part of the story. More than the knowledge, people want *personal understanding*. And in our credit-driven system, they also want and need *theorem-credits*

I'll skip ahead a few years, to the subject that Jaffe and Quinn alluded to, when I began studying 3-dimensional manifolds and their relationship to hyperbolic geometry. (Again, it matters little if you know what this is about.) I gradually built up over a number of years a certain intuition for hyperbolic three-manifolds, with a repertoire of constructions, examples and proofs. (This process actually started when I was an undergraduate, and was strongly bolstered by applications of foliations.) After a while, I conjectured or speculated that all three-manifolds have a certain geometric structure; this conjecture eventually became known as the geometrization conjecture. About two or three years later, I proved the geometrization theorem for Haken manifolds. It was a hard theorem, and I spent a tremendous amount of effort thinking about it. When I completed the proof, I spent a lot more effort checking the proof, searching for difficulties and testing it against independent information.

I'd like to spell out more what I mean when I say I proved this theorem. It meant that I had a clear and complete flow of ideas, including details, that withstood a great deal of scrutiny by myself and by others. Mathematicians have many different styles of thought. My style is not one of making broad sweeping but careless generalities, which are merely hints of inspirations: I make clear mental models, and I think things through. My proofs have turned out to be quite reliable. I have not had trouble backing up claims or producing details for things I have proven. I am good in detecting flaws in my own reasoning as well as in the reasoning of others.

However, there is sometimes a huge expansion factor in translating from the encoding in my own thinking to something that can be conveyed to someone else. My mathematical education was rather independent and idiosyncratic, where for a number of years I learned things on my own, developing personal mental models for how to think about mathematics. This has often been a big advantage for me in thinking about mathematics, because it's easy to pick up later the standard mental models shared by groups of mathematicians. This means that some concepts that I use freely and naturally in my personal thinking are foreign to most mathematicians I talk to. My personal mental models and structures are similar in character to the kinds of models groups of mathematicians share—but they are often different models. At the time of the formulation of the geometrization conjecture, my understanding of hyperbolic geometry was a good example. A random continuing example is an understanding of finite topological spaces, an oddball topic that can lend good insight to a variety of questions but that is generally not worth developing in any one case because there are standard circumlocutions that avoid it.

Neither the geometrization conjecture nor its proof for Haken manifolds was in the path of any group of mathematicians at the time—it went against the trends in topology for the preceding 30 years, and it took people by surprise. To most topologists at the time, hyperbolic geometry was an arcane side branch of mathematics, although there were other groups of mathematicians such as differential geometers who did understand it from certain points of view. It took topologists a while just to understand what the geometrization conjecture meant, what it was good for, and why it was relevant.

At the same time, I started writing notes on the geometry and topology of 3-manifolds, in conjunction with the graduate course I was teaching. I distributed them to a few people, and before long many others from around the world were writing for copies. The mailing list grew to about 1200 people to whom I was sending notes every couple of months. I tried to communicate my real thoughts in these notes. People ran many seminars based on my notes, and I got lots of feedback. Overwhelmingly, the feedback ran something like "Your notes are really inspiring and beautiful, but I have to tell you that we spent 3 weeks in our seminar working out the details of §*n.n* More explanation would sure help."

I also gave many presentations to groups of mathematicians about the ideas of studying 3-manifolds from the point of view of geometry, and about the proof of the geometrization conjecture for Haken manifolds. At the beginning, this subject was foreign to almost everyone. It was hard to communicate—the infrastructure was in my head, not in the mathematical community. There were several mathematical theories that fed into the cluster of ideas: three-manifold topology, Kleinian groups, dynamical systems, geometric topology, discrete subgroups of Lie groups, foliations, Teichmüller spaces, pseudo-Anosov diffeomorphisms, geometric group theory, as well as hyperbolic geometry.

We held an AMS summer workshop at Bowdoin in 1980, where many mathematicians in the subfields of low-dimensional topology, dynamical systems and Kleinian groups came.

It was an interesting experience exchanging cultures. It became dramatically clear how much proofs depend on the audience. We prove things in a social context and address them to a certain audience. Parts of this proof I could communicate in two minutes to the topologists, but the analysts would need an hour lecture before they would begin to understand it. Similarly, there were some things that could be said in two minutes to the analysts that would take an hour before the topologists would begin to get it. And there were many other parts of the proof which should take two minutes in the abstract, but that none of the audience at the time had the mental infrastructure to get in less than an hour.

At that time, there was practically no infrastructure and practically no context for this theorem, so the expansion from how an idea was keyed in my head to what I had to say to get it across, not to mention how much energy the audience had to devote to understand it, was very dramatic.

In reaction to my experience with foliations and in response to social pressures, I concentrated most of my attention on developing and presenting the infrastructure in what I wrote and in what I talked to people about. I explained the details to the few people who were "up" for it I wrote some papers giving the substantive parts of the proof of the geometrization theorem for Haken manifolds—for these papers, I got almost no feedback. Similarly, few people actually worked through the harder and deeper sections of my notes until much later.

