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CS281-F17

Assignment #2 v 1.1

Due: 5:00pm October 9, 2017

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NOTE: you must show derivations for your answers unless a question explicitly mentions that no justification is required.

Problem 1 (Spherical Gaussian, 10pts)

One intuitive way to summarize a probability density is via the mode, as this is the "most likely" value in some sense. A common example of this is using the maximum *a posteriori* (MAP) estimate of a model's parameters. In high dimensions, however, the mode becomes less and less representative of typical samples. Consider variates from a D-dimensional zero mean spherical Gaussian with unit variance:

$$\mathbf{x} \sim \mathcal{N}(\mathbf{0}_D, \mathbb{I}_D),$$

where $\mathbf{0}_D$ indicates a column vector of D zeros and $\mathbb{I}_D$ is a $D \times D$ identity matrix.

1. Compute the distribution that this implies over the distance of these points from the origin. That is, compute the distribution over $\sqrt{\mathbf{x}^\mathsf{T}\mathbf{x}}$, if $\mathbf{x}$ is a realization from $\mathcal{N}(\mathbf{0}_D, \mathbb{I}_D)$. (Note: Consider transformations of a Gamma distribution described in Murphy 2.4.5.)
2. Make a plot that shows this probability density function for several different values of D, up to $D = 100$.
3. Make a plot of the cumulative distribution function (CDF) over this distance distribution for $D = 100$. A closed-form solution may be difficult to compute, so you can do this numerically.)
4. From examining the CDF we can think about where most of the mass lives as a function of radius. For example, most of the mass for $D = 100$ is within a thin spherical shell. From eyeballing the plot, what are the inner and outer radii for the shell that contains 90% of the mass in this case?

1. We have $\mathbf{x} = \langle x_1, \ldots, x_D \rangle$, where the x_i are i.i.d. and distributed $\mathcal{N}(0, 1)$ by the properties of MVN's. We wish to find the distribution of

$$Y = \sqrt{\mathbf{x}^\mathsf{T}\mathbf{x}} = \sqrt{\sum_{i=1}^{D} x_i^2}$$

   As described in Murphy 2.4.4, the sum of squared standard normals $S = \sum_{i=1}^{D} x_i^2$ follows the chi-squared distribution χ^2_D. Taking the square root with the change of variables formula, $\sqrt{S}$ has the pdf

$$f_Y(y) = 2y f_S(y^2) = 2y \frac{y^{D-2} e^{-y^2/2}}{2^{D/2}\Gamma(D/2)} = \frac{2^{1-D/2} y^{D-1} e^{-y^2/2}}{\Gamma(D/2)},$$

   which is the chi distribution with D degrees of freedom.

2. See figure 1.

3. See figure 2.

4. From examining the CDF, we see that 90% of the mass lies between $r = 9$ and $r = 12$, and 80% between $r = 9$ and $r = 11$.

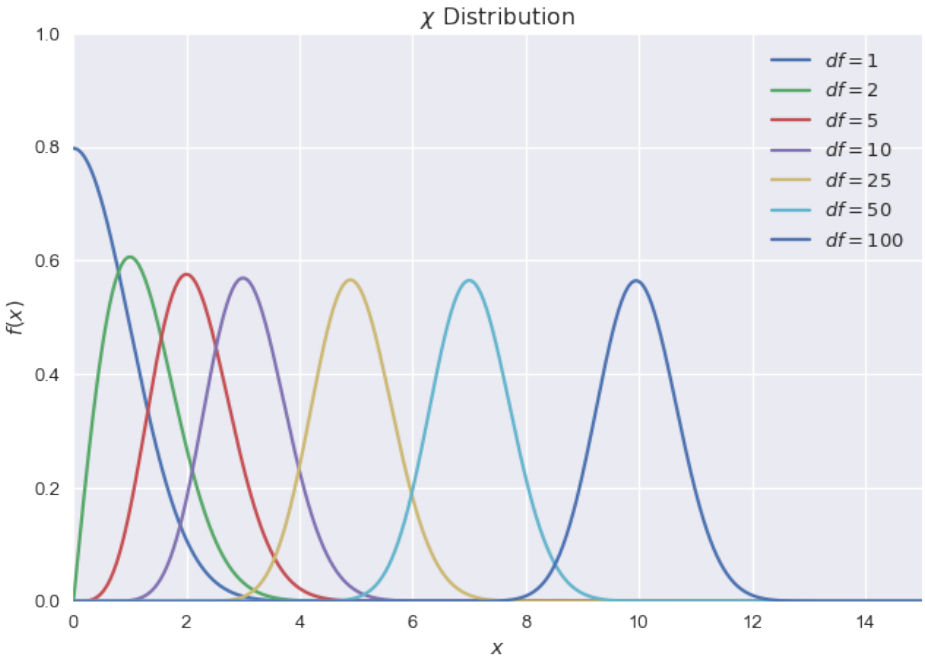


Figure 1: χ PDF for various df's.

