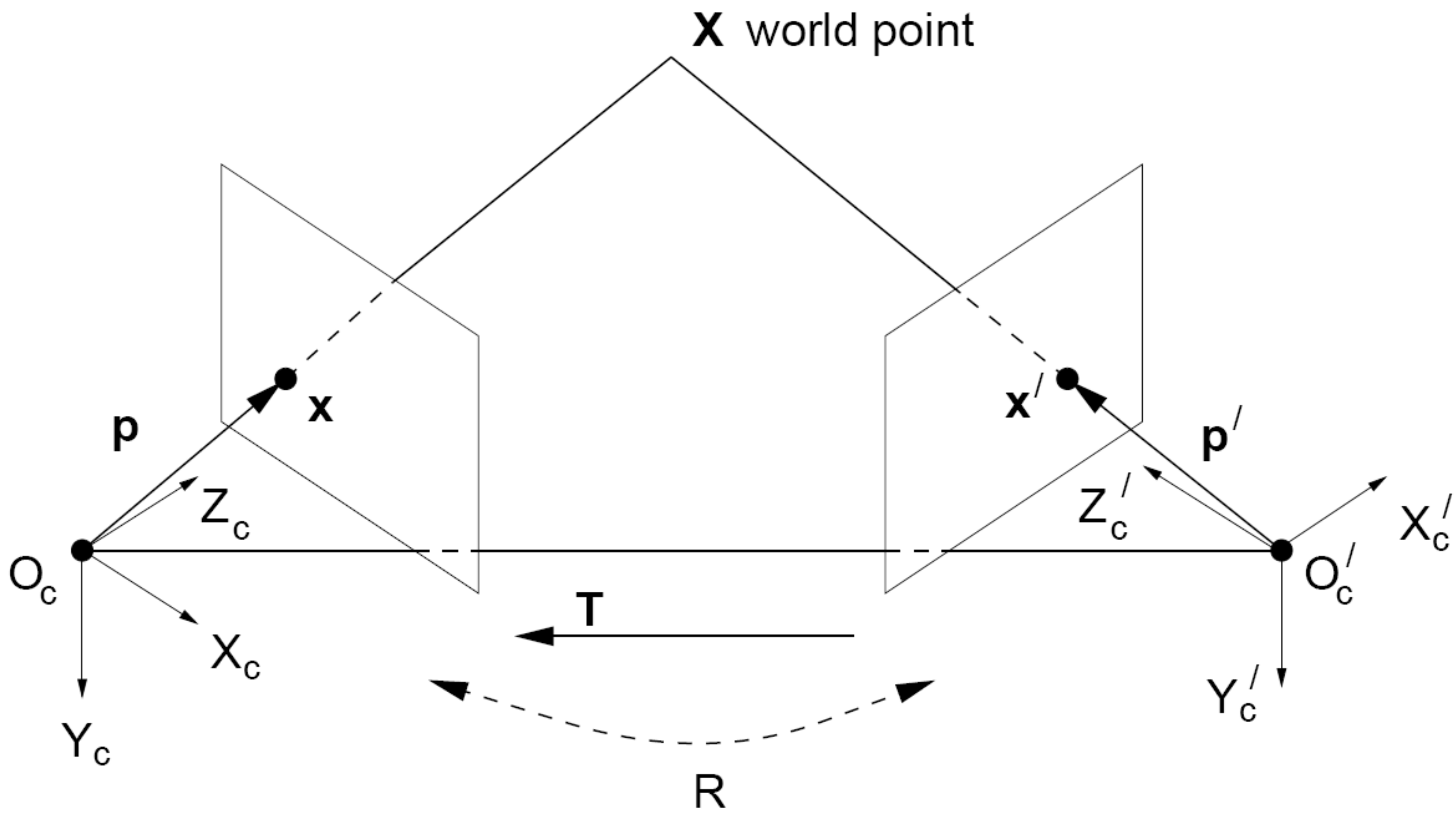
- All epipolar lines intersect at the epipole
- An epipolar plane intersects the left and right image planes in corresponding epipolar lines



# From Geometry to Algebra

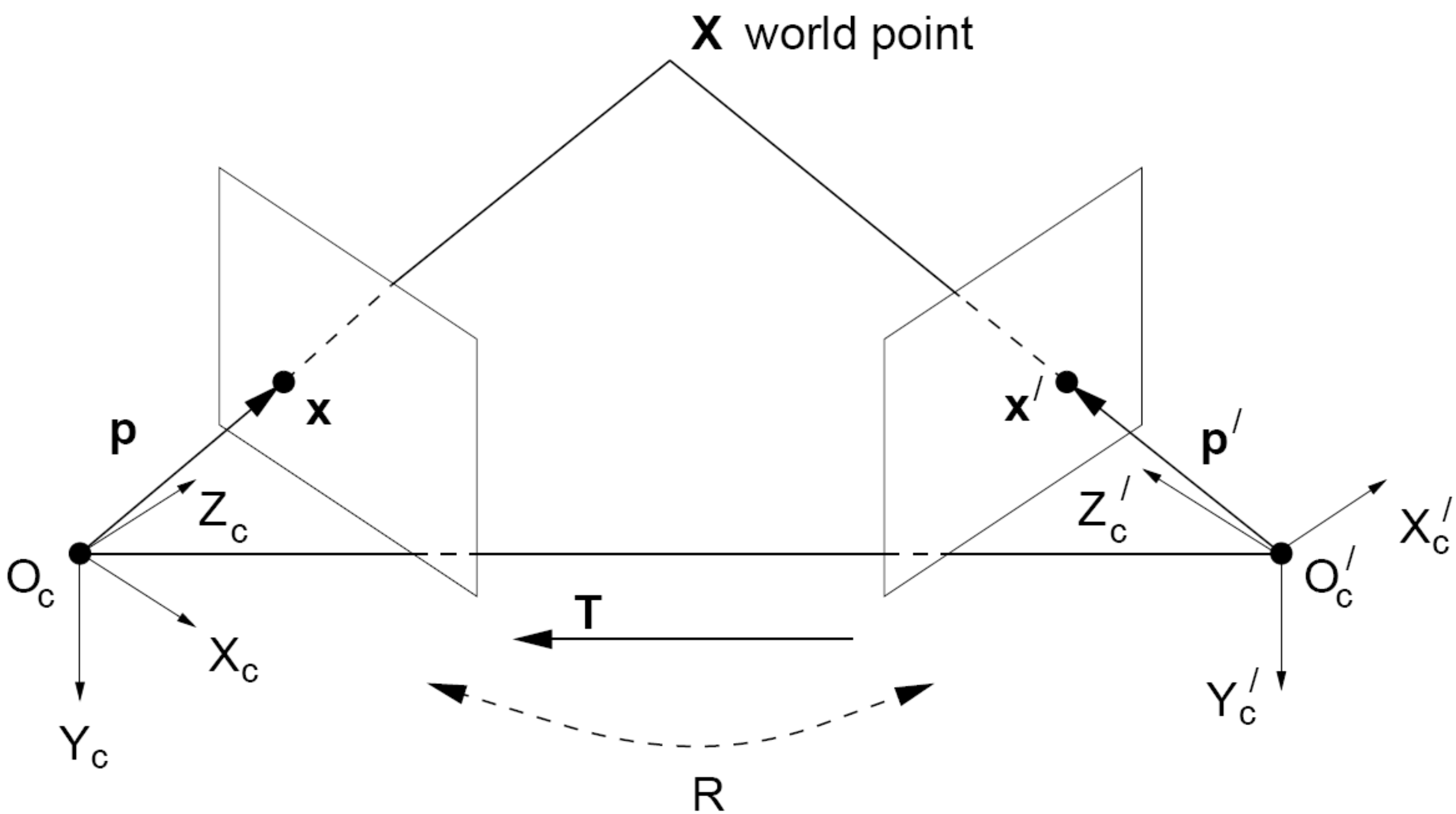
- So far, we have the explanation in terms of geometry.
- Now, how to express the epipolar constraints algebraically?