The result has been that now quite a number of mathematicians have what was dramatically lacking in the beginning: a working understanding of the concepts and the infrastructure that are natural for this subject There has been and there continues to be a great deal of thriving mathematical activity. By concentrating on building the infrastructure and explaining and publishing definitions and ways of thinking but being slow in stating or in publishing proofs of all the "theorems" I knew how to prove, I left room for many other people to pick up credit. There has been room for people to discover and publish other proofs of the geometrization theorem. These proofs helped develop mathematical concepts which are quite interesting in themselves, and lead to further mathematics

What mathematicians most wanted and needed from me was to learn my ways of thinking, and not in fact to learn my proof of the general geometrization conjecture for Haken manifolds. It is unlikely that the proof of the general geometrization conjecture will consist of pushing the same proof further.

A further issue is that people sometimes need or want an accepted and validated result not in order to learn it, but so that they can quote it and rely on it.

Mathematicians were actually very quick to accept my proof, and to start quoting it and using it based on what documentation there was, based on their experience and belief in me, and based on acceptance by opinions of experts with whom I spent a lot of time communicating the proof The theorem now is documented, through published sources authored by me and by others, so most people feel secure in quoting it; people in the field certainly have not challenged me about its validity, or expressed to me a need for details that are not available

Not all proofs have an identical role in the logical scaffolding we are building for mathematics. This particular proof probably has only temporary logical value, although it has a high motivational value in helping support a certain vision for the structure of 3-manifolds. The full geometrization conjecture is still a conjecture. It has been proven for many cases, and is supported by a great deal of computer evidence as well, but it has not been proven in generality. I am convinced that the general proof will be discovered; I hope before too many more years. At that point, proofs of special cases are likely to become obsolete.

Meanwhile, people who want to use the geometric technology are better off to start off with the assumption "Let M^3 be a manifold that admits a geometric decomposition," since this is more general than "Let M^3 be a Haken manifold." People who don't want to use the technology or who are suspicious of it can avoid it. Even when a theorem about Haken manifolds can be proven using geometric techniques, there is a high value in finding purely topological techniques to prove it.

In this episode (which still continues) I think I have managed to avoid the two worst possible outcomes: either for me not to let on that I discovered what I discovered and proved what I proved, keeping it to myself (perhaps with the hope of proving the Poincaré conjecture), or for me to present an unassailable and hard-to-learn theory with no practitioners to keep it alive and to make it grow.

I can easily name regrets about my career. I have not published as much as I should. There are a number of mathematical projects in addition to the geometrization theorem for Haken manifolds that I have not delivered well or at all to the mathematical public. When I concentrated more on developing the infrastructure rather than the top-level theorems in the geometric theory of 3-manifolds, I became somewhat disengaged as the subject continued to evolve; and I have not actively or effectively promoted the field or the careers of the excellent people in it (But some degree of disengagement seems to me an almost inevitable byproduct of the mentoring of graduate students and others: in order to really turn genuine research directions over to others, it's necessary to really let go and stop oneself from thinking about them very hard.)

On the other hand, I have been busy and productive, in many different activities. Our system does not create extra time for people like me to spend on writing and research; instead, it inundates us with many requests and opportunities for extra work, and my gut reaction has been to say 'yes' to many of these requests and opportunities. I have put a lot of effort into non-credit-producing activities that I value just as I value proving theorems: mathematical politics, revision of my notes into a book with a high standard of communication, exploration of computing in mathematics, mathematical education, development of new forms for communication of mathematics through the Geometry Center (such as our first experiment, the "Not Knot" video), directing MSRI, *etc.*

I think that what I have done has not maximized my "credits" I have been in a position not to feel a strong need to compete for more credits Indeed, I began to feel strong challenges from other things besides proving new theorems.

I do think that my actions have done well in stimulating mathematics.

Notes

[1] Reprinted from "Proof and Progress in Mathematics", by William P. Thurston, *Bulletin of the American Mathematical Society* 30, Number 7, April 1994, pp. 161-177, by permission of the Author

[2] Arthur Jaffee and Frank Quinn, "Theoretical Mathematics: Towards a Cultural Synthesis of Mathematics and Theoretical Physics", *Bulletin of the American Mathematical Society* 29, Number 1, July 1993

Technological Pedagogical Content Knowledge in the Mathematics Classroom

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Abstract

Teacher knowledge has long been a focus of many educational researchers. Current conceptualizations of teacher knowledge are beginning to reflect the knowledge and skills teachers need to successfully navigate increasingly technologically-rich mathematical classrooms with the addition of knowledge domains such as technological pedagogical content knowledge (TPACK). This article situates TPACK in the mathematics classroom by developing four central components of knowledge necessary for technology-using mathematics teachers. This article concludes by presenting a portrait of effective TPACK in action and posing questions for technology-using teachers to consider as they embark on technology use in support of mathematics instruction. The intent of this article is not to offer a one-size-fits-all solution to the many issues surrounding technology use, but to provide the impetus for discussion and reflection among mathematics educators at all levels. (Keywords: TPACK, teacher knowledge, technology, mathematics)

Research on teachers' knowledge and beliefs has long remained a substantial area of inquiry in our explorations of the nature of teaching. How teachers' understanding of teaching, learning, students, and subject matter affects their everyday practice is an important aspect of our quest to understand the complex nature of teaching and the professional knowledge necessary for effective teaching. The influence of cognitive psychology on our understanding of learning has resulted in a number of ways to characterize the structure of knowledge in the individual (Borko & Putnam, 1996).