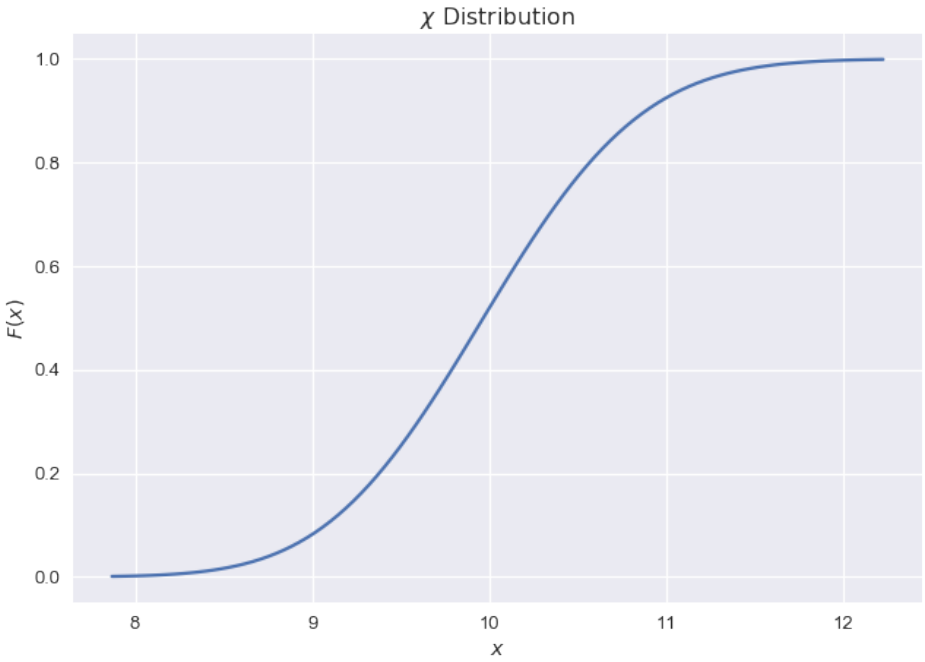


Figure 2: χ CDF for 100 degrees of freedom.

Problem 2 (Hurdle Models for Count Data, 10pts)

In this problem we consider predictive models of count data. For instance given information about the student x, can we predict how often they went to the gym that week y? A natural choice is to use a Poisson GLM i.e. y conditioned on x is modeled as a Poisson distribution.

However, in practice, it is common for count data of this form to follow a bi-modal distribution over count data. For instance, our data may come from a survey asking students how often they went to the gym in the past week. Some would do so frequently, some would do it occasionally but not in the past week (a random zero), and a substantial percentage would never do so.

When modeling this count data with generalized linear models, we may observe more zero examples than expected from our model. In the case of a Poisson, the mode of the distribution is the integer part of the mean. A Poisson GLM may therefore be inadequate when means can be relatively large but the mode of the output is 0. Such data is common when many data entries have 0 outputs and many also have much larger outputs, so the mode of output is 0 but the overall mean is not near 0. This problem is known as *zero-inflation*.

This problem considers handling zero-inflation with a two-part model called a *hurdle model*. One part is a binary model such as a logistic model for whether the output is zero or positive. Conditional on a positive output, the "hurdle is crossed" and the second part uses a truncated model that modifies an ordinary distribution by conditioning on a positive output. This model can handle both zero inflation and zero deflation.

Suppose that the first part of the process is governed by probabilities $p(y > 0 \mid x) = \pi$ and $p(y = 0 \mid x) = 1 - \pi$; and the second part depends on $\{y \in \mathbb{Z} \mid y > 0\}$ and follows a probability mass function $f(y \mid \mathbf{x})$ that is truncated-at-zero. The complete distribution is therefore:

$$P(y = 0 \mid x) = 1 - \pi$$
$$P(y = j \mid x) = \pi \frac{f(j \mid \mathbf{x})}{1 - f(0 \mid \mathbf{x})},\ j = 1, 2, ...$$

One choice of parameterization is to use a logistic regression model for π:

$$\pi = \sigma(\mathbf{x}^\top \mathbf{w}_1)$$

and use a Poisson GLM for f with mean parameters λ (see Murphy 9.3):

$$\lambda = \exp(\mathbf{x}^\top \mathbf{w}_2)$$

(a) Suppose we observe N data samples $\{(x_n, y_n)\}_{n=1}^N$. Write down the log-likelihood for the hurdle model assuming an unspecified mass function f. Give an maximum likelihood estimation approach for the specified parts of the model.

