Exercise 20

陈志杰 524531910034

In this text, MST stands for minimum spanning tree.

Lemma 1. *Every rooted tree has a leaf.*

Proof. Suppose $G = (V, E)$ is a rooted tree with root r. Let $r = x_0, \ldots, x_n = v$ be a simple path with maximal length. Assume v is not a leaf, i.e. v has a child u. Then $r = x_0, \ldots, v = x_n, u$ is the simple path connecting r and u, contradicting the maximality of $r = x_0, \ldots, x_n = v$. Thus v is a leaf, completing the proof. $\square$

Lemma 2. *Suppose $G = (V, E)$ and $|V| > 1$. Then there exists an internal vertex all of whose children are leaves.*

Proof. Let r be the root and v_0 be the vertex with maximal level. $v_0 \neq r$ follows from $|V| > 1$, and thus v_0 has parent u. Suppose $v_0, \ldots, v_n$ are all children of u, which have the same levels. Because v_0 has a maximal level, $v_0, \ldots, v_n$ all have no children, i.e., are leaves. u satisfies the desired property. $\square$

Problem 1.

Proof. Let $G = (V, E)$. We use induction on $|V|$. For the base case $|V| = 1$, there is 1 leaf and 0 internal vertices, which satisfies the desired property. Suppose $n > 1$ and the desired property holds for all cases where $|V| < n$. When $|V| = n$, let u the internal vertex with the property in Lemma 2. Let $v_1, \ldots, v_n$ be all children of u where $n \geq 2$. Consider the induced subgraph $G' = (V \backslash \{v_1, \ldots, v_n\}, E')$ of G. It has no simple circuits. Because $v_1, \ldots, v_n$ are leaves,

$$\left|E'\right| = |E| - n = |V \backslash \{v_1, \ldots, v_n\}| .$$

Thus G' is a tree, with every internal vertex having at least 2 children. By induction hypothesis, G' has more leaves than internal vertices. The internal vertices of G are precisely the internal vertices of G' along with u; the leaves of G are precisely the leaves of G' except u, along with $v_1, \ldots, v_n$. By counting and $n \geq 2$, G has more leaves than internal vertices, closing the induction and completing the proof. $\square$

Problem 2.

Solution. The total weight of the MST is 28.

(1) Add the edge with endpoints a, b.

(2) Add the edge with endpoints a, e.

(3) Add the edge with endpoints a, d.

(4) Add the edge with endpoints d, c.

(5) Add the edge with endpoints d, h.

(6) Add the edge with endpoints d, p.

(7) Add the edge with endpoints e, f.

(8) Add the edge with endpoints e, i.

(9) Add the edge with endpoints p, m.

(10) Add the edge with endpoints p, l.

(11) Add the edge with endpoints h, g.

(12) Add the edge with endpoints m, n.

(13) Add the edge with endpoints n, o.

(14) Add the edge with endpoints f, j.

(15) Add the edge with endpoints o, k.

□

Problem 3.

Solution. The total weight of the MST is 28.

(1) Add the edge with endpoints a, b.

(2) Add the edge with endpoints c, d.

(3) Add the edge with endpoints d, h.

(4) Add the edge with endpoints a, e.

(5) Add the edge with endpoints b, c.

(6) Add the edge with endpoints a, m.

(7) Add the edge with endpoints m, p.

(8) Add the edge with endpoints l, p.

(9) Add the edge with endpoints e, i.

(10) Add the edge with endpoints n, o.

(11) Add the edge with endpoints e, f.

(12) Add the edge with endpoints g, h.

(13) Add the edge with endpoints m, n.

(14) Add the edge with endpoints f, j.

(15) Add the edge with endpoints k, o.

□

Problem 4.

Proof. Let G be a connected weighted graph. Let $\ell(e)$ denote the weight of an edge e. That G has a spanning tree follows from its connectivity. We use proof by contradiction and assume that $T = (V, E)$ and $T' = (V, E')$ are two distinct MSTs, i.e., $S = (E \backslash E') \cup (E' \backslash E) \neq \emptyset$. Let e be the unique element of S with least weight. Without loss of generality, suppose $e \in E \backslash E'$.

Add e to T' and we have a simple circuit $e, e_0, \ldots, e_n$, where $e_0, \ldots, e_n \in E'$. Because removing an edge in a simple circuit from a connected graph cannot disconnect it, $T'' = (V, E' \cup \{e\} \backslash \{e_i\})$ is a (spanning) tree for each $0 \leq i \leq n$. By the minimality of T' (in comparison to T''),

$$\sum_{e' \in E'} \ell(e') \leq \sum_{e' \in E'} \ell(e') + \ell(e) - \ell(e_i),$$

which implies that $\ell(e_i) < \ell(e)$ for each $0 \leq i \leq n$ because weights of edges are pairwise distinct.

Because T contains no simple circuits, $e, e_0, \ldots, e_n$ cannot all belong to E. Suppose $e_k \in E' \backslash E$. Now e is the element of S with least weight, and $e_k \in E' \backslash E \subseteq S$ has strictly less weight than e, a contradiction. The proof is completed. $\square$