

AN EXPONENTIAL DIOPHANTINE EQUATION

MAOHUA LE

Let p be an odd prime with $p > 3$. In this paper we give all positive integer solutions (x, y, m, n) of the equation $x^2 + p^{2m} = y^n$, $\gcd(x, y) = 1$, $n > 2$ satisfying $2 \mid n$ or $2 \nmid n$ and $p \not\equiv (-1)^{(p-1)/2} \pmod{4n}$.

1. INTRODUCTION

Let $\mathbb{Z}, \mathbb{N}, \mathbb{Q}$ be the sets of all integers, positive integers and rational numbers respectively. Let p be a prime. There have been many papers concerned with solutions (x, y, m, n) of the equation

$$(1) \quad x^2 + p^m = y^n, \quad x, y, m, n \in \mathbb{N}, \quad \gcd(x, y) = 1, \quad n > 2.$$

All solutions of (1) for $p \in \{2, 3\}$ have been determined. The known results include the following:

1. (Nagell [12].) If $p = 2$, then the only solution of (1) with $m = 2$ is $(x, y, m, n) = (11, 5, 2, 3)$.
2. (Cohn [3].) If $p = 2$, then the only solution of (1) with $2 \nmid m$ are $(x, y, m, n) = (5, 3, 1, 3)$ and $(7, 3, 5, 4)$.
3. (Le [5, 6].) If $p = 2$, then (1) has no solutions (x, y, m, n) satisfying $2 \mid m$ and $m > 2$.
4. (Arif and Muriefah [1].) If $p = 3$, then the only solution of (1) with $2 \nmid m$ is $(x, y, m, n) = (10, 7, 5, 3)$.
5. (Luca [9].) If $p = 3$, then the only solution of (1) with $2 \mid m$ is $(x, y, m, n) = (46, 13, 4, 3)$.

In this paper we investigate the solutions (x, y, m, n) of (1) for m even. Then (1) may be written as

$$(2) \quad x^2 + p^{2m} = y^n, \quad x, y, m, n \in \mathbb{N}, \quad \gcd(x, y) = 1, \quad n > 2.$$

We prove the following two results.

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THEOREM 1. *If $p > 3$, then all the solutions (x, y, m, n) of (2) with $2 \mid m$ are given as follows:*

$$(i) \quad p = 239, (x, y, m, n) = (28560, 13, 1, 8).$$

$$(ii) \quad p = E(q), (x, y, m, n) = \left(((E(q))^2 - 1)/2, F(q), 1, 4 \right), \text{ where } q \text{ is an odd prime, and}$$

$$(3) \quad E(q) = \frac{1}{2} \left((1 + \sqrt{2})^q + (1 - \sqrt{2})^q \right), \quad F(q) = \frac{1}{2\sqrt{2}} \left((1 + \sqrt{2})^q - (1 - \sqrt{2})^q \right).$$

THEOREM 2. *If $p > 3$ and $p \not\equiv (-1)^{(p-1)/2} \pmod{4n}$, then (2) has no solutions (x, y, m, n) with $2 \nmid n$.*

By the above theorems, we can completely determine all solutions of (2) for the case that p is either a Fermat prime or a Mersenne prime.

COROLLARY 1. *If p is a Fermat prime with $p > 3$, then (2) has no solutions (x, y, m, n) .*

COROLLARY 2. *If $p = 7$, then the only solution of (2) is $(x, y, m, n) = (24, 5, 1, 4)$. If p is a Mersenne prime with $p > 7$, then (2) has no solutions (x, y, m, n) .*

2. PRELIMINARIES

LEMMA 1. [11, pp.12–13] *Every solution (X, Y, Z) of the equation*

$$(4) \quad X^2 + Y^2 = Z^2, \quad X, Y, Z \in \mathbb{N}, \quad \gcd(X, Y) = 1, \quad 2 \mid X$$

can be expressed as

$$(5) \quad X = 2AB, \quad Y = A^2 - B^2, \quad Z = A^2 + B^2,$$

where A, B are positive integers satisfying

$$(6) \quad A > B, \quad \gcd(A, B) = 1, \quad 2 \mid AB.$$

LEMMA 2. [11, pp.122–123] *Let n be an odd integer with $n > 1$. Then every solution (X, Y, Z) of the equation*

$$(7) \quad X^2 + Y^2 = Z^n, \quad X, Y, Z \in \mathbb{N}, \quad \gcd(X, Y) = 1$$

can be expressed as

$$(8) \quad Z = A^2 + B^2, \quad X + Y\sqrt{-1} = \lambda_1(A + \lambda_2 B\sqrt{-1})^n, \quad \lambda_1, \lambda_2 \in \{-1, 1\},$$

where A, B are coprime positive integers.

