# Meeting Notes: Multipole Comparisons

My code uses Holstein's Eq. (52), rather than Eq. (51). In his notation, I'm using F_i(E)'s rather than f_i(E)'s. I need to convert between them.

This is because:

(a) Coulomb/Radiative corrections ( some terms, up to f_15(E) ):

f_1, ~~f_2~~, f_4, ~~f_6~~, f_7, ~~f_12~~, ~~f_14~~, ~~f_15~~

Of the terms where Holstein works out the radiative correction, many vanish on integration.

(b) C_S/C_T inclusion (JTW has equivalents for only some terms, up to f_12(E) ):

f_1, f_2, f_4, f_6, f_8, f_12.

$\xi \quad a_{\beta\nu} \quad A_\beta \quad B_\nu \quad D_{\mathrm{TR}} \quad c_{\mathrm{align}}$

...But also, most of these go away upon integration, too. The *only* remaining Holstein terms that have a JTW equivalent are f_1(E) and f_4(E) [xi and A_beta]. So those are the only terms that get to have adjustments from C_S and C_T terms.

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Holstein and JTW terms have *this* relationship:

$$
\begin{aligned}
\xi &= f_1(E) \qquad \text{(? times some constant? doesn't matter.)} \\
a_{\beta\nu} &= f_2(E) \,/\, f_1(E) \\
\frac{\langle \vec{J} \rangle}{J} \cdot \frac{\vec{p}}{E} \, A_\beta &= \Lambda_1 \hat{n} \cdot \frac{\vec{p}}{E} \, f_4(E) \,/\, f_1(E) \\
\frac{\langle \vec{J} \rangle}{J} \cdot \frac{\vec{p}_\nu}{E_\nu} \, B_\nu &= \Lambda_1 \hat{n} \cdot \vec{k} \;\; f_6(E) \,/\, f_1(E) \\
\frac{\langle \vec{J} \rangle}{J} \cdot \frac{(\vec{p} \times \vec{p}_\nu)}{E E_\nu} \, D_{\mathrm{TR}} &= \Lambda_1 \hat{n} \cdot \left(\frac{\vec{p}}{E} \times \hat{k}\right) \; f_8(E) \,/\, f_1(E) \\
\left[\frac{J(J+1) - 3\langle (\vec{J} \cdot \hat{j})^2 \rangle}{J(2J-1)}\right]\left[\frac{1}{3}\,\frac{\vec{p} \cdot \vec{p}_\nu}{E E_\nu} - \frac{(\vec{p} \cdot \hat{j})(\vec{p}_\nu \cdot \hat{j})}{E E_\nu}\right] c_{\mathrm{align}} &= \Lambda_2 \left[(\hat{n} \cdot \frac{\vec{p}}{E})(\hat{n} \cdot \hat{k}) - \frac{1}{3}\left(\frac{\vec{p}}{E} \cdot \hat{k}\right)\right] f_{12}(E) \,/\, f_1(E)
\end{aligned}
$$

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**JTW Monopole Terms:**

$\xi, \quad \xi \frac{m}{E} * b_{\text{Fierz}}, \quad \xi \frac{\vec{p}\cdot\vec{p}_\nu}{EE_\nu} * a_{\beta\nu}$

**JTW Dipole Terms:**

$\xi \frac{\langle\vec{J}\rangle}{J} \cdot \frac{\vec{p}}{E} * A_\beta, \quad \xi \frac{\langle\vec{J}\rangle}{J} \cdot \frac{\vec{p}_\nu}{E_\nu} * B_\nu, \quad \xi \frac{\langle\vec{J}\rangle}{J} \cdot \frac{\vec{p}\times\vec{p}_\nu}{EE_\nu} * D_{\text{TR}}$

