

# Notes – Linear Algebra

Zhijie Chen

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Don't just read it; fight it!  
Ask your own questions.  
Look for your own examples.  
Discover your own proofs.  
Is the hypothesis necessary?  
Is the converse true?  
What happens in the classical special case?  
What about the degenerate cases?  
Where does the proof use the hypothesis?

— Paul Holmos

# Linear Maps

## Products and Quotients of Vector Spaces

**Proposition 1.** *A nonempty subset  $A$  of  $V$  is a translate of some subspace of  $V$  if and only if  $\lambda v + (1 - \lambda)w \in A$  for all  $v, w \in A$  and all  $\lambda \in \mathbf{F}$ .*

**Exercise 2.** Suppose  $U$  is a subspace of  $V$  such that  $V/U$  is finite-dimensional.

- (a) Prove that if  $W$  is a finite-dimensional subspace of  $V$  and  $V = U + W$ , then  $\dim W \geq \dim V/U$ .
- (b) Prove that there exists a finite-dimensional subspace  $W$  of  $V$  such that  $V = U \oplus W$  and  $\dim W = \dim V/U$ .

*Proof.* Let  $\bar{w}_1 + U, \dots, \bar{w}_m + U$  be a basis of  $V/U$ . It can be shown that  $V = \text{span}(\bar{w}_1, \dots, \bar{w}_m) \oplus U$ . Let  $W_0 = \text{span}(\bar{w}_1, \dots, \bar{w}_m)$ . Then  $V = U \oplus W_0$ , as desired.

Now we prove that for each subspace  $W$  of  $V$  such that  $V = U + W$ , we have  $\dim W \geq m = \dim V/U$ . For each  $\bar{w}_k$  above, by definition we have  $\bar{w}_k = u_k + w_k$  for some  $u_k \in U$  and  $w_k \in W$ . It can be shown from the linear independence of  $\bar{w}_1 + U, \dots, \bar{w}_m + U$  that  $\bar{w}_1 - u_1, \dots, \bar{w}_m - u_m$  are independent vectors in  $W$ . Hence  $\dim W \geq m$ .  $\square$

## Duality

**Proposition 3.** *Suppose  $V$  is finite-dimensional and  $U$  is a subspace of  $V$ . Then<sup>1</sup>*

$$U = \left\{ v \in V : \varphi(v) = 0 \text{ for every } \varphi \in U^0 \right\}.$$

<sup>1</sup> Compare this result with  $(U^\perp)^\perp = U$  where  $U$  is a finite-dimensional subspace of inner product space  $V$ .

**Exercise 4.** Suppose  $V$  is finite-dimensional and  $\varphi_1, \dots, \varphi_n$  is a basis of  $V'$ . Prove that there exists a basis of  $V$  whose dual basis is precisely  $\varphi_1, \dots, \varphi_n$ .

*Proof.* We start from an arbitrary basis  $u_1, \dots, u_n$  of  $V$ . Let  $\psi_1, \dots, \psi_n$  be its dual basis. In this proof, we take the standard basis  $e_1, \dots, e_n$  as the basis of  $\mathbf{F}^n$ .

Define  $S, T \in \mathcal{L}(V, \mathbf{F}^n)$  by

$$T(v) = (\varphi_1(v), \dots, \varphi_n(v)), \quad S(v) = (\psi_1(v), \dots, \psi_n(v)).$$

Then  $\mathcal{M}(S, (u_1, \dots, u_n)) = I$ .

Let  $A$  be the change of basis matrix from  $\psi$ 's to  $\varphi$ 's, i.e.,

$$\begin{pmatrix} \varphi_1 & \cdots & \varphi_n \end{pmatrix} = \begin{pmatrix} \psi_1 & \cdots & \psi_n \end{pmatrix} A.$$

Then by the definition of change of basis matrix, we have

$$\mathcal{M}(T, (u_1, \dots, u_n)) = \begin{pmatrix} \mathcal{M}(\varphi_1, (u_1, \dots, u_n)) \\ \vdots \\ \mathcal{M}(\varphi_n, (u_1, \dots, u_n)) \end{pmatrix} = A^t \begin{pmatrix} \mathcal{M}(\psi_1, (u_1, \dots, u_n)) \\ \vdots \\ \mathcal{M}(\psi_n, (u_1, \dots, u_n)) \end{pmatrix}$$

$$= A^t \cdot \mathcal{M}(S, (u_1, \dots, u_n)) = A^t.$$

Consider basis  $v_1, \dots, v_n$  of  $V$  such that the change of basis matrix from  $u$ 's to  $v$ 's is  $(A^t)^{-1}$ .<sup>2</sup> Thus

$$\mathcal{M}(T, (v_1, \dots, v_n)) = \mathcal{M}(T, (u_1, \dots, u_n)) \cdot \mathcal{M}(I, (v_1, \dots, v_n), (u_1, \dots, u_n)) = I.$$

Then the dual basis of  $v_1, \dots, v_n$  is precisely  $\varphi_1, \dots, \varphi_n$ , as desired.  $\square$

**Exercise 5** (A natural isomorphism from primal space onto double dual space). Define  $\Lambda : V \rightarrow V''$  by

$$(\Lambda v)(\varphi) = \varphi(v)$$

for each  $v \in V$  and  $\varphi \in V'$ .