(b) Assume now that we select Poisson distribution for f. Show that the truncated-at-zero Poisson distribution (as used in the hurdle model) is a member of the exponential family. Give its the sufficient statistics, natural parameters and log-partition function.

(c) What is the mean and variance of a truncated Poisson model with mean parameter λ? If we observe n i.i.d. samples from a truncated Poisson distribution, what is the maximum likelihood estimate of λ? (Note: Give an equation which could be solved numerically to obtain the MLE.)

(d) Now assume that we using a hurdle model as a GLM with f as a Poisson distribution. Show that this is a valid GLM (exponential family for y), derive its log-likelihood, and give its sufficient statistics.

(a) Let $g(y) = \mathbb{I}(y \neq 0)$ be the binary indicator of the response y being non-zero, and let $h(y, x)$ be the PDF of the truncated-at-zero distribution, where

$$h(y, x) = \frac{f(y \mid \mathbf{x})}{1 - f(0 \mid \mathbf{x})}$$

Then, the log-likelihood of one example is:

$$\begin{aligned}
\log p(y_i \,|\, x_i, \mathbf{w}_1) &= \log \left[(1 - \pi_i)^{1 - g(y_i)} \left(\pi_i h(y_i, x_i) \right)^{g(y_i)} \right] \\
&= (1 - g(y_i)) \log(1 - \pi_i) + g(y_i) \left(\log \pi_i + \log h(y_i, x_i) \right) \\
&= \left[(1 - g(y_i)) \log(1 - \pi_i) + g(y_i) \log \pi_i \right] + g(y_i) \log h(y_i, x_i)
\end{aligned}$$

We have the following total log-likelihood:

$$\begin{aligned}
\log p(\mathcal{D} \,|\, \theta) &= \sum_{i=1}^{N} \log p(y_i \,|\, x_i, \mathbf{w}_1) \\
&= \left[\sum_{i=1}^{N} (1 - g(y_i)) \log(1 - \pi_i) + g(y_i) \log \pi_i \right] + \left[\sum_{i=1}^{N} g(y_i) \log h(y_i, x_i) \right] \\
&= \left[\sum_{i=1}^{N} (1 - g(y_i)) \log(1 - \sigma(\mathbf{x}_i^\top \mathbf{w}_1)) + g(y_i) \log(\sigma(\mathbf{x}_i^\top \mathbf{w}_1)) \right] + \left[\sum_{i=1}^{N} g(y_i) (\log f(y_i \mid \mathbf{x}_i) + const) \right] \\
&= \ell_1(\mathbf{w}_1 \,|\, \mathcal{D}) + \ell_2(f \,|\, \mathcal{D})
\end{aligned}$$

We see that the log-likelihood breaks into two terms, and thus the parameters for π and f can be optimized separately. The first term is simply the log-likelihood for a logistic regression model, and we can use second-order optimization methods like IRLS to find $\mathbf{w}_1$. Similarly, we can calculate the Hessian of $\log f(y_i \mid \mathbf{x}_i)$ and estimate the MLE parameters of f using second-order methods.

(b) We have $f(y \mid \mathbf{x}) = Po(y \,|\, \lambda)$. Rewriting the pdf for the truncated Poisson, we have

$$\begin{aligned}
g(y) &= \frac{f(y \mid \mathbf{x})}{1 - f(0 \mid \mathbf{x})} \\
&= \frac{\lambda^y}{y!(e^\lambda - 1)} \\
&= \frac{1}{y!} \exp\{y \log \lambda - \log(e^\lambda - 1)\} \\
&= h(y) \exp\{\theta \phi(y) - A(\theta)\},
\end{aligned}$$

where $h(y) = \frac{1}{y!}$, $\theta = \log \lambda$, $\phi(y) = y$, and $A(\theta) = \log(e^{e^\theta} - 1)$.

