

## HOMEWORK 1: Exercises for Monte Carlo Methods

March 5, 2019

### Exercise 1.

The Monte Carlo method can be used to generate an approximate value of  $\pi$ . The figure below shows a unit square with a quarter of a circle inscribed. The area of the square is 1 and the area of the quarter circle is  $\pi/4$ . Write a script to generate random points that are distributed uniformly in the unit square. The ratio between the number of points that fall inside the circle (red points) and the total number of points thrown (red and green points) gives an approximation to the value of  $\pi/4$ . This process is a Monte Carlo simulation approximating  $\pi$ . Let  $N$  be the total number of points thrown. When  $N=50, 100, 200, 300, 500, 1000, 5000$ , what are the estimated  $\pi$  values, respectively? For each  $N$ , repeat the throwing process 100 times, and report the mean and variance. Record the means and the corresponding variances in a table.

蒙特卡洛方法可以用于产生接近  $\pi$  的近似值。图 1 显示了一个带有  $1/4$  内切圆在内的边长为 1 的正方形。正方形的面积是 1，该  $1/4$  圆的面积为  $\pi/4$ 。通过编程实现在这个正方形中产生均匀分布的点。落在圈内（红点）的点和总的投在正方形（红和绿点）上的点的比率给出了  $\pi/4$  的近似值。这一过程称为使用蒙特卡洛方法来仿真逼近  $\pi$  实际值。令  $N$  表示总的投在正方形的点。当投点个数分别是 20, 50, 100, 200, 300, 500, 1000, 5000 时， $\pi$  值分别是多少？对于每个  $N$ ，每次实验算出  $\pi$  值，重复这个过程 20 次，并在表中记下均值和方差。

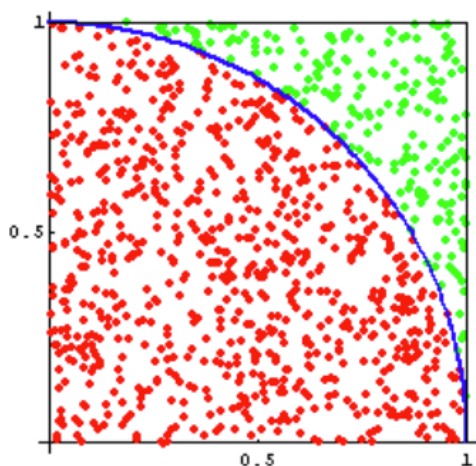


Figure 1 蒙特卡洛方法求解  $\pi$

Times			Repetition	Mean	Deviation
20			20	3.11	0.13389999999999996
50			20	3.112	0.03769599999999999
100			20	3.13	0.05134000000000001
200			20	3.111	0.008859000000000016

300			20	3.1113333333333335	0.008662666666666664
500			20	3.1355999999999993	0.006460639999999998
1000			20	3.1401999999999999	0.002821560000000005
5000			20	3.13888	0.0003840255999999993

```
.py =====
Estimate PI
20    times sampling:
[2.6, 3.0, 2.8, 2.8, 3.0, 3.0, 3.0, 3.4, 3.4, 2.6, 3.0, 3.0, 3.8, 2.6, 3.8, 3.0,
 3.6, 3.0, 3.2, 3.6]
mean:      3.11
deviation: 0.13389999999999996
50    times sampling:
[3.6, 3.12, 3.04, 3.36, 3.12, 2.8, 2.96, 3.2, 3.36, 2.88, 2.88, 3.28, 3.28, 3.12
, 3.04, 2.88, 3.12, 3.2, 2.96, 3.04]
mean:      3.112
deviation: 0.03769599999999999
100    times sampling:
[3.48, 3.04, 3.12, 2.84, 3.12, 3.08, 3.04, 2.64, 3.44, 3.32, 3.16, 3.0, 2.8, 3.2
4, 3.4, 3.44, 3.32, 2.92, 3.24, 2.96]
mean:      3.13
deviation: 0.051340000000000001
200    times sampling:
[3.08, 3.0, 3.16, 3.2, 3.12, 3.08, 3.16, 3.16, 2.92, 3.16, 3.18, 3.18, 3.0, 3.2,
 2.9, 3.02, 3.22, 3.16, 3.1, 3.22]
mean:      3.111
deviation: 0.0088590000000000016
300    times sampling:
[3.1466666666666665, 3.0266666666666667, 3.32, 3.24, 3.013333333333333, 3.1066666
666666665, 3.2133333333333334, 3.12, 3.1466666666666665, 3.04, 3.093333333333333
3, 3.0533333333333332, 2.973333333333333, 3.1466666666666665, 3.0266666666666667,
 2.9866666666666667, 3.2666666666666666, 3.1066666666666665, 3.04, 3.16]
mean:      3.1113333333333335
deviation: 0.008662666666666664
500    times sampling:
[3.024, 3.136, 2.984, 3.128, 3.104, 3.136, 3.2, 3.096, 3.144, 3.184, 3.256, 3.16
, 3.312, 3.04, 3.032, 3.152, 3.152, 3.056, 3.184, 3.232]
mean:      3.1355999999999993
deviation: 0.006460639999999998
1000    times sampling:
[3.156, 3.192, 3.048, 3.208, 3.188, 3.232, 3.168, 3.064, 3.156, 3.068, 3.1, 3.17
6, 3.16, 3.052, 3.124, 3.12, 3.108, 3.12, 3.204, 3.16]
mean:      3.1401999999999999
deviation: 0.0028215600000000005
5000    times sampling:
[3.124, 3.1312, 3.1032, 3.128, 3.1248, 3.1432, 3.1064, 3.1384, 3.1472, 3.1296, 3
.1696, 3.1656, 3.1224, 3.1784, 3.1264, 3.1448, 3.1368, 3.1608, 3.1592, 3.1376]
mean:      3.13888
deviation: 0.0003840255999999993
```

## Exercise 2.

We are now trying to integrate the another function by Monte Carlo method:

$$\int_0^1 x^3$$

A simple analytic solution exists here:  $\int_{x=0}^1 x^3 = 1/4$ . If you compute this integration using Monte Carlo method, what distribution do you use to sample  $x$ ? How good do you get when  $N = 5, 10, 20, 30, 40, 50, 60, 70, 80, 100$ , respectively? For each  $N$ , repeat

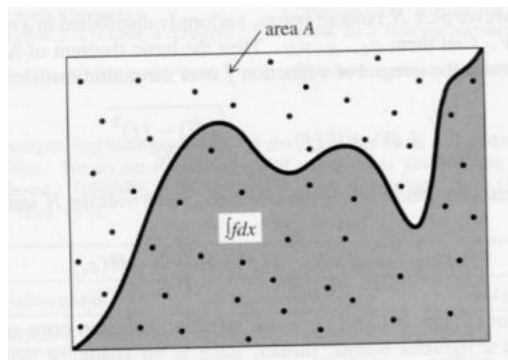
the Monte Carlo process 20 times, and report the mean and variance of the integrate in a table.