The development and analysis of teachers' formal knowledge by various researchers and educators is an effort to explain various areas of expertise related to teaching and to develop rationale theories on which to base practice (Friedson, 1986). Koehler and Mishra (2008) argue that teachers are autonomous agents with the power to significantly influence appropriate and inappropriate teaching. Thus, an understanding of the knowledge teachers must possess and access in various instructional settings has the potential to impact both teacher training and instructional practices.

With the emergence of technology's integral role in our daily lives and educational landscape at the beginning of the 21st century, researchers have begun to address the impact technology has on teacher knowledge. Although some researchers have begun by looking at the intersection of pedagogy and technology in the development of non-content-specific knowledge domains such as pedagogical technology knowledge (PTK) (Guerrero, 2005), others have examined the intersection of pedagogy and technology in the development of content-specific technological pedagogical content knowledge (TPACK) (Koehler & Mishra, 2008; Koehler, Mishra, & Yahya, 2007; Pierson, 2001).

TPACK

The TPACK (formerly TPCK) framework expands on Shulman's (1986a, 1986b, 1987) conceptualization of pedagogical content knowledge "…to describe how teachers' understanding of technologies and pedagogical content knowledge interact with one another to produce effective teaching with technology" (Koehler & Mishra, 2008, p. 12). In this model (see Figure 1), pedagogical knowledge, content knowledge, and technology knowledge intersect, interact, and influence one another to form and inform not only a teacher's understanding of content, pedagogy, and technology, but also combinations of these three knowledge domains. Together, these multiple knowledge domains intersect in the realm of TPACK to represent

> …an understanding of the representation of concepts using technologies; pedagogical techniques that use technologies in constructive ways to teach content; knowledge of what makes concepts difficult or easy to learn and how technology can help redress some of the problems that students face; knowledge of students' prior knowledge and theories of epistemology; and knowledge of how technologies can be used to build on existing knowledge and to develop new epistemologies or strengthen old ones. (Koehler & Mishra, 2008, p. 17–18).

In short, TPACK is a rich understanding of how teaching and learning within a specific content area occur and change as a result of authentic, meaningful application of appropriate technologies. "A teacher capable of negotiating these relationships represents a form of expertise different from, and greater than, the knowledge of a disciplinary expert (say a mathematician or historian), a technology expert (a computer scientist), and a pedagogical expert (an experienced educator)" (Koehler, Mishra, & Polly, 2008, p. 1). Such an expert understands how technology influences decisions about content and pedagogy while also recognizing that content and pedagogy influence decisions about and uses of technology. As teachers think about teaching specific concepts, they must concurrently be thinking about how and if technology can be used to make the concept more accessible and understandable to their students. This type of knowledge domain requires deep content knowledge, fluid pedagogical knowledge, and knowledge not only of technology tools, but knowledge about how to teach with these tools.

TPACK and Mathematics

One area that has seen dramatic growth in the influence and applications of technology on the development of content and the evolution of instruction is mathematics. Math continues to evolve as a body of knowledge as technology[1] influences what we know, how we know it, what we teach, and how we teach it. Technology has had considerable impact on the development and expansion of new and existing mathematical concepts and applications in the past few decades. For example, technology has allowed us to apply computer-like algorithms to create, analyze, and recursively define fractals, fragmented geometric shapes, objects, or quantities that are reduced-size copies (or self-similar structures) of the whole. Fractals have emerged as especially useful applications in defining and measuring geographic and meteoric features and phenomenon. Similarly, technology has influenced content development and exploration in areas such as statistics, combinatorics, algebra, probability, geometry, and matrices by providing novice and expert mathematicians increased access, understanding, and application of advanced mathematical concepts through concrete modeling, iterative applications, and recursive functioning (Grandgenett, 2008).

Technology has also had considerable impact on how we think about teaching mathematics, especially at the K–12 level. The National Council of Teachers of Mathematics (NCTM), in its *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics, 2000), states:

> Electronic technologies—calculators and computers—are essential tools for teaching, learning, and doing mathematics. They furnish visual images of mathematical ideas, they facilitate organizing and analyzing data, and they compute efficiently and accurately.… When technological tools are available, students can focus on decision making, reflection, reasoning, and problem solving. (p. 24).

The Association of Mathematics Teacher Educators (AMTE) (2006), in a recent position paper, agreed with the NCTM and further stated that "technology has become an essential tool for doing mathematics in today's world, and thus … it is essential for the teaching and learning of mathematics" (p. 1).

