Exercise 2

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Problem 1.

Solution. (b) It is not reflexive. It is symmetric. It is not antisymmetric. It is transitive.

(f) It is reflexive. It is symmetric. It is not antisymmetric. It is transitive. ☐

Problem 2.

Solution. (a) R_1; (b) R_1; (c) $\mathbb{R} \times \mathbb{R}$; (e) R_1; (f) $\mathbb{R} \times \mathbb{R}$; (g) $\mathbb{R} \times \mathbb{R}$; (h) R_3. ☐

Problem 3.

Proof.

(a) For every $(a, b) \in (R \circ S) \circ T$, there exists $c \in A$ such that $(a, c) \in T$ and $(c, b) \in R \circ S$. Then there exists $d \in A$ such that $(c, d) \in S$ and $(d, b) \in R$. $(a, c) \in T$ and $(c, d) \in S$ imply that $(a, d) \in S \circ T$. Then $(a, b) \in R \circ (S \circ T)$ follows from $(d, b) \in R$.

For every $(a, b) \in R \circ (S \circ T)$, there exists $c \in A$ such that $(a, c) \in S \circ T$ and $(c, b) \in R$. Then there exists $d \in A$ such that $(a, d) \in T$ and $(d, c) \in S$. $(d, c) \in S$ and $(c, b) \in R$ imply that $(d, b) \in R \circ S$. Then $(a, b) \in (R \circ S) \circ T$ follows from $(a, d) \in T$.

That completes the proof for $(R \circ S) \circ T = R \circ (S \circ T)$.

(b) For every $(a, b) \in R \circ (S \cup T)$, there exists $c \in A$ such that $(a, c) \in S \cup T$ and $(c, b) \in R$. Then $(a, c) \in S$ or $(a, c) \in T$. If $(a, c) \in S$ then $(a, b) \in R \circ S$ follows from $(c, b) \in R$; if $(a, c) \in T$ then $(a, b) \in R \circ T$ follows from $(c, b) \in R$. That implies that $(a, b) \in (R \circ S) \cup (R \circ T)$.

For every $(a, b) \in (R \circ S) \cup (R \circ T)$, $(a, b) \in R \circ S$ or $(a, b) \in R \circ T$. If $(a, b) \in R \circ S$, then there exists $c \in A$ such that $(a, c) \in S$ and $(c, b) \in R$, which implies that $(a, c) \in S \cup T$. That, together with $(c, b) \in R$, implies that $(a, b) \in R \circ (S \cup T)$. If $(a, b) \in R \circ T$, similarly $(a, b) \in R \circ (S \cup T)$. Either way, $(a, b) \in R \circ (S \cup T)$.

That completes the proof for $R \circ (S \cup T) = (R \circ S) \cup (R \circ T)$. ☐

Problem 4.

Solution. We disprove the proposition by constructing a counterexample. Let R_1 and R_2 be two binary relations on $\{1, 2, 3\}$ defined by

$$R_1 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\},$$
$$R_2 = \{(1, 1), (2, 2), (3, 3), (2, 3), (3, 2)\}.$$

Because R_1 and R_2 are both reflexive, symmetric and transitive, they are two equivalence relations. However, $(1, 2), (2, 3) \in R_1 \cup R_2$ and $(1, 3) \notin R_1 \cup R_2$, which implies that $R_1 \cup R_2$ is not transitive, and thus not an equivalence relation. That disproves the proposition. ☐

Problem 5.

Solution. We prove the proposition. Suppose R_1 and R_2 are two equivalence relations on A. For every $a \in A$, because $(a,a) \in R_1$ and $(a,a) \in R_2$ by reflexivity, $(a,a) \in R_1 \cap R_2$, which implies that $R_1 \cap R_2$ is reflexive. For every $(a,b) \in R_1 \cap R_2$, we have $(a,b) \in R_1$ and $(a,b) \in R_2$, which, respectively, imply that $(b,a) \in R_1$ and $(b,a) \in R_2$ by symmetry. That implies that $(b,a) \in R_1 \cap R_2$, which implies that $R_1 \cap R_2$ is symmetric. For every $(a,b),(b,c) \in R_1 \cap R_2$, we have $(a,b),(b,c) \in R_1$ and $(a,b),(b,c) \in R_2$, which, respectively, imply that $(a,c) \in R_1$ and $(a,c) \in R_2$ by transitivity. That implies that $(a,c) \in R_1 \cap R_2$, which implies that $R_1 \cap R_2$ is transitive. Because $R_1 \cap R_2$ is reflexive, symmetric and transitive, it is an equivalence relation, as desired. $\square$

Problem 6.

