Exercise 5

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In this text: *countable* means countably infinite; $(\cdot,\cdot)_k$ means the kth element in the pair; $\mathcal{L}(V,W)$ means the vector space of all linear maps from V to W; $V \cong W$ means V and W are isomorphic vector spaces.

Problem 1.

Solution. Let S denote the set given in each problem.

(a) Countable. Define $F : \mathbb{Z}^+ \to S$ by $F(n) = n + 10$.

(b) Countable. Define $F : \mathbb{Z}^+ \to S$ by $F(n) = -2n + 1$.

(d) Uncountable.

(e) Countable. Define $F : A \times \mathbb{Z}^+ \to \mathbb{Z}^+$ by $F(a, n) = 2n - a + 2$.

□

Problem 2.

Proof. We only prove (b). Then (a) follows immediately because A is not finite and no bijections exist between A and $\mathbb{N}$. Let x_k denote the kth term of the sequence x, where $k \in \mathbb{N}$. We construct a sequence a of natural numbers in the following way. $a_0 = f(0)_0 + 1$. Inductively, $a_k = a_{k-1} + f(k)_k + 1$. Then $a_k > a_{k-1}$ and $a_k > f(k)_k$. Hence a is increasing and $f(n) \neq a$ for each n because $f(n)_n < a_n$, as desired. □

Problem 3.

Proof. $(A \to (B \to C)) \approx (A \times B \to C) \approx (B \times A \to C) \approx (B \to (A \to C))$. □

Problem 4.

Proof. Define $H : (B \times C)^A \to B^A \times C^A$ by $\left(H(F)_k\right)(a) = F(a)_k$, where $k = 1, 2$. Suppose $H(F) = H(G)$. Then $F(a)_k = (H(F)_k)(a) = (H(G)_k)(a) = G(a)_k, k = 1, 2$, which implies that $F = G$. Thus H is injective. For every $\varphi_1 \in B^A, \varphi_2 \in C^A$, let $F(a) = (\varphi_1(a), \varphi_2(a))$. Then $(H(F)_k)(a) = F(a)_k = \varphi_k(a)$, which implies that $H(F) = (\varphi_1, \varphi_2)$. Thus H is surjective. Hence the desired result holds from the bijectivity of H. □

Problem 5.

Proof. It suffices to show that $2^{\mathbb{R}\times\mathbb{R}} \approx \mathbb{R}^{\mathbb{R}}$. $\mathbb{R} \times \mathbb{R} \approx \mathbb{R}$ follows from

$$\mathbb{R} \times \mathbb{R} \approx (2^{\mathbb{N}})^2 \approx 2^{\mathbb{N}\times 2} \approx 2^{\mathbb{N}} \approx \mathbb{R}.$$

Then $2^{\mathbb{R}\times\mathbb{R}} \preceq \mathbb{R}^{\mathbb{R}}$ follows from

$$2^{\mathbb{R}\times\mathbb{R}} \approx 2^{\mathbb{R}} \preceq \mathbb{R}^{\mathbb{R}}.$$

$\mathbb{R}^{\mathbb{R}} \preceq 2^{\mathbb{R}\times\mathbb{R}}$ follows from

$$\mathbb{R}^{\mathbb{R}} \approx (2^{\mathbb{N}})^{2^{\mathbb{N}}} \approx 2^{\mathbb{N}\times 2^{\mathbb{N}}} \preceq 2^{2^{\mathbb{N}}\times 2^{\mathbb{N}}} \approx 2^{\mathbb{R}\times\mathbb{R}}.$$

Because $2^{\mathbb{R}\times\mathbb{R}} \preceq \mathbb{R}^{\mathbb{R}}$ and $\mathbb{R}^{\mathbb{R}} \preceq 2^{\mathbb{R}\times\mathbb{R}}$, by the Bernstein theorem $2^{\mathbb{R}\times\mathbb{R}} \approx \mathbb{R}^{\mathbb{R}}$, as desired. □

Problem 6.

Lemma 1. $\mathbb{R}^{\mathbb{N}} \approx \mathbb{R}$.

Proof.
$$\mathbb{R} \preceq \mathbb{R}^{\mathbb{N}} \approx 2^{\mathbb{N}\times\mathbb{N}} \approx 2^{\mathbb{N}} \approx \mathbb{R}. \qquad \square$$

Lemma 2 (from real analysis)**.** *If $f : \mathbb{R} \to \mathbb{R}$ is monotonically increasing, then f has at most countable discontinuities, and all of them are jump discontinuities.*

Proof of the Problem. Let $\mathcal{E}$ denote the set of all monotonically increasing functions from $\mathbb{R}$ to $\mathbb{R}$. An element of $\mathcal{E}$ is uniquely determined by the following: all its discontinuities (at most countable), its values at all discontinuities, and its values at all $x \in \mathbb{Q}$, because after that all undefined values of f are uniquely determined by continuity and limit (Heine's definition). Hence
$$\mathcal{E} \preceq \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{Q}}.$$
The first term corresponds to "all its discontinuities", represented by the set of functions from $\mathbb{N}$ to $\mathbb{R}$; the second term corresponds to "its values at all discontinuities", represented by the set of functions from $\mathbb{N}$ to $\mathbb{R}$; the third term corresponds to "$f(x)$'s at all $x \in \mathbb{Q}$", represented by the set of functions from $\mathbb{Q}$ to $\mathbb{R}$. "$\preceq$" is used because not all combinations are allowed or the function might be overdefined, but every element of $\mathcal{E}$ can be described as such.