- (a) Prove that if  $T \in \mathcal{L}(V)$ , then  $T'' \circ \Lambda = \Lambda \circ T$ .
- (b) Prove that if  $V$  is finite-dimensional, then  $\Lambda$  is an isomorphism from  $V$  onto  $V''$ .

**Remarks 6.** (a) Suppose  $V$  is finite-dimensional. Then  $V$  and  $V'$  are isomorphic, but finding an isomorphism from  $V$  onto  $V'$  generally requires choosing a basis of  $V$ . In contrast, the isomorphism  $\Lambda$  from  $V$  onto  $V''$  does not require a choice of basis and thus is more natural.

- (b) Another natural isomorphism is  $\pi' \in \mathcal{L}((V/U)', V')$  where  $\pi$  is the normal quotient map.

## Polynomials

**Theorem 7.** Suppose  $p \in \mathcal{P}(\mathbf{F})$  is a nonconstant polynomial of degree  $m$ . Then  $p$  has at most  $m$  zeros in  $\mathbf{F}$ .<sup>3</sup>

**Remark 8.** Theorem 7 implies that the coefficients of a polynomial are uniquely determined. In particular, the degree of a polynomial is well-defined.

**Theorem 9** (Division algorithm for polynomials). Suppose that  $p, s \in \mathcal{P}(\mathbf{F})$ , with  $s \neq 0$ . Then there exist unique polynomials  $q, r \in \mathcal{P}(\mathbf{F})$  such that  $p = sq + r$ .

*Proof.* Let  $n = \deg p$  and  $m = \deg s$ . The case where  $n < m$  is trivial. Thus we now assume that  $n \geq m$ .

The list

$$1, z, \dots, z^{m-1}, s, zs, \dots, z^{n-m}s$$

is linearly independent in  $\mathcal{P}_n(\mathbf{F})$ . And it also has length  $n + 1$ . Hence the list is a basis of  $\mathcal{P}_n(\mathbf{F})$ .

Because  $p \in \mathcal{P}_n(\mathbf{F})$ , there exist unique constants  $a_0, \dots, a_{m-1}, b_0, \dots, b_{n-m} \in \mathbf{F}$  such that

$$\begin{aligned} p &= a_0 + a_1z + \dots + a_{m-1}z^{m-1} + b_0s + b_1zs + \dots + b_{n-m}z^{n-m}s \\ &= (a_0 + a_1z + \dots + a_{m-1}z^{m-1}) + s(b_0 + b_1z + \dots + b_{n-m}z^{n-m}). \end{aligned} \quad \square$$

**Exercise 10.** Suppose  $p, q \in \mathcal{P}(\mathbf{C})$  are nonconstant polynomials with no zeros in common. Let  $m = \deg p$  and  $n = \deg q$ . Prove that there exist  $r \in \mathcal{P}_{n-1}(\mathbf{C})$  and  $s \in \mathcal{P}_{m-1}(\mathbf{C})$  such that  $rp + sq = 1$ .<sup>4</sup>

<sup>2</sup> *Proof Idea.* The change of basis for  $V' \rightarrow V'$  corresponds to the transpose of  $V \leftarrow V$ , where transpose and inverse both come from duality.

<sup>3</sup> A useful corollary of this theorem: when a polynomial  $p$  has too many zeros,  $p = 0$ .

<sup>4</sup> *Proof Idea.* Define  $T : \mathcal{P}_{n-1}(\mathbf{C}) \times \mathcal{P}_{m-1}(\mathbf{C}) \rightarrow \mathcal{P}_{m+n-1}(\mathbf{C})$  by  $T(r, s) = rp + sq$  and prove the surjectivity of  $T$ .

## Eigenvalues and Eigenvectors

### Invariant Subspaces

**Exercise 11.** Suppose that  $\lambda_1, \dots, \lambda_n \in \mathbf{R}$  are pairwise distinct. Prove that the list  $e^{\lambda_1 x}, \dots, e^{\lambda_n x}$  is linearly independent in  $\mathbf{R}^{\mathbf{R}}$ .

*Proof.* Let  $V = \text{span}(e^{\lambda_1 x}, \dots, e^{\lambda_n x})$ .<sup>5</sup> Define  $D \in \mathcal{L}(V)$  by  $Df = f'$ . Then  $e^{\lambda x}$  is an eigenvector of  $D$  corresponding to  $\lambda$ . A list of eigenvectors corresponding to distinct eigenvalues is linearly independent.  $\square$

<sup>5</sup> Alternatively we can let  $V = D(\mathbf{R})$ .

**Exercise 12.** Suppose  $V$  is finite-dimensional,  $T \in \mathcal{L}(V)$ , and  $U$  is a subspace of  $V$  invariant under  $T$ . Prove that each eigenvalue of the quotient operator  $T/U$  is an eigenvalue of  $T$ .