# Stereo geometry, with calibrated cameras




Main idea

# Stereo geometry, with calibrated cameras

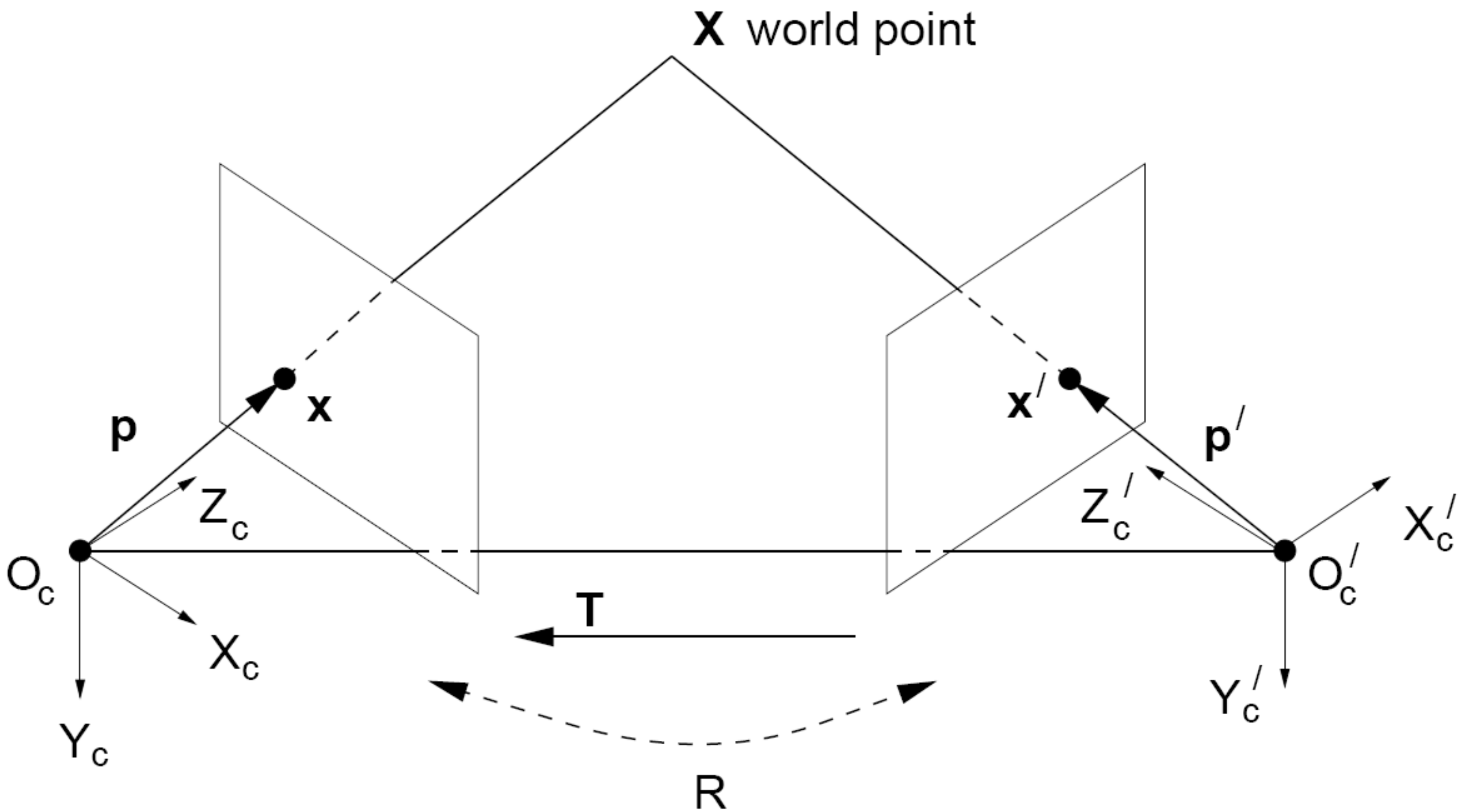



If the stereo rig is calibrated, we know :

how to **rotate** and **translate** camera reference frame 1 to get to camera reference frame 2.

Rotation: 3 x 3 matrix **R**; translation: 3 vector **T**.

# Stereo geometry, with calibrated cameras




If the stereo rig is calibrated, we know :

how to **rotate** and **translate** camera reference frame 1 to get to camera reference frame 2.

$$\mathbf{X}'_c = \mathbf{R}\mathbf{X}_c + \mathbf{T}$$

# An aside: cross product

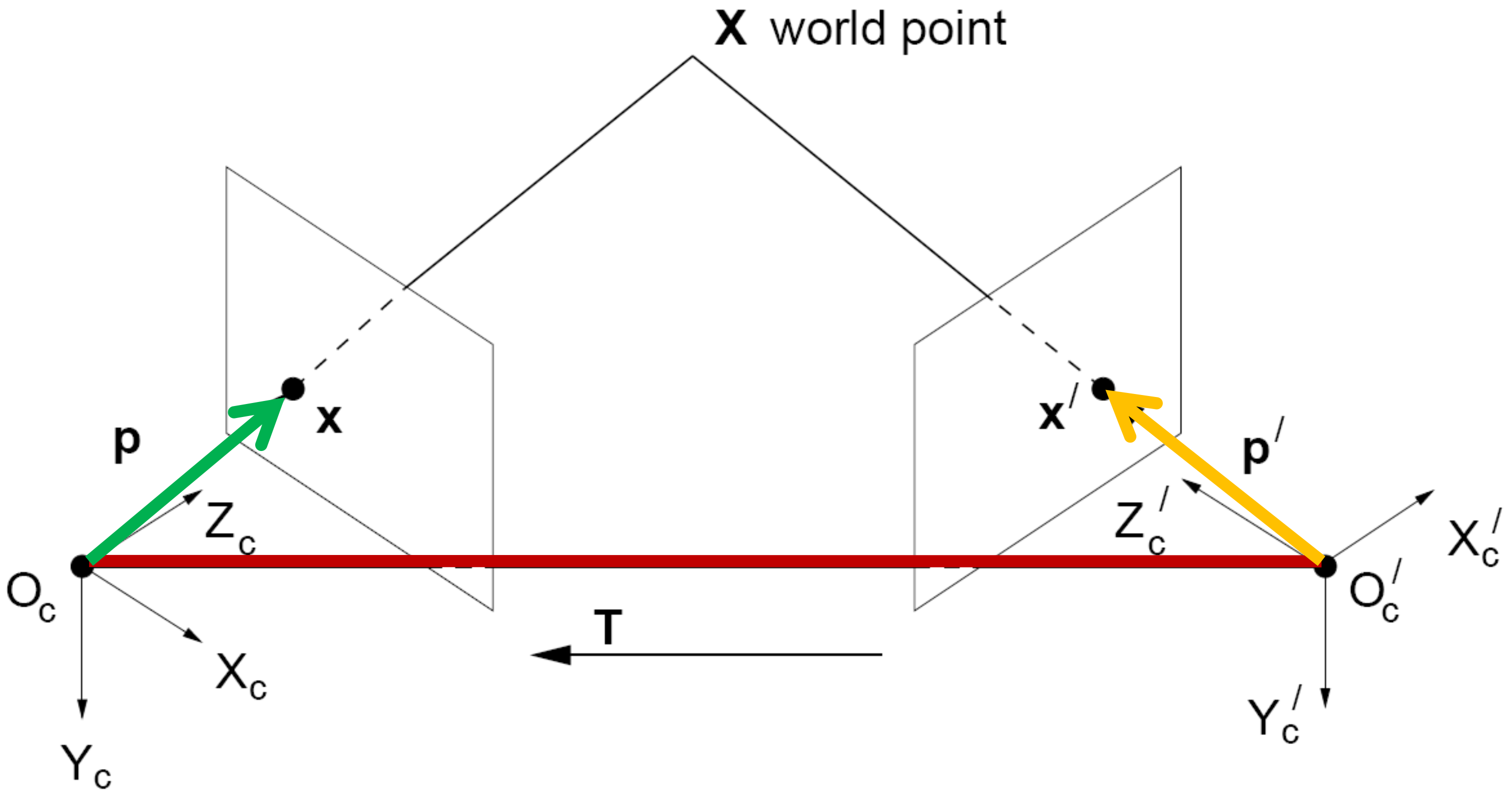
$$\vec{a} \times \vec{b} = \vec{c}$$

Vector cross product takes two vectors and returns a third vector that's perpendicular to both inputs.

