

## Exercise 6

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In this text, a class means the collection of all sets that satisfy a given property (first-order formula);  $V$  is the universe  $\{x : x = x\}$ ;  $n^+$  means  $n \cup \{n\}$ .

### Problem 1.

*Proof.* (a)  $\forall y(y \notin \emptyset)$ . Thus  $\forall x \forall y((x \in y \wedge y \in \emptyset) \rightarrow x \in \emptyset)$  holds.

- (b) Suppose  $n$  is  $\in$ -transitive. Let  $x, y$  be such that  $x \in y$  and  $y \in n \cup \{n\}$ . If  $y \in n$ , then  $x \in n \subseteq n \cup \{n\}$  by the  $\in$ -transitivity of  $n$ . If  $y \in \{n\}$ , then  $x \in y$  implies  $x = n$ . Thus  $x \in n \cup \{n\}$  trivially. By proof by cases,  $n \cup \{n\}$  is  $\in$ -transitive.  $\square$

### Problem 2.

*Proof.* (a)  $0 = \emptyset$  has no nonempty subsets. Thus  $0$  is  $\in$ -well-ordered by definition.

- (b) Suppose  $n$  is  $\in$ -well-ordered. Let  $x \subseteq n \cup \{n\}$ . Let  $z = x \setminus \{n\} \subseteq n$ . Let  $s$  be an  $\in$ -least element of  $z$ . For each  $a \in x$ , if  $a \in z$  then  $s = a$  or  $s \in a$  by definition; if<sup>1</sup>  $a = n$  then  $s \in z \subseteq n = a$ . Hence  $s$  is an  $\in$ -least element of  $x$ , completing the proof.  $\square$

### Problem 3.

*Proof.* Suppose  $x, y$  are such that  $x, y$  are inductive;  $x \subseteq z$  for every inductive set  $z$ ;  $y \subseteq z$  for every inductive set  $z$ . Then taking  $z = y$  leads to  $x \subseteq y$ , and taking  $z = x$  leads to  $y \subseteq x$ . Thus  $x = y$ , as desired.  $\square$

### Problem 4.

*Proof.*  $\emptyset \in u$  and  $\emptyset \in v$  leads to  $\emptyset \in u \cap v$ . For every  $x \in u \cap v$ , we have  $x \in u$  and  $x \in v$ , which implies that  $x \cup \{x\} \in u$  and  $x \cup \{x\} \in v$ , which imply that  $x \cup \{x\} \in u \cap v$ . Thus  $u \cap v$  is inductive.  $\square$

### Problem 5.

*Proof.* (a) For each inductive set  $w \subseteq u$ ,  $\emptyset \in w$ . Thus  $\emptyset \in v$ . For each  $x \in v$ ,  $x \in w$  for all inductive set  $w \subseteq u$ , which implies that  $x \cup \{x\} \in w$  for all inductive set  $w \subseteq u$ , which implies that  $x \cup \{x\} \in v$ . Hence  $v$  is inductive.

- (b) For any inductive set  $w \subseteq u$ ,  $v \subseteq w$  by the definition of  $v$ . The desired result follows from that  $v$  is inductive.  $\square$

### Problem 6.

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<sup>1</sup>Note that we do not need to distinguish whether  $n \in x$  or not.

*Proof.* Suppose there exists an inductive set  $u$ . Let<sup>2</sup>

$$N = \bigcap_{X \text{ is inductive}} X.$$

$N$  is indeed a set because  $N \subseteq u$ . We prove that  $N$  is the smallest inductive set.

For each inductive set  $X$ ,  $\emptyset \in X$ . Thus  $\emptyset \in N$ . For each  $x \in N$ ,  $x \in X$  for all inductive set  $X$ , which implies that  $x \cup \{x\} \in X$  for all inductive set  $X$ , which implies that  $x \cup \{x\} \in N$ . Hence  $N$  is inductive. For any inductive set  $X$ ,  $X \subseteq N$  by definition. Hence  $N$  is *the* smallest inductive set.  $\square$

**Problem 7.**

*Proof.* (a)  $\mathbb{N}$  and  $X$  are both inductive by definition. Then by Problem 4,  $\mathbb{N} \cap X$  is inductive.

(b) By definition of  $\mathbb{N}$  (the smallest inductive set),  $\mathbb{N} \subseteq \mathbb{N} \cap X \subseteq \mathbb{N}$ , which implies that  $\mathbb{N} \subseteq X$ , completing the proof.  $\square$

**Problem 8.**<sup>3</sup>

**Lemma 1.** We define that a set  $T$  is transitive if  $\forall x(x \in T \rightarrow x \subseteq T)$ . Every  $n \in \mathbb{N}$  is transitive.

