Exercise 13

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Problem 1.

Solution. (a)

$$\phi = (p \to \neg q) \wedge (q \to r) \wedge (\neg p \to \neg r).$$

(b) Let $\mathcal{J}$ be a truth assignment. If $\mathcal{J}(p) = \mathbf{F}$, then $[\![(\neg p \to \neg r)]\!]_{\mathcal{J}} = \mathbf{T}$ implies that $\mathcal{J}(r) = \mathbf{F}$. $\mathcal{J}(q) = \mathbf{F}$ follows from that $[\![(q \to r)]\!]_{\mathcal{J}} = \mathbf{T}$. But it is not reasonable to invite nobody to a party. Thus $\mathcal{J}(p) = \mathbf{T}$, and $\mathcal{J}(q) = \mathbf{F}$ follows from that $[\![(p \to \neg q)]\!]_{\mathcal{J}} = \mathbf{T}$. Now $[\![\phi]\!]_{\mathcal{J}} = \mathbf{T}$ regardless of $\mathcal{J}(r)$.

Let $\mathcal{J}$ be such that $\mathcal{J}(p) = \mathbf{T}, \mathcal{J}(q) = \mathbf{F}, \mathcal{J}(r) = \mathbf{T}$. It can be verified that $[\![\phi]\!]_{\mathcal{J}} = \mathbf{T}$. □

Problem 2.

Solution. Let propositional variables $p_1, p_2, p_3, q_1, q_2, q_3$ be such that p_1, p_2, p_3 respectively represent that Smith, Jones, Williams is innocent, q_1 represents that Cooper and Jones were friends, q_2 represents that Williams disliked Smith, and q_3 represents that Jones was out of the town the day Cooper was killed. Define compound propositions ϕ, ψ, χ by

$$\begin{aligned}
\phi &= (\neg p_1 \wedge p_2 \wedge p_3 \wedge (\neg q_1 \wedge q_3) \wedge (\neg q_3 \wedge (\neg p_1 \vee \neg p_2))) \\
\psi &= (p_1 \wedge \neg p_2 \wedge p_3 \wedge (q_1 \wedge q_2) \wedge (\neg q_3 \wedge (\neg p_1 \vee \neg p_2))) \\
\chi &= (p_1 \wedge p_2 \wedge \neg p_3 \wedge (q_1 \wedge q_2) \wedge (\neg q_1 \wedge q_3))
\end{aligned}$$

Then the desired truth assignment $\mathcal{J}$ should satisfy $[\![\phi \vee \psi \vee \chi]\!]_{\mathcal{J}} = \mathbf{T}$. ϕ is a contradiction because it is logically equivalent to the conjunction of q_3 and $\neg q_3$ and another compound proposition. χ is a contradiction because it is logically equivalent to the conjunction of q_1 and $\neg q_1$ and another compound proposition. Thus $[\![\psi]\!]_{\mathcal{J}} = \mathbf{T}$, which implies that $[\![p_1 \wedge \neg p_2 \wedge p_3]\!]_{\mathcal{J}} = \mathbf{T}$. That implies that Jones was the murderer. □

Problem 3.

Solution. (a)

$$\begin{aligned}
\phi &\equiv \neg p \vee (q \leftrightarrow r) \\
&\equiv \neg p \vee ((q \wedge r) \vee (\neg q \wedge \neg r)) \\
&\equiv (q \wedge r) \vee ((\neg q \wedge \neg r) \vee \neg p) \\
&\equiv (q \vee ((\neg q \wedge \neg r) \vee \neg p)) \wedge (r \vee ((\neg q \wedge \neg r) \vee \neg p)) \\
&\equiv ((\neg q \wedge \neg r) \vee \neg p \vee q) \wedge ((\neg q \wedge \neg r) \vee \neg p \vee r) \\
&\equiv (\neg q \vee \neg p \vee q) \wedge (\neg r \vee \neg p \vee q) \wedge (\neg q \vee \neg p \vee r) \wedge (\neg r \vee \neg p \vee r) \\
&\equiv (\neg r \vee \neg p \vee q) \wedge (\neg q \vee \neg p \vee r).
\end{aligned}$$