**JTW Quadrupole Terms:**

$\xi \left(\frac{J(J+1)-3\langle(\vec{J}\cdot\hat{j})^2\rangle}{J(2J-1)}\right)\left(\frac{1}{3}\frac{\vec{p}\cdot\vec{p}_\nu}{EE_\nu} - \frac{(\vec{p}\cdot\hat{j})(\vec{p}_\nu\cdot\hat{j})}{EE_\nu}\right) * c_{\text{align}}$

**Holstein (52) Monopole Term:**

$F_0(E) = f_1(E)$

**Holstein (52) Dipole Term:**

$$\Lambda_1 (\hat{n}\cdot\frac{\vec{p}}{E}) * F_1(E) = \Lambda_1 (\hat{n}\cdot\frac{\vec{p}}{E}) * \left(f_4(E) + \frac{1}{3} f_7(E)\right)$$

**Holstein (52) Quadrupole Term:**

$$\Lambda_2 \left((\hat{n}\cdot\frac{\vec{p}}{E})^2 - \frac{1}{3}\frac{p^2}{E^2}\right) * F_2(E) = \Lambda_2 \left((\hat{n}\cdot\frac{\vec{p}}{E})^2 - \frac{1}{3}\frac{p^2}{E^2}\right) * \left(f_{10}(E) + \frac{1}{3} f_{13}(E)\right) = \Lambda_2 T_2(\hat{n}) : [\frac{\vec{p}}{E}, \frac{\vec{p}}{E}] * \left(f_{10}(E) + \frac{1}{3} f_{13}(E)\right)$$

**Holstein (52) Octopole Term:**

$$\Lambda_3 \left((\hat{n}\cdot\frac{\vec{p}}{E})^3 - \frac{3}{5}\frac{p^2}{E^2}(\hat{n}\cdot\frac{\vec{p}}{E})\right) * F_3(E) = \Lambda_3 \left((\hat{n}\cdot\frac{\vec{p}}{E})^3 - \frac{3}{5}\frac{p^2}{E^2}(\hat{n}\cdot\frac{\vec{p}}{E})\right) * f_{18}(E) = \Lambda_3 T_3(\hat{n}) : [\frac{\vec{p}}{E}, \frac{\vec{p}}{E}, \frac{\vec{p}}{E}] * f_{18}(E)$$

**Holstein (52) Hexadecapole Term:**

(none)

**Holstein (51) Monopole Terms:**

$f_1(E), \quad \frac{\vec{p}\cdot\hat{k}}{E} * f_2(E), \quad \left(\frac{(\vec{p}\cdot\hat{k})^2}{E^2} - \frac{1}{3}\frac{p^2}{E^2}\right) * f_3(E)$

**Holstein (51) Dipole Terms:**

$\Lambda_1 (\hat{n}\cdot\frac{\vec{p}}{E}) * f_4(E), \quad \Lambda_1 (\hat{n}\cdot\frac{\vec{p}}{E})\frac{\vec{p}\cdot\hat{k}}{E} * f_5(E),$

$\Lambda_1 (\hat{n}\cdot\hat{k}) * f_6(E), \quad \Lambda_1 (\hat{n}\cdot\hat{k})\frac{\vec{p}\cdot\hat{k}}{E} * f_7(E),$

$\Lambda_1 \hat{n}\cdot\left(\frac{\vec{p}}{E}\times\hat{k}\right) * f_8(E) \quad \Lambda_1 \hat{n}\cdot\left(\frac{\vec{p}}{E}\times\hat{k}\right)\frac{\vec{p}\cdot\hat{k}}{E} * f_9(E)$

**Holstein (51) Quadrupole Terms:**

$\Lambda_2 T_2(\hat{n}) : [\frac{\vec{p}}{E}, \frac{\vec{p}}{E}] * f_{10}(E), \quad \Lambda_2 T_2(\hat{n}) : [\frac{\vec{p}}{E}, \frac{\vec{p}}{E}](\frac{\vec{p}\cdot\hat{k}}{E}) * f_{11}(E),$

$\Lambda_2 T_2(\hat{n}) : [\frac{\vec{p}}{E}, \hat{k}] * f_{12}(E), \quad \Lambda_2 T_2(\hat{n}) : [\frac{\vec{p}}{E}, \hat{k}](\frac{\vec{p}\cdot\hat{k}}{E}) * f_{13}(E),$

$\Lambda_2 T_2(\hat{n}) : [\hat{k}, \hat{k}] * f_{14}(E), \quad \Lambda_2 T_2(\hat{n}) : [\hat{k}, \hat{k}](\frac{\vec{p}\cdot\hat{k}}{E}) * f_{15}(E)$ (?)

