Exercise 17

陈志杰 524531910034

Problem 1.

Proof. Let $G = (V, E)$ be a directed graph. Note that an edge has exactly one initial vertex.

$$\begin{aligned}\sum_{v\in V}\deg^+(v) &= \sum_{v\in V}|\{e \in E : v \text{ is the initial vertex of } e\}| \\ &= \sum_{v\in V}\sum_{e\in E \text{ such that } v \text{ is the initial vertex of } e} 1 \\ &= \sum_{e\in E \text{ such that the initial vertex of } e \text{ is } v \text{ for some (unique) } v\in V} 1 \\ &= |E|\,.\end{aligned}$$

Similarly $\sum_{v\in V}\deg^-(v) = |E|$. Thus $\sum_{v\in V}\deg^+(v) = \sum_{v\in V}\deg^-(v)$, as desired. □

Problem 2.

Proof. By definition of connectivity, there exists a sequence $u = w_0, w_1, \ldots, w_{n-1}, w_n = v$ of vertices such that w_i, w_{i+1} are adjacent for each $0 \le i < n$. Execute the following:

1. If there exists $0 \le i < j \le n$ such that $w_i = w_j$, consider the new sequence of vertices below
$$u = w_0, \ldots, w_i = w_j, \ldots, w_n = v$$
and return to the start of Step 1. Note that every two consecutive vertices in the sequence above are still adjacent.
2. Else, terminate the process.

Because every execution of Step 1 decreases the number of vertices in the sequence by at least 1, the process will halt after at most n executions of Step 1 (no vertices occur more than once in a sequence consisting of only one vertex). Now we obtain a sequence of pairwise distinct vertices $u = x_0, \ldots, x_{m-1}, x_m = v$, every two consecutive vertices of which are adjacent. Let e_i be incident with x_i, x_{i+1}, $0 \le i < m$. For every $0 \le i < j < m$, e_i and e_j have distinct initial vertex and thus are distinct. The path $e_0, \ldots, e_{m-1}$ is a simple path from u to v, as desired. □

Problem 3.

Solution. The number of connected components are, respectively, $3, 1, 2$.

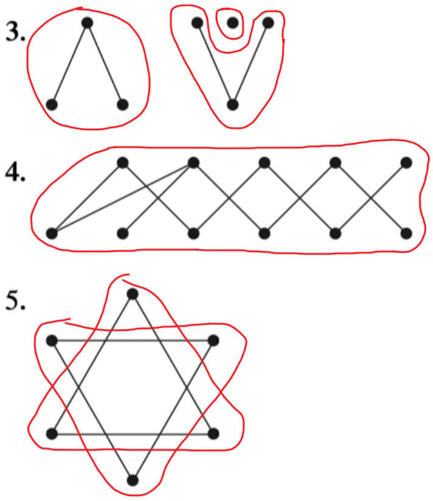

Figure 1: Connected components in Exercise 3.

□

Problem 4.

Solution. The number of strongly connected components are, respectively, $3, 4, 2$.

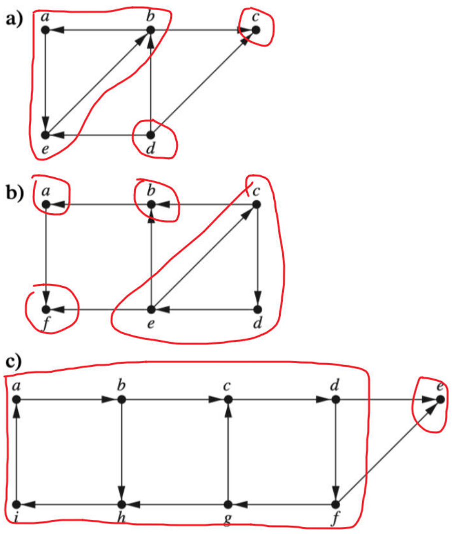

Figure 2: Strongly connected components in Exercise 4.

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