Exercise 11

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In this text, $[\![\to]\!]$ denotes the truth table of the logical connective $\to$.

Problem 1.

Solution. (a) If Φ does not talk about x, and $\Phi, \psi \vdash \phi$, then $\Phi \vdash \forall x(\psi \to \phi)$.

(b) No logic. This is a mathematical definition.

(c) If Φ and ϕ do not talk about x, and $\Phi, \psi \vdash \phi$, then $\Phi, \exists x\psi \vdash \phi$.

$\square$

Problem 2.

Proof. Let $\mathfrak{I}$ be an interpretation with domain A such that $[\![\phi[x \to t]]\!]_{\mathfrak{I}} = \mathbf{T}$ and $[\![\forall x(\phi \to \psi)]\!]_{\mathfrak{I}} = \mathbf{T}$. Thus for any $a \in A$, $[\![\phi \to \psi]\!]_{\mathfrak{I}[x\to a]} = \mathbf{T}$. Let $a = [\![t]\!]_{\mathfrak{I}}$ and we have $[\![\phi \to \psi]\!]_{\mathfrak{I}[x\to[\![t]\!]_{\mathfrak{I}}]} = \mathbf{T}$, which implies that $[\![\to]\!]([\![\phi]\!]_{\mathfrak{I}[x\to[\![t]\!]_{\mathfrak{I}}]}, [\![\psi]\!]_{\mathfrak{I}[x\to[\![t]\!]_{\mathfrak{I}}]}) = \mathbf{T}$, which implies that $[\![\to]\!]([\![\phi[x \to t]]\!]_{\mathfrak{I}}, [\![\psi[x \to t]]\!]_{\mathfrak{I}} = \mathbf{T}$. Now that $[\![\phi[x \to t]]\!]_{\mathfrak{I}} = \mathbf{T}$, by definition of $[\![\to]\!]$ we have $[\![\psi[x \to t]]\!]_{\mathfrak{I}} = \mathbf{T}$, completing the proof. $\square$

Problem 3.

Proof. (a) Let $\mathfrak{I}$ be an interpretation with domain A such that $[\![\forall x\phi]\!]_{\mathfrak{I}} = \mathbf{T}$. For every $a \in A$, we have $[\![\phi]\!]_{\mathfrak{I}[x\to a]} = \mathbf{T}$. Then $[\![\psi]\!]_{\mathfrak{I}[x\to a]} = \mathbf{T}$ follows from that $\phi \models \psi$. Now that we have proved that for every $a \in A$, $[\![\psi]\!]_{\mathfrak{I}[x\to a]} = \mathbf{T}$, we conclude that $[\![\forall x\psi]\!]_{\mathfrak{I}} = \mathbf{T}$, completing the proof.

(b) Let $\mathfrak{I}$ be an interpretation with domain A such that $[\![\forall x\phi]\!]_{\mathfrak{I}} = \mathbf{T}$ and $[\![\chi]\!]_{\mathfrak{I}} = \mathbf{T}$ for each $\chi \in \Phi$. For every $a \in A$, we have $[\![\phi]\!]_{\mathfrak{I}[x\to a]} = \mathbf{T}$. For each $\chi \in \Phi$, because x does not freely occur in Φ, x does not freely occur in χ, which implies that $[\![\chi]\!]_{\mathfrak{I}[x\to a]} = [\![\chi]\!]_{\mathfrak{I}} = \mathbf{T}$. Then $[\![\psi]\!]_{\mathfrak{I}[x\to a]} = \mathbf{T}$ follows from that $\Phi, \phi \models \psi$. Now we conclude that $[\![\forall x\psi]\!]_{\mathfrak{I}} = \mathbf{T}$, completing the proof.

(c) Let $S = \{P, Q, R\}$ where P, Q, R are unary predicate symbols. Let $\mathfrak{A} = (\mathbb{Z}^+, \mathfrak{a})$ be an S-structure such that $\mathfrak{a}(P)(x) = \mathbf{T}$ if and only if $x = 2$, $\mathfrak{a}(Q)(x) = \mathbf{T}$ if and only if x is even, and $\mathfrak{a}(R)(x) = \mathbf{T}$ if and only if x is composite. Let $\Phi = \{P(x)\}$, $\phi = ((\neg P(x) \land Q(x)) \to R(x))$, and $\psi = (\neg P(x) \to R(x))$. x obviously occurs freely in Φ.

First we show that $\Phi, \phi \models \psi$. Let $\mathfrak{I}$ be an S-interpretation such that $[\![P(x)]\!]_{\mathfrak{I}} = \mathbf{T}$ and $[\![\phi]\!]_{\mathfrak{I}} = \mathbf{T}$. Then $[\![\neg P(x)]\!]_{\mathfrak{I}} = \mathbf{F}$, which implies that $[\![\psi]\!]_{\mathfrak{I}} = T$ by the truth table of the logical connective $\to$. Hence $\Phi, \phi \models \psi$.

Now we show that $\Phi, \forall x\phi \not\models \forall x\psi$. Let $\mathfrak{I} = (\mathfrak{A}, \beta)$ be an S-interpretation such that $\beta(x) = 2$. By definition $[\![P(x)]\!]_{\mathfrak{I}} = \mathbf{T}$. Notice the mathematical fact that "if $n \neq 2$ and n is even, then n is composite". Let $n \in \mathbb{Z}^+$. If $[\![(\neg P(x) \land Q(x))]\!]_{\mathfrak{I}[x\to n]} = \mathbf{T}$, then $n \neq 2$ and n is even. Thus n is composite, which implies that $[\![R(x)]\!]_{\mathfrak{I}[x\to n]} = \mathbf{T}$. Thus $[\![\phi]\!]_{\mathfrak{I}[x\to n]} = \mathbf{T}$. Because n is arbitrary, $[\![\forall x\phi]\!]_{\mathfrak{I}} = \mathbf{T}$. Consider $[\![\psi]\!]_{\mathfrak{I}[x\to 3]}$. By definition $[\![\neg P(x)]\!]_{\mathfrak{I}[x\to 3]} = \mathbf{T}$ and $[\![R(x)]\!]_{\mathfrak{I}[x\to 3]} = \mathbf{F}$. Thus $[\![\psi]\!]_{\mathfrak{I}[x\to 3]} = \mathbf{F}$, which implies that $[\![\forall x\psi]\!]_{\mathfrak{I}} = \mathbf{F}$. Hence $\Phi, \forall x\phi \not\models \forall x\psi$.

$\square$