Exercise 3

陈志杰 524531910034

Problem 1.

Proof. First suppose $R_1 \subseteq R_2$. Let $[a]_{R_1} \in P_1$, where $a \in A$. For any $x \in [a]_{R_1}$, aR_2x because aR_1x and $R_1 \subseteq R_2$. Thus $x \in [a]_{R_2}$, which implies that $[a]_{R_1} \subseteq [a]_{R_2}$. Then P_1 is a refinement of P_2 because $[a]_{R_2} \in P_2$.

Now suppose P_1 is a refinement of P_2. For any $(a, b) \in R_1$, $a, b \in [a]_{R_1}$. Because $[a]_{R_1} \in P_1$, $\exists [c]_{R_2} \in P_2$ such that $[a]_{R_1} \subseteq [c]_{R_2}$. Thus $a, b \in [c]_{R_2}$, which implies that cR_2a and cR_2b, which implies that aR_2c. Thus aR_2b. That implies that $R_1 \subseteq R_2$, completing the proof. □

Problem 2.

Proof. We prove by induction on n that $(R^n)^{-1} = (R^{-1})^n$. The base case $n = 1$ is trivial. Suppose the desired result holds for $n-1$. Then $(R^n)^{-1} = (R^{n-1} \circ R)^{-1} = R^{-1} \circ (R^{n-1})^{-1} = R^{-1} \circ (R^{-1})^{n-1} = (R^{-1})^n$, which closes the induction. Because R is symmetric, $R = R^{-1}$. Thus $(R^n)^{-1} = (R^{-1})^n = R^n$, which implies that R^n is symmetric. □

Lemma 1. *Suppose $R, S \subseteq A \times A$ and $R \circ S \subseteq S$. Then $\forall T \subseteq S, R \circ T \subseteq S$.*

Proof. $\forall (a, b) \in R \circ T, \exists c, aTc$ and cRb. Then aSc and cRb, which implies that $(a, b) \in R \circ S \subseteq S$. Thus $R \circ T \subseteq S$. □

Lemma 2. *Suppose $R, S \subseteq A \times A$ and $R \circ S \subseteq S$. Then $\forall n \in \mathbb{Z}^+, R^n \circ S \subseteq S$.*

Proof. We use induction on n. The base case $n = 1$ is trivial. Suppose the desired result holds for $n - 1$. Then $R^n \circ S = R \circ (R^{n-1} \circ S)$ (by associativity) $\subseteq S$ (by Lemma 1). □

Problem 3.

Proof. Suppose $R \circ S \subseteq S$. For any $(a, b) \in R^+ \circ S$, $\exists c, aSc$ and cR^+b, which implies that $\exists n$, cR^nb. Thus $(a, b) \in R^n \circ S \subseteq S$ by Lemma 2.

Suppose $R^+ \circ S \subseteq S$. For any $(a, b) \in R \circ S$, $\exists c, aSc$ and cRb, which implies that cR^+b. Thus $(a, b) \in R^+ \circ S \subseteq S$, completing the proof. □

Lemma 3. *If $R \in A \times A$ is transitive, then $R^+ = R$.*

Proof. $R \subseteq R$. R is transitive. For every transitive S such that $S \supseteq R$, $R \subseteq S$ trivially. Thus by definition $R^+ = R$. □

Problem 4.

Proof. R^+ is transitive by definition. By Lemma 3, $(R^+)^+ = R^+$. □

Problem 5.

Proof. Because $S \subseteq R^+$ and R^+ is transitive, $S^+ \subseteq R^+$. We prove by induction on n that $R^n \subseteq S^n$. The base case $n = 1$ is trivial. Suppose the desired result holds for $n - 1$. Then $\forall (a, b) \in R^n = R^{n-1} \circ R, \exists c, aRc$ and $cR^{n-1}b$. Then aSc and $cS^{n-1}b$ follow from $R \subseteq S$ and induction hypothesis. Thus aS^nb, which closes the induction. Because $R^n \subseteq S^n$ for all $n \in \mathbb{Z}^+$, $R^+ \subseteq S^+$ by definition, which, together with $S^+ \subseteq R^+$, proves that $R^+ = S^+$. □

Problem 6.