$\Lambda_2 T_2(\hat{n}) : [\frac{\vec{p}}{E}, \frac{\vec{p}}{E}\times\hat{k}] * f_{16}(E),$

$\Lambda_2 T_2(\hat{n}) : [\hat{k}, \frac{\vec{p}}{E}\times\hat{k}] * f_{17}(E)$

**Holstein (51) Octopole Terms:**

$\Lambda_3 T_3(\hat{n}) : [\frac{\vec{p}}{E}, \frac{\vec{p}}{E}, \frac{\vec{p}}{E}] * f_{18}(E)$ (also some other stuff, but this is the only term that doesn't integrate to zero.)

**Holstein (51) Hexadecapole Terms:**

(some stuff. don't care.)

# Integrals By 'Inspection'

**Monopole:**

$$\int 1 \, \mathrm{d}\hat{\Omega}_k = 4\pi \qquad \text{--} \qquad f_1(E)$$

$$\int \left(\frac{\vec{p}\cdot\hat{k}}{E}\right) \mathrm{d}\hat{\Omega}_k = 0 \qquad \text{--} \qquad f_2(E)$$

$$\int \left[\left(\frac{\vec{p}\cdot\hat{k}}{E}\right)^2 - \frac{1}{3}\frac{p^2}{E^2}\right] \mathrm{d}\hat{\Omega}_k = 0 \qquad \text{--} \qquad f_3(E)$$

**Dipole:**

$$\int \left(\hat{n}\cdot\frac{\vec{p}}{E}\right) \mathrm{d}\hat{\Omega}_k = 4\pi\left(\hat{n}\cdot\frac{\vec{p}}{E}\right) \qquad \text{--} \qquad f_4(E)$$

$$\int \left(\hat{n}\cdot\frac{\vec{p}}{E}\right)\left(\frac{\vec{p}\cdot\hat{k}}{E}\right) \mathrm{d}\hat{\Omega}_k = 0 \qquad \text{--} \qquad f_5(E)$$

$$\int \left(\hat{n}\cdot\hat{k}\right) \mathrm{d}\hat{\Omega}_k = 0 \qquad \text{--} \qquad f_6(E)$$

$$\int \left(\hat{n}\cdot\hat{k}\right)\left(\frac{\vec{p}\cdot\hat{k}}{E}\right) \mathrm{d}\hat{\Omega}_k = \frac{1}{3}4\pi\left(\hat{n}\cdot\frac{\vec{p}}{E}\right) \qquad \text{--} \qquad f_7(E)$$

$$\int \hat{n}\cdot\left(\frac{\vec{p}\times\hat{k}}{E}\right) \mathrm{d}\hat{\Omega}_k = 0 \qquad \text{--} \qquad f_8(E)$$

$$\int \hat{n}\cdot\left(\frac{\vec{p}\times\hat{k}}{E}\right)\left(\frac{\vec{p}\cdot\hat{k}}{E}\right) \mathrm{d}\hat{\Omega}_k = 0 \qquad \text{--} \qquad f_9(E)$$

