Exercise 19

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Lemma. *Suppose G is a simple undirected graph, and $u = x_0, x_1, \ldots, x_n = v$ is a path connecting u and v, where $u \neq v$. Then there exists a simple path $u = x_0, x_{i_1}, \ldots, x_{i_{m-1}}, x_n = v$ connecting u and v such that $0 < i_1 < i_2 < \cdots < i_m < n$.*

Proof. Because n is finite and during each iteration the length of P decreases, we can execute the following procedure which will eventually halt.

$P \leftarrow (u = x_0, x_1, \ldots, x_n = v)$[1] ▷ Here P is a path.
while there exists duplicate vertices in P **do**
  Denote P by $u = y_0, \ldots, w = y_i, \ldots, w = y_j, \ldots, y_m = v$ where $0 \leq i < j \leq m$.
  $P \leftarrow (u, \ldots, y_{i-1}, y_i, y_{j+1}, \ldots, v)$
end while
return P

The returned path satisfies the desired properties. □

Problem 1.

Proof. Suppose the unique simple path connecting u and v passes through w. Let this very path be $u = x_0, x_1, \ldots, w = x_i, \ldots, x_n = v$, where $0 \leq i \leq n$. We have $d(u, v) = n$. $u = x_0, x_1, \ldots, x_i = w$ is a simple path connecting u and w; thus $d(u, w) = i$. $w = x_i, x_{i+1}, \ldots, x_n = v$ is a simple path connecting w and v; thus $d(w, v) = n - i$. Now we have $d(u, v) = d(u, w) + d(w, v)$.

Suppose $d(u, v) = d(u, w) + d(w, v)$. Let $u = x_0, x_1, \ldots, x_m = w$ be the unique simple path connecting u and w, and $w = x_m, x_{m+1}, \ldots, x_n = v$ be the unique simple path connecting w and v, where $m \leq n$, $d(u, w) = m$ and $d(w, v) = n - m$. Then $u = x_0, \ldots, x_m = w, \ldots, x_n = v$ is a path of length $d(u, v)$ connecting u and v. By the lemma, we can remove some vertices from this path and make it a simple path. However, the length of the unique simple path connecting u and v is exactly $d(u, v)$. Hence no vertices are removed from this path before it becomes a simple path, i.e., it is a simple path. Because this simple path passes through w, the proof is completed. □

Problem 2.

Proof. (a) Let r be the root. Suppose uR_1v, i.e., u is v's ancestor in G. Let $r = x_0, x_1, \ldots, x_n = v$ be the unique simple path (of length n) connecting r and v. By definition $u = x_k$ for some $0 \leq k < n$. Hence $r = x_0, x_1, \ldots, x_k = u$ is the unique simple path (of length k) connecting r and u. That u's level is less than v's level follows from $k < n$. Thus uR_2v, proving that $R_1 \subseteq R_2$.

(b) Let $V = \{r\}$, $E = \emptyset$, and $G = (V, E)$. Trivially G is a rooted tree with root r, and $R_1 = R_2 = \emptyset$, as desired.

(c) Let $V = \{r, u, v_0, v_1\}$, $E = \{e_0, e_1, e_2\}$ where e_0 has endpoints r, u, e_1 has endpoints r, v_0, and e_2 has endpoints v_0, v_1. Let $G = (V, E)$ be a rooted tree with root r. $R_1 = \{(r, u), (r, v_0), (r, v_1), (v_0, v_1)\}$. Because the levels of r, u, v_0, v_1 are respectively $0, 1, 1, 2$, $R_2 = \{(r, u), (r, v_0), (r, v_1), (v_0, v_1), (r, v_1)\}$. $R_1 \neq R_2$, as desired.

□

[1]The parentheses here are used only to avoid ambiguity about precedence of operations.

Problem 3.

Proof. Let r be the root. Let $r = x_0, x_1, \ldots, x_n = w$ is the unique simple path connecting r and w. By definition $u = x_i, v = x_j$ for some $0 \leq i, j < n$. We use proof by cases.

(a) $i = j$. Then $u = v$.

(b) $i < j$. Because $r = x_0, \ldots, u = x_i, \ldots, x_j = v$ is a simple path connecting r and v, u is v's ancestor.

(c) $i > j$. Because $r = x_0, \ldots, v = x_j, \ldots, x_i = u$ is a simple path connecting r and u, u is v's descendent.

Now the proof is completed. □