So here, c is perpendicular to both a and b, which means the dot product = 0.

$$\vec{a} \cdot \vec{c} = 0$$
$$\vec{b} \cdot \vec{c} = 0$$

# From geometry to algebra




$$\boxed{\mathbf{X}'} = \mathbf{R}\boxed{\mathbf{X}} + \boxed{\mathbf{T}}$$

$$\underbrace{\mathbf{T} \times \mathbf{X}'}_{\text{Normal to the plane}} =$$

$$= \mathbf{T} \times \mathbf{R}\mathbf{X}$$

$$\mathbf{X}' \cdot (\mathbf{T} \times \mathbf{X}') = \mathbf{X}' \cdot (\mathbf{T} \times \mathbf{R}\mathbf{X})$$

$$0 = \mathbf{X}' \cdot (\mathbf{T} \times \mathbf{R}\mathbf{X})$$

# Another aside: Matrix form of cross product

$$\vec{a} \times \vec{b} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \vec{c}$$

***Can be expressed as a matrix multiplication!!!***

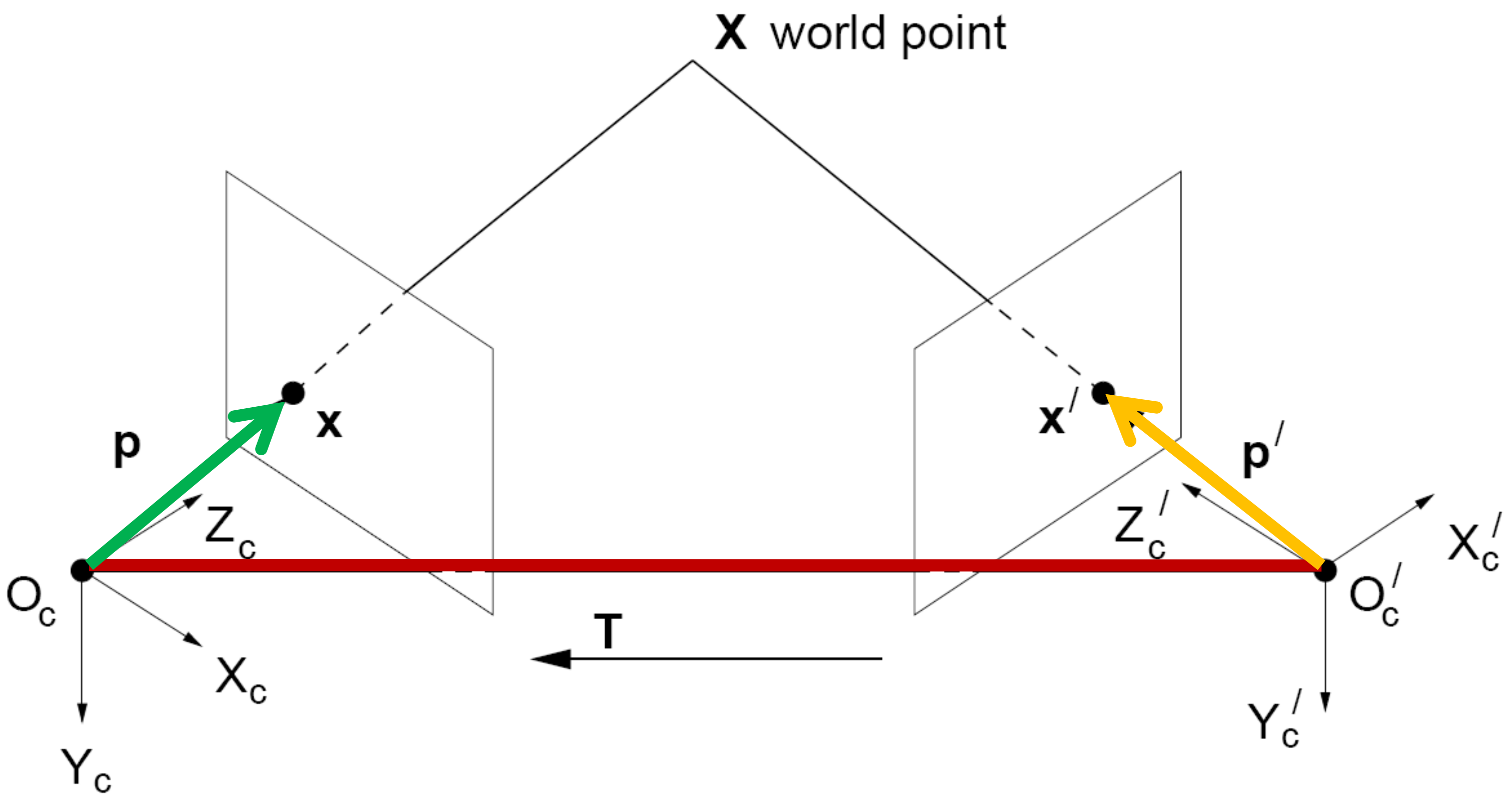
$$[a_x] = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

Notation:

$$\vec{a} \times \vec{b} = [a_\times]\vec{b}$$

***Has rank 2!***

# From geometry to algebra




$$\boxed{\mathbf{X}'} = \mathbf{R}\boxed{\mathbf{X}} + \boxed{\mathbf{T}}$$

$$\underbrace{\mathbf{T} \times \mathbf{X}'}_{\text{Normal to the plane}} = \mathbf{T} \times \mathbf{R}\mathbf{X}$$

