Exercise 7

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Problem 1.

Solution. (a) $p \wedge q$; (b) $p \wedge \neg q$; (c) $\neg p \wedge \neg q$; (d) $p \vee q$. □

Problem 2.

Proof. (a)

$$\neg(p \wedge \neg q) \equiv \neg p \vee \neg\neg q \equiv \neg p \vee q.$$

(b) From the truth table, we conclude that the truth values of LHS and RHS are the same for all truth assignments.

p	q	r	$p \vee q$	$\neg p \vee r$	$\neg p \wedge q$	$p \wedge r$	$(p \vee q) \wedge (\neg p \vee r)$	$(\neg p \wedge q) \vee (p \wedge r)$
T	**T**	**T**	**T**	**T**	**F**	**T**	**T**	**T**
T	**T**	**F**	**T**	**F**	**F**	**F**	**F**	**F**
T	**F**	**T**	**T**	**T**	**F**	**T**	**T**	**T**
T	**F**	**F**	**T**	**F**	**F**	**F**	**F**	**F**
F	**T**	**T**	**T**	**T**	**T**	**F**	**T**	**T**
F	**T**	**F**	**T**	**T**	**T**	**F**	**T**	**T**
F	**F**	**T**	**F**	**T**	**F**	**F**	**F**	**F**
F	**F**	**F**	**F**	**T**	**F**	**F**	**F**	**F**

□

Problem 3.

Proof. Let $\mathcal{J}$ be a truth assignment such that $\forall \phi \in \Phi, [\![\phi]\!]_{\mathcal{J}} = \mathbf{T}$. If $[\![p]\!]_{\mathcal{J}} = \mathbf{T}$, then $[\![\neg p]\!]_{\mathcal{J}} = \mathbf{F}$, and then $[\![r]\!]_{\mathcal{J}} = \mathbf{T}$ follows from $[\![\neg p \vee r]\!]_{\mathcal{J}} = \mathbf{T}$. If $[\![q]\!]_{\mathcal{J}} = \mathbf{T}$, then $[\![\neg q]\!]_{\mathcal{J}} = \mathbf{F}$, and then $[\![r]\!]_{\mathcal{J}} = \mathbf{T}$ follows from $[\![\neg q \vee r]\!]_{\mathcal{J}} = \mathbf{T}$. By proof by cases, $[\![r]\!]_{\mathcal{J}} = \mathbf{T}$. □

Problem 4.

Proof. Let $\mathcal{J} : \{p, q\} \to \{\mathbf{T}, \mathbf{F}\}$ be such that $[\![p]\!]_{\mathcal{J}} = \mathbf{F}, [\![q]\!]_{\mathcal{J}} = \mathbf{T}$. Then by definition $[\![p \vee q]\!]_{\mathcal{J}} = \mathbf{T}, [\![\neg p \vee \neg q]\!]_{\mathcal{J}} = \mathbf{T}, [\![p \wedge \neg q]\!]_{\mathcal{J}} = \mathbf{F}$. Thus $\Phi \not\models p \wedge \neg q$. □

Problem 5.

Proof. (a) For each $\mathcal{J}$ such that $[\![\phi]\!]_{\mathcal{J}} = \mathbf{T}$ and $\forall \chi \in \Phi, [\![\chi]\!]_{\mathcal{J}} = \mathbf{T}$, $[\![\psi]\!]_{\mathcal{J}} = \mathbf{T}$ follows from that $\Phi \models \psi$. Thus $\Phi, \phi \models \psi$ by definition.

(b) Suppose $\Phi, \psi \models \chi$. For any $\mathcal{J}$ such that $\forall \phi \in \Phi, [\![\phi]\!]_{\mathcal{J}} = \mathbf{T}$, if $[\![\psi]\!]_{\mathcal{J}} = \mathbf{T}$ then $[\![\chi]\!]_{\mathcal{J}} = \mathbf{T}$ follows from that $\Phi, \psi \models \chi$, and then $[\![\neg\psi \vee \chi]\!]_{\mathcal{J}} = \mathbf{T}$ by definition; if $[\![\psi]\!]_{\mathcal{J}} = \mathbf{F}$ then $[\![\neg\psi]\!]_{\mathcal{J}} = \mathbf{T}$, and then $[\![\neg\psi \vee \chi]\!]_{\mathcal{J}} = \mathbf{T}$ by definition. By proof by cases, $\Phi \models \neg\psi \vee \chi$.

Suppose $\Phi \models \neg\psi \vee \chi$. For any $\mathcal{J}$ such that $[\![\psi]\!]_{\mathcal{J}} = \mathbf{T}$ and $\forall \phi \in \Phi, [\![\phi]\!]_{\mathcal{J}} = \mathbf{T}$, $[\![\neg\psi \vee \chi]\!]_{\mathcal{J}} = \mathbf{T}$ follows from that $\Phi \models \neg\psi \vee \chi$. $[\![\psi]\!]_{\mathcal{J}} = \mathbf{T}$ implies that $[\![\neg\psi]\!]_{\mathcal{J}} = \mathbf{F}$. Thus $[\![\chi]\!]_{\mathcal{J}} = \mathbf{T}$ follows from that $[\![\neg\psi \vee \chi]\!]_{\mathcal{J}} = \mathbf{T}$. That implies that $\Phi, \psi \models \chi$. The proof is completed.

□

Problem 6.

Proof. (a) Suppose $\phi \models \psi$. Then we only consider the black rows of the truth table (and disgard the gray row). We conclude that $\phi \wedge \psi$ and ϕ have the same truth values under all truth assignments, and so do $\phi \vee \psi$ and ψ, as desired.

ϕ	ψ	$\phi \wedge \psi$	$\phi \vee \psi$
T	**T**	**T**	**T**
T	F	F	T
F	**T**	**F**	**T**
F	**F**	**F**	**F**

(b) $\phi \wedge \psi \models \phi$ implies that $\phi \vee (\phi \wedge \psi) \equiv (\phi \wedge \psi) \vee \phi \equiv \phi$. $\phi \models \phi \vee \psi$ implies that $\phi \wedge (\phi \vee \psi) \equiv \phi$. That proves the absorption rules.

$\square$