*Proof.* It suffices to show that  $T/U - \lambda I = (T - \lambda I)/U$  is not injective  $\implies T - \lambda I$  is not injective. We prove that  $T - \lambda I$  is invertible  $\implies (T - \lambda I)/U$  is injective.

Suppose  $T - \lambda I$  is invertible.  $U$  being invariant under  $T$  implies that  $U$  is invariant under  $T - \lambda I$ . Thus  $(T - \lambda I)v \in U \iff v \in U$ . Suppose  $((T - \lambda I)/U)(v + U) = 0$ . Then  $(T - \lambda I)v \in U$ , which implies that  $v \in U$ , i.e.,  $v + U = 0$ . That proves the injectivity of  $(T - \lambda I)/U$ .  $\square$

**Exercise 13.** Suppose  $V$  is finite-dimensional and  $T \in \mathcal{L}(V)$ . Prove that  $T$  has an eigenvalue if and only if there exists a subspace of  $V$  of dimension  $\dim V - 1$  that is invariant under  $T$ .<sup>6</sup>

<sup>6</sup> *Proof Idea.* Consider the zero entries in  $\mathcal{M}(T)$  and its transpose matrix  $\mathcal{M}(T')$ .

*Proof.* We first suppose that  $T$  has an eigenvalue  $\lambda$ . Then  $\lambda$  is an eigenvalue of  $T'$ . There exists  $\varphi \in V'$  such that  $\varphi \circ T = T'\varphi = \lambda\varphi$ . Extend  $\varphi$  to a basis  $\varphi, \varphi_2, \dots, \varphi_n$  of  $V'$  and let  $v, v_2, \dots, v_n$  be the basis of  $V$  whose dual basis is  $\varphi, \varphi_2, \dots, \varphi_n$ . Then  $(\varphi \circ T)v_k = \lambda\varphi(v_k) = 0$  for every  $k$ . Because  $\varphi(Tv_k) = 0$  for every  $k$ , we have  $Tv_k \in \text{span}(v_2, \dots, v_n)$ . That proves that  $\text{span}(v_2, \dots, v_n)$  is invariant under  $T$ .

Reversing the steps above leads to an eigenvector of  $T'$ , completing the proof.  $\square$

### The Minimal Polynomial

**Exercise 14** (Companion matrix of a polynomial). Suppose  $a_0, \dots, a_{n-1} \in \mathbf{F}$ . Let  $T \in \mathcal{L}(\mathbf{F}^n)$  be such that  $\mathcal{M}(T)$  (with respect to the standard basis) is

$$\begin{pmatrix} 0 & & & & -a_0 \\ 1 & 0 & & & -a_1 \\ & 1 & 0 & & -a_2 \\ & & \ddots & & \vdots \\ & & & 0 & -a_{n-2} \\ & & & 1 & -a_{n-1} \end{pmatrix}$$

Prove that the minimal polynomial of  $T$  is the polynomial

$$a_0 + a_1 z + \dots + a_{n-1} z^{n-1} + z^n.$$

**Remark 15.** This exercise implies that every monic polynomial is the minimal polynomial of some operator. Hence an algorithm that could produce exact eigenvalues for each operators on each  $\mathbf{F}^n$  does not exist.

**Exercise 16.** Prove that every operator on a finite-dimensional vector space of dimension at least 2 has an invariant subspace of dimension 2.

*Proof.* Let  $T \in \mathcal{L}(V)$  and  $n = \dim V$ . We use induction on  $n$ . The base case  $n = 2$  is trivial. Now suppose  $n > 2$  and the desired result holds for all smaller positive integers. Let  $p$  be the minimal polynomial of  $T$ .

If  $T$  has an eigenvalue  $\lambda$ , then  $p(z) = q(z)(z - \lambda)$  for some monic polynomial  $q$  with  $\deg q = \deg p - 1$ . Because  $q(T)|_{\text{im}(T - \lambda I)} = 0$ , the desired result holds by induction hypothesis if  $\dim \text{im}(T - \lambda I) \geq 2$ . If  $T - \lambda I = 0$  the desired result trivially holds. If  $\dim \text{im}(T - \lambda I) = 1$ , then  $(T - \lambda I)v$  is a scalar multiple of some fixed  $u \in V$  for all  $v \in V$ . Take  $w \in V \setminus \text{span}(u)$  and  $\text{span}(u, w)$  will satisfy the desired property.

