On the Ridge Detection Algorithm

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Ridge detection algorithms are important in various applications [1],[2].

Local *height definition* of the ridge[1] for a smooth function $f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$ is a set of points such that:

1. the gradient $\mathbf{v} = \nabla f$ is the eigenvector of the Hessian $\boldsymbol{H} = \partial^2 f / \partial x^i \, \partial x^j$, corresponding to the eigenvalue Λ;

2. all other eigenvalues λ_α are

    (a) smaller $\lambda_\alpha < \Lambda$,

    (b) negative $\lambda_\alpha < 0$.

Present note suggests an optimization of the straightforward approach to explicitly test the gradient against the diagonalized Hessian and its spectrum.

Namely, it should be more efficient to test if the gradient $\mathbf{v}$ is indeed an eigenvector $\boldsymbol{H}\mathbf{v} = \Lambda\mathbf{v}$. This statement is equivalent to:

$$e^{jk\cdots} \frac{\partial^2 f}{\partial x^i \, \partial x^j} \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^k} = 0, \tag{1}$$

where $e^{jk\cdots}$ is the n-dimensional Levi-Civita symbol and the Einstein summation convention is adopted. This equation can be transformed as follows:

$$0 = e^{jk\cdots} \frac{\partial v^i}{\partial x^j} v^i v^k = \frac{e^{jk\cdots}}{2} \frac{\partial(v^i v^i)}{\partial x^j} v^k = \frac{e^{jk\cdots}}{2} \frac{\partial(|\mathbf{v}|^2 v^k)}{\partial x^j}, \tag{2}$$

or

$$0 = \mathrm{curl}\left(|\mathbf{v}|^2 \mathbf{v}\right), \tag{3}$$

where curl is a scalar for 2-dimensional images and a vector for 3-dimensional fields. This "sign-changing" condition can be efficiently bi-sected.

The Hessian spectral analysis is only necessary where Eq.(3) is satisfied. It can also be performed without computationally heavy explicit diagonalization. We begin with the eigenvalue Λ, corresponding to the eigenvector $\mathbf{v}$:

$$\Lambda = \frac{\mathbf{v}^T \boldsymbol{H} \mathbf{v}}{|\mathbf{v}|^2}. \tag{4}$$

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[1] Make adjustments for d-dimensional ridges...

If Λ turns out to be negative, then it remains to check whether the

$$\boldsymbol{H} - \Lambda \boldsymbol{E} \preceq 0, \tag{5}$$

(where $\boldsymbol{E}$ is the identity matrix) is negative semi-definite.

Otherwise, the same requirement is imposed on

$$\boldsymbol{H} \left(\boldsymbol{E} - \frac{\mathbf{v}\,\mathbf{v}^T}{|\mathbf{v}|^2} \right) = \boldsymbol{H} - \Lambda \frac{\mathbf{v}\,\mathbf{v}^T}{|\mathbf{v}|^2} \preceq 0, \tag{6}$$

where the expression in parentheses is the projection to the subspace orthogonal to $\mathbf{v}$.

Both inequalities (5),(6) are efficiently checked by Cholesky decomposition algorithm.

References

[1] D. Eberly, R. Gardner, B. Morse, S. Pizer, and C. Scharlach, *Ridges for Image Analysis*, J Math Imaging Vis **4**, 353 (1994).

[2] D. Eberly, *Ridges in Image and Data Analysis*, Computational Imaging and Vision **7** (Springer Netherlands, 1996).