我们现在尝试通过蒙特卡洛的方法求解如下的积分：

$$\int_0^1 x^3$$

该积分的求解我们可以直接求解，即有 $\int_{x=0}^1 x^3 = 1/4$ 。如果你用蒙特卡洛的方法求解该积分，你认为 $x$ 可以通过什么分布采样获得？如果采样次数是分别是 $N = 5, 10, 20, 30, 40, 50, 60, 70, 80, 100$ ，积分结果有多好？对于每个采样次数 $N$ ，重复蒙特卡洛过程 100 次，求出均值和方差，然后在表格中记录对应的均值和方差。

用包含该函数的最小矩形，这个地方就是 $(0, 0)$ 到 $(1, 1)$ ，在该矩形中生成二维的点 $(x_0, y_0)$ ，如果点 $(x_0, y_0)$ 在函数 $x^3$ 下方的点的个数和总的点的个数的比率乘以这个矩形的面积就是该积分



Times		Repetition	Mean	Deviation
5		100	0.2539999999999998	0.03908399999999998
10		100	0.23	0.01929999999999996
20		100	0.24850000000000005	0.009772750000000002
30		100	0.24566666666666678	0.006681222222222219
40		100	0.24500000000000008	0.004237500000000002
50		100	0.24840000000000001	0.004049440000000002
60		100	0.24216666666666678	0.003035861111111111
70		100	0.25128571428571417	0.0020983469387755105
80		100	0.24825000000000005	0.0018563125000000003
100		100	0.24070000000000003	0.0014965100000000002

```

Estimate the integral of  $x^3$ 
5      times sampling:
[0.4, 0.0, 0.2, 0.4, 0.2, 0.2, 0.0, 0.0, 0.0, 0.0, 0.4, 0.2, 0.2, 0.2, 0.4, 0.4,
 0.2, 0.4, 0.4, 0.2, 0.4, 0.4, 0.2, 0.6, 0.4, 0.2, 0.0, 0.0, 0.2, 0.4, 0.0, 0.2,
 0.2, 0.2, 0.2, 0.2, 0.2, 0.4, 0.2, 0.2, 0.2, 0.6, 0.6, 0.4, 0.0, 0.0, 0.2, 0.2,
 0.4, 0.2, 0.2, 0.2, 0.4, 0.6, 0.2, 0.6, 0.6, 0.2, 0.2, 0.0, 0.6, 0.6, 0.6, 0.0,
 0.2, 0.4, 0.4, 0.0, 0.0, 0.0, 0.2, 0.4, 0.6, 0.0, 0.4, 0.2, 0.4, 0.0, 0.6, 0.0,
 0.2, 0.4, 0.4, 0.2, 0.0, 0.0, 0.0, 0.2, 0.0, 0.4, 0.2, 0.0, 0.2, 0.2, 0.2, 0.6,
 0.4, 0.2, 0.8, 0.0]
mean:      0.25399999999999998
deviation: 0.039083999999999998
10     times sampling:
[0.2, 0.0, 0.1, 0.3, 0.4, 0.1, 0.2, 0.2, 0.3, 0.0, 0.2, 0.0, 0.1, 0.3, 0.0, 0.0,
 0.4, 0.1, 0.2, 0.3, 0.1, 0.3, 0.0, 0.8, 0.4, 0.1, 0.2, 0.2, 0.2, 0.3, 0.1, 0.4,
 0.2, 0.2, 0.1, 0.3, 0.2, 0.3, 0.2, 0.1, 0.3, 0.4, 0.2, 0.3, 0.2, 0.1, 0.4, 0.3,
 0.0, 0.3, 0.0, 0.3, 0.2, 0.0, 0.2, 0.2, 0.4, 0.2, 0.2, 0.4, 0.3, 0.1, 0.3,
 0.2, 0.3, 0.6, 0.1, 0.2, 0.2, 0.2, 0.3, 0.1, 0.3, 0.4, 0.4, 0.1, 0.1, 0.1, 0.1,
 0.4, 0.2, 0.4, 0.1, 0.3, 0.1, 0.2, 0.2, 0.3, 0.4, 0.5, 0.2, 0.4, 0.4, 0.3, 0.1,
 0.3, 0.2, 0.3, 0.3]
mean:      0.23
deviation: 0.019299999999999996
20     times sampling:
[0.4, 0.3, 0.15, 0.25, 0.35, 0.45, 0.25, 0.1, 0.2, 0.15, 0.25, 0.3, 0.3, 0.15, 0
.35, 0.2, 0.4, 0.1, 0.25, 0.35, 0.35, 0.15, 0.2, 0.35, 0.3, 0.4, 0.1, 0.3, 0.2,
0.25, 0.35, 0.15, 0.15, 0.15, 0.2, 0.25, 0.15, 0.1, 0.15, 0.15, 0.25, 0.1, 0.3,
0.4, 0.2, 0.3, 0.25, 0.15, 0.3, 0.25, 0.35, 0.2, 0.3, 0.1, 0.15, 0.2, 0.1, 0.15,
0.15, 0.1, 0.2, 0.1, 0.2, 0.55, 0.3, 0.15, 0.35, 0.25, 0.25, 0.45, 0.3, 0.4, 0.
25, 0.25, 0.1, 0.3, 0.1, 0.25, 0.25, 0.2, 0.4, 0.35, 0.15, 0.2, 0.35, 0.3, 0.1,
0.2, 0.4, 0.25, 0.3, 0.35, 0.4, 0.2, 0.25, 0.35, 0.25, 0.3, 0.3, 0.3]
mean:      0.248500000000000005
deviation: 0.0097727500000000002
30     times sampling:
[0.23333333333333334, 0.06666666666666667, 0.26666666666666666, 0.166666666666666
66, 0.2, 0.36666666666666664, 0.36666666666666664, 0.23333333333333334, 0.26666
66666666666, 0.13333333333333333, 0.23333333333333334, 0.2, 0.23333333333333334
, 0.36666666666666664, 0.23333333333333334, 0.3333333333333333, 0.23333333333333
334, 0.26666666666666666, 0.13333333333333333, 0.2, 0.2, 0.23333333333333334, 0.
13333333333333333, 0.3, 0.2, 0.23333333333333334, 0.26666666666666666, 0.1666666
6666666666, 0.16666666666666666, 0.4, 0.23333333333333334, 0.3, 0.166666666666666
66, 0.13333333333333333, 0.3, 0.36666666666666664, 0.36666666666666664, 0.26666
66666666666, 0.26666666666666666, 0.3, 0.13333333333333333, 0.3, 0.1666666666666
6666, 0.33333333333333333, 0.23333333333333334, 0.16666666666666666, 0.3, 0.2, 0
.06666666666666667, 0.36666666666666664, 0.33333333333333333, 0.16666666666666666
, 0.2, 0.3, 0.3, 0.4, 0.26666666666666666, 0.33333333333333333, 0.266666666666666
66, 0.26666666666666666, 0.23333333333333334, 0.3, 0.3, 0.2, 0.3, 0.3, 0.3, 0.33
33333333333333333, 0.23333333333333334, 0.26666666666666666, 0.16666666666666666, 0
.43333333333333335, 0.16666666666666666, 0.13333333333333333, 0.36666666666666666
4, 0.2, 0.26666666666666666, 0.26666666666666666, 0.16666666666666666, 0.1, 0.06
66666666666666667, 0.26666666666666666, 0.3, 0.13333333333333333, 0.3, 0.2, 0.1333
33333333333333333, 0.43333333333333335, 0.33333333333333333, 0.26666666666666666, 0.2
, 0.23333333333333334, 0.13333333333333333, 0.36666666666666664, 0.1666666666666
6666, 0.26666666666666666, 0.3, 0.2, 0.16666666666666666, 0.23333333333333334]
mean:      0.245666666666666678
deviation: 0.006681222222222219

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40      times sampling:
[0.3, 0.275, 0.325, 0.25, 0.2, 0.3, 0.2, 0.2, 0.275, 0.175, 0.1, 0.225, 0.2, 0.2
5, 0.325, 0.225, 0.125, 0.3, 0.2, 0.2, 0.4, 0.15, 0.275, 0.3, 0.225, 0.375, 0.2,
0.325, 0.275, 0.275, 0.175, 0.3, 0.325, 0.3, 0.175, 0.125, 0.3, 0.15, 0.2, 0.25
, 0.2, 0.25, 0.3, 0.275, 0.25, 0.3, 0.2, 0.3, 0.175, 0.275, 0.3, 0.25, 0.25, 0.2
5, 0.125, 0.325, 0.35, 0.075, 0.2, 0.175, 0.3, 0.275, 0.225, 0.25, 0.225, 0.275,
0.35, 0.225, 0.275, 0.25, 0.3, 0.125, 0.35, 0.225, 0.3, 0.2, 0.3, 0.275, 0.25,
0.325, 0.125, 0.25, 0.175, 0.275, 0.275, 0.15, 0.25, 0.25, 0.3, 0.25, 0.3, 0.225
, 0.325, 0.225, 0.125, 0.225, 0.275, 0.2, 0.125, 0.3]
mean:      0.245000000000000008
deviation: 0.004237500000000002
50      times sampling:
[0.26, 0.34, 0.24, 0.3, 0.18, 0.22, 0.2, 0.28, 0.28, 0.18, 0.26, 0.24, 0.3, 0.26
, 0.24, 0.34, 0.22, 0.18, 0.2, 0.2, 0.26, 0.38, 0.18, 0.24, 0.42, 0.2, 0.26, 0.1
8, 0.14, 0.38, 0.2, 0.22, 0.16, 0.22, 0.24, 0.26, 0.24, 0.26, 0.28, 0.36, 0.24,