(c) Using the results derived in Murphy 9.2.3, we use the log partition function of the truncated Poisson

to derive the first and second cumulants of the sufficient statistics (in this case, the variable itself):

$$\begin{aligned}
\mathbb{E}(y) = \mathbb{E}(\phi(y)) &= \frac{dA}{d\theta} = \frac{e^{e^\theta} e^\theta}{e^{e^\theta} - 1} = \frac{\lambda e^\lambda}{e^\lambda - 1} \\
\mathrm{var}(y) = \mathrm{var}(\phi(y)) &= \frac{d^2 A}{d\theta^2} \\
&= \left(\frac{d}{d\lambda} \frac{\lambda e^\lambda}{e^\lambda - 1} \right) \frac{d\lambda}{d\theta} \\
&= \left(-\frac{\lambda e^{2\lambda}}{(e^\lambda - 1)^2} + \frac{\lambda e^\lambda + e^\lambda}{e^\lambda - 1} \right) \lambda \\
&= \frac{\lambda^2 + \lambda}{1 - e^{-\lambda}} - \frac{\lambda^2}{(1 - e^{-\lambda})^2}
\end{aligned}$$

The log-likelihood of n samples $\mathcal{D} = \{y_i\}_{i=1}^n$ is

$$\begin{aligned}
\log p(\mathcal{D} \,|\, \theta) &= \log \left(\left[\prod_{i=1}^n h(y_i) \right] \exp\{\theta \sum_{i=1}^n \phi(y_i) - nA(\theta)\} \right) \\
&= \sum_{i=1}^n \log h(y_i) + \theta \sum_{i=1}^n \phi(y_i) - nA(\theta) \\
&= \theta \sum_{i=1}^n \phi(y_i) - nA(\theta) + const
\end{aligned}$$

Taking the derivative w.r.t. θ, we obtain

$$\frac{d}{d\theta} \log p(\mathcal{D} \,|\, \theta) = \sum_{i=1}^n \phi(y_i) - nA'(\theta) = \sum_{i=1}^n \phi(y_i) - n\mathbb{E}(\phi(y))$$

Thus, at the MLE estimate $\hat{\theta}$,

$$\mathbb{E}(\phi(y)) = \frac{1}{n} \sum_{i=1}^n \phi(y_i)$$

Substituting in the problem's specifications, we have the following relation for the MLE estimate for $\hat{\lambda}$:

$$\frac{\hat{\lambda} e^{\hat{\lambda}}}{e^{\hat{\lambda}} - 1} = \frac{1}{n} \sum_{i=1}^n y_i$$

(d) Let $\tilde{y} = \mathbb{I}(y \neq 0)$. We write the pdf of the hurdle model as

$$\begin{aligned}
f(y) &= (1 - \pi)^{1 - \tilde{y}} \left(\pi \frac{\lambda^y}{y!(e^\lambda - 1)} \right)^{\tilde{y}} \\
&= \frac{1}{y!} \exp\left\{ y \log \lambda + \tilde{y} \log \frac{\pi}{(1 - \pi)(e^\lambda - 1)} + \log(1 - \pi) \right\} \\
&= h(y) \exp\{\boldsymbol{\theta}^\mathsf{T} \boldsymbol{\phi}(y) - A(\boldsymbol{\theta})\},
\end{aligned}$$

where

$$\begin{aligned}
\boldsymbol{\theta} &= \left[\log \lambda,\ \log \frac{\pi}{(1 - \pi)(e^\lambda - 1)} \right] \\
\boldsymbol{\phi}(y) &= \left[y,\ \mathbb{I}(y \neq 0) \right] \\
A(\boldsymbol{\theta}) &= \log(1 - \pi)
\end{aligned}$$

Using the general results derived earlier in (c), the log-likelihood for the hurdle model is:

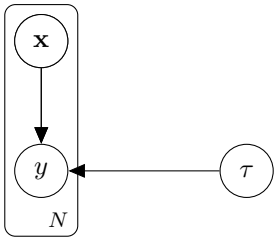
$$\begin{aligned}\log p(\mathcal{D} \mid \boldsymbol{\theta}) &= \boldsymbol{\theta}^{\mathsf{T}} \sum_{i=1}^{n} \boldsymbol{\phi}(y_i) - nA(\boldsymbol{\theta}) + const \\ &= \log \lambda \left[\sum_{i=1}^{n} y_i\right] + \log \frac{\pi}{(1-\pi)(e^{\lambda}-1)} \left[\sum_{i=1}^{n} \mathbb{I}(y_i \neq 0)\right] - n \log(1-\pi) + const,\end{aligned}$$

and the sufficient statistics are n and

$$\boldsymbol{\phi}(\mathcal{D}) = \left[\sum_{i=1}^{n} y_i,\ \sum_{i=1}^{n} \mathbb{I}(y_i \neq 0)\right]$$

Problem 3 (Directed Graphical and Naive Bayes, 10pts)

To draw the DGMs for this problem, we recommend using the `tikzbayesnet` *library. For example the following is drawn in LaTeX:*



This problem focuses on modeling a joint distribution of random variables, $p(y, x_1, \ldots, x_V)$, consisting of discrete variables. These variables represent a class label $y \in \{1, \ldots, C\}$ and features $x_1 \ldots, x_V$ each of each can take on a values $x_v \in \{0, 1\}$.

