Exercise 4

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In the following text we adopt the following notations.

(a) $A - B = A \backslash B$.

(b) $F(A) = \{F(a) : a \in A\}$.

Problem 1.

Solution.

(a) $C_0 = \mathbb{N} - G(\mathbb{N}) = \{2n+1 : n \in \mathbb{N}\}$, $D_0 = F(C_0) = \{2n+1 : n \in \mathbb{N}\}$. By induction, $C_i = D_i = \{2^i(2n+1) : n \in \mathbb{N}\}$ for all $i \in \mathbb{N}$. Thus $\bigcup_{i\in\mathbb{N}} C_i = \mathbb{Z}^+$ because every positive integer becomes an odd number when divided by 2 sufficiently many times and $0 \notin \bigcup_{i\in\mathbb{N}} C_i$. Hence the bijection $H : \mathbb{N} \to \mathbb{N}$ is $H(x) = x$.

(b) $C_0 = \mathbb{N} - G(\mathbb{N}) = \emptyset$. Thus $\forall i \in \mathbb{N}, C_i = D_i = \emptyset$. The bijection $H : \mathbb{N} \to \mathbb{N}$ is

$$H(x) = (\text{the unique } y \in \mathbb{N} \text{ such that } G(y) = x) = x.$$

□

Problem 2.

Solution. $C_0 = [0,1] - G((0,1)) = \{0,1\}$, $D_0 = F(C_0) = \{\frac{1}{3}, \frac{2}{3}\}$, $C_1 = G(D_0) = \{\frac{1}{3}, \frac{2}{3}\}$. By induction, $C_i = \{\frac{1}{2}\left(1 \pm \frac{1}{3^i}\right)\}$ for all $i \in \mathbb{N}$. Thus the bijection $H : \mathbb{N} \to \mathbb{N}$ is

$$H(x) = \begin{cases} \dfrac{1+x}{3}, & \text{if } x = \dfrac{1}{2}\left(1 \pm \dfrac{1}{3^i}\right) \text{ for some } i \in \mathbb{N}; \\ x, & \text{otherwise.} \end{cases}$$

□

Problem 3.

Proof. (a) $\forall a \in A, (a, [a]_R) \in F$. If $(a, [a]_R), (a, [b]_R) \in F$, because F contains precisely the ordered pairs of an element of A and *the* equivalence class it is in, $[a]_R = [b]_R$. That proves that F is a function.

(b) For any $X \in B$, $X = [a]_R$ for some $a \in A$ by definition. Thus $(a, X) = (a, [a]_R) \in F$, which implies that $F(a) = X$. That proves that F is surjective.

(c) For each $(a,b) \in R$, aRb implies that $[a]_R = [b]_R$. By definition $F(a) = [a]_R = [b]_R = F(b)$. Thus $(a,b) \in \{(a,b) : F(a) = F(b)\}$. For each $(a,b) \in \{(a,b) : F(a) = F(b)\}$, $F(a) = F(b)$. Because $(a, [a]_R), (b, [b]_R) \in F$ and F is a function, $[a]_R = F(a) = F(b) = [b]_R$. Thus $(a,b) \in R$. That proves that $R = \{(a,b) : F(a) = F(b)\}$. □

Problem 4.

Solution. Define $F : A \to A \times A$ by

$$F(a_0b_0a_1b_1a_2b_2\dots) = (a_0a_1a_2\dots, b_0b_1b_2\dots)$$

for each $a_0b_0a_1b_1a_2b_2\dots \in A$. Trivially $a_0a_1a_2\dots, b_0b_1b_2\dots \in A$ and F is a function. Let $d(a, n)$ denote a_n, where $a = a_0a_1a_2\dots \in A$ and $n \in \mathbb{N}$. We claim that F is a bijection.

Suppose $F(s) = F(s')$, where $s = a_0b_0a_1b_1a_2b_2\dots$ and $s' = a'_0b'_0a'_1b'_1a'_2b'_2\dots$. Then $a_0a_1a_2\dots = a'_0a'_1a'_2\dots$ and $b_0b_1b_2\dots = b'_0b'_1b'_2\dots$ by definition. That implies that $s = s'$ because $d(s, n) = d(s', n)$ for every $n \in \mathbb{N}$. Hence F is injective.

For each pair $(a_0a_1a_2\dots, b_0b_1b_2\dots) \in A \times A$, let $s = a_0b_0a_1b_1a_2b_2\dots$. Then $s \in A$ and $F(s) = (a_0a_1a_2\dots, b_0b_1b_2\dots)$ by definition. Hence F is surjective.

The injectivity and surjectivity of F imply that F is a bijection. Thus $A \approx A \times A$. $\square$