If  $T$  has no eigenvalues, then  $\mathbf{F} = \mathbf{R}$  and  $p(z) = q(z)(z^2 + bz + c)$  for some  $b, c \in \mathbf{R}$  with  $b^2 < 4c$  and monic polynomial  $q$  with  $\deg q = \deg p - 2$ . If  $\dim \text{im}(T^2 + bT + cI) \geq 2$  the desired result holds by induction hypothesis. If  $T^2 + bT + cI = 0$ , then let  $w \in V$  be such that  $w \neq 0$ . It can be verified that  $\text{span}(w, Tw)$  is invariant under  $T$ . If  $\dim \text{im}(T^2 + bT + cI) = 1$ , because  $\dim \ker(T^2 + bT + cI)$  is even,  $n$  is odd, which implies that  $T$  has an eigenvalue. That completes the proof.  $\square$

## Commuting Operators

**Exercise 17.** Suppose  $\mathcal{E} \subseteq \mathcal{L}(V)$  and every element of  $\mathcal{E}$  is diagonalizable. Prove that there exists a basis of  $V$  with respect to which every element of  $\mathcal{E}$  has a diagonal matrix if and only if every pair of elements of  $\mathcal{E}$  commutes.<sup>7</sup>

*Proof.* Suppose every pair of elements of  $\mathcal{E}$  commutes. We use induction on  $n = \dim V$ . The base case  $n = 1$  is trivial. Now suppose  $n > 1$  and the desired result holds for all smaller integers. Without loss of generality, suppose  $\mathcal{E} \cap \{\lambda I : \lambda \in \mathbf{F}\} = \emptyset$ , or else consider  $\mathcal{E} \setminus \{\lambda I : \lambda \in \mathbf{F}\}$ .

Let  $\lambda_1, \dots, \lambda_m$  be the distinct eigenvalues of  $T \in \mathcal{E}$ . Then  $V = E(\lambda_1, T) \oplus \dots \oplus E(\lambda_m, T)$ . Because  $E(\lambda_k, T)$  is invariant under every  $S \in \mathcal{E}$ , it suffices to show that the desired result holds on  $E(\lambda_1, T)$ . Because  $E(\lambda_1, T) \subsetneq V$ , it holds by induction hypothesis. The other direction is trivial.  $\square$

**Exercise 18.** Suppose  $V$  is a finite-dimensional nonzero complex vector space. Suppose that  $\mathcal{E} \subseteq \mathcal{L}(V)$  is such that  $S$  and  $T$  commute for all  $S, T \in \mathcal{E}$ .

- Prove that there is a vector in  $V$  that is an eigenvector for every element of  $\mathcal{E}$ .
- Prove that there exists a basis of  $V$  with respect to which every element of  $\mathcal{E}$  has an upper-triangular matrix.<sup>8</sup>

<sup>7</sup> This exercise extends simultaneous diagonalizability to more than 2 (possibly infinitely many) operators.

<sup>8</sup> This exercise extends simultaneous upper triangularizability to more than 2 (possibly infinitely many) operators.

## Inner Product Spaces

### Inner Products and Norms

**Theorems 19** (Polarization identities). (a) Suppose  $V$  is a real inner product space. Then

$$\langle u, v \rangle = \frac{\|u + v\|^2 - \|u - v\|^2}{4}.$$

(b) Suppose  $V$  is a complex inner product space. Then

$$\langle u, v \rangle = \frac{\|u + v\|^2 - \|u - v\|^2 + \|u + iv\|^2 i - \|u - iv\|^2 i}{4}.$$

**Exercise 20.** Prove that if  $\|\cdot\|$  is a norm on  $U$  satisfying the parallelogram equality, then there is an inner product  $\langle \cdot, \cdot \rangle$  on  $U$  such that  $\|u\| = \langle u, u \rangle^{1/2}$  for all  $u \in U$ .<sup>9</sup>

<sup>9</sup> Recall Theorems 19.

### Orthonormal Bases

**Exercise 21.** Suppose  $v_1, \dots, v_m$  is a linearly independent list in  $V$ . Prove that the orthonormal list produced by the formulas of the Gram-Schmidt procedure is the only orthonormal list  $e_1, \dots, e_m$  in  $V$  such that  $\langle v_k, e_k \rangle > 0$  and  $\text{span}(v_1, \dots, v_k) = \text{span}(e_1, \dots, e_k)$  for each  $k$ .

**Exercise 22.** Suppose  $v_1, \dots, v_n$  is a basis of  $V$ . Prove that there exists a basis  $u_1, \dots, u_n$  of  $V$  such that<sup>10</sup>

$$\langle u_j, v_k \rangle = \begin{cases} 0 & \text{if } j \neq k, \\ 1 & \text{if } j = k. \end{cases}$$

<sup>10</sup> *Proof Idea.* Linearity in inner product and patterns of 0, 1 inspire the use of linear functionals.

*Proof.* Define  $\varphi_k \in V'$  for each  $k$  by  $\varphi_k(u) = \langle u, v_k \rangle$ . By the Riesz representation theorem,  $\varphi_1, \dots, \varphi_n$  is a spanning list in  $V'$ . Thus it is a basis of  $V'$ . Let  $u_1, \dots, u_n$  be the basis of  $V$  whose dual basis is  $\varphi_1, \dots, \varphi_n$ . Then  $u_1, \dots, u_n$  satisfies the desired property.  $\square$

### Orthogonal Complements and Minimization Problems

**Theorem 23** (Riesz representation theorem). Suppose  $V$  is finite-dimensional. For each  $v \in V$ , define  $\varphi_v \in V'$  by

$$\varphi_v(u) = \langle u, v \rangle$$

for each  $u \in V$ . Then  $v \mapsto \varphi_v$  is a one-to-one map from  $V$  onto  $V'$ .