Solution. (a) $[1]_R = \{x \in \mathbb{R} : x - 1 \text{ is an integer}\} = \{x \in \mathbb{R} : x \text{ is an integer}\} = \mathbb{Z}$.

(b) $[1/2]_R = \left\{x \in \mathbb{R} : x - \frac{1}{2} \text{ is an integer}\right\} = \left\{x + \frac{1}{2} : x \in \mathbb{Z}\right\}$. $\square$

Problem 7.

Proof. Let $P = \{[a]_R : a \in \mathbb{R}\}$ be the set of all equivalence classes of R.

(a) For every $[a]_R \in P$, $([a]_R, [a]_R) \in S_2$. Thus S_2 is reflexive.

For every $([a]_R, [b]_R) \in S_2$, $a = b$ or $a - b = 1/2$.[1] If $a = b$ then $([b]_R, [a]_R) = ([a]_R, [a]_R) \in S_2$. If $a - b = 1/2$ then $(b+1) - a = 1/2$, which implies that $([b+1]_R, [a]_R) \in S_2$. Because $(b+1)Rb$, we have $[b+1]_R = [b]_R$. Thus $([b]_R, [a]_R) \in S_2$. That proves that S_2 is symmetric.

For every $([a]_R, [b]_R), ([b']_R, [c]_R) \in S_2$ where $[b]_R = [b']_R$, we have $([b]_R, [c]_R) \in S_2$. Then $(a = b \text{ or } a - b = 1/2)$, and $(b = c \text{ or } b - c = 1/2)$.[2] By enumerating all possible cases, we have $a = c$ or $a - c = 1$ (which implies that $[a]_R = [c]_R$ because aRc) or $a - c = 1/2$. In all cases, $([a]_R, [c]_R) \in S_2$ by definition. That proves that S_2 is transitive.

Because S_2 is reflexive, symmetric and transitive, it is an equivalence relation on P, as desired.

(b) For every $[a]_R \in P$, $([a]_R, [a]_R) \in S_3$. Thus S_3 is reflexive.

For every $([a]_R, [b]_R) \in S_3$, we have $a = b$ or $|a - b| = 1/3$. If $a = b$ then $([b]_R, [a]_R) = ([a]_R, [a]_R) \in S_3$. If $|a - b| = 1/3$ then $|b - a| = 1/3$, which implies that $([b]_R, [a]_R) \in S_3$. That proves that S_3 is symmetric.

For every $([a]_R, [b]_R), ([b']_R, [c]_R) \in S_3$ where $[b]_R = [b']_R$, we have $([b]_R, [c]_R) \in S_3$. Then $(a = b \text{ or } |a - b| = 1/3)$, and $(b = c \text{ or } |b - c| = 1/3)$. By enumerating all possible cases, we have $a = c$ or $|a - c| = 1/3$ or $|a - c| = 2/3$. In the first two cases, $([a]_R, [c]_R) \in S_3$ by definition. In the last case, if $a - c = 2/3$ then $|(a-1) - c| = 1/3$, which implies that $([a-1]_R, [c]_R) \in S_3$. Because $(a-1)Ra$, we have $[a-1]_R = [a]_R$. Thus $([a]_R, [c]_R) \in S_3$. If $c - a = 2/3$, similarly we have $([a]_R, [c]_R) \in S_3$. That proves that S_3 is transitive.

Because S_3 is reflexive, symmetric and transitive, it is an equivalence relation on P, as desired.

(c) $([0]_R, [1/4]_R), ([1/4]_R, [1/2]_R) \in S_4$ follows from $|0 - 1/4| = |1/4 - 1/2| = 1/4$. Now we prove by contradiction that $([0]_R, [1/2]_R) \notin S_4$. Assume $([0]_R, [1/2]_R) \in S_4$. Then there exists $a, b \in \mathbb{R}$ such that $aR0$ and $bR(1/2)$ and $(a = b \text{ or } |a - b| = 1/4)$. By

[1] Erratum: we cannot conclude $a = b \vee a - b = 1/2$ from that; instead we should write "For every $(\alpha, \beta) \in S_2, \exists a, b, a = b \vee a - b = 1/2$". Similar mistakes happen below.

[2] The parentheses here are used to avoid ambiguity of precedence of logical connectives.

definition of R suppose $a = m + 0, b = n + 1/2$, where $m, n \in \mathbb{Z}$. Then $m = n + 1/2$ or $|m - n - 1/2| = 1/4$, i.e., $2(m - n) = 1$ or $|2(m - n) - 1| = 1/2$. However, the former case is impossible because $2(m - n)$ is even but 1 is odd, and the latter case is impossible because $|2(m - n) - 1|$ is an integer but $1/2$ is not. The contradiction proves that S_4 is not transitive, and thus not an equivalence relation, as desired. $\square$