LEMMA 3. [7] *The only solutions of the operation*

$$(9) \quad X^2 + 1 = 2Y^4, \quad X, Y \in \mathbb{N}$$

are $(X, Y) = (1, 1)$ and $(239, 13)$.

LEMMA 4. [8] *Let D be a positive integer which is not a square. Then the equation*

$$(10) \quad X^4 - DY^2 = -1, \quad X, Y \in \mathbb{N}$$

has at most one solution (X, Y) . Moreover, if (X, Y) is a solution of (10), then the fundamental solution $U_1 + V_1\sqrt{D}$ of the Pell equation

$$(11) \quad U^2 - DV^2 = -1, \quad U, V \in \mathbb{N}$$

satisfies

$$(12) \quad U_1 = dt^2, \quad X^2 + Y\sqrt{D} = \left(U_1 + V_1\sqrt{D} \right)^d, \quad d, t \in \mathbb{N}, \quad 2 \nmid d, \quad d \text{ is square free.}$$

LEMMA 5. [13] *The equation*

$$(13) \quad X^2 + 1 = 2Y^r, \quad X, Y, r \in \mathbb{N}, \quad X > Y > 1, \quad r > 1, \quad 2 \nmid r$$

has no solutions (X, Y, r) .

LEMMA 6. [4, Lemma 15] *The equation*

$$(14) \quad X^{2r} + 1 = 2Y^2, \quad X, Y, r \in \mathbb{N}, \quad X > 1, \quad Y > 1, \quad r > 1, \quad 2 \nmid r$$

has no solutions (X, Y, r) .

Let α, β be algebraic integers. If $\alpha + \beta$ and $\alpha\beta$ are nonzero coprime integers and α/β is not a root of unity, then (α, β) is called a Lucas pair. Further, let $a = \alpha + \beta$ and $c = \alpha\beta$. Then we have

$$(15) \quad \alpha = \frac{1}{2}(a + \lambda\sqrt{b}), \quad \beta = \frac{1}{2}(a - \lambda\sqrt{b}), \quad \lambda \in \{-1, 1\},$$

where $b = a^2 - 4c$. We call (a, b) the parameters of the Lucas pair (α, β) . Two Lucas pairs (α_1, β_1) and (α_2, β_2) are equivalent if $\alpha_1/\alpha_2 = \beta_1/\beta_2 = \pm 1$. Given a Lucas pair (α, β) , one defines the corresponding sequence of Lucas numbers by $u_t = u_t(\alpha, \beta) = (\alpha^t - \beta^t)/(\alpha - \beta)$ for $t = 0, 1, 2, \dots$. For equivalent Lucas pairs (α_1, β_1) and (α_2, β_2) , we have $u_t(\alpha_1, \beta_1) = \pm u_t(\alpha_2, \beta_2)$ for any $t \geq 0$. A prime p is a primitive divisor of $u_t(\alpha, \beta)$ if $p \mid u_t$ and $p \nmid bu_1 \cdots u_{t-1}$.

LEMMA 7. [10] Let (α, β) be a Lucas pair with parameters (a, b) . If p is a primitive divisor of $u_t(\alpha, \beta)$ ($t > 2$), then $p - \left(\frac{b}{p}\right) \equiv 0 \pmod{t}$ where $\left(\frac{b}{p}\right)$ is the Legendre symbol.

A Lucas pair (α, β) such that $u_t(\alpha, \beta)$ has no primitive divisors will be called a t -defective Lucas pair.