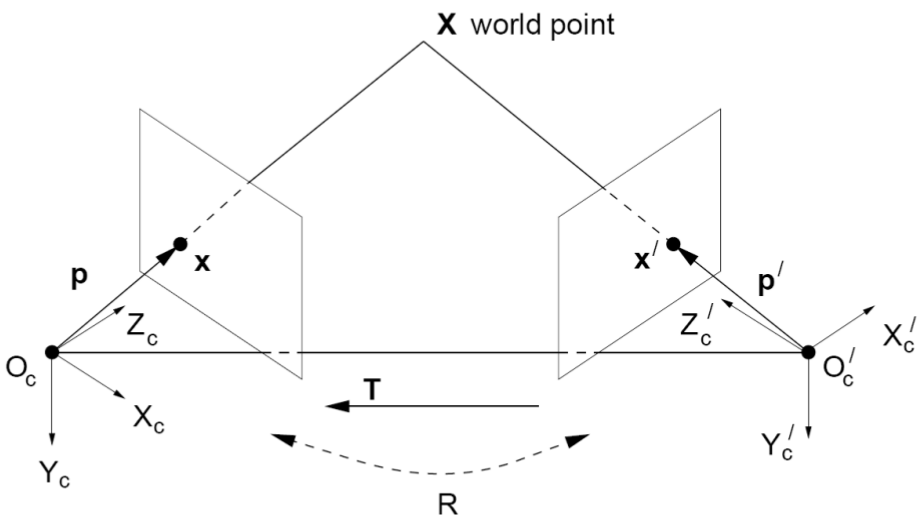
$$\mathbf{X}' \cdot (\mathbf{T} \times \mathbf{X}') = \mathbf{X}' \cdot (\mathbf{T} \times \mathbf{R}\mathbf{X})$$

$$\boxed{0 = \mathbf{X}' \cdot (\mathbf{T} \times \mathbf{R}\,\mathbf{X})}$$

# Essential matrix

$$\boxed{\mathbf{X}' \cdot (\mathbf{T} \times \mathbf{RX}) = 0}$$

$$\mathbf{X}' \cdot ([\mathrm{T}_x]\mathbf{RX}) = 0$$




Let $\mathbf{E} = [\mathrm{T}_x]\mathbf{R}$

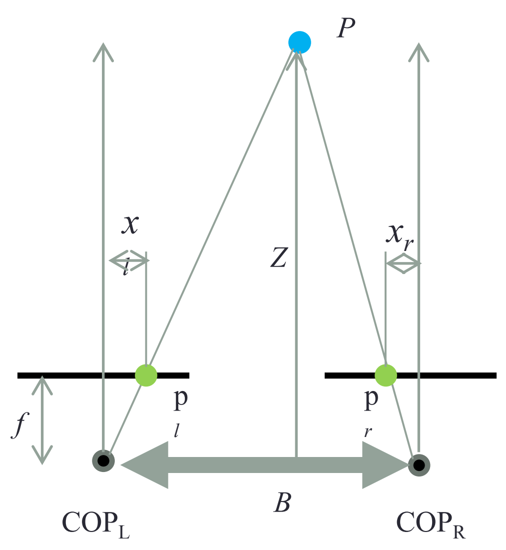
$$\mathbf{X}'^{T}\mathbf{EX} = 0$$

**E** is called the **essential matrix**, and it relates corresponding image points between both cameras, given the rotation and translation.

Note: these points are in **each camera coordinate systems**.

We know if we observe a point in one image, its position in other image is constrained to lie on line defined by above.

# Essential matrix example: parallel cameras




$$\mathbf{R} =$$

$$\mathbf{T} =$$

$$\mathbf{E} = [\mathrm{T}_{\times}]\mathbf{R} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & B \\ 0 & -B & 0 \end{pmatrix}$$

$$\mathbf{p} = [Zx, Zy, \frac{Z}{f}]$$

$$\mathbf{p}' = [Zx', Zy', \frac{Z}{f}]$$

$$\mathbf{p}'^{T}\mathbf{E}\mathbf{p} = 0$$

$$\begin{bmatrix} x' & y' & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & B \\ 0 & -B & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} x' & y' & 1 \end{bmatrix} \begin{bmatrix} 0 \\ B \\ -By \end{bmatrix} = 0$$

Given a known point (x,y) in the original image, this is a *line* in the (x',y') image.

$$By' = By \Rightarrow \boxed{\mathbf{y}' = \mathbf{y}}$$

**For the parallel cameras, image of any point must lie on same horizontal line in each image plane.**

# Weak calibration

- Want to estimate world geometry without requiring calibrated cameras
  - Archival videos (already have the pictures)
  - Photos from multiple unrelated users
  - Dynamic camera system

- **Main idea**:
  - Estimate epipolar geometry from a (redundant) set of point correspondences between two uncalibrated cameras

# From before: Projection matrix

- This can be rewritten as a matrix product using homogeneous coordinates:

$$\begin{bmatrix} w\,x_{im} \\ w\,y_{im} \\ w \end{bmatrix} = \mathbf{K}_{int}\,\mathbf{\Phi}_{ext} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

where:

$$\mathbf{\Phi}_{ext} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & -\mathbf{R}_1^{\mathrm{T}}\mathbf{T} \\ r_{21} & r_{22} & r_{23} & -\mathbf{R}_2^{\mathrm{T}}\mathbf{T} \\ r_{31} & r_{32} & r_{33} & -\mathbf{R}_3^{\mathrm{T}}\mathbf{T} \end{bmatrix}$$

$$\mathbf{K}_{int} = \begin{bmatrix} -f/s_x & 0 & o_x \\ 0 & -f/s_y & o_y \\ 0 & 0 & 1 \end{bmatrix}$$

***Note: Invertible, scale x and y, assumes no skew***

# From before: Projection matrix

- This can be rewritten as a matrix product using homogeneous coordinates:

$$\begin{bmatrix} w x_{im} \\ w y_{im} \\ w \end{bmatrix} = \mathbf{K}_{int} \mathbf{\Phi}_{ext} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

$$\mathbf{p}_{im} = \mathbf{K}_{int} \underbrace{\mathbf{\Phi}_{ext} \mathbf{P}_w}_{\mathbf{p}_c}$$

$$\mathbf{p}_{im} = \mathbf{K}_{int} \mathbf{p}_c$$

# Uncalibrated case

For a given camera:

$$\mathbf{p}_{im} = \mathbf{K}_{int}\mathbf{p}_{c}$$

So, for **two** cameras (left and right):

$$\mathbf{p}_{c,left} = \mathbf{K}_{int,left}^{-1}\mathbf{p}_{im,left}$$

$$\mathbf{p}_{c,right} = \underbrace{\mathbf{K}_{int,right}^{-1}}_{}\mathbf{p}_{im,right}$$

Internal calibration matrices, one per camera

# Uncalibrated case

$$\mathbf{p}_{c,right} = \mathbf{K}_{int,right}^{-1}\mathbf{p}_{im,right} \qquad \mathbf{p}_{c,left} = \mathbf{K}_{int,left}^{-1}\mathbf{p}_{im,left}$$

*From before, the* ***essential*** *matrix* ***E.***

$$\boxed{\mathbf{p}_{c,right}^{\mathrm{T}}\mathbf{E}\mathbf{p}_{c,left} = 0}$$

$$\left(\mathbf{K}_{int,right}^{-1}\mathbf{p}_{im,right}\right)^{\mathrm{T}}\mathbf{E}\left(\mathbf{K}_{int,left}^{-1}\mathbf{p}_{im,left}\right) = 0$$

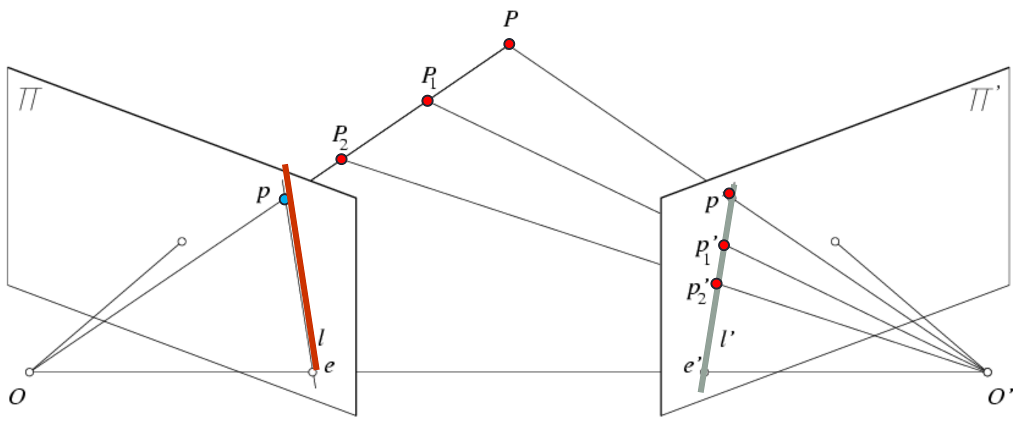
$$\mathbf{p}_{im,right}^{\mathrm{T}}\left(\underbrace{\left(\mathbf{K}_{int,right}^{-1}\right)^{T}\mathbf{E}\,\mathbf{K}_{int,left}^{-1}}_{\mathbf{F}}\right)\mathbf{p}_{im,left} = 0$$