*Proof Idea.* If  $\varphi(u) = \langle u, v \rangle$  holds for all  $u \in V$ , then  $v \in (\ker \varphi)^\perp$ . However,  $(\ker \varphi)^\perp$  has dimension 1 (except when  $\varphi = 0$ ). Hence we can obtain the right  $v$  by choosing an arbitrary nonzero  $w \in (\ker \varphi)^\perp$  and then multiplying by an appropriate scalar.  $\square$

**Exercise 24.** Suppose  $V$  is finite-dimensional and  $P \in \mathcal{L}(V)$  is such that  $P^2 = P$ .

- (a) Suppose every vector in  $\ker P$  is orthogonal to every vector in  $\operatorname{im} P$ . Prove that there exists a subspace  $U$  of  $V$  such that  $P = P_U$ .<sup>11</sup>
- (b) Suppose  $\|Pv\| \leq \|v\|$  for every  $v \in V$ . Prove that there exists a subspace  $U$  of  $V$  such that  $P = P_U$ .<sup>12</sup>

**Propositions 25** (Algebraic properties of the pseudoinverse). *Suppose  $V$  is finite-dimensional and  $T \in \mathcal{L}(V, W)$ .*

- (a) *The pseudoinverse of an orthogonal projection is the operator itself.*
- (b)  $\ker T^\dagger = (\operatorname{im} T)^\perp$  and  $\operatorname{im} T^\dagger = (\ker T)^\perp$ .
- (c)  $TT^\dagger T = T$  and  $T^\dagger TT^\dagger = T^\dagger$ .
- (d)  $(T^\dagger)^\dagger = T$ .

<sup>11</sup> *Proof Idea.* Observe that  $\ker P = (\operatorname{im} P)^\perp$ . Thus we prove that  $P$  and  $P_{\operatorname{im} P}$  agree on  $\ker P$  and  $\operatorname{im} P$ .

<sup>12</sup> *Proof Idea.* Observe that among all  $v$ 's with the same  $Pv$ , the one in  $(\ker P)^\perp$  is the shortest. Applying this  $v$  makes the best use of the inequality. Thus we prove that  $P$  and  $P_{(\ker P)^\perp}$  agree on  $\ker P$  and  $(\ker P)^\perp$ .

## Operators on Inner Product Spaces

### Self-Adjoint and Normal Operators

**Exercise 26.** Suppose  $T \in \mathcal{L}(V)$  and  $\lambda \in \mathbb{F}$ . Prove that

$$\lambda \text{ is an eigenvalue of } T \iff \bar{\lambda} \text{ is an eigenvalue of } T^*.$$

**Exercise 27.** Suppose  $T \in \mathcal{L}(V)$  and  $U$  is a subspace of  $V$ . Prove that

$$U \text{ is invariant under } T \iff U^\perp \text{ is invariant under } T^*.$$

**Exercise 28.** Suppose  $T \in \mathcal{L}(V)$  is normal. Prove that

$$\ker T^k = \ker T \quad \text{and} \quad \operatorname{im} T^k = \operatorname{im} T.$$

**Exercise 29.** Suppose  $T \in \mathcal{L}(V, W)$ . Prove that under the standard identification of  $V$  with  $V'$  (by Riesz representation theorem) and  $W$  with  $W'$ , the adjoint map  $T^*$  corresponds to the dual map  $T'$ . More precisely, prove that

$$T'(\varphi_w) = \varphi_{T^*w}$$

for all  $w \in W$ .

*Remark 30.* Furthermore, under this identification of  $V$  with  $V'$ , the orthogonal complement corresponds to the annihilator; the formulas for  $\ker T^*$  and  $\operatorname{im} T^*$  become identical to the formulas for  $\ker T'$  and  $\operatorname{im} T'$ . Note that orthogonal complements and adjoints are easier to deal with than annihilators and dual maps.

### Spectral Theorem

**Exercise 31.** (a) Suppose  $\mathbb{F} = \mathbb{C}$  and  $\mathcal{E} \subseteq \mathcal{L}(V)$ . Prove that there is an orthonormal basis of  $V$  with respect to which every element of  $\mathcal{E}$  has a diagonal matrix if and only if  $S$  and  $T$  are commuting normal operators for all  $S, T \in \mathcal{E}$ .

(b) Suppose  $\mathbb{F} = \mathbb{R}$  and  $\mathcal{E} \subseteq \mathcal{L}(V)$ . Prove that there is an orthonormal basis of  $V$  with respect to which every element of  $\mathcal{E}$  has a diagonal matrix if and only if  $S$  and  $T$  are commuting self-adjoint operators for all  $S, T \in \mathcal{E}$ .<sup>13</sup>

<sup>13</sup> This exercise extends the spectral theorems to the context of a collection of commuting operators.



## Positive Operators

**Exercise 32.** For  $T \in \mathcal{L}(V)$  and  $u, v \in V$ , define  $\langle u, v \rangle_T$  by  $\langle u, v \rangle_T = \langle Tu, v \rangle$ .