Properly implemented, technology changes how mathematics is taught by allowing teachers and students to focus on deep conceptual understanding over rote procedural skills through problem solving, reasoning, and decision making. There are many ways in which technology can be used to foster this type of mathematical thinking. For example, dynamic software environments, such as Geometer's Sketchpad, Cabri, Fathom, or Tinkerplots, make the exploration of core mathematical concepts tangible and interactive for students. These type of environments allow students to "…build and investigate mathematical models, objects, figures, diagrams, and graphs," (Key Curriculum Press, 2008, para. 1) in ways that bridge the gap between concrete and abstract. Handheld graphing devices allow students, through explorations and applications, to develop a deeper understanding of mathematical concepts and use higher-level approaches to solve mathematical problems. Handhelds also promote assimilation between mathematical concepts and their multiple representations (e.g., functions and their graphical, tabular, and symbolic representations). Wireless network technologies, such as the TI Navigator, promote improved student engagement, understanding, and performance by allowing for real-time tracking of student progress, collaborative lesson engagement, and instant feedback. Finally, virtual learning environments actively involve students in interactive mathematics instruction. Students are able to manipulate "physical" objects to visualize relationships and applications, form and test conjectures, and connect abstract concepts to concrete representations.

As researchers continue to pay attention to pedagogically appropriate uses of technology in professional development and classroom settings (e.g.,

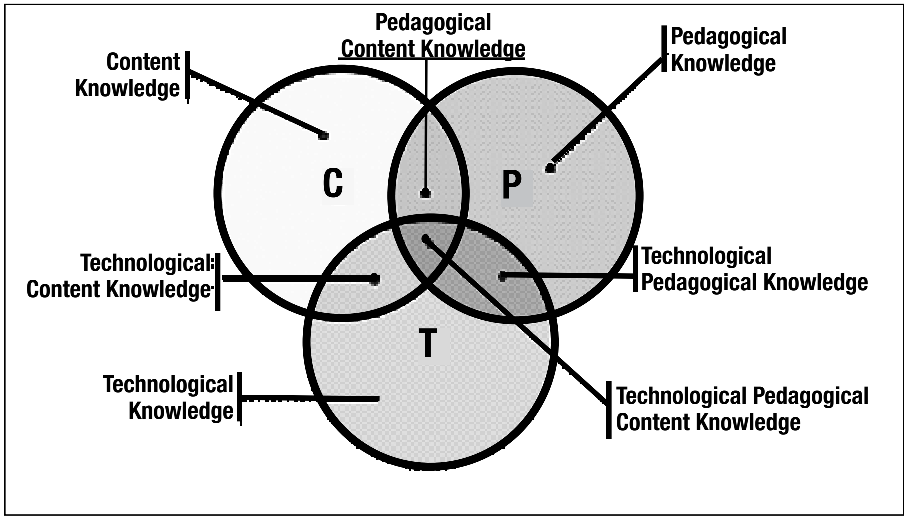

Figure 1. TPACK framework (source: www.tpck.org).

1 Here and throughout this discussion, the term technology refers to contemporary instructional and learning technologies, typically available in desktop or handheld form, that engage teachers and students in the teaching/learning process by promoting "…interactivity, multimodality, various new forms of communication, access to expertise, new varieties of resources, opportunities for stimulation, enhanced productivity, and so on" (Herman, 1994, p. 133).

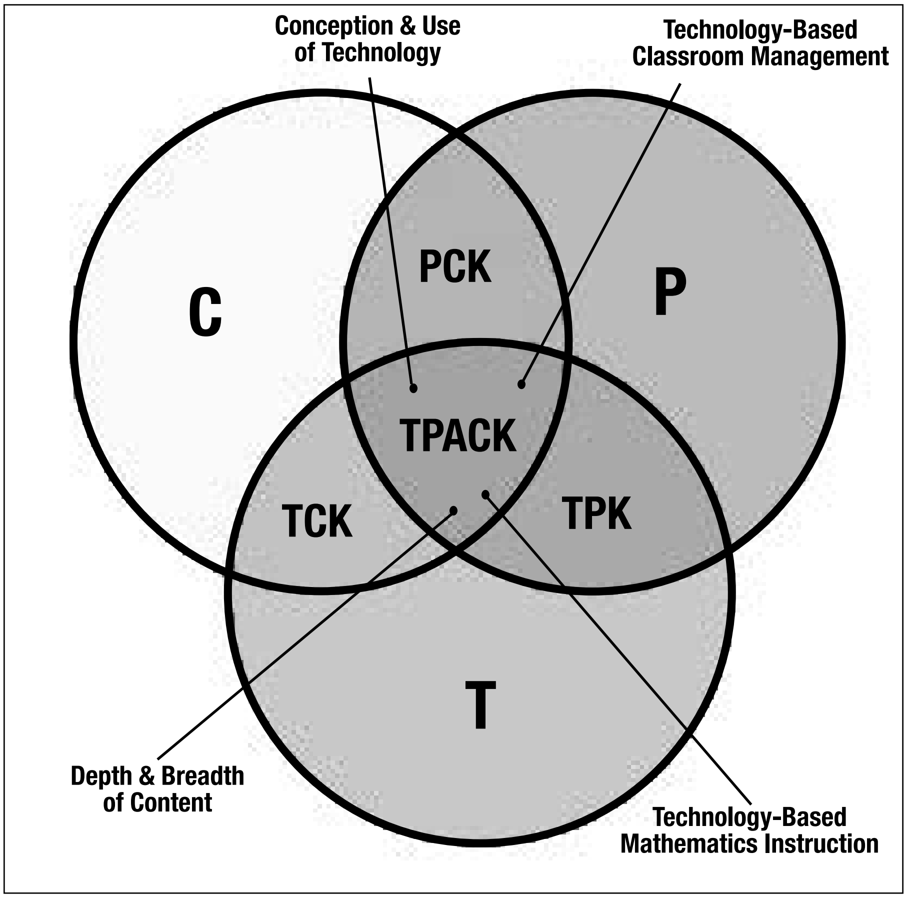

Figure 2: Four central components of mathematics-related TPACK and their derivation from pedagogical content knowledge (PCK), technological pedagogical knowledge (TPK), and technological content knowledge (TCK).