0.22, 0.2, 0.38, 0.26, 0.16, 0.24, 0.28, 0.14, 0.28, 0.28, 0.16, 0.32, 0.24, 0.2
2, 0.38, 0.4, 0.24, 0.24, 0.22, 0.16, 0.22, 0.32, 0.18, 0.26, 0.22, 0.26, 0.28,
0.26, 0.26, 0.26, 0.26, 0.28, 0.3, 0.16, 0.28, 0.22, 0.16, 0.26, 0.1, 0.22, 0.26
, 0.34, 0.22, 0.22, 0.18, 0.2, 0.4, 0.2, 0.28, 0.32, 0.12, 0.24, 0.28, 0.22, 0.2
4, 0.26, 0.2, 0.34, 0.3]
mean:      0.248400000000000001
deviation: 0.004049440000000002
60      times sampling:
[0.233333333333333334, 0.233333333333333334, 0.35, 0.25, 0.18333333333333332, 0.23
333333333333333334, 0.15, 0.3, 0.18333333333333332, 0.3, 0.2833333333333333, 0.2, 0
.18333333333333332, 0.2, 0.25, 0.23333333333333334, 0.2, 0.36666666666666664, 0.
15, 0.13333333333333333, 0.2, 0.2833333333333333, 0.18333333333333332, 0.2, 0.2,
0.3, 0.25, 0.23333333333333334, 0.26666666666666666, 0.26666666666666666, 0.216
666666666666667, 0.2, 0.13333333333333333, 0.2, 0.2833333333333333, 0.2, 0.266666
66666666666, 0.3, 0.3, 0.2833333333333333, 0.13333333333333333, 0.3, 0.266666666
66666666, 0.25, 0.16666666666666666, 0.21666666666666667, 0.26666666666666666, 0
.18333333333333332, 0.2833333333333333, 0.2, 0.18333333333333332, 0.216666666666
66667, 0.21666666666666667, 0.26666666666666666, 0.16666666666666666, 0.3, 0.25,
0.26666666666666666, 0.3, 0.26666666666666666, 0.25, 0.2, 0.26666666666666666,
0.21666666666666667, 0.26666666666666666, 0.2833333333333333, 0.11666666666666666
7, 0.16666666666666666, 0.23333333333333334, 0.2, 0.35, 0.2833333333333333, 0.23
333333333333334, 0.21666666666666667, 0.2833333333333333, 0.23333333333333334, 0
.35, 0.3, 0.21666666666666667, 0.21666666666666667, 0.31666666666666665, 0.3, 0.
23333333333333334, 0.21666666666666667, 0.3, 0.21666666666666667, 0.35, 0.2, 0.1
8333333333333332, 0.26666666666666666, 0.2833333333333333, 0.31666666666666665,
0.38333333333333336, 0.23333333333333334, 0.18333333333333332, 0.233333333333333
34, 0.3, 0.21666666666666667, 0.21666666666666667, 0.3]
mean:      0.242166666666666678
deviation: 0.00303586111111111
70      times sampling:
[0.2857142857142857, 0.35714285714285715, 0.24285714285714285, 0.171428571428571
43, 0.22857142857142856, 0.18571428571428572, 0.2857142857142857, 0.285714285714
2857, 0.21428571428571427, 0.2857142857142857, 0.24285714285714285, 0.2, 0.25714
28571428571, 0.2857142857142857, 0.15714285714285714, 0.32857142857142857, 0.214
28571428571427, 0.3, 0.22857142857142856, 0.22857142857142856, 0.214285714285714
27, 0.24285714285714285, 0.18571428571428572, 0.22857142857142856, 0.35714285714
285715, 0.24285714285714285, 0.3, 0.2, 0.3, 0.2, 0.2714285714285714, 0.242857142
85714285, 0.2714285714285714, 0.2, 0.24285714285714285, 0.2571428571428571, 0.25
71428571428571, 0.37142857142857144, 0.2857142857142857, 0.3142857142857143, 0.3
4285714285714286, 0.32857142857142857, 0.24285714285714285, 0.22857142857142856,
0.2, 0.24285714285714285, 0.2571428571428571, 0.14285714285714285, 0.3142857142
857143, 0.3, 0.18571428571428572, 0.17142857142857143, 0.2571428571428571, 0.242
85714285714285, 0.2571428571428571, 0.2857142857142857, 0.2571428571428571, 0.25
71428571428571, 0.2, 0.22857142857142856, 0.3, 0.18571428571428572, 0.2428571428
5714285, 0.2571428571428571, 0.24285714285714285, 0.2571428571428571, 0.24285714
285714285, 0.24285714285714285, 0.2571428571428571, 0.2571428571428571, 0.242857
14285714285, 0.2857142857142857, 0.3, 0.2857142857142857, 0.3142857142857143, 0.
24285714285714285, 0.21428571428571427, 0.22857142857142856, 0.3, 0.214285714285
71427, 0.3, 0.22857142857142856, 0.3, 0.2714285714285714, 0.2714285714285714, 0.
2571428571428571, 0.2571428571428571, 0.2, 0.2857142857142857, 0.185714285714285
72, 0.17142857142857143, 0.17142857142857143, 0.24285714285714285, 0.3, 0.257142
8571428571, 0.2571428571428571, 0.22857142857142856, 0.2, 0.22857142857142856, 0
.2857142857142857]
mean:      0.25128571428571417
deviation: 0.0020983469387755105
```



```

80      times sampling:
[0.2375, 0.2125, 0.2625, 0.25, 0.2, 0.225, 0.2375, 0.25, 0.2375, 0.2875, 0.2375,
 0.25, 0.275, 0.3125, 0.3125, 0.275, 0.2125, 0.2, 0.2625, 0.225, 0.275, 0.2375,
 0.275, 0.3, 0.2375, 0.2875, 0.2, 0.3, 0.1875, 0.275, 0.225, 0.2375, 0.3, 0.275,
 0.25, 0.3, 0.2125, 0.2375, 0.3125, 0.2, 0.35, 0.25, 0.2875, 0.225, 0.2875, 0.325
, 0.2, 0.3, 0.2, 0.275, 0.2, 0.175, 0.225, 0.2, 0.3, 0.225, 0.3125, 0.225, 0.312
5, 0.225, 0.25, 0.3625, 0.2, 0.175, 0.25, 0.2, 0.2375, 0.2625, 0.2375, 0.2625, 0
.3, 0.2375, 0.2375, 0.275, 0.2375, 0.225, 0.2375, 0.225, 0.3125, 0.2875, 0.225,
 0.25, 0.2625, 0.2, 0.2, 0.3, 0.2375, 0.3125, 0.275, 0.225, 0.2375, 0.2375, 0.162
5, 0.3, 0.2, 0.1625, 0.125, 0.25, 0.2125, 0.25]
mean:      0.24825000000000005
deviation: 0.0018563125000000003
100      times sampling:
[0.25, 0.25, 0.24, 0.23, 0.22, 0.26, 0.24, 0.2, 0.22, 0.23, 0.21, 0.3, 0.19, 0.2
4, 0.22, 0.29, 0.27, 0.23, 0.25, 0.2, 0.2, 0.22, 0.19, 0.24, 0.21, 0.22, 0.25, 0
.28, 0.23, 0.33, 0.19, 0.23, 0.18, 0.23, 0.28, 0.25, 0.28, 0.22, 0.26, 0.16, 0.2
8, 0.23, 0.26, 0.25, 0.24, 0.21, 0.2, 0.24, 0.16, 0.21, 0.25, 0.22, 0.29, 0.22,
 0.24, 0.25, 0.22, 0.25, 0.26, 0.29, 0.21, 0.29, 0.24, 0.17, 0.23, 0.22, 0.25, 0
.22, 0.23, 0.22, 0.25, 0.21, 0.18, 0.33, 0.22, 0.36, 0.21, 0.23, 0.28, 0.31, 0.22
, 0.31, 0.17, 0.23, 0.24, 0.25, 0.28, 0.26, 0.32, 0.21, 0.27, 0.29, 0.25, 0.25,
 0.28, 0.19, 0.27, 0.27, 0.31, 0.18]
mean:      0.24070000000000003
deviation: 0.0014965100000000002