(b) ϕ is satisfiable if and only if χ is satisfiable, where

$$\chi = ((q \leftrightarrow r) \leftrightarrow p_1) \wedge ((p \to p_1) \leftrightarrow p_2) \wedge p_2.$$

$$\begin{aligned}
((p \to p_1) \leftrightarrow p_2) &\equiv ((p \to p_1) \to p_2) \wedge (p_2 \to (p \to p_1)) \\
&\equiv (\neg(p \to p_1) \vee p_2) \wedge (\neg p_2 \vee (p \to p_1)) \\
&\equiv ((p \wedge \neg p_1) \vee p_2) \wedge (\neg p_2 \vee \neg p \vee p_1) \\
&\equiv (p \vee p_2) \wedge (\neg p_1 \vee p_2) \wedge (\neg p_2 \vee \neg p \vee p_1).
\end{aligned}$$

$$\begin{aligned}
((q \leftrightarrow r) \leftrightarrow p_1) &\equiv ((q \leftrightarrow r) \to p_1) \wedge (p_1 \to (q \leftrightarrow r)) \\
&\equiv (\neg(q \leftrightarrow r) \vee p_1) \wedge (\neg p_1 \vee (q \leftrightarrow r)) \\
&\equiv ((q \wedge \neg r) \vee (\neg q \wedge r) \vee p_1) \wedge (\neg p_1 \vee (q \wedge r) \vee (\neg q \wedge \neg r)).
\end{aligned}$$

$$\begin{aligned}
&((q \wedge \neg r) \vee (\neg q \wedge r) \vee p_1) \\
\equiv\ &(q \vee (\neg q \wedge r) \vee p_1) \wedge (\neg r \vee (\neg q \wedge r) \vee p_1) \\
\equiv\ &(q \vee \neg q \vee p_1) \wedge (q \vee r \vee p_1) \wedge (\neg r \vee \neg q \vee p_1) \wedge (\neg r \vee r \vee p_1) \\
\equiv\ &(q \vee r \vee p_1) \wedge (\neg r \vee \neg q \vee p_1).
\end{aligned}$$

$$\begin{aligned}
&(\neg p_1 \vee (q \wedge r) \vee (\neg q \wedge \neg r)) \\
\equiv\ &(\neg p_1 \vee q \vee (\neg q \wedge \neg r)) \wedge (\neg p_1 \vee r \vee (\neg q \wedge \neg r)) \\
\equiv\ &(\neg p_1 \vee q \vee \neg q) \wedge (\neg p_1 \vee q \vee \neg r) \wedge (\neg p_1 \vee r \vee \neg q) \wedge (\neg p_1 \vee r \vee \neg r) \\
\equiv\ &(\neg p_1 \vee q \vee \neg r) \wedge (\neg p_1 \vee r \vee \neg q).
\end{aligned}$$

Now that $\chi \equiv (q \vee r \vee p_1) \wedge (\neg r \vee \neg q \vee p_1) \wedge (\neg p_1 \vee q \vee \neg r) \wedge (\neg p_1 \vee r \vee \neg q) \wedge (p \vee p_2) \wedge (\neg p_1 \vee p_2) \wedge (\neg p_2 \vee \neg p \vee p_1) \wedge p_2$, $\psi = (q \vee r \vee p_1) \wedge (\neg r \vee \neg q \vee p_1) \wedge (\neg p_1 \vee q \vee \neg r) \wedge (\neg p_1 \vee r \vee \neg q) \wedge (p \vee p_2) \wedge (\neg p_1 \vee p_2) \wedge (\neg p_2 \vee \neg p \vee p_1) \wedge p_2$.

□