Means, 1994), theoretical development of content-specific technology-related knowledge domains continue to emerge. This article adds to the development of mathematical TPACK by identifying key components of technology use in the mathematics classroom, demonstrating robust TPACK through a classroom vignette, and addressing instructional implications of technology use through the development of questions to guide the technology-using mathematics teacher. The skills and knowledge related to technology use in the mathematics classroom are very similar to those referred to in other knowledge domains, and yet they are vastly different because of their reliance on technology as an instructional and educational tool. Technology has the unique characteristic of being an instructional tool that requires a specific set of operational skills while also necessitating a diverse array of instructional strategies and procedures.

Components of Mathematical TPACK

The knowledge needed to effectively employ technology as part of mathematics instruction includes technology-specific management, instructional, and pedagogical knowledge; increased mathematics subject-matter knowledge; and knowledge of when and how best to use technology to support mathematics instruction. TPACK in mathematics goes beyond knowledge of learning a technology tool and its operation, per se, to the dimension of how to operate a piece of technology to improve mathematics teaching and learning. Although this knowledge includes learning the basic operational skills, it embodies the aspects of technology most relevant to its capacity for use in instruction to improve teaching and learning. Nowhere are these intricacies of technology's effect on content and instruction more varied and applicable than in the mathematics classroom, where technology has the potential to change not only what we teach but how we teach it.

This article argues that TPACK in mathematics can be characterized by four central components (see Figure 2). The first component, a teacher's conception and use of technology, relates technology to pedagogical content knowledge by focusing on how the teacher can use technology to make the subject matter more comprehensible and accessible to students. The next two components of TPACK include elements associated with general pedagogical knowledge. These components encompass the general principles of instruction, organization, and classroom management specific to the application of technology in the mathematics classroom. The final component of TPACK relies on a teachers' subject-matter knowledge and deals with the increased responsibility teachers have to understand their content areas with both breadth and depth as a result of using technology as part of their instruction.

It should be noted that the number of TPACK components should not be thought of as a one-to-one correspondence with the subdomains of knowledge (i.e., PCK, TCK, and TPK) presented in the TPACK model. Rather, it is the result of careful analysis of seminal factors involved in technology-related pedagogy and subject matter specific to the content area under discussion. Consequently, this mathematical TPACK model posits one component related to each of PCK and TCK and two components related to TPK. This is not to say that TPK carries a greater weight or more importance in the development of TPACK in mathematics, but the issues related to instruction and management warranted their own consideration. Other content areas or conceptualizations of TPACK may include greater or fewer components, depending on the roles and interactions of technology, content, and pedagogy in the development of TPACK. As mentioned previously, the way that technology influences both pedagogy and content in mathematics is vastly different than the ways it may influence other content areas, so core components of TPACK may differ between mathematics and other content areas.

Conception and Use of Technology

The first component of TPACK includes a teacher's overarching conception of the use of technology in support of teaching and learning mathematics. It includes what the teacher believes about mathematics as a field, how he or she feels mathematics can best be addressed through the use of technology, and what is important for students to learn about mathematics through the use of technology. This component serves as the basis for the instructional and curricular decisions teachers make in rendering the subject matter more accessible to students.

The teacher must decide how to best use technology (if at all) to address the needs of the students, the content, and instruction and then decide which technology best accomplishes all these goals. Is exploratory data analysis (EDA) best taught through the use of spreadsheets, graphing calculators, or one of the many statistical programs available for all levels of students? There is no single correct answer to this question; the answer will depend largely on the skills of the teacher with various types of technology; the specific topic he or she is investigating within EDA; and the students' skills, needs, motivations, and prior understanding.

Most important, this component includes the knowledge of how to use technology in pedagogically appropriate ways that support instruction authentically rather than as a side-show tool. Although researchers are able to identify what is not "pedagogically appropriate," such as using technology for drill-and-skill or rote computation, they are less willing to specify exact elements of what appropriateness looks like. Because so much of appropriate technology use depends on the specific needs and nuances of each student, class, and teacher, pedagogically appropriate use of technology may vary from classroom to classroom. However, most researchers agree that, in general, pedagogically appropriate use implies a seamless integration of technology that promotes inquiry, reasoning, contextualized learning, and sense-making (Guerrero, 2005).

In mathematics, pedagogically appropriate uses of technology encompass the ways technology is used to represent and formulate mathematics so that it is understandable to students through the most useful representations, demonstrations, examples, and applications. For example, when teaching fractals, how should a teacher use technology to represent them and most effectively demonstrate their applications? Would an online virtual manipulative demonstrate their creation most efficiently? Would a technical computing environment make investigation into their iterative nature more accessible? Would dynamic geometry software demonstrate their applications most effectively? Of course, the answer depends on the exact nature of the content being covered, the instructional objectives, which tools are available to the teacher, and the tools with which he or she is most skilled.