```

### Exercise 3:

We are now trying to integrate a more difficult function by Monte Carlo method that may not be analytically computed:

$$\int_{x=2}^4 \int_{y=-1}^1 f(x,y) = \frac{y^2 * e^{-y^2} + x^4 * e^{-x^2}}{x * e^{-x^2}}$$

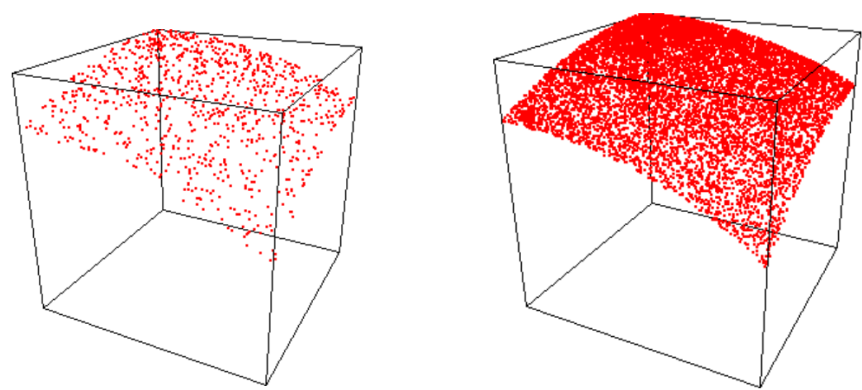
Can you compute the above integration analytically? If you compute this integration using Monte Carlo method, what distribution do you use to sample (x,y)? How good do you get when the sample sizes are N = 5, 10, 20, 30, 40, 50, 60, 70, 80, 100, 200 respectively? For each N, repeat the Monte Carlo process 100 times, and report the mean and variance of the integrate.

