Exercise 1

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Problem 1.

Solution. (1) No. $\{\emptyset\}$ is not an element of $\{\emptyset\}$.

(2) Yes. $\{\emptyset\}$ is indeed an element of $\{\{\emptyset\}\}$.

(3) No. Here is a counterexample. Let $A = \{1\}$. Then $\mathcal{P}(A) = \{\emptyset, \{1\}\}$. Because $1 \notin \mathcal{P}(A)$, A is not a subset of $\mathcal{P}(A)$.

(4) Yes. Because A itself is a subset of A (trivially $\forall x \in A, x \in A$), $A \in \mathcal{P}(A)$. $\square$

Problem 2.

Solution. Let $A = \emptyset, B = A \cup \{A\}$. Trivially $A \subseteq B$, and $A \in B$ follows from $A \in \{A\}$, as desired. $\square$

Problem 3.

Solution. Let $A = \emptyset$, $B = A \cup \{A\}$, and $C = B \cup \{B\}$. Trivially $A \subseteq B$, and $A \in B$ follows from $A \in \{A\}$. Similarly we have $B \subseteq C$ and $B \in C$, as desired. $\square$

Problem 4.

Proof. For any $X \in \mathcal{P}(A) \cap \mathcal{P}(B)$, $X \in \mathcal{P}(A)$ and $X \in \mathcal{P}(B)$, which implies that $X \subseteq A$ and $X \subseteq B$, which implies that $\forall x \in X, x \in A$ and $x \in B$. Thus $\forall x \in X, x \in A \cap B$, which implies that $X \subseteq A \cap B$, i.e., $X \in \mathcal{P}(A \cap B)$.

For any $X \in \mathcal{P}(A \cap B)$, $X \subseteq A \cap B$, which implies that $\forall x \in X, x \in A \cap B$, which implies that $\forall x \in X, x \in A$ and $\forall x \in X, x \in B$. This implies that $X \subseteq A$ and $X \subseteq B$. Thus $X \in \mathcal{P}(A)$ and $X \in \mathcal{P}(B)$, i.e., $X \in \mathcal{P}(A) \cap \mathcal{P}(B)$. That completes the proof. $\square$

Problem 5.

Proof. For any $(a, b) \in A \times \bigcup B$, we have $a \in A$ and $\exists z \in B, b \in z$. Thus $(a, b) \in A \times z$. Because $z \in B$, $A \times z \in \{A \times X | X \in B\}$. Thus $(a, b) \in \bigcup\{A \times X | X \in B\}$ by definition.

For any $(a, b) \in \bigcup\{A \times X | X \in B\}$, there exists $z \in B$ such that $(a, b) \in A \times z$, i.e., $a \in A$ and $b \in z$. Because $z \in B$, we have $b \in \bigcup B$. Thus $(a, b) \in A \times \bigcup B$. That completes the proof. $\square$