- (a) Suppose  $T \in \mathcal{L}(V)$ . Prove that  $\langle \cdot, \cdot \rangle_T$  is an inner product on  $V$  if and only if  $T$  is an invertible positive operator (with respect to the original inner product).
- (b) Prove that every inner product on  $V$  is of the form  $\langle \cdot, \cdot \rangle_T$  for some invertible positive operator  $T \in \mathcal{L}(V)$ .

*Proof Idea.* (b) Let  $\langle \cdot, \cdot \rangle_1$  be an inner product on  $V$ . We need to find  $T$  such that  $\langle u, v \rangle_1 = \langle Tu, v \rangle$ . Fixing  $v$  is not sufficient to find the unique  $Tu$ . Thus we fix  $u$  and consider  $\langle v, u \rangle_1 = \langle v, Tu \rangle$ . The map  $v \mapsto \langle v, u \rangle_1$  is in  $V'$ . Hence we use the Riesz representation theorem to define  $Tu$ .  $\square$

## Singular Value Decomposition

**Exercise 33.** Suppose  $T \in \mathcal{L}(V, W)$  and  $s > 0$ . Prove that  $s$  is a singular value of  $T$  if and only if there exist nonzero vectors  $v \in V$  and  $w \in W$  such that<sup>14</sup>

$$Tv = sw \quad \text{and} \quad T^*w = sv.$$

<sup>14</sup> The vectors  $v, w$  satisfying both equations above are called a *Schmidt pair*.

**Exercise 34.** Suppose  $T \in \mathcal{L}(V, W)$ . Suppose  $s_1 \geq s_2 \geq \cdots \geq s_m > 0$  and  $e_1, \dots, e_m$  is an orthonormal list in  $V$  and  $f_1, \dots, f_m$  is an orthonormal list in  $W$  such that

$$Tv = s_1 \langle v, e_1 \rangle f_1 + \cdots + s_m \langle v, e_m \rangle f_m$$

for every  $v \in V$ .

- (a) Prove that  $s_1, \dots, s_m$  are the positive singular values of  $T$ .
- (b) Prove that if  $k \in \{1, \dots, m\}$ , then  $e_k$  is an eigenvector of  $T^*T$  with corresponding eigenvalue  $s_k^2$ .

**Exercise 35.** Suppose  $T \in \mathcal{L}(V, W)$ . Let  $n = \dim V$  and let  $s_1 \geq \cdots \geq s_n$  denote the singular values of  $T$ . Prove that if  $1 \geq k \geq n$ , then

$$\min\{\|T|_U\| : U \text{ is a subspace of } V \text{ with } \dim U = k\} = s_{n-k+1}.$$

*Remark 36.* Compare the result above to that

$$\min\{\|T - S\| : S \in \mathcal{L}(V, W) \text{ and } \dim \operatorname{im} S \leq k\} = s_{k+1}.$$

**Exercise 37.** Suppose  $T \in \mathcal{L}(V)$ ,  $S \in \mathcal{L}(V)$  is a unitary operator, and  $R \in \mathcal{L}(V)$  is a positive operator such that  $T = SR$ . Prove that  $R = \sqrt{T^*T}$ .

## Operators on Complex Vector Spaces

### Generalized Eigenvectors and Nilpotent Operators

**Exercise 38.** Suppose  $T \in \mathcal{L}(V)$ .  $m \in \mathbb{Z}^+$  is such that the minimal polynomial of  $T$  is a polynomial multiple of  $(z - \lambda)^m$ . Prove that

$$\dim \ker(T - \lambda I)^m \geq m.$$

*Proof.* Let the minimal polynomial of  $T$  be  $(z - \lambda)^m q(z)$ . Observe that  $m$  is the least integer (If  $k < m$  satisfies the property then  $(T - \lambda I)^k q(T) = 0$ .) such that

$$(T - \lambda I)^m|_{\text{im } q(T)} = 0.$$

Thus  $T - \lambda I$  is nilpotent on  $\text{im } q(T)$ , which implies that  $m \leq \dim \text{im } q(T)$ .

On the other hand,

$$q(T)|_{\text{im}(T - \lambda I)^m} = 0.$$

Thus  $\ker q(T) \supseteq \text{im}(T - \lambda I)^m$ . Hence

$$\dim \ker(T - \lambda I)^m = n - \dim \text{im}(T - \lambda I)^m \geq n - \dim \ker q(T) = \dim \text{im } q(T) \geq m.$$

□

**Proposition 39.** Suppose that  $T \in \mathcal{L}(V)$ . Then there is a basis consisting of generalized eigenvectors of  $T$  if and only if the minimal polynomial of  $T$  equals  $(z - \lambda_1) \dots (z - \lambda_m)$  for some  $\lambda_1, \dots, \lambda_m \in \mathbb{F}$ .

## Generalized Eigenspace Decomposition

**Exercise 40.** Suppose  $T \in \mathcal{L}(V)$  and  $\lambda$  is an eigenvalue of  $T$ . Prove that the exponent of  $z - \lambda$  in the factorization of the minimal polynomial of  $T$  is the least positive integer  $m$  such that  $(T - \lambda I)^m|_{G(\lambda, T)} = 0$ .

