Exercise 9

陈志杰 524531910034

Problem 1.

Solution. (a), (d). □

Problem 2.

Solution. (a), (c), (d), (e). □

Problem 3.

Proof.

(a)
- For every $a, b \in \mathbb{N}$, $a + b = b + a$.
- For every $a, b \in \mathbb{N}$, $\mathfrak{I}_1(f)(a, b) = \mathfrak{I}_1(f)(b, a)$.
- For every $a, b \in \mathbb{N}$, $\mathfrak{I}_1(R)(\mathfrak{I}_1(f)(a, b), \mathfrak{I}_1(f)(b, a)) = \mathbf{T}$.
- For every $a, b \in \mathbb{N}$, $\mathfrak{I}_1[x \mapsto a, y \mapsto b](R)(\mathfrak{I}_1[x \mapsto a, y \mapsto b](f)(a, b), \mathfrak{I}_1[x \mapsto a, y \mapsto b](f)(b, a)) = \mathbf{T}$.
- For every $a, b \in \mathbb{N}$, $\mathfrak{I}'_1(R)(\mathfrak{I}'_1(f)(\llbracket x \rrbracket_{\mathfrak{I}'_1}, \llbracket y \rrbracket_{\mathfrak{I}'_1}), \mathfrak{I}'_1(f)(\llbracket y \rrbracket_{\mathfrak{I}'_1}, \llbracket x \rrbracket_{\mathfrak{I}'_1})) = \mathbf{T}$, where $\mathfrak{I}'_1 = \mathfrak{I}_1[x \mapsto a, y \mapsto b]$.
- For every $a, b \in \mathbb{N}$, $\llbracket R(f(x, y), f(y, x)) \rrbracket_{\mathfrak{I}_1[x \mapsto a, y \mapsto b]} = \mathbf{T}$.
- For every $a \in \mathbb{N}$, $\llbracket \forall y R(f(x, y), f(y, x)) \rrbracket_{\mathfrak{I}_1[x \mapsto a]} = \mathbf{T}$.
- $\llbracket \forall x \forall y R(f(x, y), f(y, x)) \rrbracket_{\mathfrak{I}_1} = \mathbf{T}$.

(b)
- For every $a, b \in \mathbb{N}$, $a * b = b * a$.
- For every $a, b \in \mathbb{N}$, $\mathfrak{I}_2(f)(a, b) = \mathfrak{I}_2(f)(b, a)$.
- For every $a, b \in \mathbb{N}$, $\mathfrak{I}_2(R)(\mathfrak{I}_2(f)(a, b), \mathfrak{I}_2(f)(b, a)) = \mathbf{T}$.
- For every $a, b \in \mathbb{N}$, $\mathfrak{I}_2[x \mapsto a, y \mapsto b](R)(\mathfrak{I}_2[x \mapsto a, y \mapsto b](f)(a, b), \mathfrak{I}_2[x \mapsto a, y \mapsto b](f)(b, a)) = \mathbf{T}$.
- For every $a, b \in \mathbb{N}$, $\mathfrak{I}'_2(R)(\mathfrak{I}'_2(f)(\llbracket x \rrbracket_{\mathfrak{I}'_2}, \llbracket y \rrbracket_{\mathfrak{I}'_2}), \mathfrak{I}'_2(f)(\llbracket y \rrbracket_{\mathfrak{I}'_2}, \llbracket x \rrbracket_{\mathfrak{I}'_2})) = \mathbf{T}$, where $\mathfrak{I}'_2 = \mathfrak{I}_2[x \mapsto a, y \mapsto b]$.
- For every $a, b \in \mathbb{N}$, $\llbracket R(f(x, y), f(y, x)) \rrbracket_{\mathfrak{I}_2[x \mapsto a, y \mapsto b]} = \mathbf{T}$.
- For every $a \in \mathbb{N}$, $\llbracket \forall y R(f(x, y), f(y, x)) \rrbracket_{\mathfrak{I}_2[x \mapsto a]} = \mathbf{T}$.
- $\llbracket \forall x \forall y R(f(x, y), f(y, x)) \rrbracket_{\mathfrak{I}_2} = \mathbf{T}$.

(c)
- For every $a, b \in \{\mathbf{T}, \mathbf{F}\}$, $\llbracket \wedge \rrbracket(a, b) = \llbracket \wedge \rrbracket(b, a)$.
- For every $a, b \in \{\mathbf{T}, \mathbf{F}\}$, $\mathfrak{I}_3(f)(a, b) = \mathfrak{I}_3(f)(b, a)$.
- For every $a, b \in \{\mathbf{T}, \mathbf{F}\}$, $\mathfrak{I}_3(R)(\mathfrak{I}_3(f)(a, b), \mathfrak{I}_3(f)(b, a)) = \mathbf{T}$.
- For every $a, b \in \{\mathbf{T}, \mathbf{F}\}$, $\mathfrak{I}_3[x \mapsto a, y \mapsto b](R)(\mathfrak{I}_3[x \mapsto a, y \mapsto b](f)(a, b), \mathfrak{I}_3[x \mapsto a, y \mapsto b](f)(b, a)) = \mathbf{T}$.
- For every $a, b \in \{\mathbf{T}, \mathbf{F}\}$, $\mathfrak{I}'_3(R)(\mathfrak{I}'_3(f)(\llbracket x \rrbracket_{\mathfrak{I}'_3}, \llbracket y \rrbracket_{\mathfrak{I}'_3}), \mathfrak{I}'_3(f)(\llbracket y \rrbracket_{\mathfrak{I}'_3}, \llbracket x \rrbracket_{\mathfrak{I}'_3})) = \mathbf{T}$, where $\mathfrak{I}'_3 = \mathfrak{I}_3[x \mapsto a, y \mapsto b]$.