(a) Let $V = 4$. Use the chain rule to select any valid factorization of this joint distribution into univariate distributions. Draw the directed graphical model corresponding to this factorization.

(b) What is the sum of the sizes of the *conditional probability tables* associated with this graphical model. Can you reduce the order of magnitude of this value with a different DGM?

(c) Now consider a naive Bayes factorization of this model, given by,

$$p(y, x_1, \ldots, x_V) \approx p(y) \prod_v p(x_v|y).$$

Draw a directed graphical model for this factorization. What is the size of the conditional probability tables required to fully express any factored distribution of this form?

(d) In class, we parameterized naive Bayes such that the class distribution is Categorical with a Dirichlet prior, and the class-conditional distributions are Bernoulli with a Beta prior. Extend the graphical model above to show the generative model of N data points and include the parameters and hyper-parameters as random variables.

(e) Assuming the data obeys the naive Bayes assumptions, answer the following questions as true/false using your directed graphical model. Justify your answer.

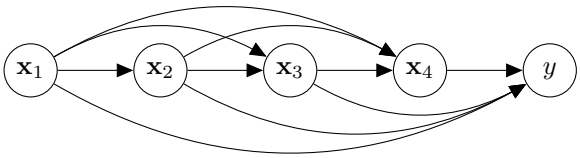
- For a given example, features x_1 and x_2 are independent.
- The class labels y are always conditionally independent of the class-conditional parameters.
- Upon observing the class distribution parameters, the class labels are conditionally independent.
- Upon observing the class distribution parameters, the features are conditionally independent.
- Upon observing the class distribution hyper-parameters, the class labels are conditionally independent.

(f) For the next problem, we will utilize naive Bayes for a problem where each example has a *bag* or multiset of items. A bag is a set that may contain multiple instances of the same value. One approach is to ignore this property and use x_v as an indicator function for each item type. An alternative is to model x_v with sample space $\{0, \ldots, D\}$, where D is the maximum times an item appears and to use a Dirichlet-Categorical for the class-conditional. Give one benefit and one drawback of this approach. Propose a third option for modeling this distribution.

(a) We choose the factorization

$$p(y, x_{1:4}) = p(x_1)p(x_2 \,|\, x_1)p(x_3 \,|\, x_{1:2})p(x_4 \,|\, x_{1:3})p(y \,|\, x_{1:4}),$$

where the MATLAB-like notation $x_{1:V}$ denotes $x_1, \ldots, x_V$. The DGM is shown below.



(b) The sum of the sizes of the CPT's for the DGM factored as in (a) with V binary variables x_i is

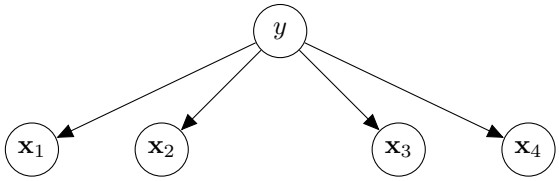
$$C2^V + \sum_{i=1}^{V} 2^i = C2^V + 2^{V+1} - 1$$

Thus, for $V = 4$, we have a total CPT size of $16C + 31$. Note that all possible factorizations of the joint result in CPT's with size $O(C2^V)$, since the only difference in the factorizations is at what point y is added to the chain rule.

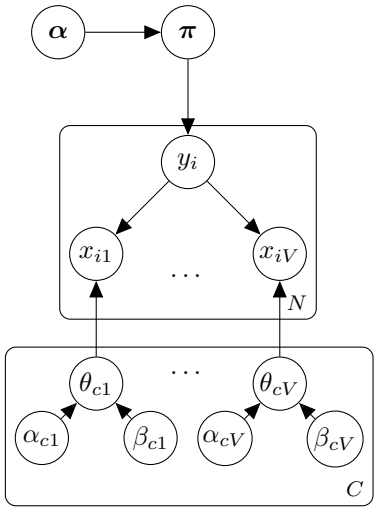
(c) The size of the CPT's for this factorization is

$$C + \sum_{i=1}^{V} 2C = C(1 + 2V)$$

The DGM is shown below.



