Exercise 10

陈志杰 524531910034

Problem 1.

Solution. Let $\mathcal{J}$ be the truth assignment such that $\mathcal{J}(p) = \mathbf{T}, \mathcal{J}(q) = \mathbf{F}, \mathcal{J}(r) = \mathbf{F}$. Then $[\![(p \wedge q) \to r]\!]_{\mathcal{J}} = \mathbf{T}$. However, $[\![p \to r]\!]_{\mathcal{J}} = \mathbf{F}, [\![q \to r]\!]_{\mathcal{J}} = \mathbf{T}$, which implies that $[\![(p \to r) \wedge (q \to r)]\!]_{\mathcal{J}} = \mathbf{F}$. Thus $(p \wedge q) \to r$ and $(p \to r) \wedge (q \to r)$ have different truth values under $\mathcal{J}$, proving that the two propositions are not logically equivalent. □

Problem 2.

Solution.

- Let $\mathfrak{A} = (A, \mathfrak{a})$ be an S-structure where $A = \{0, 1\}$, $\mathfrak{a}(P) = P^{\mathfrak{A}}$ is such that $P^{\mathfrak{A}}(0) = \mathbf{F}, P^{\mathfrak{A}}(1) = \mathbf{T}$, and $\mathfrak{a}(Q) = Q^{\mathfrak{A}}$ is such that $Q^{\mathfrak{A}}(x) = \mathbf{F}$ for all $x \in A$. Let $\mathfrak{I} = (\mathfrak{A}, \beta)$ be an S-interpretation where β is some fixed assignment in $\mathfrak{A}$.
- Let $a = 1$. Then $[\![P(x) \to Q(x)]\!]_{\mathfrak{I}[x \mapsto 1]} = \mathbf{F}$ follows from that $\mathfrak{I}[x \mapsto 1](P)(1) = \mathbf{T}$ and $\mathfrak{I}[x \mapsto 1](Q)(1) = \mathbf{F}$. Thus $[\![\forall x(P(x) \to Q(x))]\!]_{\mathfrak{I}} = \mathbf{F}$.
- $[\![\forall x(P(x))]\!]_{\mathfrak{I}} = \mathbf{F}$ follows from that $[\![P(x)]\!]_{\mathfrak{I}[x \mapsto 0]} = \mathbf{F}$. That implies that $[\![\forall x(P(x)) \to \forall x(Q(x))]\!]_{\mathfrak{I}} = \mathbf{T}$.

The two propositions have different truth values on $\mathfrak{I}$, and hence are not logically equivalent. □

Problem 3.

Proof. We use proof by contradiction and assume that the proposition is false on interpretation $\mathfrak{I}$ with domain A.

- There does not exist $a \in A$ such that $[\![(P(x) \to \forall y(P(y)))]\!]_{\mathfrak{I}[x \mapsto a]} = \mathbf{T}$.
- For any $a \in A$, $[\![(P(x) \to \forall y(P(y)))]\!]_{\mathfrak{I}[x \mapsto a]} = \mathbf{F}$.
- For any $a \in A$, $[\![P(x)]\!]_{\mathfrak{I}[x \mapsto a]} = \mathbf{T}$ and $[\![\forall y(P(y))]\!]_{\mathfrak{I}[x \mapsto a]} = \mathbf{F}$, i.e.,
  - for any $a \in A$, $[\![P(x)]\!]_{\mathfrak{I}[x \mapsto a]} = \mathbf{T}$;
    - $[\![\forall x(P(x))]\!]_{\mathfrak{I}} = \mathbf{T}$. $(\star)$
  - for any $a \in A$, $[\![\forall y(P(y))]\!]_{\mathfrak{I}[x \mapsto a]} = \mathbf{F}$.
    - Apply $a = \mathfrak{I}(x)$. $[\![\forall y(P(y))]\!]_{\mathfrak{I}[x \mapsto \mathfrak{I}(x)]} = \mathbf{F}$, i.e., $[\![\forall y(P(y))]\!]_{\mathfrak{I}} = \mathbf{F}$.
    - This does not hold: for all $b \in A$, $[\![P(y)]\!]_{\mathfrak{I}[y \mapsto b]} = \mathbf{T}$.
    - This does not hold: for all $b \in A$, $\mathfrak{I}(P)(b) = \mathbf{T}$.
    - This does not hold: for all $b \in A$, $[\![P(x)]\!]_{\mathfrak{I}[x \mapsto b]} = \mathbf{T}$.
    - $[\![\forall x(P(x))]\!]_{\mathfrak{I}} = \mathbf{F}$. $(\star)$

The two $(\star)$'s trigger a contradiction. Thus the proposition in question is true under any interpretation. □

Remark. In the following two problems, when claiming "no logic", we need to additionally write

Logic in the bigger picture: if $\Phi \vdash \psi$ and $\psi \vdash \phi$, then $\Phi \vdash \phi$.

Otherwise it is considered wrong by Q. Cao. TAs actually do not have clear criteria for these problems; same answers from different students did receive different scores.

Problem 4.

Solution. (a) If Φ does not talk about x, and $\Phi, \psi \vdash \phi$, then $\Phi \vdash \forall x(\psi \to \phi)$.

(b) No logic. It is a mathematical definition.

(c) No logic. It is a mathematical fact.

□

Problem 5.

Solution. (a) If Φ and ψ do not talk about x, and $\Phi, \phi \vdash \psi$, then $\Phi, \exists x \phi \vdash \psi$.

(b) No logic. It is a mathematical fact.

□