*Proof.* Let  $E = \{n \in \mathbb{N} : n \text{ is transitive}\}$ . We prove by induction that  $E = \mathbb{N}$ .  $\forall x(x \in 0 \rightarrow x \subseteq 0)$  follows from  $0 = \emptyset$ . Assume that  $n \in \mathbb{N}$  is transitive. For each  $x \in n^+ = n \cup \{n\}$ , either  $x \in n$  or  $x = n$ . If  $x \in n$  then  $x \subseteq n \subseteq n^+$  by induction hypothesis; if  $x = n$  then  $x \subseteq n^+$  trivially. That closes the induction and completes the proof.  $\square$

**Lemma 2.** Suppose  $n, m \in \mathbb{N}$ . If  $n^+ = m^+$  then  $n = m$ .

*Proof.* Suppose  $n \cup \{n\} = m \cup \{m\}$ . Then  $n \in m \cup \{m\}$ . Thus  $n = m$  or  $n \in m$ . Similarly  $m = n$  or  $m \in n$ . If  $n \neq m$  then  $n \in m$  and  $m \in n$ . By Lemma 1,  $n \subseteq m$  and  $m \subseteq n$ , which implies that  $n = m$ .  $\square$

*Proof of the Problem.* Fix  $x$ . Let  $R$  be the property defined by

$$R(A) = ((0, x) \in A \wedge \forall n \forall y ((n, y) \in A \rightarrow (n^+, y \times x) \in A)).$$

Let class  $F = \{(u, v) : P(u, v)\}$ , which is equivalent to

$$F = (\mathbb{N} \times V) \cap \bigcap_{R(A)} A.$$

First we prove that  $R(F)$ . For every  $A$  such that  $R(A)$ ,  $(0, x) \in A$ . Thus  $(0, x) \in F$ . For any  $(n, y) \in F$ ,  $(n, y) \in A$  for each  $A$  with property  $R$ , which implies that  $(n^+, y \times x) \in A$  for each  $A$  with property  $R$ , which implies that  $(n^+, y \times x) \in F$ . Hence  $R(F)$ .

Let

$$X = \{(n, y) \in F : ((n, y) = (0, x) \vee \exists n' \exists y' ((n', y') \in F \wedge n = n'^+ \wedge y = y' \times x))\}.$$

<sup>2</sup>Note that we have not claimed that  $N = \mathbb{N}$ .

<sup>3</sup>**Erratum:** the seemingly fancy proof below is actually a false one (I applied a property ( $P$  in this case) to classes ( $F$  in this case)). None have found a proper solution to this problem yet. I have proved the problem with the Recursion Theorem, which is not meaningful.

Now we prove that  $R(X)$ .  $(0, x) \in X$  trivially. For every  $(n, y) \in X$ ,  $(n, y) \in F$ . Thus  $(n^+, y \times x)$  satisfies the second condition in the definition of  $X$ , which implies that  $(n^+, y \times x) \in X$ . Hence  $R(X)$ .

By definition of  $F$ ,  $F \subseteq X$ . Let  $E = \{n \in \mathbb{N} : \forall y \forall z ((n, y) \in F \wedge (n, z) \in F) \rightarrow y = z\}$ . We prove by induction that  $E = \mathbb{N}$ .

First we prove that  $0 \in E$ . Suppose  $(0, y), (0, z) \in F$ . Because  $\forall n \in \mathbb{N}, n^+ \neq \emptyset$ ,  $(0, y), (0, z)$  must satisfy the first condition in the definition of  $X$ , which implies that  $y = z = x$ .

Assume  $n \in E$  for some  $n \in \mathbb{N}$ . Suppose  $(n^+, y), (n^+, z) \in F$ . Because  $n^+ \neq \emptyset$ , then  $(n^+, y), (n^+, z)$  must satisfy the second condition in the definition of  $X$ . Thus  $\exists (m_1, y'), (m_2, z') \in F$  such that  $n^+ = m_1^+ = m_2^+$  and  $y = y' \times x$  and  $z = z' \times x$ . By Lemma 1,  $m_1 = m_2 = n$ . Furthermore, by induction hypothesis,  $y' = z'$ , which implies that  $y = z$ . That closes the induction, proving that  $E = \mathbb{N}$ , completing the proof.  $\square$