Exercise 18

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Problem 1.

Solution. (a), (b), (d), (f). □

Lemma. *In a connected graph, removing an edge in a simple circuit cannot disconnect the graph.*

Proof. Let $G = (V, E)$ be a connected graph and $e_j \in E$ with endpoints u, v be an edge in a simple circuit $e_1, \ldots, e_n$, where $1 \leq j \leq n$. Let $G' = (V, E \backslash \{e_j\})$. Because there exists a path $e_{j+1}, \ldots, e_n, e_1, \ldots, e_{j-1}$ in G' between u, v, adding edge e_j to G' does not change the number of connected components. In other words, G' is connected, as desired. □

Problem 2.

Proof. Suppose $G' = (V', E')$. Because E' is finite, the number of simple circuits in G' is finite. Removing an edge cannot create new simple circuits. Thus we can execute the following process, which will eventually terminate:

$G'' \leftarrow G'$
while there exists a simple circuit in G'' **do**
    $e \leftarrow$ an edge in a simple circuit in G''
    $G'' \leftarrow (V', E' \backslash \{e\})$
end while
return G''

Now $G'' = (V', E'')$ is a connected subgraph of G' without simple circuits, i.e., a tree. Thus

$$\left|E'\right| \geq \left|E''\right| = \left|V'\right| - 1 = n - 1.$$

□

Problem 3.

Proof. Let $G_k = (V_k, E_k)$ be the (pairwise distinct) connected components of G, where $1 \leq k \leq n$. By Exercise 2, $|E_k| \geq |V_k| - 1$ for each $1 \leq k \leq n$. Because connected components are induced subgraphs of equivalence classes of the connectivity relation, V is precisely the disjoint union of V_k's. Two edges in different connected components cannot have common endpoints. Thus E_k's are pairwise disjoint. The endpoints (not necessarily distinct) of every $e \in E$ fall in the same equivalence class of the connectivity relation, and thus in the same connected component. Thus E is precisely the disjoint union of E_k's. Now we have

$$|E| = \sum_{k=1}^{n} |E_k| \geq \sum_{k=1}^{n} (|V_k| - 1) = \left(\sum_{k=1}^{n} |V_k| \right) - n = |V| - n.$$

□