我们现在尝试通过蒙特卡洛的方法求解如下的更复杂的积分：

$$\int_{x=2}^4 \int_{y=-1}^1 f(x,y) = \frac{y^2 * e^{-y^2} + x^4 * e^{-x^2}}{x * e^{-x^2}}$$

你能够通过公式直接求解上述的积分吗？如果你用蒙特卡洛的方法求解该积分，你认为(x, y)可以通过什么分布采样获得？如果点 ( x, y ) 的采样次数是分别是 N = 10, 20, 30, 40, 50, 60, 70, 80, 100, 200, 500, 积分结果有多好？对于每个采样次数 N，重复蒙特卡洛过程 100 次，求出均值和方差，然后在表格中记录对应的均值和方差。

计算该函数  $f(x, y)$  的最大最小值，可以得出包含该函数的最小的三维矩体，在长方体中随机生成点  $(x_0, y_0)$ ，在  $f(x, y)$  下方的点的个数与总的点的个数的比率乘以矩体的体积就是该函数的积分



Times		Repetition	Mean	Deviation
10		100	107884.9652915795	32181384785.11628
20		100	112788.82735028774	16403314612.739002
30		100	112243.95378820904	8895928020.417952
40		100	123413.86181082216	9551677620.44123
50		100	132077.3514478732	9403676383.041075
60		100	114968.32159860246	5693999582.952734
70		100	108819.034255143	5280932546.868621
80		100	106658.99977690249	4705866208.588251
100		100	108865.73770332123	4062592550.8703184
200		100	115404.22044826539	2168689733.2274776
500		100	113246.52114243388	691620817.6681459

```

Estimate the double integral of  $(y^2 * e^{(-y^2)} + x^4 * e^{(-x^2)}) / (x * e^{(-x^2)})$ 
10 times sampling:
[0.0, 326924.13724721083, 326924.13724721083, 0.0, 0.0, 0.0, 0.0, 326924.1372472
1083, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 326924.13724721083, 326924.13724721083, 3269
24.13724721083, 0.0, 326924.13724721083, 980772.4117416323, 0.0, 0.0, 0.0, 0.0,
326924.13724721083, 0.0, 0.0, 0.0, 326924.13724721083, 0.0, 0.0, 0.0, 0.0, 32692
4.13724721083, 653848.2744944217, 0.0, 0.0, 0.0, 326924.13724721083, 0.0, 326924
.13724721083, 0.0, 0.0, 0.0, 0.0, 326924.13724721083, 326924.13724721083, 0.0, 3
26924.13724721083, 0.0, 326924.13724721083, 0.0, 0.0, 0.0, 0.0, 326924.137247210
83, 326924.13724721083, 326924.13724721083, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0
.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 326924.13724721083, 0.0, 0.0, 326924
.13724721083, 326924.13724721083, 0.0, 326924.13724721083, 0.0, 0.0, 0.0, 0.0, 0
.0, 0.0, 0.0, 326924.13724721083, 0.0, 0.0, 0.0, 0.0, 326924.13724721083, 0.0, 0
.0, 326924.13724721083, 0.0, 326924.13724721083, 326924.13724721083]
mean: 107884.9652915795
deviation: 32181384785.11628
20 times sampling:
[0.0, 326924.13724721083, 326924.13724721083, 0.0, 0.0, 163462.06862360542, 1634
62.06862360542, 163462.06862360542, 0.0, 0.0, 163462.06862360542, 0.0, 0.0, 0.0,
163462.06862360542, 0.0, 163462.06862360542, 0.0, 326924.13724721083, 0.0, 1634
62.06862360542, 163462.06862360542, 326924.13724721083, 0.0, 0.0, 0.0, 326924.13
724721083, 0.0, 490386.20587081613, 0.0, 0.0, 0.0, 163462.06862360542, 0.0, 0.0,
163462.06862360542, 0.0, 163462.06862360542, 0.0, 163462.06862360542, 0.0, 1634
62.06862360542, 0.0, 163462.06862360542, 326924.13724721083, 163462.06862360542,
0.0, 326924.13724721083, 163462.06862360542, 0.0, 0.0, 163462.06862360542, 0.0,
326924.13724721083, 0.0, 0.0, 0.0, 326924.13724721083, 326924.13724721083, 0.0,
163462.06862360542, 0.0, 163462.06862360542, 163462.06862360542, 0.0, 163462.06
862360542, 0.0, 326924.13724721083, 0.0, 0.0, 326924.13724721083, 163462.0686236
0542, 163462.06862360542, 163462.06862360542, 0.0, 163462.06862360542, 326924.13
724721083, 163462.06862360542, 163462.06862360542, 0.0, 0.0, 0.0, 163462.0686236
0542, 163462.06862360542, 163462.06862360542, 163462.06862360542, 163462.0686236
0542, 0.0, 163462.06862360542, 326924.13724721083, 163462.06862360542, 0.0, 0.0,
0.0, 0.0, 163462.06862360542, 0.0, 490386.20587081613, 0.0]
mean: 112788.82735028774
deviation: 16403314612.739002
30 times sampling:
[0.0, 0.0, 217949.42483147385, 108974.71241573693, 0.0, 0.0, 0.0, 217949.4248314
7385, 108974.71241573693, 108974.71241573693, 108974.71241573693, 0.0, 0.0, 1089
74.71241573693, 108974.71241573693, 108974.71241573693, 0.0, 217949.42483147385,
108974.71241573693, 108974.71241573693, 108974.71241573693, 217949.42483147385,
108974.71241573693, 108974.71241573693, 108974.71241573693, 0.0, 0.0, 108974.71
241573693, 326924.13724721083, 0.0, 217949.42483147385, 0.0, 217949.42483147385,
0.0, 108974.71241573693, 108974.71241573693, 0.0, 217949.42483147385, 0.0, 0.0,
108974.71241573693, 0.0, 108974.71241573693, 217949.42483147385, 0.0, 217949.42
483147385, 0.0, 108974.71241573693, 108974.71241573693, 0.0, 217949.42483147385,
217949.42483147385, 108974.71241573693, 108974.71241573693, 217949.42483147385,
0.0, 108974.71241573693, 0.0, 108974.71241573693, 326924.13724721083, 108974.71
241573693, 217949.42483147385, 217949.42483147385, 217949.42483147385, 0.0, 0.0,
0.0, 217949.42483147385, 108974.71241573693, 326924.13724721083, 0.0, 217949.42
483147385, 217949.42483147385, 108974.71241573693, 217949.42483147385, 108974.71
241573693, 0.0, 0.0, 217949.42483147385, 0.0, 108974.71241573693, 217949.4248314
7385, 217949.42483147385, 217949.42483147385, 108974.71241573693, 217949.4248314
7385, 108974.71241573693, 0.0, 108974.71241573693, 0.0, 326924.13724721083, 2179
49.42483147385, 108974.71241573693, 108974.71241573693, 108974.71241573693, 0.0,
217949.42483147385, 108974.71241573693, 108974.71241573693, 217949.42483147385]
mean: 112243.95378820904
deviation: 8895928020.417952