LEMMA 8. [14] Let t satisfy $4 < t < 30$ and $t \neq 6$. Then, up to equivalence, all parameters of t -defective Lucas pairs are given as follows:

- (i) $t = 5$, $(a, b) = (1, 5), (1, -7), (2, -40), (1, -11), (1, -15), (12, -76), (12, -1364)$;
- (ii) $t = 7$, $(a, b) = (1, -7), (1, -19)$;
- (iii) $t = 8$, $(a, b) = (2, -24), (1, -7)$;
- (iv) $t = 10$, $(a, b) = (2, -8), (5, -3), (5, -47)$;
- (v) $t = 12$, $(a, b) = (1, 5), (1, -7), (1, -11), (2, -56), (1, -15), (1, -19)$;
- (vi) $t \in \{13, 18, 30\}$, $(a, b) = (1, -7)$.

A positive integer t is called totally non-defective if no Lucas pair is t -defective.

LEMMA 9. [2] If $t > 30$, then t is totally non-defective.

3. PROOFS

PROOF OF THEOREM 1: Let (x, y, m, n) be a solution of (2). Since $p > 3$ and $n > 2$, we have $2 \mid x$ and $2 \nmid y$. If $2 \mid n$, since $\gcd(y^{n/2} + x, y^{n/2} - x) = 1$, then from (2) we get $y^{n/2} + x = p^{2m}$ and $y^{n/2} - x = 1$. This implies that

$$(16) \quad p^{2m} + 1 = 2y^{n/2},$$

$$(17) \quad p^{2m} - 1 = 2x.$$

Since $n/2 > 1$, by Lemma 5, we see from (16) that $n/2$ has no odd prime divisors. So we have $n = 2^{s+1}$, where s is a positive integer.

When $s = 1$, (16) can be written as

$$(18) \quad p^{2m} + 1 = 2y^2.$$

Then $(u, v) = (p^m, y)$ is a solution of the Pell equation

$$(19) \quad u^2 - 2v^2 = -1, \quad u, v \in \mathbb{N}.$$

Since $1 + \sqrt{2}$ is the fundamental solution of (19), we get

$$(20) \quad \begin{aligned} p^m &= \frac{1}{2} \left((1 + \sqrt{2})^l + (1 - \sqrt{2})^l \right), \\ y &= \frac{1}{2\sqrt{2}} \left((1 + \sqrt{2})^l - (1 - \sqrt{2})^l \right), \quad l \in \mathbb{N}, \quad 2 \nmid l. \end{aligned}$$

On the other hand, if m has an odd prime divisor r , then $(X, Y) = (p^{m/r}, y)$ is a solution of (14). However, by Lemma 6, this is impossible. Therefore, if $m > 1$, then m is a power of 2 and $(X, Y) = (p^{m/2}, y)$ is a solution of (10) for $D = 2$. But, by Lemma 4, this is impossible too. So we have $m = 1$. Then the positive integer l in (20) must be an odd prime. Thus, by (17) and (20), we obtain the solution (ii).

When $s > 1$, we see from (16) that $(X, Y) = (p^m, y^{n/8})$ is a solution of (9). Therefore, by Lemma 3, we get the solution (i). Thus, the theorem is proved. \square

PROOF OF THEOREM 2: Let (x, y, m, n) be a solution of (2) with $2 \nmid n$. Then $(X, Y, Z) = (x, p^m, y)$ is a solution of (7). By Lemma 2, we get

$$(21) \quad x + p^m \sqrt{-1} = \lambda_1 (A + \lambda_2 B \sqrt{-1})^n, \quad \lambda_1, \lambda_2 \in \{-1, 1\},$$

where A, B are positive integers satisfying

$$(22) \quad A^2 + B^2 = y, \quad \gcd(A, B) = 1.$$

From (21), we get

$$(23) \quad p^m = \lambda_1 \lambda_2 B \sum_{i=0}^{(n-1)/2} \binom{n}{2i+1} A^{n-2i-1} (-B^2)^i.$$

Let

$$(24) \quad \alpha = A + B\sqrt{-1}, \quad \beta = A - B\sqrt{-1}.$$

We see from (22) and (24) that (α, β) is a Lucas pair with parameters $(2A, -4B^2)$. Further, let $u_t(\alpha, \beta)$ ($t = 0, 1, 2, \dots$) denote the corresponding Lucas numbers. By (23), we get

$$(25) \quad p^m = \pm B u_n(\alpha, \beta).$$

Notice that $\left(\frac{-4B^2}{p}\right) = \left(\frac{-1}{p}\right) = (-1)^{(p-1)/2}$, where $\left(\frac{*}{p}\right)$ is the Legendre symbol.