Proof. (a) $\forall a \in \mathbb{R}$, trivially there is no integer in $(a, a]$. Thus R is reflexive. For all $(a,b),(b,c) \in R$, there is no integer in $(b,a]$ or $(c,b]$. Thus there is no integer between $(b,a] \cup (c,b] = (c,a]$, which implies that aRc. Thus R is transitive. It can be verified that $\frac{1}{3}R\frac{2}{3}$ and $\frac{2}{3}R\frac{1}{3}$, but $\frac{1}{3} \neq \frac{2}{3}$. Thus R is not antisymmetric. That completes the proof.

(b) R is transitive by definition. By Lemma 3, $R^+ = R$.

(c) $\forall a \in A, aRa$ and $aR^{-1}a$ by reflexivity. Thus $(a,a) \in R \cap R^{-1}$. Thus $R \cap R^{-1}$ is reflexive.

$\forall (a,b) \in R \cap R^{-1}$, aRb and $aR^{-1}b$, which respectively imply that $bR^{-1}a$ and bRa. Thus $(b,a) \in R \cap R^{-1}$, which implies that $R \cap R^{-1}$ is symmetric.

Suppose $(a,b),(b,c) \in R \cap R^{-1}$. $(a,b),(b,c) \in R$ implies by transitivity that $(a,c) \in R$. $(a,b),(b,c) \in R^{-1}$ implies that $(b,a),(c,b) \in R$, which implies by transitivity that $(c,a) \in R$, which implies that $(a,c) \in R^{-1}$. Thus $(a,c) \in R \cap R^{-1}$, which implies that $R \cap R^{-1}$ is transitive.

Because $R \cap R^{-1}$ is reflexive, symmetric and transitive, it is an equivalence relation on A.

(d) $\forall [a]_{R\cap R^{-1}} \in B$, aRa implies that $([a]_{R\cap R^{-1}}, [a]_{R\cap R^{-1}}) \in S$. Thus S is reflexive.

Suppose $([a]_{R\cap R^{-1}}, [b]_{R\cap R^{-1}}), ([b]_{R\cap R^{-1}}, [a]_{R\cap R^{-1}}) \in S$. Then aRb and bRa by definition. [1] Thus $(a,b) \in R \cap R^{-1}$, which implies that $[a]_{R\cap R^{-1}} = [b]_{R\cap R^{-1}}$. Thus S is antisymmetric.

For all $([a]_{R\cap R^{-1}}, [b]_{R\cap R^{-1}}), ([b']_{R\cap R^{-1}}, [c]_{R\cap R^{-1}}) \in S$ where $[b]_{R\cap R^{-1}} = [b']_{R\cap R^{-1}}$, we have $([b]_{R\cap R^{-1}}, [c]_{R\cap R^{-1}}) \in S$. Thus aRb and bRc by definition. Thus aRc because R is transitive, which implies that $([a]_{R\cap R^{-1}}, [c]_{R\cap R^{-1}}) \in S$. That implies that S is transitive.

Because S is reflexive, antisymmetric and transitive, it is a partial ordering on B. □

Problem 7.

Proof. Let $R = \{(x,y) \in A \times A : F(x) = F(y)\}$. $\forall x \in A$, $F(x) = F(x)$, which implies that xRx. Thus R is reflexive. $\forall (x,y) \in R, F(x) = F(y)$. Thus $F(y) = F(x)$, which implies that yRx. Thus R is symmetric. $\forall (x,y),(y,z) \in R, F(x) = F(y) = F(z)$. Then xRz follows from $F(x) = F(z)$, which implies that R is transitive. Because R is reflexive, symmetric and transitive, it is an equivalence relation. □

Problem 8.

Proof. Let $G = \{([x]_R, F(x)) : x \in a\}$ and $P = \{[x]_R : x \in A\}$.

First suppose $G : P \to B$. For any $x, y \in A$ such that xRy, $[x]_R = [y]_R$. Thus $F(x) = G([x]_R) = G([y]_R) = F(y)$.

Now suppose that for any $x, y \in A$ such that xRy, $F(x) = F(y)$. Suppose $[x]_R \in P$. Then there exists an element $F(x)$ in B such that $([x]_R, F(x)) \in G$. Suppose that there are 2 elements $a, b \in B$ such that $([x]_R, a), ([x]_R, b) \in G$. Then $\exists y, z \in [x]_R, F(y) = a, F(z) = b$. Because xRy and xRz, we have yRx. Then yRz by transitivity of R, which implies that $a = F(y) = F(z) = b$. That proves that G is a function from P to B. That completes the proof. □

[1] Erratum: there are similar mistakes to those in hw2.