**Exercise 41.** Suppose  $T \in \mathcal{L}(V)$  and  $\lambda$  is an eigenvalue of  $T$  with multiplicity  $d$ . Prove that  $G(\lambda, T) = \ker(T - \lambda I)^d$ .

**Exercise 42.** Suppose  $\mathbb{F} = \mathbb{C}$  and  $T \in \mathcal{L}(V)$ . Prove that there exist  $D, N \in \mathcal{L}(V)$  such that  $T = D + N$ ,  $D$  is diagonalizable,  $N$  is nilpotent, and  $DN = ND$ .<sup>15</sup>

<sup>15</sup> The generalized eigenspace decomposition allows us to restrict our attention to a single generalized eigenspace.

## Trace: A Connection Between Matrices and Operators

**Exercise 43.** Prove that the trace is the only linear functional  $\tau : \mathcal{L}(V) \rightarrow \mathbb{F}$  such that

$$\tau(ST) = \tau(TS)$$

for all  $S, T \in \mathcal{L}(V)$  and  $\tau(I) = \dim V$ .

*Proof Idea.* Define  $P_{j,k} \in \mathcal{L}(V)$  by  $P_{j,k}(a_1 v_1 + \dots + a_n v_n) = a_k v_j$ , where  $v_1, \dots, v_n$  is a basis of  $V$ . Prove that  $\tau(P_{j,k}) = 1$  if  $j = k$  and 0 otherwise. □

*Proof.* Taking  $S = P_{j,j}$ ,  $T = P_{j,k}$  leads to the desired result for  $j \neq k$ . For  $P_{j,j}$ , consider its matrix version and use  $\tau(D) = \tau(P^{-1}DP)$  where  $P$  is a permutation matrix. □

**Exercise 44.** Suppose  $V$  and  $W$  are finite-dimensional inner product spaces.

(a) Prove that

$$\text{tr}(T^*T) = \sum_{k=1}^n \sum_{j=1}^m |\langle T e_k, f_j \rangle|^2,$$

where  $e_1, \dots, e_n$  is an orthonormal basis of  $V$  and  $f_1, \dots, f_m$  is an orthonormal basis of  $W$ .

- (b) Prove that  $\langle S, T \rangle = \text{tr}(T^*S)$  defines an inner product on  $\mathcal{L}(V, W)$ , and it is the same as the standard inner product on  $\mathbf{F}^{mn}$ .

*Remarks 45.* The right-hand side of (a) is the sum of the squares of the absolute values of the matrix entries, and does not depend on the chosen orthonormal bases. The norm on  $\mathcal{L}(V, W)$  induced by (b) is called the *Frobenius norm* or the *Hilbert-Schmidt norm*.

## Multilinear Algebra and Determinants

### Determinants

**Proposition 46** (Vandermonde matrix). *Suppose  $n > 1$  and  $\beta_1, \dots, \beta_n \in \mathbf{F}$ . Then*

$$\det \begin{pmatrix} 1 & \beta_1 & \beta_1^2 & \cdots & \beta_1^{n-1} \\ 1 & \beta_2 & \beta_2^2 & \cdots & \beta_2^{n-1} \\ & & \ddots & & \\ 1 & \beta_n & \beta_n^2 & \cdots & \beta_n^{n-1} \end{pmatrix} = \prod_{1 \leq j < k \leq n} (\beta_k - \beta_j).$$

*Proof.* Let  $e_1, \dots, e_n$  denote the standard basis of  $\mathbf{F}^n$ . Then

$$A = \mathcal{M}(S, (1, z, \dots, z^{n-1}), (e_1, \dots, e_n)),$$

where  $A$  is the Vandermonde matrix and  $S \in \mathcal{L}(\mathcal{P}_{n-1}(\mathbf{F}), \mathbf{F}^n)$  is defined by  $Sp = (p(\beta_1), \dots, p(\beta_n))$ .

Let  $T \in \mathcal{L}(\mathcal{P}_{n-1}(\mathbf{F}))$  be such that  $T1 = 1$  and

$$Tz^k = (z - \beta_1) \cdots (z - \beta_k)$$

for  $k = 1, \dots, n-1$ . Then  $B = \mathcal{M}(T, (1, z, \dots, z^{n-1}))$  has determinant 1. Now

$$\det A = (\det A)(\det B) = \det \mathcal{M}(ST, (1, z, \dots, z^{n-1}), (e_1, \dots, e_n)).$$

Calculate  $C$  and we get the desired result.  $\square$

**Exercise 47.** Suppose  $n$  is a positive integer and  $\delta : \mathbf{C}^{n,n} \rightarrow \mathbf{C}$  is a function such that

$$\delta(AB) = \delta(A) \cdot \delta(B)$$

for all  $A, B \in \mathbf{C}^{n,n}$  and  $\delta(A)$  equals the product of the diagonal entries of  $A$  for each diagonal matrix  $A \in \mathbf{C}^{n,n}$ . Prove that

$$\delta(A) = \det A$$

for all  $A \in \mathbf{C}^{n,n}$ .<sup>16</sup>

<sup>16</sup> *Proof Idea.* Use the Gaussian elimination and diagonalization.

