

Exercise 10

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Problem 1.

Solution. Let \mathcal{J} be the truth assignment such that $\mathcal{J}(p) = \mathbf{T}$, $\mathcal{J}(q) = \mathbf{F}$, $\mathcal{J}(r) = \mathbf{F}$. Then $\llbracket (p \wedge q) \rightarrow r \rrbracket_{\mathcal{J}} = \mathbf{T}$. However, $\llbracket p \rightarrow r \rrbracket_{\mathcal{J}} = \mathbf{F}$, $\llbracket q \rightarrow r \rrbracket_{\mathcal{J}} = \mathbf{T}$, which implies that $\llbracket (p \rightarrow r) \wedge (q \rightarrow r) \rrbracket_{\mathcal{J}} = \mathbf{F}$. Thus $(p \wedge q) \rightarrow r$ and $(p \rightarrow r) \wedge (q \rightarrow r)$ have different truth values under \mathcal{J} , proving that the two propositions are not logically equivalent. \square

Problem 2.

Solution. • Let $\mathfrak{A} = (A, \mathfrak{a})$ be an S -structure where $A = \{0, 1\}$, $\mathfrak{a}(P) = P^{\mathfrak{A}}$ is such that $P^{\mathfrak{A}}(0) = \mathbf{F}$, $P^{\mathfrak{A}}(1) = \mathbf{T}$, and $\mathfrak{a}(Q) = Q^{\mathfrak{A}}$ is such that $Q^{\mathfrak{A}}(x) = \mathbf{F}$ for all $x \in A$. Let $\mathfrak{J} = (\mathfrak{A}, \beta)$ be an S -interpretation where β is some fixed assignment in \mathfrak{A} .

- Let $a = 1$. Then $\llbracket P(x) \rightarrow Q(x) \rrbracket_{\mathfrak{J}[x \mapsto 1]} = \mathbf{F}$ follows from that $\mathfrak{J}[x \mapsto 1](P)(1) = \mathbf{T}$ and $\mathfrak{J}[x \mapsto 1](Q)(1) = \mathbf{F}$. Thus $\llbracket \forall x(P(x) \rightarrow Q(x)) \rrbracket_{\mathfrak{J}} = \mathbf{F}$.
- $\llbracket \forall x(P(x)) \rrbracket_{\mathfrak{J}} = \mathbf{F}$ follows from that $\llbracket P(x) \rrbracket_{\mathfrak{J}[x \mapsto 0]} = \mathbf{F}$. That implies that $\llbracket \forall x(P(x)) \rightarrow \forall x(Q(x)) \rrbracket_{\mathfrak{J}} = \mathbf{T}$.

The two propositions have different truth values on \mathfrak{J} , and hence are not logically equivalent. \square

Problem 3.

Proof. We use proof by contradiction and assume that the proposition is false on interpretation \mathfrak{J} with domain A .

- There does not exist $a \in A$ such that $\llbracket (P(x) \rightarrow \forall y(P(y))) \rrbracket_{\mathfrak{J}[x \mapsto a]} = \mathbf{T}$.
- For any $a \in A$, $\llbracket (P(x) \rightarrow \forall y(P(y))) \rrbracket_{\mathfrak{J}[x \mapsto a]} = \mathbf{F}$.
- For any $a \in A$, $\llbracket P(x) \rrbracket_{\mathfrak{J}[x \mapsto a]} = \mathbf{T}$ and $\llbracket \forall y(P(y)) \rrbracket_{\mathfrak{J}[x \mapsto a]} = \mathbf{F}$, i.e.,
 - for any $a \in A$, $\llbracket P(x) \rrbracket_{\mathfrak{J}[x \mapsto a]} = \mathbf{T}$;
 - * $\llbracket \forall x(P(x)) \rrbracket_{\mathfrak{J}} = \mathbf{T}$. (★)
 - for any $a \in A$, $\llbracket \forall y(P(y)) \rrbracket_{\mathfrak{J}[x \mapsto a]} = \mathbf{F}$.
 - * Apply $a = \mathfrak{J}(x)$. $\llbracket \forall y(P(y)) \rrbracket_{\mathfrak{J}[x \mapsto \mathfrak{J}(x)]} = \mathbf{F}$, i.e., $\llbracket \forall y(P(y)) \rrbracket_{\mathfrak{J}} = \mathbf{F}$.
 - * This does not hold: for all $b \in A$, $\llbracket P(y) \rrbracket_{\mathfrak{J}[y \mapsto b]} = \mathbf{T}$.
 - * This does not hold: for all $b \in A$, $\mathfrak{J}(P)(b) = \mathbf{T}$.
 - * This does not hold: for all $b \in A$, $\llbracket P(x) \rrbracket_{\mathfrak{J}[x \mapsto b]} = \mathbf{T}$.
 - * $\llbracket \forall x(P(x)) \rrbracket_{\mathfrak{J}} = \mathbf{F}$. (★)

The two (★)'s trigger a contradiction. Thus the proposition in question is true under any interpretation. \square

Remark. In the following two problems, when claiming “no logic”, we need to additionally write

Logic in the bigger picture: if $\Phi \vdash \psi$ and $\psi \vdash \phi$, then $\Phi \vdash \phi$.

Otherwise it is considered wrong by Q. Cao. Actually, TAs do not have clear criteria for these problems; *such ridiculous things did happen that same answers from different students receive different scores.*

Problem 4.

Solution. (a) If Φ does not talk about x , and $\Phi, \psi \vdash \phi$, then $\Phi \vdash \forall x(\psi \rightarrow \phi)$.

(b) No logic. It is a mathematical definition.

(c) No logic. It is a mathematical fact.

□

Problem 5.

Solution. (a) If Φ and ψ do not talk about x , and $\Phi, \phi \vdash \psi$, then $\Phi, \exists x\phi \vdash \psi$.

(b) No logic. It is a mathematical fact.

□