solution to induction

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1 Solutions

1. use induction to prove that $\forall n \in \mathbf{N}$:
   $1 * 2 + 2 * 2^2 + 3 * 2^3 + ... + n * 2^n = (n-1) * 2^{n+1} + 2$

solution:
let P(n) be statement that:
for $n \in \mathbf{N}$ $\quad 1 * 2 + 2 * 2^2 + 3 * 2^3 + ... + n * 2^n = (n-1) * s^{n+1} + 2$

for base case P(1):

$LHS = 1 * 2 = 2$
$RHS = (1-1) * 2^{1+1} + 2 = 2$
$LHS = RHS$ thus P(1) is true

assume for some $k \in \mathbf{N}$ P(k) is true

for P(k+1)

$LHS = 1 * 2 + 2 * 2^2 + ... + k * 2^k + (k+1) * 2^{k+1}$
$RHS = (k) * 2^{k+2} + 2$

since P(k) is true, subin RHS for P(k)

$1 * 2 + 2 * 2^2 + ... + k * 2^k + (k+1) * 2^{k+1} = (k-1) * 2^{k+1} + 2 + (k+1) * 2^{k+1}$

$(k-1) * 2^{k+1} + 2 + (k+1) * 2^{k+1}$
$= (k+1+k-1) * 2k + 1 + 2$
$= (2k) * s^{k+1} + 2$
$= (k) * 2^{k+2} + 2$

$LHS = RHS$
thus P(k+1) is true given P(k) is true

Hence by principle of mathematic induction P(n) is true for all $n \in \mathbf{N}$

2. Use Principle of Mathematical Induction to show that for all $n \in \mathbf{N}, a_n = 2^{n+2} \cdot 5^{2n+1} + 3^{n+2} \cdot 2^{2n+1}$ is divisible by 19.

solution:

let P(n) be statement that:

for $n \in \mathbf{N}, a_1 = 2^{n+2} \cdot 5^{2n+1} + 3^{n+2} \cdot 2^{2n+1}, a_n = 19x, x \in \mathbf{N}$

for P(1) $a_n = 2^3 * 5^3 + 3^3 * 2^3 = 1216 = 16 * 64$

thus P(1) is true as $64 \in \mathbf{N}$

assume for some $k \in \mathbf{N}$ P(k) is true. i.e $a_k = 19x$, $x \in \mathbf{N}$

for P(k+1):

$a_{k+1} = 2^{k+3} * 5^{2k+3} + 3^{k+3} * 2^{2k+3}$
$= 2 * 5^2 * 2^{k+2} * 5^{2k+1} + 3 * 2^2 * 3^{k+2} * 2^{2k+1}$
$= 50 * 2^{k+2} * 5^{2k+1} + 12 * 3^{k+2} * 2^{2k+1}$
$= 38 * 2^{k+2} * 5^{2k+1} + 12 * (2^{k+2} * 5^{2k+1} + 3^{k+2} * 2^{2k+1})$

given by P(k) we get:

$= 19 * 2 * 2^{k+2} * 5^{2k+1} + 12 * 19x$
$= 19 * (2 * 2^{k+2} * 5^{2k+1} + 12)$

since $2 * 2^{k+2} * 5^{2k+1} + 12 \in \mathbf{N}$. P(k+1) is true given by P(k)

Hence by principle of mathematic induction P(n) is true for all $n \in \mathbf{N}$

1. Let $a_1, a_2, a_3, \ldots$ be the sequence of numbers defined by

$$a_n = \begin{cases} 1 & \text{if } n = 1, \\ \sqrt{2 + a_{n-1}} & \text{if } n \geq 2. \end{cases}$$

Prove by induction that $0 < a_n < 2$ for all $n \geq 1$.

solution:
let P(n) be statement that:
$0 < a_n < 2$ for all $n \geq 1$

for P(1):
$0 < 1 < 2$
Thus P(1) is true

assume for some $k > 1$, P(k) is true
for P(k+1)
since $k \geq 2$, $k + 1 \geq 2$
$a_{k+1} = \sqrt{2 + a_k}$
by P(k) $0 < a_n < 2$
$2 < 2 + a_k < 4$
$\sqrt{2} < \sqrt{2 + a_k} < 2$
$0 < \sqrt{2}$ thus $0 < \sqrt{2 + a_k}$

thus P(k+1) is true given by P(k)

Hence by principle of mathematic induction P(n) is true for all $n \in \mathbf{N}$

4.Given a recursive sequence (x_n) defined by $x_1 = 3$, $x_2 = 5$ and $x_n = 3x_{n-1}$ + $2x_{n-2}$ for n $\geq$ 3. prove that $x_n < 4^n$ for all $n \in \mathbf{N}$

solution:
let P(n) be statement that:
$x_n < 4^n$ for all $x \in \mathbf{N}$

for P(1): $LHS = x_1 = 3$ $RHS = 4^1 = 4$ $3 < 4$ is true, thus P(1) is true

for P(2): $LHS = X_2 = 5$ $RHS = 4^2 = 16$ $5 < 16$ is true, thus P(2) is true

assume for some $k \geq 3, k \in \mathbf{N}$ $P(1) \wedge P(2) \wedge P(3) \wedge ...P(k-1) \wedge P(k)$ are true

for P(k+1)

$x_{k+1} = 3x_k + 2x_{k-1}$

since P(k) and P(k-1) are true
$x_k < 4^k$
$x_{k-1} < 4^{k-1}$
are true

subin for x_{k+1} we get
$3x_k + 2x_{k-1} < 3 * 4^k + 2 * 4^{k-1}$ is true
$3x_k + 2x_{k-1} < 14 * 4^{k-1}$ is true

since $4^{k+1} = 4^2 * 4^{k-1} = 16 * 4^{k-1} > 14 * 4^{k-1}$

$x_{k+1} = 3x_k + 2x_{k-1} < 16 * 4^{k-1} = 4^{k+1}$ is true

hence by principle of strong mathematical induction P(n) is true for all $n \in \mathbf{N}$

the rest are left as an exercise to the reader