### Tensor Products

**Exercise 48.** Suppose  $m$  and  $n$  are positive integers. For  $v \in \mathbf{F}^m$  and  $w \in \mathbf{F}^n$ , identify  $v \otimes w$  with an  $m$ -by- $n$  matrix. With that identification, show that the set

$$\{v \otimes w : v \in \mathbf{F}^m \text{ and } w \in \mathbf{F}^n\}$$

is the set of  $m$ -by- $n$  matrices that have rank at most one.

**Exercise 49.** Suppose  $\dim V > 1$  and  $\dim W > 1$ . Prove that

$$\{v \otimes w : (v, w) \in (V, W)\}$$

is not a subspace of  $V \otimes W$ .

## Complexification

**Propositions 50** (Properties of complexification). (a)  $\lambda \in \mathbf{R}$  is an eigenvalue of  $T$  if and only if  $\lambda$  is an eigenvalue of  $T_{\mathbf{C}}$ .

(b)  $\lambda \in \mathbf{C}$  is an eigenvalue of  $T_{\mathbf{C}}$  if and only if  $\bar{\lambda}$  is an eigenvalue of  $T_{\mathbf{C}}$ .

(c) The minimal polynomial of  $T_{\mathbf{C}}$  equals the minimal polynomial of  $T$ .

(d) Suppose  $V$  is a real inner product space. For  $u, v, w, x \in V$ , define

$$\langle u + iv, w + ix \rangle_{\mathbf{C}} = \langle u, w \rangle + \langle v, x \rangle + (\langle v, w \rangle - \langle u, x \rangle)i.$$

Then  $\langle \cdot, \cdot \rangle_{\mathbf{C}}$  makes  $V_{\mathbf{C}}$  into a complex inner product space. If  $u, v \in V$ , then

$$\langle u, v \rangle_{\mathbf{C}} = \langle u, v \rangle \quad \text{and} \quad \|u + iv\|_{\mathbf{C}}^2 = \|u\|^2 + \|v\|^2.$$

**Proposition 51.** Every operator on a finite-dimensional nonzero vector space has an invariant subspace of dimension 1 or 2.

*Proof.* The case where  $\mathbf{F} = \mathbf{C}$  is trivial. Now assume  $V$  is a real vector space and  $T \in \mathcal{L}(V)$ . Then  $T_{\mathbf{C}}$ <sup>17</sup> has an eigenvalue  $a + bi$  with  $a, b \in \mathbf{R}$ . Thus there exist  $u, v \in V$ , not both 0, such that

$$Tu + iTv = (au - bv) + (av + bu)i.$$

Hence  $\text{span}(u, v)$  is invariant under  $T$ , as desired.  $\square$

**Exercise 52.** Suppose  $\mathbf{F} = \mathbf{R}$ ,  $T \in \mathcal{L}(V)$ , and  $\lambda \in \mathbf{C}$ .

(a) Prove that  $u + iv \in G(\lambda, T_{\mathbf{C}})$  if and only if  $u - iv \in G(\bar{\lambda}, T_{\mathbf{C}})$ .

(b) Prove that the multiplicity of  $\lambda$  as an eigenvalue of  $T_{\mathbf{C}}$  equals that of  $\bar{\lambda}$ .

(c) Use (b) to show that if  $\dim V$  is an odd number, then  $T_{\mathbf{C}}$  has a real eigenvalue.

(d) Use (c) to show that if  $\dim V$  is an odd number, then  $T$  has an eigenvalue.

## Primary Decomposition

**Exercise 53** (Primary Decomposition). (a) Suppose  $f, g$  are coprime polynomials and  $T \in \mathcal{L}(V)$ . Prove that<sup>18</sup>

$$\ker f(T) \oplus \ker g(T) = \ker(fg)(T).$$

<sup>17</sup> *Proof Idea.* Field extension.

<sup>18</sup> *Proof Idea.* Use Bezout's lemma.

(b) Suppose  $T \in \mathcal{L}(V)$  and the minimal polynomial of  $T$

$$p = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m},$$

where  $p_k$ 's are pairwise coprime. Prove that

$$V = \bigoplus_{k=1}^m \ker p_k^{\alpha_k}(T).$$

**Exercise 54.** Suppose  $\mathbf{F} = \mathbf{C}$ . Prove the generalized eigenspace decomposition, and prove that

$$G(\lambda, T) = \ker(T - \lambda I)^m,$$

where  $T \in \mathcal{L}(V)$ ,  $\lambda \in \mathbf{F}$ , and  $m$  is the exponent of  $z - \lambda$  in the factorization of the minimal polynomial of  $T$ .

**Exercise 55.** Suppose  $V$  is a finite-dimensional inner product space. Suppose  $P \in \mathcal{L}(V)$  is such that  $P^2 = P$  and every vector in  $\ker P$  is orthogonal to every vector in  $\operatorname{im} P$ . Prove that there exists a subspace  $U$  of  $V$  such that  $P = P_U$ .