Now the desired result follows from the lemma and
$$\mathbb{R} \preceq \mathcal{E} \preceq \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{Q}} \approx \mathbb{R} \times \mathbb{R} \times \mathbb{R} \approx \mathbb{R}. \qquad \square$$

Problem 7.

Proof. Treat $\mathbb{R}$ as a vector space over scalar field $\mathbb{Q}$: $\forall \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}, \boldsymbol{u} + \boldsymbol{v} \in \mathbb{R}$; $\forall \boldsymbol{u} \in \mathbb{R}, \forall c \in \mathbb{Q}, c\boldsymbol{u} \in \mathbb{R}$. If $\mathbb{R}$ is finite-dimensional, there exists a basis of finite length m. Then every element of $\mathbb{R}$ can be uniquely written as a linear combination of the m vectors in the basis. That implies that $\mathbb{R} \approx \mathbb{Q}^m \approx \mathbb{N}$, a contradiction. Hence $\mathbb{R}$ is not finite-dimensional.

Let $\mathcal{C} = \{\boldsymbol{v}_k\}_{k\in\mathbb{N}} \subseteq \mathbb{R}$ be a linearly independent list in $\mathbb{R}$, where $\boldsymbol{v}_0 = 1$. By Zorn's lemma, let $\mathcal{B} \supseteq \mathcal{C}$ be a basis of $\mathbb{R}$. Let $\boldsymbol{U} = \operatorname{span}(1)$. Then P is the quotient space $\mathbb{R}/\boldsymbol{U}$. Let $\{\varphi_{\boldsymbol{v}} : \boldsymbol{v} \in \mathcal{B}\}$ be the dual basis of $\mathcal{B}$. Note that for each $\boldsymbol{u} \in \mathbb{R}$,
$$\boldsymbol{u} = \sum_{\boldsymbol{v}\in\mathcal{B}} \varphi_{\boldsymbol{v}}(\boldsymbol{u})\boldsymbol{v}.$$

Define $T \in \mathcal{L}(\mathbb{R}, \mathbb{R}/\boldsymbol{U})$ by
$$T\boldsymbol{u} = \left[\sum_{k\in\mathbb{N}} \varphi_{\boldsymbol{v}_k}(\boldsymbol{u})\boldsymbol{v}_{k+1} + \left(\boldsymbol{u} - \sum_{k\in\mathbb{N}} \varphi_{\boldsymbol{v}_k}(\boldsymbol{u})\boldsymbol{v}_k\right)\right] + \boldsymbol{U}.$$
Suppose $T\boldsymbol{u} = \boldsymbol{0}$. Then
$$\sum_{k\in\mathbb{N}} \varphi_{\boldsymbol{v}_k}(\boldsymbol{u})\boldsymbol{v}_{k+1} + \left(\boldsymbol{u} - \sum_{k\in\mathbb{N}} \varphi_{\boldsymbol{v}_k}(\boldsymbol{u})\boldsymbol{v}_k\right) \in \boldsymbol{U} = \operatorname{span}(1).$$
That implies $\boldsymbol{u} = 0$ because applying φ_1 to the term above leads to 0. Thus T is injective. For each $A \in \mathbb{R}/\boldsymbol{U}$, fix $\boldsymbol{w} \in \mathbb{R}$ such that $A = \boldsymbol{w} + \boldsymbol{U}$. Let
$$\boldsymbol{u} = \sum_{k\in\mathbb{N}} \varphi_{\boldsymbol{v}_{k+1}}(\boldsymbol{w})\boldsymbol{v}_k + \left(\boldsymbol{w} - \sum_{k\in\mathbb{N}} \varphi_{\boldsymbol{v}_k}(\boldsymbol{w})\boldsymbol{v}_k\right).$$

It can be verified that $T\boldsymbol{u} = \boldsymbol{w} + \boldsymbol{U}$. Thus T is surjective. Hence we can conclude that T is an isomorphism from $\mathbb{R}$ onto $\mathbb{R}/\boldsymbol{U}$.

Now we have

$$\mathbb{R} \cong \mathbb{R}/\boldsymbol{U} = P.$$

The desired result follows from that isomorphic vector spaces are equinumerous by definition. $\square$