```



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40      times sampling:
[81731.03431180271, 81731.03431180271, 163462.06862360542, 163462.06862360542, 8
1731.03431180271, 0.0, 81731.03431180271, 81731.03431180271, 0.0, 81731.03431180
271, 245193.10293540807, 245193.10293540807, 163462.06862360542, 163462.06862360
542, 81731.03431180271, 81731.03431180271, 163462.06862360542, 163462.0686236054
2, 0.0, 245193.10293540807, 245193.10293540807, 245193.10293540807, 326924.13724
721083, 0.0, 0.0, 0.0, 81731.03431180271, 0.0, 163462.06862360542, 81731.0343118
0271, 326924.13724721083, 163462.06862360542, 81731.03431180271, 163462.06862360
542, 81731.03431180271, 163462.06862360542, 163462.06862360542, 0.0, 81731.03431
180271, 0.0, 0.0, 0.0, 0.0, 163462.06862360542, 163462.06862360542, 0.0, 81731.0
3431180271, 81731.03431180271, 81731.03431180271, 163462.06862360542, 81731.0343
1180271, 81731.03431180271, 163462.06862360542, 0.0, 81731.03431180271, 245193.1
0293540807, 81731.03431180271, 81731.03431180271, 326924.13724721083, 0.0, 0.0,
81731.03431180271, 81731.03431180271, 163462.06862360542, 163462.06862360542, 16
3462.06862360542, 81731.03431180271, 81731.03431180271, 81731.03431180271, 16346
2.06862360542, 163462.06862360542, 81731.03431180271, 163462.06862360542, 163462
.06862360542, 0.0, 163462.06862360542, 81731.03431180271, 245193.10293540807, 40
8655.1715590135, 163462.06862360542, 163462.06862360542, 0.0, 245193.10293540807
, 0.0, 163462.06862360542, 163462.06862360542, 0.0, 163462.06862360542, 81731.03
431180271, 163462.06862360542, 408655.1715590135, 81731.03431180271, 0.0, 0.0, 4
08655.1715590135, 245193.10293540807, 163462.06862360542, 163462.06862360542, 16
3462.06862360542, 245193.10293540807]
mean:      123413.86181082216
deviation: 9551677620.44123
50      times sampling:
[65384.82744944216, 65384.82744944216, 261539.30979776863, 65384.82744944216, 26
1539.30979776863, 392308.96469665295, 130769.65489888431, 196154.48234832648, 13
0769.65489888431, 130769.65489888431, 65384.82744944216, 0.0, 326924.13724721083
, 196154.48234832648, 130769.65489888431, 130769.65489888431, 261539.30979776863
, 65384.82744944216, 196154.48234832648, 65384.82744944216, 65384.82744944216, 1
96154.48234832648, 65384.82744944216, 0.0, 0.0, 130769.65489888431, 0.0, 196154.
48234832648, 130769.65489888431, 65384.82744944216, 130769.65489888431, 65384.82
744944216, 196154.48234832648, 65384.82744944216, 130769.65489888431, 130769.654
89888431, 326924.13724721083, 65384.82744944216, 196154.48234832648, 65384.82744
944216, 65384.82744944216, 65384.82744944216, 65384.82744944216, 65384.827449442
16, 130769.65489888431, 130769.65489888431, 130769.65489888431, 196154.482348326
48, 65384.82744944216, 196154.48234832648, 65384.82744944216, 0.0, 196154.482348
32648, 130769.65489888431, 261539.30979776863, 130769.65489888431, 0.0, 196154.4
8234832648, 0.0, 130769.65489888431, 0.0, 196154.48234832648, 65384.82744944216,
130769.65489888431, 0.0, 261539.30979776863, 0.0, 130769.65489888431, 261539.30
979776863, 261539.30979776863, 196154.48234832648, 196154.48234832648, 0.0, 6538
4.82744944216, 0.0, 0.0, 130769.65489888431, 261539.30979776863, 65384.827449442
16, 65384.82744944216, 130769.65489888431, 130769.65489888431, 130769.6548988843
1, 196154.48234832648, 196154.48234832648, 0.0, 130769.65489888431, 65384.827449
44216, 65384.82744944216, 523078.61959553725, 130769.65489888431, 196154.4823483
2648, 65384.82744944216, 65384.82744944216, 196154.48234832648, 261539.309797768
63, 326924.13724721083, 130769.65489888431, 130769.65489888431, 261539.309797768
63]
mean:      132077.3514478732
deviation: 9403676383.041075

```