"Fundamental matrix" $\mathbf{F}$

$$\boxed{\mathbf{p}_{im,right}^{\mathrm{T}}\mathbf{F}\mathbf{p}_{im,left} = 0} \qquad or \qquad \mathbf{p}^{T}\mathbf{F}\,\mathbf{p}' = 0$$

# Properties of the Fundamental Matrix

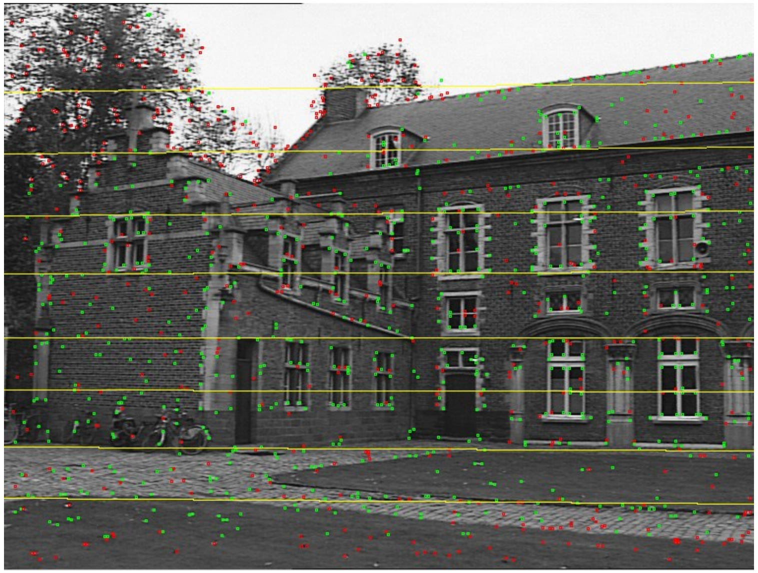
$$\mathbf{p}^T \mathbf{F} \mathbf{p}' = 0$$



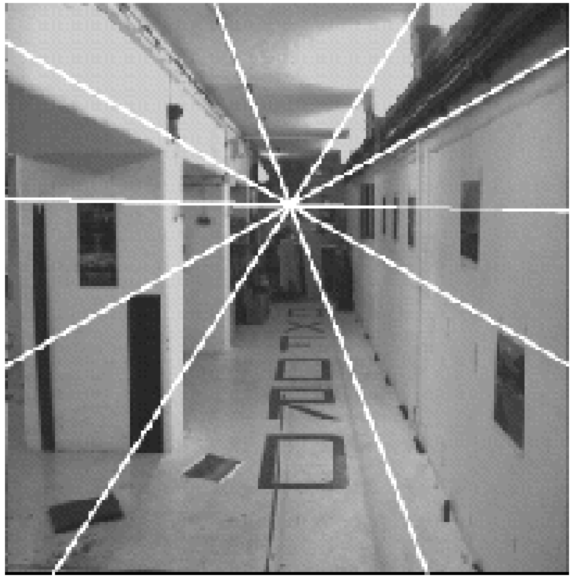

- $\mathbf{l} = \mathbf{F}\mathbf{p}'$ is the epipolar line associated with $\mathbf{p}'$
- $\mathbf{l}' = \mathbf{F}^T\mathbf{p}$ is the epipolar line associated with $\mathbf{p}$
- Epipoles found by $\mathbf{F}\mathbf{p}' = \mathbf{0}$ and $\mathbf{F}^T\mathbf{p} = \mathbf{0}$
  - You'll see more one these on the problem set to explain
- $\mathbf{F}$ is singular (mapping from 2-D point to 1-D family so rank 2 – more later)

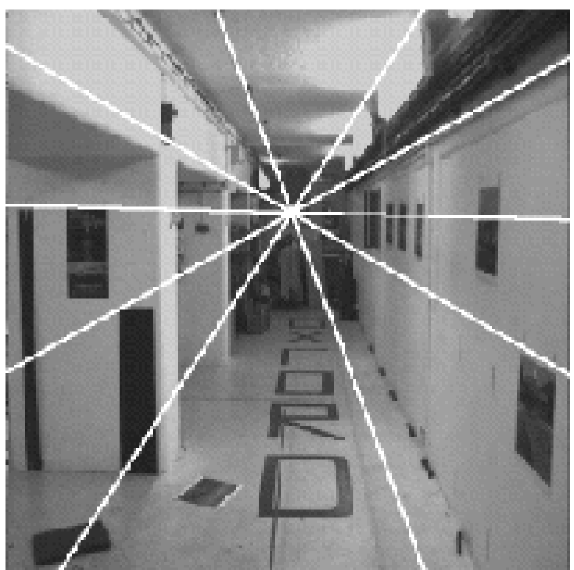
# Fundamental matrix

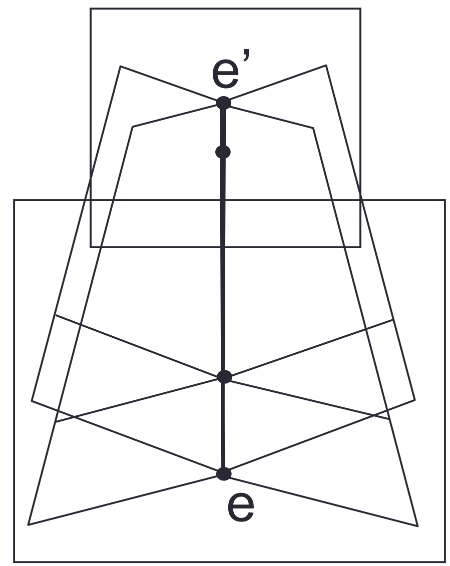
- Relates pixel coordinates in the two views
- More general form than essential matrix: we remove need to know intrinsic parameters
- If we estimate fundamental matrix from correspondences in pixel coordinates, can reconstruct epipolar geometry without intrinsic or extrinsic parameters.



# Different Example: forward motion








courtesy of Andrew Zisserman

# Computing F from correspondences

Each point correspondence generates one constraint on F

$$\mathbf{p}_{im,right}^{T}\,\mathbf{F}\mathbf{p}_{im,left} = 0$$

$$\begin{bmatrix} u' & v' & 1 \end{bmatrix} \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = 0$$

Collect n of these constraints

$$\begin{bmatrix} u'_1u_1 & u'_1v_1 & u'_1 & v'_1u_1 & v'_1v_1 & v'_1 & u_1 & v_1 & 1 \end{bmatrix} \begin{bmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \\ f_{33} \end{bmatrix} = \mathbf{0}$$

Solve for f , vector of parameters.