As there is no single technological tool that is best for all instructional purposes, teachers must also be aware of the growing variety of tools available for their mathematics instruction. Within the mathematics classroom, this includes a thorough understanding of the uses and applications of, among other things, graphing calculators and their various programs and applications; data collection devices such as calculator-based rangers (CRBs), computer-based laboratories (CBLs), and other related scientific probes; spreadsheets; statistical software programs; dynamic geometry software programs; mathematical modeling and technical computing environments; online dynamic manipulatives; and the burgeoning use of networking tools such as TI Navigator. Though cost, time, and other factors limit teachers' instructional decisions, especially where technology is concerned, technology-using teachers must have a solid foundation for the decision of choosing one technology tool over another. Teachers must possess not only the knowledge of how to use the various features of each of these technologies, but also a thorough conceptualization of when and how to use them as instructional tools.

Technology-Based Mathematics Instruction

The second component of TPACK includes teachers' knowledge of and ability to maneuver through various instructional issues specifically related to the use of technology in support of mathematics teaching and learning. From this point of view, teachers need to understand that technology should be viewed as one instructional tool among many. It does not replace the teacher or any type of instruction, but should be included as part of a teacher's instructional repertoire.

Technology's success as a learning and instructional tool depends upon it being integrated into a meaningful curricular and instructional framework, and it should be used only when it is the most appropriate means of reaching an instructional goal (Sandholtz, Ringstaff, & Dwyer, 1997). Is an investigation of symmetry better suited to miras, geoboards, or dynamic geometry software? Then, once technology has been selected as the tool of choice, the teacher must decide which of various types of technology are best suited to the learning objectives and content of a given lesson. Is an investigation of linear functions better suited to graphing calculators, spreadsheets, virtual manipulatives, or dynamic geometry software?

Also included in this component is the teacher's ability to orchestrate the classroom environment in light of new demands and opportunities created by the use of technology. With the flourishing number of mobile and networking technologies, such as SmartBoards, interactive slates, and the TI Navigator system, teachers must have the ability to manage collaborative inquiry and share control of the technology with students and among students (Goss, Renshaw, Galbraith, & Geiger, 2000). As such, technology-using mathematics teachers need to be aware of and comfortable with a didactic shift in attention from them to the topic the class is exploring, and in their role as part ofteacher-directed instruction versus student-centered collaboration. They also need to recognize that technology may disrupt their instructional plans by uncovering insights into new and unexpected areas, and teachers should be comfortable with adapting to and making spontaneous changes in instruction. Now, more than ever, teachers need to be comfortable with and knowledgeable enough to go with that "teachable moment." By the same token, though, teachers should be aware of their responsibility to set boundaries on how far students can and should go in their investigations and individual work.

Finally, this instructional component of TPACK includes the ability to adjust the use of technology to serve the needs of a diverse array of students in terms of mathematical ability, affect, and interest. Just as students may lose interest in an assignment when it is too easy or too difficult, they may lose interest in using technology when its use does not take their personal needs into account (Sandholtz, Ringstaff, & Dwyer, 1997). Technology has long been used as a remediation tool, but some technology tools, such as the TI Navigator system, may actually make modified instruction to serve individual needs even easier by allowing the teacher to send, via a classroom network, different sets of students different sets of problems, instructions, or tasks.

Management

The third component of TPACK in mathematics covers management issues specifically related to teaching and learning with technology. The use of technology in instruction introduces a number of management variables and issues that teachers seldom encounter when their instruction does not use technology. Included here is a teacher's understanding of how to handle students' attitudes toward technology and their behavior as a result of using technology. How does one deal with issues such as students sending games and messages via graphing calculators, abusing data gathering probes, or using computers as physical shields to hide off-task behavior? On the other hand, teachers need to understand that some technologies may actually make management of instruction and behavior easier by providing constant access to student activity, progress, and understanding. The TI Navigator system allows teachers to grab screenshots of student work on the calculator at any point in time and provides teachers with instantaneous, formative assessment capabilities at any point in the lesson.

Management also encompasses teachers' understanding of how to deal with the physical environment (e.g., lighting, glare, setup of equipment, physical layout of room) and technical problems (e.g., broken hard drives, jammed printers, network problems, software restrictions, worn-out batteries) that arise as a result of using technology. Although ability to deal with such logistical elements of technology use improves over time, early on there is often a steep learning curve associated with managing all the physical and technical aspects of various technology tools in the mathematics classroom.

A final element of the management component of TPACK is a teacher's ability to maintain student engagement once the novelty effect has worn off. The use of technology has been shown to have positive effects on student attitudes, on-task behavior, initiative, engagement, and experimentation, but when used too often, too infrequently, or inappropriately, it can also result in student frustration, boredom, distraction, and unwillingness to transition to other activities (Sandholtz, Ringstaff, & Dwyer, 1997). As with any instructional tool, teachers need to know when and how to use technology to provide students with the most authentic learning environment possible. Using technology with every activity and for every instructional purpose is just as futile as using direct instruction for every topic and lesson.