```

60      times sampling:
[108974.71241573693, 108974.71241573693, 108974.71241573693, 54487.35620786846,
108974.71241573693, 108974.71241573693, 108974.71241573693, 54487.35620786846, 5
4487.35620786846, 108974.71241573693, 54487.35620786846, 108974.71241573693, 163
462.06862360542, 54487.35620786846, 108974.71241573693, 54487.35620786846, 54487
.35620786846, 108974.71241573693, 108974.71241573693, 0.0, 108974.71241573693, 1
08974.71241573693, 54487.35620786846, 163462.06862360542, 54487.35620786846, 108
974.71241573693, 217949.42483147385, 217949.42483147385, 54487.35620786846, 3269
24.13724721083, 108974.71241573693, 54487.35620786846, 108974.71241573693, 0.0,
54487.35620786846, 54487.35620786846, 108974.71241573693, 108974.71241573693, 16
3462.06862360542, 0.0, 54487.35620786846, 108974.71241573693, 54487.35620786846,
108974.71241573693, 217949.42483147385, 272436.7810393423, 54487.35620786846, 0
.0, 54487.35620786846, 108974.71241573693, 217949.42483147385, 54487.35620786846
, 108974.71241573693, 54487.35620786846, 54487.35620786846, 108974.71241573693,
217949.42483147385, 163462.06862360542, 108974.71241573693, 0.0, 108974.71241573
693, 163462.06862360542, 108974.71241573693, 108974.71241573693, 217949.42483147
385, 217949.42483147385, 54487.35620786846, 54487.35620786846, 163462.0686236054
2, 163462.06862360542, 54487.35620786846, 217949.42483147385, 108974.71241573693
, 217949.42483147385, 381411.49345507927, 163462.06862360542, 54487.35620786846,
108974.71241573693, 108974.71241573693, 108974.71241573693, 163462.06862360542,
163462.06862360542, 217949.42483147385, 108974.71241573693, 54487.35620786846,
0.0, 272436.7810393423, 163462.06862360542, 108974.71241573693, 0.0, 163462.0686
2360542, 54487.35620786846, 326924.13724721083, 54487.35620786846, 217949.424831
47385, 163462.06862360542, 108974.71241573693, 163462.06862360542, 54487.3562078
6846, 0.0]

```

```

mean:      114968.32159860246

```

```

deviation: 5693999582.952734

```

```

70      times sampling:
[280220.68906903785, 186813.79271269188, 140110.34453451892, 140110.34453451892,
186813.79271269188, 46703.44817817297, 46703.44817817297, 233517.24089086484, 1
40110.34453451892, 186813.79271269188, 140110.34453451892, 140110.34453451892, 4
6703.44817817297, 46703.44817817297, 0.0, 140110.34453451892, 93406.89635634594,
140110.34453451892, 93406.89635634594, 0.0, 46703.44817817297, 140110.344534518
92, 140110.34453451892, 93406.89635634594, 93406.89635634594, 93406.89635634594,
93406.89635634594, 140110.34453451892, 280220.68906903785, 46703.44817817297, 1
86813.79271269188, 186813.79271269188, 46703.44817817297, 46703.44817817297, 186
813.79271269188, 46703.44817817297, 46703.44817817297, 93406.89635634594, 140110
.34453451892, 186813.79271269188, 186813.79271269188, 0.0, 46703.44817817297, 23
3517.24089086484, 140110.34453451892, 93406.89635634594, 186813.79271269188, 934
06.89635634594, 140110.34453451892, 186813.79271269188, 93406.89635634594, 0.0,
140110.34453451892, 280220.68906903785, 93406.89635634594, 46703.44817817297, 93
406.89635634594, 46703.44817817297, 0.0, 46703.44817817297, 93406.89635634594, 1
40110.34453451892, 46703.44817817297, 46703.44817817297, 46703.44817817297, 9340
6.89635634594, 0.0, 0.0, 140110.34453451892, 93406.89635634594, 93406.8963563459
4, 46703.44817817297, 93406.89635634594, 186813.79271269188, 140110.34453451892,
93406.89635634594, 140110.34453451892, 46703.44817817297, 46703.44817817297, 46
703.44817817297, 186813.79271269188, 233517.24089086484, 0.0, 93406.89635634594,
93406.89635634594, 233517.24089086484, 0.0, 140110.34453451892, 140110.34453451
892, 233517.24089086484, 46703.44817817297, 233517.24089086484, 186813.792712691
88, 140110.34453451892, 233517.24089086484, 46703.44817817297, 0.0, 46703.448178
17297, 46703.44817817297, 0.0]

```

```

mean:      108819.034255143

```

```

deviation: 5280932546.868621

```

```

80      times sampling:
[81731.03431180271, 0.0, 40865.517155901354, 81731.03431180271, 0.0, 40865.51715
5901354, 40865.517155901354, 81731.03431180271, 40865.517155901354, 81731.034311
80271, 0.0, 81731.03431180271, 122596.55146770403, 122596.55146770403, 122596.55
146770403, 81731.03431180271, 163462.06862360542, 81731.03431180271, 163462.0686
2360542, 163462.06862360542, 122596.55146770403, 81731.03431180271, 122596.55146
770403, 40865.517155901354, 81731.03431180271, 81731.03431180271, 245193.1029354
0807, 40865.517155901354, 163462.06862360542, 81731.03431180271, 81731.034311802
71, 204327.58577950674, 286058.6200913094, 0.0, 286058.6200913094, 163462.068623
60542, 81731.03431180271, 40865.517155901354, 163462.06862360542, 0.0, 81731.034
31180271, 81731.03431180271, 122596.55146770403, 81731.03431180271, 122596.55146
770403, 122596.55146770403, 40865.517155901354, 81731.03431180271, 163462.068623
60542, 163462.06862360542, 81731.03431180271, 163462.06862360542, 122596.5514677
0403, 122596.55146770403, 81731.03431180271, 286058.6200913094, 40865.5171559013
54, 122596.55146770403, 204327.58577950674, 204327.58577950674, 163462.068623605
42, 40865.517155901354, 40865.517155901354, 81731.03431180271, 40865.51715590135
4, 40865.517155901354, 40865.517155901354, 0.0, 40865.517155901354, 0.0, 204327.
58577950674, 204327.58577950674, 204327.58577950674, 163462.06862360542, 81731.0
3431180271, 163462.06862360542, 245193.10293540807, 163462.06862360542, 81731.03
431180271, 40865.517155901354, 122596.55146770403, 163462.06862360542, 122596.55
146770403, 163462.06862360542, 81731.03431180271, 286058.6200913094, 81731.03431
180271, 40865.517155901354, 122596.55146770403, 81731.03431180271, 81731.0343118
0271, 122596.55146770403, 40865.517155901354, 122596.55146770403, 122596.5514677
0403, 81731.03431180271, 40865.517155901354, 122596.55146770403, 163462.06862360
542, 0.0]
mean:      106658.99977690249
deviation: 4705866208.588251
100      times sampling:
[32692.41372472108, 32692.41372472108, 65384.82744944216, 32692.41372472108, 130
769.65489888431, 98077.24117416324, 98077.24117416324, 98077.24117416324, 98077.
24117416324, 0.0, 65384.82744944216, 98077.24117416324, 163462.06862360542, 9807
7.24117416324, 163462.06862360542, 65384.82744944216, 65384.82744944216, 65384.8
2744944216, 65384.82744944216, 163462.06862360542, 65384.82744944216, 130769.654
89888431, 0.0, 196154.48234832648, 163462.06862360542, 98077.24117416324, 196154
.48234832648, 228846.89607304757, 65384.82744944216, 98077.24117416324, 196154.4
8234832648, 98077.24117416324, 163462.06862360542, 65384.82744944216, 196154.482
34832648, 65384.82744944216, 98077.24117416324, 294231.7235224897, 65384.8274494
4216, 98077.24117416324, 130769.65489888431, 32692.41372472108, 98077.2411741632
4, 130769.65489888431, 98077.24117416324, 130769.65489888431, 196154.48234832648
, 130769.65489888431, 98077.24117416324, 98077.24117416324, 65384.82744944216, 9
8077.24117416324, 65384.82744944216, 130769.65489888431, 130769.65489888431, 653
84.82744944216, 196154.48234832648, 0.0, 65384.82744944216, 98077.24117416324, 0
.0, 65384.82744944216, 98077.24117416324, 196154.48234832648, 32692.41372472108,
65384.82744944216, 163462.06862360542, 163462.06862360542, 163462.06862360542,
196154.48234832648, 32692.41372472108, 130769.65489888431, 98077.24117416324, 29
4231.7235224897, 65384.82744944216, 196154.48234832648, 98077.24117416324, 19615
4.48234832648, 32692.41372472108, 0.0, 32692.41372472108, 163462.06862360542, 65
384.82744944216, 98077.24117416324, 228846.89607304757, 65384.82744944216, 13076
9.65489888431, 65384.82744944216, 65384.82744944216, 130769.65489888431, 65384.8
2744944216, 228846.89607304757, 261539.30979776863, 65384.82744944216, 98077.241
17416324, 65384.82744944216, 130769.65489888431, 130769.65489888431, 130769.6548
9888431, 32692.41372472108]
mean:      108865.73770332123
deviation: 4062592550.8703184