- For every $a, b \in \{\mathbf{T}, \mathbf{F}\}$, $[\![R(f(x,y), f(y,x))]\!]_{\mathfrak{I}_3[x \mapsto a, y \mapsto b]} = \mathbf{T}$.
- For every $a \in \{\mathbf{T}, \mathbf{F}\}$, $[\![\forall y R(f(x,y), f(y,x))]\!]_{\mathfrak{I}_3[x \mapsto a]} = \mathbf{T}$.
- $[\![\forall x \forall y R(f(x,y), f(y,x))]\!]_{\mathfrak{I}_3} = \mathbf{T}$.

(d) Let $A = \{0\}$, β be the assignment that maps every variable to 0, $f^{\mathfrak{A}} : A \times A \to A$ be the function $f^{\mathfrak{A}}(x,y) = x$ for every $x, y \in A$, $R^{\mathfrak{A}} : A \times A \to \{\mathbf{T}, \mathbf{F}\}$ be the function $R^{\mathfrak{A}}(x,y) = \mathbf{F}$ for every $x, y \in A$. Let $\mathfrak{a}$ be the map that $\mathfrak{a}(f) = f^{\mathfrak{A}}$, $\mathfrak{a}(R) = R^{\mathfrak{A}}$. Let $\mathfrak{A}$ be the structure $(A, \mathfrak{a})$ and $\mathfrak{I}$ be the interpretation $(\mathfrak{A}, \beta)$.

Let $\phi = \forall x \forall y R(f(x,y), f(y,x))$. We now prove by contradiction that $[\![\phi]\!]_{\mathfrak{I}} = \mathbf{F}$. Assume that $[\![\phi]\!]_{\mathfrak{I}} = \mathbf{T}$, i.e., for every $a, b \in A$, $[\![R(f(x,y), f(y,x))]\!]_{\mathfrak{I}[x \mapsto a, y \mapsto b]} = \mathbf{T}$. Let $\mathfrak{I}' = \mathfrak{I}[x \mapsto a, y \mapsto b]$ where $a = b = 0$. Now $[\![x]\!]_{\mathfrak{I}'} = [\![y]\!]_{\mathfrak{I}'} = 0$. Then $\mathfrak{I}'(R)(\mathfrak{I}'(f)(0,0), \mathfrak{I}'(f)(0,0)) = \mathbf{T}$, which implies that $R^{\mathfrak{A}}(f^{\mathfrak{A}}(0,0), f^{\mathfrak{A}}(0,0)) = \mathbf{T}$, contradicting the definition of $R^{\mathfrak{A}}$. Thus $[\![\phi]\!]_{\mathfrak{I}} = \mathbf{F}$, proving that ϕ is not valid.

□

Problem 4.

Proof. Consider an S-interpretation $\mathfrak{I}$ with domain A such that $[\![\exists x \forall y (R(x,y))]\!]_{\mathfrak{I}} = \mathbf{T}$.

- There exists $a \in A$ such that $[\![\forall y (R(x,y))]\!]_{\mathfrak{I}[x \mapsto a]} = \mathbf{T}$.
- For every $b \in A$, $[\![R(x,y)]\!]_{\mathfrak{I}[x \mapsto a, y \mapsto b]} = \mathbf{T}$, of course including $b = a$.
- $[\![R(x,y)]\!]_{\mathfrak{I}[x \mapsto a, y \mapsto a]} = \mathbf{T}$.
- $\mathfrak{I}[x \mapsto a, y \mapsto a](R)([\![x]\!]_{\mathfrak{I}[x \mapsto a, y \mapsto a]}, [\![y]\!]_{\mathfrak{I}[x \mapsto a, y \mapsto a]}) = \mathbf{T}$.
- $\mathfrak{I}(R)(a,a) = \mathbf{T}$.
- $\mathfrak{I}[x \mapsto a](R)([\![x]\!]_{\mathfrak{I}[x \mapsto a]}, [\![x]\!]_{\mathfrak{I}[x \mapsto a]}) = \mathbf{T}$.
- There exists $a \in A$ such that $[\![R(x,x)]\!]_{\mathfrak{I}[x \mapsto a]} = \mathbf{T}$.
- $[\![\exists x (R(x,x))]\!]_{\mathfrak{I}} = \mathbf{T}$.

Thus $\exists x \forall y (R(x,y)) \models \exists x (R(x,x))$, as desired. □