(d) The DGM is shown below.



(e)
- False. For a given example, features x_1 and x_2 are conditionally independent, given the class y. If y is in the evidence, then features x_1 and x_2 are d-separated.
- False. The class labels y are not always conditionally independent of the class-conditional parameters. E.g., given the class distribution parameter $\boldsymbol{\pi}$, the node y_i and the node θ_{c1} are not d-separated.
- True. Upon observing the class distribution parameters, the class labels are conditionally independent. Each y_i is d-separated by $\boldsymbol{\pi}$, which blocks all paths between nodes.
- False. Upon observing the class distribution parameters, the features are not d-separated, since y_i is not a blocking node.
- False. Upon observing the class distribution hyper-parameters, the class labels are not d-separated by any evidence nodes, since $\boldsymbol{\pi}$ is unobserved.

(f) The bag-of-words model is simple and memory-efficient way to represent a text as a histogram over words. However, this representation throws away the syntax and ordering of the words, e.g. so that the phrases "toy poodle" and "poodle toy" are equivalent. An alternative way to model the distribution is to use the tf-idf statistic instead of the raw count of word i. The tf-idf statistic grows proportionally with the count of the word in the document, but scales inversely with the fraction of the documents that contain the word, so that words that are common across the corpus are assigned low values, while terms "unique" to a document are given greater weight.

Problem 4 (Naive Bayes Implementation, 10pts)

You will now implement a naive Bayes classifier for for sentiment classification. For this problem you will use the IMDB Movie Reviews dataset which consists of positive and negative movie reviews . Here are two example reviews:

> `there is no story! the plot is hopeless! a filmed based on a car with a stuck accelerator, no brakes, and a stuck automatic transmission gear lever cannot be good! ... i feel sorry for the actors ... poor script ... heavily over-dramatized ... this film was nothing but annoying, stay away from it!` [negative review]
>
> `i had forgotten both how imaginative the images were, and how witty the movie ... anyone interested in politics or history will love the movie's offhand references - anyone interested in romance will be moved - this one is superb.` [positive review]

As noted in the last problem, it is common to think of the input data as a bag/multiset. In text applications, sentences are often represented as a *bag-of-words*, containing how many times each word appears in the sentence. For example, consider two sentences:

- `We like programming. We like food.`
- `We like CS281.`

A vocabulary is constructed based on these two sentences:

```
["We", "like", "programming", "food", "CS281"]
```

Then the two sentences are represented as the number of occurrences of each word in the vocabulary (starting from position 1):

- `[0, 2, 2, 1, 1, 0]`
- `[0, 1, 1, 0, 0, 1]`

We have included a utility file `utils.py` that does this mapping. For these problems you can therefore treat text in this matrix representation.

- Implement a Naive Bayes classifier using a Bernoulli class-conditional with a Beta prior where each feature is an indicator that a word appears at least once in the bag.
- Implement a Naive Bayes classifier using a Categorical class-conditional with a Dirichlet prior. Here the features represent that count of each word in the bag.
- For both models, experiment with various settings for the priors. For the Dirichlet prior on the class, begin with $\boldsymbol{\alpha} = \mathbf{1}$ (Laplace Smoothing). Do the same for the class-conditional prior (be it Dirichlet or Beta). Keeping uniformity, vary the magnitude to .5 and smaller. If the classes are unbalanced in the dataset, does it help to use a larger α for the less-often occuring class? Optionally, choose class-conditional priors based on an outside text source. Validate your choices on the validation set, and report accuracy on the test set.
- (Optional) With the bag-of-words representation, would the model be able to capture phrases like "don't like"? An alternative to the bag-of-words model is known as the bag-of-bigrams model, where a bigram is two consecutive words in a sentence. Modify `utils.py` to include bigram features with either model and see if they increase accuracy.
- (Optional Reading) *Baselines and Bigrams: Simple, Good Sentiment and Topic Classification* `http://www.aclweb.org/anthology/P/P12/P12-2.pdf#page=118`

See jupyter notebook for code. For $\boldsymbol{\alpha} = 1$, $\boldsymbol{\beta} = 1$ we obtain a test accuracy of 86.2%. Here, we have perfectly balanced class counts, so all uniform values for $\boldsymbol{\alpha}$ will yield the same results. Thus, we only vary $\boldsymbol{\beta}$ in our tests. The validation accuracies for various values are shown in the table below; we can see that larger pseudo-counts result in higher testing accuracy, but that the difference is fairly small.