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200      times sampling:
[65384.82744944216, 163462.06862360542, 98077.24117416324, 114423.44803652378, 6
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.27548596595, 130769.65489888431, 32692.41372472108, 32692.41372472108, 179808.2
7548596595, 65384.82744944216, 98077.24117416324, 81731.03431180271, 65384.82744
944216, 130769.65489888431, 245193.10293540807, 98077.24117416324, 98077.2411741
6324, 81731.03431180271, 65384.82744944216, 130769.65489888431, 196154.482348326
48, 98077.24117416324, 163462.06862360542, 114423.44803652378, 65384.82744944216
, 114423.44803652378, 98077.24117416324, 81731.03431180271, 147115.86176124486,
81731.03431180271, 65384.82744944216, 163462.06862360542, 163462.06862360542, 98
077.24117416324, 98077.24117416324, 16346.20686236054, 81731.03431180271, 65384.
82744944216, 98077.24117416324, 114423.44803652378, 147115.86176124486, 114423.4
4803652378, 114423.44803652378, 65384.82744944216, 163462.06862360542, 49038.620
58708162, 114423.44803652378, 196154.48234832648, 130769.65489888431, 32692.4137
2472108, 163462.06862360542, 147115.86176124486, 98077.24117416324, 81731.034311
80271, 81731.03431180271, 212500.689210687, 130769.65489888431, 81731.0343118027
1, 65384.82744944216, 163462.06862360542, 49038.62058708162, 130769.65489888431,
114423.44803652378, 147115.86176124486, 114423.44803652378, 163462.06862360542,
114423.44803652378, 65384.82744944216, 179808.27548596595, 163462.06862360542,
147115.86176124486, 81731.03431180271, 65384.82744944216, 147115.86176124486, 81
731.03431180271, 81731.03431180271, 98077.24117416324, 179808.27548596595, 81731
.03431180271, 130769.65489888431, 163462.06862360542, 81731.03431180271, 114423.
44803652378, 114423.44803652378, 81731.03431180271, 130769.65489888431, 130769.6
5489888431, 245193.10293540807, 147115.86176124486, 212500.689210687, 114423.448
03652378, 98077.24117416324, 147115.86176124486, 98077.24117416324, 130769.65489
888431, 114423.44803652378]
mean:      115404.22044826539
deviation: 2168689733.2274776
500      times sampling:
[137308.13764382855, 117692.68940899588, 91538.75842921903, 130769.65489888431,
124231.17215394009, 91538.75842921903, 137308.13764382855, 78461.7929393306, 111
154.20666405167, 111154.20666405167, 183077.51685843806, 104615.72391910745, 111
154.20666405167, 111154.20666405167, 85000.2756842748, 78461.7929393306, 150385.
10313371697, 104615.72391910745, 98077.24117416324, 137308.13764382855, 111154.2
0666405167, 98077.24117416324, 91538.75842921903, 117692.68940899588, 111154.206
66405167, 98077.24117416324, 111154.20666405167, 71923.31019438637, 130769.65489
888431, 65384.82744944216, 117692.68940899588, 98077.24117416324, 71923.31019438
637, 98077.24117416324, 111154.20666405167, 137308.13764382855, 137308.137643828
55, 98077.24117416324, 150385.10313371697, 91538.75842921903, 130769.65489888431
, 78461.7929393306, 91538.75842921903, 85000.2756842748, 65384.82744944216, 9807
7.24117416324, 143846.62038877275, 137308.13764382855, 117692.68940899588, 98077
.24117416324, 137308.13764382855, 130769.65489888431, 117692.68940899588, 111154
.20666405167, 85000.2756842748, 124231.17215394009, 163462.06862360542, 137308.1
3764382855, 163462.06862360542, 170000.5513685496, 124231.17215394009, 156923.58
58786612, 117692.68940899588, 130769.65489888431, 156923.5858786612, 150385.1031
3371697, 117692.68940899588, 104615.72391910745, 156923.5858786612, 150385.10313
371697, 137308.13764382855, 124231.17215394009, 78461.7929393306, 124231.1721539
4009, 104615.72391910745, 124231.17215394009, 78461.7929393306, 124231.172153940
09, 65384.82744944216, 85000.2756842748, 98077.24117416324, 65384.82744944216, 9
1538.75842921903, 117692.68940899588, 124231.17215394009, 117692.68940899588, 91
538.75842921903, 85000.2756842748, 85000.2756842748, 78461.7929393306, 137308.13
764382855, 85000.2756842748, 117692.68940899588, 124231.17215394009, 85000.27568
42748, 124231.17215394009, 150385.10313371697, 85000.2756842748, 143846.62038877
275, 104615.72391910745]
mean:      113246.52114243388
deviation: 691620817.6681459

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