Depth and Breadth of Mathematics Content

The fourth component of TPACK takes into account the increased responsibility teachers have to understand their mathematics in breadth and depth. Placing technology in the hands of students gives them the power to explore math to a depth that may be unfamiliar to the teacher (Goss et al., 2000). As a result, teachers need to be confident in their ability to handle students' investigations and inquiries. As with instructional flexibility, depth in content knowledge provides teachers with the ability and flexibility to explore, emphasize, or de-emphasize various mathematical topics that may arise in the course of instruction and investigation. When a student, using a graphing calculator, discovers an interesting fact about the slope of a tangent line while graphing quadratics, the teacher must decide if the findings are relevant and worth pursuing or tangential and best left alone. This requires content knowledge of not only functions and derivatives, but also a broader understanding of mathematics, the mathematics curricula, and where/how derivatives fit into the scope and sequence of both.

Along with having an extremely strong knowledge base in their subject matter, teachers must also possess a willingness to acknowledge their own subject-matter shortcomings. As a result of the depth and breadth of content that can be explored through technology, teachers need to understand that students may encounter topics and ideas that teachers may be unprepared to manage or address. In lieu of understanding every possible avenue a student's investigation and insight may take, the teacher needs to be able to acknowledge that they are unsure of a student's discovery, comments, or questions and must be willing to invest the time and energy to investigate these various content trails on their own.

A Portrait of TPACK in Action

Perhaps one of the best ways to grasp the complexity of TPACK in action is to examine each of the four components in the context of one teacher's technology use. Barbara, a secondary mathematics teacher at a rural high school in central California, has been teaching for 18 years and recently moved from the middle grades to her new position at her district's only high school. She has long been a proponent of technology in support of mathematics teaching and learning and spearheaded a controversial effort to require graphing calculators while at the middle school. She was an active participant in a long-term technology professional development program within the state and continues to be an ardent proponent of technology at all grade levels in her district. Barbara's technology use is, for lack of a better word, fluid. She demonstrates robust mathematical TPACK through her conceptions and use of technology, her technology-based math instruction and management skills, and her depth and breadth of mathematical content knowledge.

Conception and Use of Technology

Barbara is attracted to technology as an instructional tool because of its potential to improve her students' learning and depth of understanding. She uses technology because she feels it allows students to learn about "current math" and to learn math with more depth. She feels that technology can be applied in ways that make learning math more meaningful for the students and that technology allows her and her students to do things they would not be able to do otherwise. Technology enables her students to "see the math," make connections, and go more in depth with increased understanding. Barbara reflects her beliefs in the benefits of technology through the decisions she makes about using and implementing technology. In her own words, she sees the "big picture" and believes technology provides students with useful experiences that connect mathematics to their daily lives, and prepares them for a technologically savvy "real world."

Technology-Based Mathematics Instruction

Most of the decisions Barbara makes regarding technology use are aimed at making technology a natural part of the learning process rather than an object of study in and of itself. "I am just continually looking for how to make it seamless between teaching the math," she explains. Despite her obvious enthusiasm for and commitment to technology as an instructional tool, Barbara believes that students should be taught to think about technology as one of many tools in their mathematical repertoire. "It's a tool like anything else, like spell check or like pencils or using compasses," she says. For Barbara, technology should be used with, rather than blindly replace, other tools, such as mental math and manual computation with paper and pencil.

When she uses technology to teach a topic, it is clear that Barbara has carefully planned the lesson and knows where she wants it to end up. However, she is open to using student input to guide the nuances of how the lesson will get there. Barbara often uses questioning to guide students in a whole-class discovery-based discussion focusing on real-world theme problems that often span the course of several days. Barbara believes that students learn best when they see the connections within mathematics and the real world. For Barbara, technology is crucial to helping students make these connections by providing them with hands-on experiences and instant visuals, reinforcing concept links, and connecting math to real-world applications. In her attempts to make technology use seamless, Barbara plans her lessons so that students learn the technology as they learn the mathematics. As she introduces new material, she reinforces old technology-related skills and integrates new skills.

Depth and Breadth of Content

In a typical lesson, Barbara gives students a focus problem at the beginning of class, and, though the use of various questioning and discussion techniques, she gives students instructions for setting up their calculators and guides them through the exploration of a mathematical topic. These lessons are very interactive and involve a lot of give and take between Barbara and her students. These "theme problems" often run the course of several days. For example, students were investigating the rate of change in millimeters of the lean on the Tower of Pisa over the last 100 years. Although the problem started out as a theme problem that the class would pursue through two days of data analysis, it expanded into a problem that had students researching Tower of Pisa facts at home, arguing about an outlying point created through human interference versus natural causes, and wanting to explore center of gravity so they could determine when the tower would fall if its lean continued increasing at the current rate. This led to mini research projects, investigation of statistical outliers and their meaning, and construction of models to replicate "lean and fall" scenarios.