β	0.1	0.5	1
acc (binary feats.)	84.2	85.6	86.1
acc (categorical feats.)	84.4	85.9	86.2

When the classes in a dataset are imbalanced, the weights will be lower for the class with less training data, and classification will thus be incorrectly biased towards one class over the other. Using a non-uniform prior $\boldsymbol{\alpha}$ can substantially improve classification performance.

Problem 5 (Logistic Regression with Autograd, 15pts)

In the previous problem, you implemented a Naive Bayes classifier for sentiment classification on the IMDB Movie Reviews dataset. In this problem, you will apply logistic regression to the same task.

(a) ℓ_1-regularized logistic regression. Consider a model parameterized by $\mathbf{w}$:

$$p(\mathbf{w}) = \frac{1}{2b}\exp(-\frac{\|\mathbf{w}\|_1}{b})$$
$$p(y=1|\mathbf{x},\mathbf{w}) = \sigma(\mathbf{w}^\top \mathbf{x})$$
$$p(y=0|\mathbf{x},\mathbf{w}) = 1-\sigma(\mathbf{w}^\top \mathbf{x})$$

where $\sigma(\cdot)$ is the sigmoid function. Note that we are imposing a Laplacian prior on $\mathbf{w}$, see Murphy, 2.4.4.

(i) Given a dataset $\{(\mathbf{x}^{(i)}, y^{(i)})\}_{i=1}^N$, derive the necessary gradient updates for MAP of $\mathbf{w}$.[a]

(ii) Show that for some constant λ, MAP inference of $\mathbf{w}$ is equivalent to minimizing

$$-\frac{1}{N}\sum_{i=1}^{N}\log p(y^{(i)}|\mathbf{x}^{(i)},\mathbf{w}) + \lambda\|\mathbf{w}\|_1$$

(b) Implementation using PyTorch automatic differentiation.[b]

(i) Using the bag-of-words feature representation from the previous question, train a logistic regression model using PyTorch autograd and `torch.nn`. Report test accuracy. Select regularization strength λ based on validation accuracy.

(ii) Which 5 words correspond to the largest weight indices, per class, in the learnt weight vectors? Which 5 words correspond to the least weight indices?

(iii) Study how sparsity (i.e percentage of zero elements in a vector) of the parameter vector changes with different values of λ. Again, tune λ on the validation set and report the test accuracies on the test set. Suggested values to try are $\{0, 0.001, 0.01, 0.1, 1\}$. You can treat parameters with $< 1e-4$ absolute values as zeros.

[a] You only need to consider the case where $\forall i, w_i \neq 0$. If $\exists i, w_i = 0$, we can use its subgradients instead.

[b] `https://github.com/harvard-ml-courses/cs281/blob/master/cs281-f17/sections/04/walkthrough.ipynb`.

(a) (i) The objective function for MAP is:

$$\begin{aligned} f(\mathbf{w}) &= \log p(\mathcal{D}\,|\,\mathbf{w}) + \log p(\mathbf{w}) \\ &= \left[\sum_{i=1}^{N} y_i \log(\sigma(\mathbf{w}^\mathsf{T}\mathbf{x}_i)) + (1-y_i)\log(1-\sigma(\mathbf{w}^\mathsf{T}\mathbf{x}_i))\right] - \frac{\|\mathbf{w}\|_1}{b} - \log(2b) \end{aligned}$$

Note that the derivative of the sigmoid function is

$$\frac{d\sigma(a)}{da} = \frac{e^{-a}}{1+e^{-a}}\frac{1}{1+e^{-a}} = \sigma(a)(1-\sigma(a))$$

Then, taking gradients of the objective w.r.t. $\mathbf{w}$ yields

$$\begin{aligned}
\frac{df(\mathbf{w})}{\mathbf{w}} &= \left[\sum_{i=1}^{N} y_i \frac{\sigma(\mathbf{w}^\top \mathbf{x}_i)(1-\sigma(\mathbf{w}^\top \mathbf{x}_i))}{\sigma(\mathbf{w}^\top \mathbf{x}_i)}\mathbf{x}_i - (1-y_i)\frac{\sigma(\mathbf{w}^\top \mathbf{x}_i)(1-\sigma(\mathbf{w}^\top \mathbf{x}_i))}{1-\sigma(\mathbf{w}^\top \mathbf{x}_i)}\mathbf{x}_i\right] - \frac{\text{sgn}(\mathbf{w})}{b} \\
&= \left[\sum_{i=1}^{N} y_i(1-\mu_i)\mathbf{x}_i - (1-y_i)\mu_i \mathbf{x}_i\right] - \frac{\text{sgn}(\mathbf{w})}{b} \\
&= \left[\sum_{i=1}^{N} (y_i - \mu_i)\mathbf{x}_i\right] - \frac{\text{sgn}(\mathbf{w})}{b} \\
&= \boldsymbol{X}^\top(\boldsymbol{y} - \boldsymbol{\mu}) - \frac{\text{sgn}(\mathbf{w})}{b}
\end{aligned}$$