# The (in)famous "eight-point algorithm"

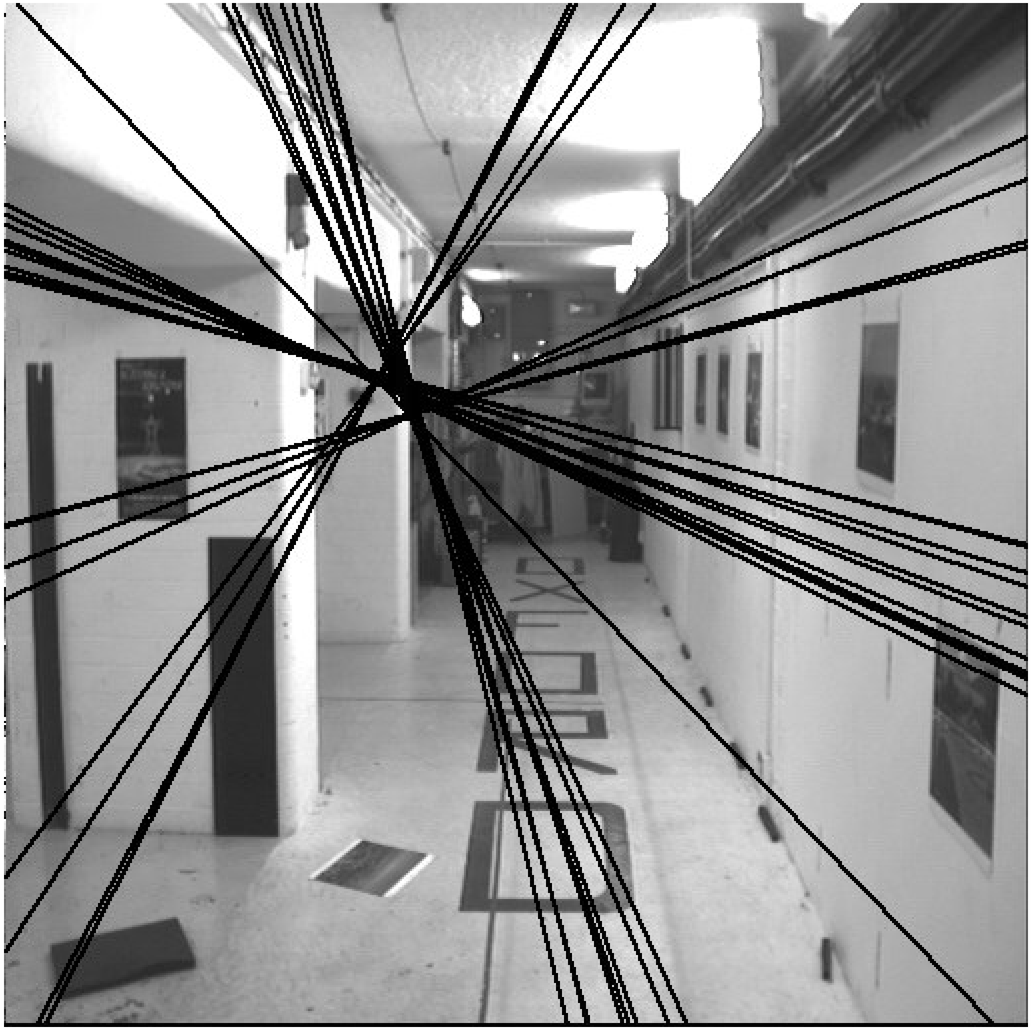
| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 250906.36 | 183269.57 | 921.81 | 200931.10 | 146766.13 | 738.21 | 272.19 | 198.81 | 1.00 |
| 2692.28 | 131633.03 | 176.27 | 6196.73 | 302975.59 | 405.71 | 15.27 | 746.79 | 1.00 |
| 416374.23 | 871684.30 | 935.47 | 408110.89 | 854384.92 | 916.90 | 445.10 | 931.81 | 1.00 |
| 191183.60 | 171759.40 | 410.27 | 416435.62 | 374125.90 | 893.65 | 465.99 | 418.65 | 1.00 |
| 48988.86 | 30401.76 | 57.89 | 298604.57 | 185309.58 | 352.87 | 846.22 | 525.15 | 1.00 |
| 164786.04 | 546559.67 | 813.17 | 1998.37 | 6628.15 | 9.86 | 202.65 | 672.14 | 1.00 |
| 116407.01 | 2727.75 | 138.89 | 169941.27 | 3982.21 | 202.77 | 838.12 | 19.64 | 1.00 |
| 135384.58 | 75411.13 | 198.72 | 411350.03 | 229127.78 | 603.79 | 681.28 | 379.48 | 1.00 |

$$\begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = 0$$

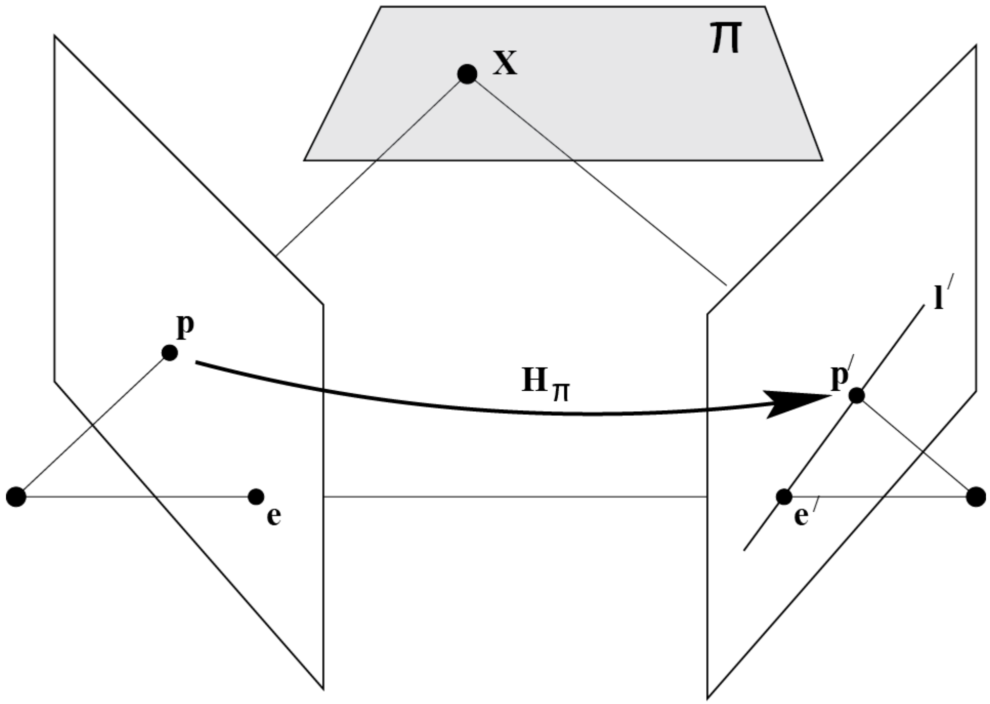
- In principal can solve with 8 points.
- Better with more – yields homogeneous linear least-squares:
  - Find unit norm vector F yielding smallest residual
  - Remember SVD or substitute a 1?

- What happens when there is noise?