By Lemma 7, if p is a primitive divisor of $u_n(\alpha, \beta)$, then $p - (-1)^{(p-1)/2} \equiv 0 \pmod{n}$. Since $2 \nmid n$ and $p - (-1)^{(p-1)/2} \equiv 0 \pmod{4}$, we get $p \equiv (-1)^{(p-1)/2} \pmod{4n}$. Therefore, by (25), if the solution (x, y, m, n) satisfies $p \not\equiv (-1)^{(p-1)/2} \pmod{4n}$, then $u_n(\alpha, \beta)$ has no primitive divisors. By Lemmas 8 and 9, we deduce that $n = 3$ and $p \mid B$. Then, by (23), we get

$$(26) \quad B = p^s, \quad 3A^2 - B^2 = \pm p^{m-s}, \quad s \in \mathbb{N}, \quad s \leq m.$$

Since $\gcd(A, B) = 1$, we see from (26) that $p = 3$. thus, if $p > 3$, then (2) has no solutions (x, y, m, n) satisfying $2 \nmid n$ and $p - (-1)^{(p-1)/2} \not\equiv 0 \pmod{4n}$. The theorem is proved. \square

PROOF OF COROLLARY 1: Let p be a Fermat prime. Then we have

$$(27) \quad p = 2^{2^s} + 1, \quad s \in \mathbb{N}.$$

Since $p - (-1)^{(p-1)/2} = 2^{2^s}$, by Theorem 2, then (2) has no solutions (x, y, m, n) with $2 \nmid n$.

On the other hand, since $p \neq 239$, by the proof of Theorem 1, if (x, y, m, n) is a solution of (2) with $2 \mid n$, then we have $m = 1$, $n = 4$ and

$$(28) \quad p^2 + 1 = 2y^2.$$

Substitute (27) into (28), and we get

$$(29) \quad 2^{2^{s+1}-2} + (2^{2^s-1} + 1)^2 = y^2.$$

Therefore, by Lemma 1, we obtain from (29) that

$$(30) \quad 2^{2^s-1} = 2AB, \quad 2^{2^s-1} + 1 = A^2 - B^2, \quad y = A^2 + B^2,$$

where A, B are positive integers satisfying (6). From (30), since $\gcd(A, B) = 1$, we get from the first equation $s > 1$, $A = 2^{2^{s-2}}$ and $B = 1$. However, by the second equation in (30), we get

$$(31) \quad 1 \equiv 2^{2^s-1} + 1 = 2^{2^{s+1}-4} - 1 \equiv 3 \pmod{4},$$

which is a contradiction. Thus, the corollary is proved. \square

PROOF OF COROLLARY 2: Let p be a Mersenne prime. Then we have

$$(32) \quad p = 2^r - 1, \quad r \text{ is an odd prime,}$$

if $p \geq 7$. Since $p - (-1)^{(p-1)/2} = 2^r$, by Theorem 2, then (2) has no solutions (x, y, m, n) with $2 \nmid n$.

By Theorem 1, if $r = 3$, then $p = 7$ and the only solution of (2) with $2 \mid n$ is $(x, y, m, n) = (24, 5, 1, 4)$. Since $p \neq 239$, by the proof of Theorem 1, if $r > 3$ and (x, y, m, n) is a solution of (2) with $2 \mid n$, then $m = 1$, $n = 4$ and (28) holds. Substitute (32) into (28), and we get

$$(33) \quad 2^{2r-2} + (2^{r-1} - 1)^2 = y^2.$$

By Lemma 1, we obtain from (33) that

$$(34) \quad 2^{r-1} = 2AB, \quad 2^{r-1} - 1 = A^2 - B^2, \quad y = A^2 + B^2,$$

whence we obtain $A = 2^{r-2}$ and $B = 1$, since $\gcd(A, B) = 1$, but these do not satisfy the second equation in (34), when $r > 3$. Thus, if $p > 7$, then (2) has no solutions (x, y, m, n) . The corollary is proved. \square

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Department of Mathematics
 Zhanjiang Normal College
 Postal Code 524048
 Zhanjiang, Guangdong
 People's Republic of China