In follow-up interviews, Barbara mentioned that the problem had "morphed out of control" in several different directions than the one she intended, but she was going with what the students wanted to do because they were engaged, exploring some really rich content, and getting at the heart of some really teachable moments. Students were, of their own volition, asking about and investigating rich crosscurricular content areas that demonstrated not only true interest, but depth of understanding. She believed that engaging them through their own questions was well worth the risk of not necessarily knowing the outcome. Though she did not know the answer to every question the students asked, Barbara demonstrated flexibility in her willingness to explore content areas beyond her immediate grasp.

Management

Because of her extensive background with all types of technology, Barbara is very comfortable and proficient with technology, likes to try new things, and is willing to take the risk of experimenting in front of her students. She has become an expert at troubleshooting graphing calculators in a multi-platform setting. Although she now uses TI graphing calculators in her teaching, some of her students still own and use Casios that were required when they were in middle school. Barbara is able to move effortlessly from helping a TI student to helping a Casio student. Because she gives instructions only on the TI, Barbara often pairs her few Casio students with one another and frequently checks in with them to make

Table 1: Questions to Guide the Development and Use of Technology by Analyzing Each Component of TPACK

Component	Question
Instruction	Is technology the most appropriate instructional tool for teaching and learning this topic? How will technology affect the collaborative nature of my classroom? Will I be able to adapt my instruction based on student feedback, progress, and/or inquiry with technology? How will I adjust my instruction and the use of technology to meet individual student needs?
Management	How will I manage the physical logistics of technology? Where will we use technology? How many students per computer/calculator? Can I troubleshoot technical and/or application problems? How do I manage student progress and behavior? How do I encourage and maintain student engagement with technology-based lessons?
Depth and Breadth of Content	Do I have the mathematical knowledge to handle student inquiries that may take us beyond the intent of this lesson? Am I willing to acknowledge my own content-related shortcomings and invest the time and energy to investigate student-generated "content trails"?
Conception and Use of Technology	Is this topic best addressed through the use of technology? If so, how? What should students learn about this topic through the use of technology? How does technology improve teaching and learning of this topic? Is the use of technology in this lesson pedagogically appropriate? Do I have the skills to operate, navigate, and apply the various features of mathematics-related technology tools? Is technology fully integrated into this lesson or an add-on? Which technology will best support teaching and learning of a specific topic?

sure they are able to follow along. By her own account, part of her success at troubleshooting comes from her strict protocol for handling technology. She tells students not to touch the calculator screens, not to use anything other than their fingers to push the buttons, and to place their calculators on their desks at all times. "I don't know how much difference it makes," she admits. "It's just that I am more comfortable staying focused on math if I can walk around the room and see what they are doing, and I can troubleshoot faster."

Barbara exemplifies a teacher with comprehensive TPACK through her balanced use of technology as one tool within her instructional repertoire and her grounded beliefs that it is her responsibility to help prepare students for the tech-savvy real world. She firmly believes in the benefits of technology for making content accessible to her students but uses it within a meaningful curricular framework. Her teaching emphasizes collaborative inquiry through student-centered discussions and activities and often focuses on larger theme problems that run the course of several days. She has a solid background in mathematics and, though she clearly has instructional and content goals in mind, she is willing to explore unfamiliar areas if that is where student inquiries take a lesson.

Discussion

When choosing to use technology as part of their instructional repertoire, teachers must understand elements and implications of technology use related to instruction, management, content, pedagogy, and technology itself. Though Barbara's illustration provides an example of one teacher's use and conception of technology, the journey toward becoming such an authentic technology-using teacher takes time, energy, and commitment. Whether a novice technology user or a more experienced one such as Barbara, technology-using teachers are continually changing and growing in their conceptions and use of technology.

Questions, such as those provided in Table 1, provide a springboard for discussion and reflection centered on each of the four components of TPACK. These questions prompt theoretical and practical deliberation by both experienced and novice teachers, individuals and groups engaged in a technology-based change process, and teacher educators. Teachers will address various components of TPACK in different ways and must rely on their own expertise to begin thinking about some of the theoretical aspects of the application of technology in support of mathematics instruction.

Conclusion

Development and understanding of TPACK, especially as it relates to specific content areas, is imperative because of the importance of technology's appropriate use in educational settings. If technology is to influence teachers' practices in reform-oriented ways and improve students' learning by having a positive impact on engagement, achievement, and confidence, it must be successfully integrated into instruction in effective, authentic, and nonroutine ways. Ensuring technology's proper use in educational settings requires the development and understanding of the characteristics of teachers' technological pedagogical content knowledge base.

This article has attempted to address some of the central components of practical TPACK for the mathematics classroom. These include, but are not limited to, conception and use of technology; technology-based mathematics instruction and management; and depth and breadth of mathematics content. Although implicit in this knowledge base is facility with basic operational skills for various types of technology, TPACK most notably embodies the aspects of technology most relevant to its ability to be used as part of a teacher's instructional repertoire to improve teaching and learning. The working technology knowledge of a mathematics teacher using graphing calculators, computer software programs, and computer-based laboratories to deeply explore a mathematical topic is vastly different than that of an English teacher using the Internet and software programs to investigate and prepare literary documents. Each content area has specific instructional goals and needs that technology can address in a variety of ways. TPACK embodies a teacher's ability to distinguish between the types of technology that are most suited to their content area and make decisions regarding its appropriate application.

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