# Doing the obvious thing



# Rank of F




- Assume we know the homography $H_\pi$ that maps from Left to Right (Full 3x3)

$$\mathbf{p}' = \mathbf{H}_\pi \mathbf{p}$$

- Let line $\mathbf{l}'$ be the epiloar line corresponding to $\mathbf{p}$ – goes through epipole $\mathbf{e}'$

- So: $\mathbf{l}' = \mathbf{e}' \times \mathbf{p}'$

$$= \mathbf{e}' \times \mathbf{H}_\pi \mathbf{p}$$

$$= [\mathbf{e}']_\times \mathbf{H}_\pi \mathbf{p}$$

$$= \mathbf{F}\ \mathbf{p}$$

- Rank of F is rank of $[\mathbf{e}']_\mathbf{x}$ = 2

# Fix the linear solution

- Use SVD or other method to do linear computation for F
- Decompose F using SVD (not the same SVD):

$$\mathbf{F} = U D V^T$$

- Set the last singular value to zero:

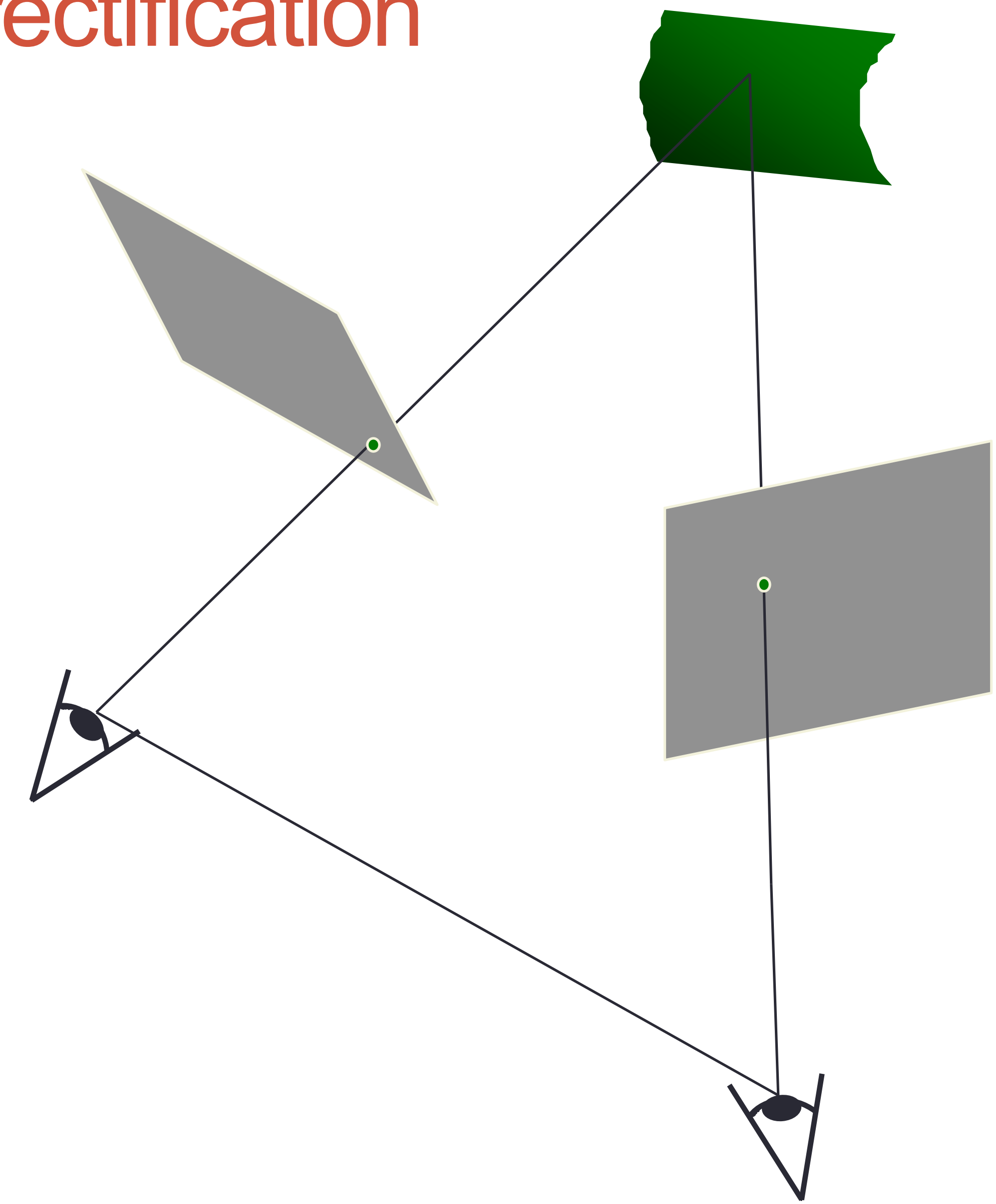
$$D = \begin{bmatrix} r & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & t \end{bmatrix} \Rightarrow \hat{D} = \begin{bmatrix} r & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Estimate new F from the new $\hat{D}$

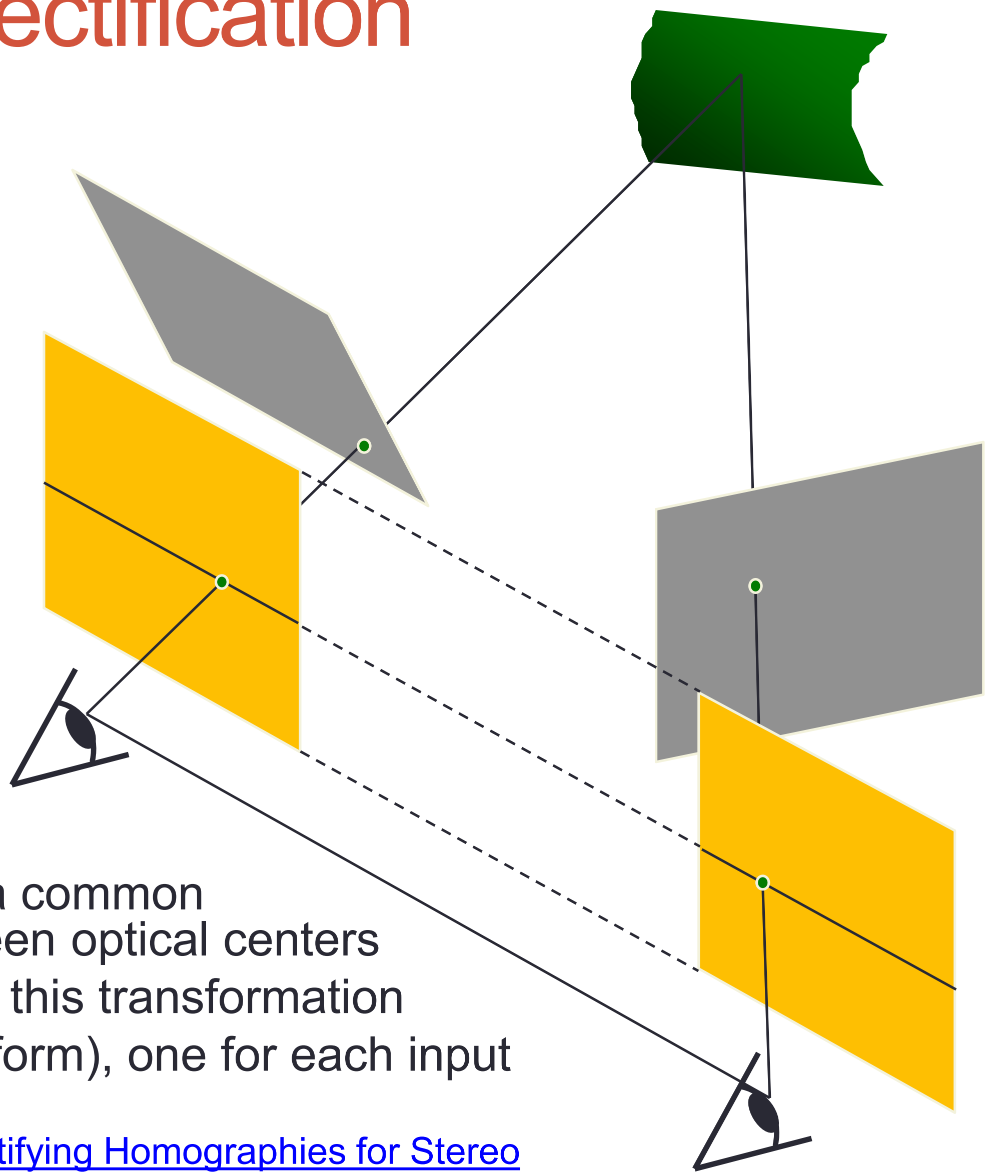
$$\hat{\mathbf{F}} = U \hat{D} V^T$$

# That's better…

# Stereo image rectification



# Stereo image rectification



- Reproject image planes onto a common plane parallel to the line between optical centers
- Pixel motion is horizontal after this transformation
- Two homographies (3x3 transform), one for each input image reprojection

➢ C. Loop and Z. Zhang. [Computing Rectifying Homographies for Stereo Vision](). IEEE Conf. Computer Vision and Pattern Recognition, 1999.

