Exercise 6

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In this text, a class means the collection of all sets that satisfy a given property (first-order formula); V is the universe $\{x : x = x\}$; n^+ means $n \cup \{n\}$.

Problem 1.

Proof. (a) $\forall y(y \notin \emptyset)$. Thus $\forall x \forall y((x \in y \wedge y \in \emptyset) \to x \in \emptyset)$ holds.

(b) Suppose n is $\in$-transitive. Let x, y be such that $x \in y$ and $y \in n \cup \{n\}$. If $y \in n$, then $x \in n \subseteq n \cup \{n\}$ by the $\in$-transitivity of n. If $y \in \{n\}$, then $x \in y$ implies $x = n$. Thus $x \in n \cup \{n\}$ trivially. By proof by cases, $n \cup \{n\}$ is $\in$-transitive.

$\square$

Problem 2.

Proof. (a) $0 = \emptyset$ has no nonempty subsets. Thus 0 is $\in$-well-ordered by definition.

(b) Suppose n is $\in$-well-ordered. Let $x \subseteq n \cup \{n\}$. Let $z = x \backslash \{n\} \subseteq n$. Let s be an $\in$-least element of z. For each $a \in x$, if $a \in z$ then $s = a$ or $s \in a$ by definition; if[1] $a = n$ then $s \in z \subseteq n = a$. Hence s is an $\in$-least element of x, completing the proof.

$\square$

Problem 3.

Proof. Suppose x, y are such that x, y are inductive; $x \subseteq z$ for every inductive set z; $y \subseteq z$ for every inductive set z. Then taking $z = y$ leads to $x \subseteq y$, and taking $z = x$ leads to $y \subseteq x$. Thus $x = y$, as desired. $\square$

Problem 4.

Proof. $\emptyset \in u$ and $\emptyset \in v$ leads to $\emptyset \in u \cap v$. For every $x \in u \cap v$, we have $x \in u$ and $x \in v$, which implies that $x \cup \{x\} \in u$ and $x \cup \{x\} \in v$, which imply that $x \cup \{x\} \in u \cap v$. Thus $u \cap v$ is inductive. $\square$

Problem 5.

Proof. (a) For each inductive set $w \subseteq u$, $\emptyset \in w$. Thus $\emptyset \in v$. For each $x \in v$, $x \in w$ for all inductive set $w \subseteq u$, which implies that $x \cup \{x\} \in w$ for all inductive set $w \subseteq u$, which implies that $x \cup \{x\} \in v$. Hence v is inductive.

(b) For any inductive set $w \subseteq u$, $v \subseteq w$ by the definition of v. The desired result follows from that v is inductive.

$\square$

Problem 6.

[1] Note that we do not need to distinguish whether $n \in x$ or not.

Proof. Suppose there exists an inductive set u. Let[2]

$$N = \bigcap_{X \text{ is inductive}} X.$$

N is indeed a set because $N \subseteq u$. We prove that N is the smallest inductive set.

For each inductive set X, $\emptyset \in X$. Thus $\emptyset \in N$. For each $x \in N$, $x \in X$ for all inductive set X, which implies that $x \cup \{x\} \in X$ for all inductive set X, which implies that $x \cup \{x\} \in N$. Hence N is inductive. For any inductive set X, $X \subseteq N$ by definition. Hence N is *the* smallest inductive set. □

Problem 7.

Proof. (a) $\mathbb{N}$ and X are both inductive by definition. Then by Problem 4, $\mathbb{N} \cap X$ is inductive.

(b) By definition of $\mathbb{N}$ (the smallest inductive set), $\mathbb{N} \subseteq \mathbb{N} \cap X \subseteq \mathbb{N}$, which implies that $\mathbb{N} \subseteq X$, completing the proof.

□

Problem 8. [3]

Lemma 1. *We define that a set T is transitive if $\forall x(x \in T \to x \subseteq T)$. Every $n \in \mathbb{N}$ is transitive.*

Proof. Let $E = \{n \in \mathbb{N} : n \text{ is transitive}\}$. We prove by induction that $E = \mathbb{N}$. $\forall x(x \in 0 \to x \subseteq 0)$ follows from $0 = \emptyset$. Assume that $n \in \mathbb{N}$ is transitive. For each $x \in n^+ = n \cup \{n\}$, either $x \in n$ or $x = n$. If $x \in n$ then $x \subseteq n \subseteq n^+$ by induction hypothesis; if $x = n$ then $x \subseteq n^+$ trivially. That closes the induction and completes the proof. □

Lemma 2. *Suppose $n, m \in \mathbb{N}$. If $n^+ = m^+$ then $n = m$.*

Proof. Suppose $n \cup \{n\} = m \cup \{m\}$. Then $n \in m \cup \{m\}$. Thus $n = m$ or $n \in m$. Similarly $m = n$ or $m \in n$. If $n \neq m$ then $n \in m$ and $m \in n$. By Lemma 1, $n \subseteq m$ and $m \subseteq n$, which implies that $n = m$. □

Proof of the Problem. Fix x. Let R be the property defined by

$$R(A) = ((0, x) \in A \wedge \forall n \forall y((n, y) \in A \to (n^+, y \times x) \in A)).$$

Let class $F = \{(u, v) : P(u, v)\}$, which is equivalent to

$$F = (\mathbb{N} \times V) \cap \bigcap_{R(A)} A.$$

First we prove that $R(F)$. For every A such that $R(A)$, $(0, x) \in A$. Thus $(0, x) \in F$. For any $(n, y) \in F$, $(n, y) \in A$ for each A with property R, which implies that $(n^+, y \times x) \in A$ for each A with property R, which implies that $(n^+, y \times x) \in F$. Hence $R(F)$.

Let

$$X = \{(n, y) \in F : ((n, y) = (0, x) \vee \exists n' \exists y'((n', y') \in F \wedge n = n'^+ \wedge y = y' \times x))\}.$$

[2] Note that we have not claimed that $N = \mathbb{N}$.

[3] Erratum: the seemingly fancy proof below is actually false. I applied a property P to a class F, whereas quantifying classes is not permitted. None have found a proper solution to this problem yet. I have proved the problem with the Transfinite Recursion Theorem (involving axiomatic definitions of ordinal numbers).

Now we prove that $R(X)$. $(0, x) \in X$ trivially. For every $(n, y) \in X$, $(n, y) \in F$. Thus $(n^+, y \times x)$ satisfies the second condition in the definition of X, which implies that $(n^+, y \times x) \in X$. Hence $R(X)$.

By definition of F, $F \subseteq X$. Let $E = \{n \in \mathbb{N} : \forall y \forall z(((n, y) \in F \wedge (n, z) \in F) \to y = z)\}$. We prove by induction that $E = \mathbb{N}$.

First we prove that $0 \in E$. Suppose $(0, y), (0, z) \in F$. Because $\forall n \in \mathbb{N}, n^+ \neq \emptyset$, $(0, y), (0, z)$ must satisfy the first condition in the definition of X, which implies that $y = z = x$.

Assume $n \in E$ for some $n \in \mathbb{N}$. Suppose $(n^+, y), (n^+, z) \in F$. Because $n^+ \neq \emptyset$, then $(n^+, y), (n^+, z)$ must satisfy the second condition in the definition of X. Thus $\exists (m_1, y'), (m_2, z') \in F$ such that $n^+ = m_1^+ = m_2^+$ and $y = y' \times x$ and $z = z' \times x$. By Lemma 1, $m_1 = m_2 = n$. Furthermore, by induction hypothesis, $y' = z'$, which implies that $y = z$. That closes the induction, proving that $E = \mathbb{N}$, completing the proof. $\square$