Thus, the MAP gradient update for a learning rate η is

$$\mathbf{w}^{(k+1)} \leftarrow \mathbf{w}^{(k)} + \eta\left(\boldsymbol{X}^\top(\boldsymbol{y} - \boldsymbol{\mu}^{(k)}) - \frac{\text{sgn}(\mathbf{w}^{(k)})}{b}\right)$$

(ii) In part (a), we showed that MAP finds the weights $\mathbf{w}$ that maximize the quantity

$$f(\mathbf{w}) = \left[\sum_{i=1}^{N} \log p(y_i \mid \mathbf{x}_i, \mathbf{w})\right] - \lambda\|\mathbf{w}\|_1 + const,$$

where $\lambda = 1/b$. Rewriting, we have

$$\begin{aligned}
\arg\max_{\mathbf{w}} f(\mathbf{w}) &= \arg\min_{\mathbf{w}} -f(\mathbf{w}) \\
&= \arg\min_{\mathbf{w}} -\left[\sum_{i=1}^{N} \log p(y_i \mid \mathbf{x}_i, \mathbf{w})\right] + \lambda\|\mathbf{w}\|_1 \\
&= \arg\min_{\mathbf{w}} -\frac{1}{N}\left[\sum_{i=1}^{N} \log p(y_i \mid \mathbf{x}_i, \mathbf{w})\right] + \lambda'\|\mathbf{w}\|_1,
\end{aligned}$$

where $\lambda' = \frac{\lambda}{N} = \frac{1}{bN}$.

(b) See jupyter notebook for code. Note that we generalize this to a softmax model and use the ADAM optimizer.

(i) Using $\lambda = 0$, which yielded the optimal validation accuracy, we obtain a testing accuracy of 84.40%.

(ii) The 5 words for negative reviews with the heaviest and lightest weights in the final fitted model are:

```
Heaviest:  'wasted', 'waste', 'save', 'tedious', 'awful.'
Lightest:  'loved', '7/10', 'superb.', 'enjoy', 'refreshing'
```

The 5 words for positive reviews with the heaviest and lightest weights in the final fitted model are:

```
Heaviest:  'enjoy', 'excellent.', 'awesome', '10/10', 'refreshing'
Lightest:  'instead.', 'costs.', 'wasted', 'waste', 'save'
```

(iii) The testing accuracies and weight sparsity are shown for various λ in the table below. As expected, sparsity increases with regularization strength. However, accuracy decreases.

λ	acc. (%)	sparsity (%)
0	84.40	0.89
0.001	80.90	34.32
0.01	71.80	39.72
0.1	52.10	41.02
1	50.00	41.20

Problem 6 (Neural Networks, 5pts)

In the previous problem, we have implemented a Logistic Regression classifier using PyTorch. Logistic Regression can be seen as a 1-layer neural network. With PyTorch automatic differentiation, implementing a multi-layer neural network only requires incremental change to our logistic regression implementation.

(a) Implement a multi-layer neural network for IMDB classification and report accuracy on the test set. You are free to design the network structure (number of hidden units, activation function) and choose the optimization methods (SGD or ADAM, regularization or not, etc.).

(b) (Optional) Implement sentiment classification based on Convolutional Neural Networks. We recommend reading Yoon Kim (2014) *Convolution Neural Networks for Sentence Classification*. Note that in this part, you need to treat the text as a sequence of word vectors instead of bag-of-words. You can do this by forwarding `batch.text[0]` to `torch.nn.Embedding(vocab_size, embedding_dim)` after setting the weights using the pretrained vectors from `text_field.vocab.vectors`.

See jupyter notebook for code. After adding one hidden layer with a reLU activation function to the model in 5., we obtain an improved testing accuracy of 85.40%. The top five negative words for the softmax activation function were `'boring', 'wasted', 'waste', 'guess', 'horrible'`, ad the top five positive words were `'great.', 'excellent', 'wonderful.', 'favorite', 'loved'`.