# Rectification Example

C. Loop and Z. Zhang, [Computing Rectifying Homographies for Stereo Vision](), IEEE Conf. Computer Vision and Pattern Recognition, 1999.

(a) Original image pair overlayed with several epipolar lines.

(b) Image pair transformed by the specialized projective mapping $H_p$ and $H'_p$. Note that the epipolar lines are now parallel to each other in each image.

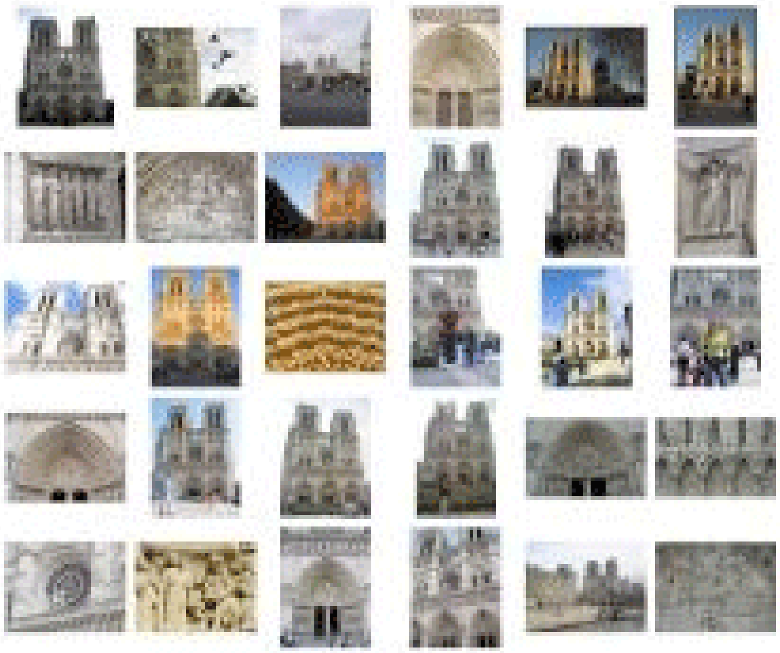
(c) Image pair transformed by the similarity $H_r$ and $H'_r$. Note that the image pair is now rectified (the epipolar lines are horizontally aligned).

(d) Final image rectification after shearing transform $H_s$ and $H'_s$. Note that the image pair remains rectified, but the horizontal distortion is reduced.

# Some example cool applications…

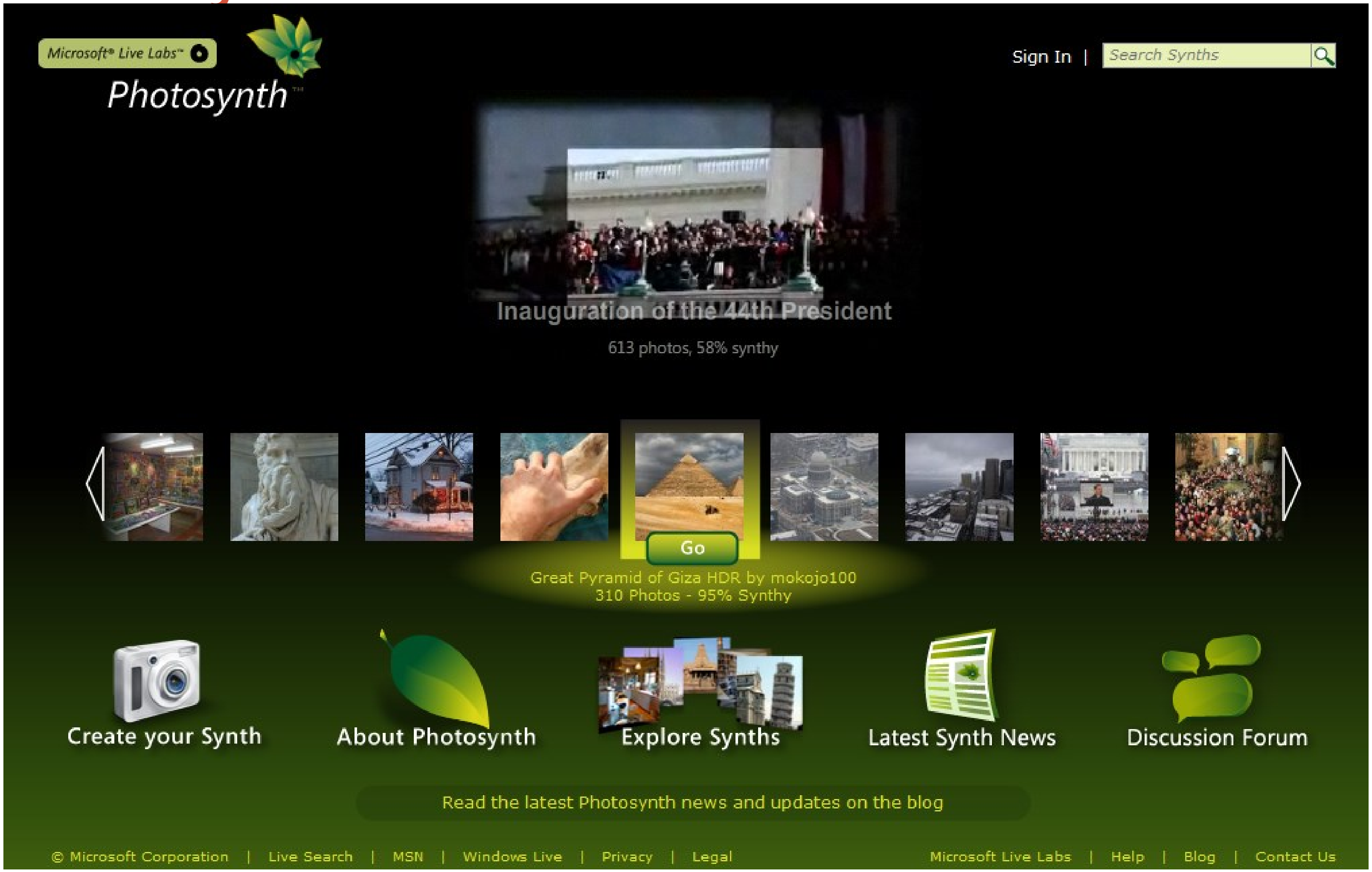
# Photo synth

Noah Snavely, Steven M. Seitz, Richard Szeliski, "Photo tourism: Exploring photo collections in 3D," SIGGRAPH 2006



http://photosynth.net/

# Photosynth.net




Based on Photo Tourism
by Noah Snavely, Steve Seitz, and Rick Szeliski

# 3D from multiple images

*Building Rome in a Day: Agarwal et al. 2009*

# Summary

- For 2-views, there is a geometric relationship that define the relations between rays in one view to rays in the other
  - Calibrated – Essential matrix
  - Uncalibrated – Fundamental matrix.
- This relation can be estimated from point correspondences – both in calibrated cases and uncalibrated.
- Extensions allow combining multiple views to get more geometric information about scenes
  - SLAM (simultaneous localization and mapping) – you'll hear about this (I hope!)