**Quadrupole:**

$$\int T_2(\hat{n}) : \left[\frac{\vec{p}}{E}, \frac{\vec{p}}{E}\right] \mathrm{d}\hat{\Omega}_k = 4\pi\, T_2(\hat{n}) : \left[\frac{\vec{p}}{E}, \frac{\vec{p}}{E}\right] \qquad \text{--} \qquad f_{10}(E)$$

$$\int T_2(\hat{n}) : \left[\frac{\vec{p}}{E}, \frac{\vec{p}}{E}\right]\left(\frac{\vec{p}\cdot\hat{k}}{E}\right) \mathrm{d}\hat{\Omega}_k = 0 \qquad \text{--} \qquad f_{11}(E)$$

$$\int T_2(\hat{n}) : \left[\frac{\vec{p}}{E}, \hat{k}\right] \mathrm{d}\hat{\Omega}_k = 0 \qquad \text{--} \qquad f_{12}(E)$$

$$\int T_2(\hat{n}) : \left[\frac{\vec{p}}{E}, \hat{k}\right]\left(\frac{\vec{p}\cdot\hat{k}}{E}\right) \mathrm{d}\hat{\Omega}_k = \frac{1}{3}4\pi T_2(\hat{n}) : \left[\frac{\vec{p}}{E}, \frac{\vec{p}}{E}\right] \qquad \text{--} \qquad f_{13}(E)$$

$$\int T_2(\hat{n}) : \left[\hat{k}, \hat{k}\right] \mathrm{d}\hat{\Omega}_k = 0 \qquad \text{--} \qquad f_{14}(E)$$

$$\int T_2(\hat{n}) : \left[\hat{k}, \hat{k}\right]\left(\frac{\vec{p}\cdot\hat{k}}{E}\right) \mathrm{d}\hat{\Omega}_k = 0 \qquad \text{--} \qquad f_{15}(E) \quad (?)$$

$$\int T_2(\hat{n}) : \left[\frac{\vec{p}}{E}, \frac{\vec{p}}{E}\times\hat{k}\right] \mathrm{d}\hat{\Omega}_k = 0 \qquad \text{--} \qquad f_{16}(E)$$

$$\int T_2(\hat{n}) : \left[\hat{k}, \frac{\vec{p}}{E}\times\hat{k}\right] \mathrm{d}\hat{\Omega}_k = 0 \qquad \text{--} \qquad f_{17}(E)$$

**Octopole:**

$$\int T_3(\hat{n}) : \left[\frac{\vec{p}}{E}, \frac{\vec{p}}{E}, \frac{\vec{p}}{E}\right] \mathrm{d}\hat{\Omega}_k = 4\pi T_3(\hat{n}) : \left[\frac{\vec{p}}{E}, \frac{\vec{p}}{E}, \frac{\vec{p}}{E}\right] \qquad \text{--} \qquad f_{18}(E)$$

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Holstein's tensor notation definitions:

$$
\begin{aligned}
T_2(\hat{n}) : [\vec{a}, \vec{b}] &= \left((\hat{n}\cdot\vec{a})(\hat{n}\cdot\vec{b}) - \tfrac{1}{3}\,\vec{a}\cdot\vec{b}\right) \\
T_3(\hat{n}) : [\vec{a}, \vec{b}, \vec{c}] &= \left((\hat{n}\cdot\vec{a})(\hat{n}\cdot\vec{b})(\hat{n}\cdot\vec{c}) - \tfrac{1}{5}\left((\hat{n}\cdot\vec{a})(\vec{b}\cdot\vec{c}) + (\hat{n}\cdot\vec{b})(\vec{a}\cdot\vec{c}) + (\hat{n}\cdot\vec{c})(\vec{a}\cdot\vec{c})\right)\right) \\
T_4(\hat{n}) : [\vec{a}, \vec{b}, \vec{c}, \vec{d}] &= \text{(some stuff)}
\end{aligned}
$$

**Additional thing(s) I want to remember to ask about:**

* 37K electric charge radius / weak charge radius. I care because coulomb corrections.
  * We've been using a uniform spherical distribution with radius R=4.637/hbarc for 37K. For both electric and weak charge.
  * By contrast, our naive default for other isotopes is R=1.2*A^(1/3)/hbarc (comes out to 3.999/hbarc for 37K).
  * Where does "4.637" come from???