Linear Algebra Done Right

Notes – Linear Algebra

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Contents

Don’t just read it; fight it!

Ask your own questions.

Look for your own examples.

Discover your own proofs.

Is the hypothesis necessary?

Is the converse true?

What happens in the classical special case?

What about the degenerate cases?

Where does the proof use the hypothesis?

— Paul Holmos

Linear Maps

Products and Quotients of Vector Spaces

Exercise 1. Suppose $V_1, \ldots, V_m$ are vector spaces. Prove that $\mathcal{L}(V_1 \times \cdots \times V_m, W)$ and $\mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$ are isomorphic vector spaces.

Proof. We construct an isomorphism T between the two vector spaces. For every $\Gamma \in \mathcal{L}(V_1 \times \cdots \times V_m, W)$, define $\varphi_k : V_k \to W$ for each k by

$$\varphi_k(v_k) = \Gamma(0, \ldots, v_k, \ldots, 0)$$

with v_k in the k^{th} slot and 0 in all other slots. It can be verified that $\varphi_k \in \mathcal{L}(V_k, W)$.

Define T by $T(\Gamma) = (\varphi_1, \ldots, \varphi_m)$. It can be verified that T is a linear map. We prove T is an isomorphism by constructing its inverse linear map S.

For every $(\varphi_1, \ldots, \varphi_m) \in \mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$, let

$$S(\varphi_1, \ldots, \varphi_m)\,(v_1, \ldots, v_m) = \varphi_1(v_1) + \cdots + \varphi_m(v_m)\,.$$

It can be shown that S is a linear map, and that $S \circ T = I$ and $T \circ S = I$. That proves T is indeed an isomorphism between the two vector spaces. $\square$

Proposition 2. *A nonempty subset A of V is a translate of some subspace of V if and only if $\lambda v + (1 - \lambda)\, w \in A$ for all $v, w \in A$ and all $\lambda \in \mathbb{F}$.*

Exercise 3. Suppose U is a subspace of V such that V/U is finite-dimensional.

(a) Prove that if W is a finite-dimensional subspace of V and $V = U + W$, then $\dim W \geq \dim V/U$.

(b) Prove that there exists a finite-dimensional subspace W of V such that $V = U \oplus W$ and $\dim W = \dim V/U$.

Proof. Let $\overline{w}_1 + U, \ldots, \overline{w}_m + U$ be a basis of V/U. It can be shown that $V = \operatorname{span}(\overline{w}_1, \ldots, \overline{w}_m) \oplus U$. Let $W_0 = \operatorname{span}(\overline{w}_1, \ldots, \overline{w}_m)$. Then $V = U \oplus W_0$, as desired.

Now we prove that for each subspace W of V such that $V = U + W$, we have $\dim W \geq m = \dim V/U$. For each $\overline{w}_k$ above, by definition we have $\overline{w}_l = u_k + w_k$ for some $u_k \in U$ and $w_k \in W$. It can be shown from the linear independence of $\overline{w}_1 + U, \ldots, \overline{w}_m + U$ that $\overline{w}_1 - u_1, \ldots, \overline{w}_m - u_m$ are independent vectors in W. Hence $\dim W \geq m$. $\square$

Duality

Proposition 4. *Suppose V is finite-dimensional and U is a subspace of V. Then* [1]

$$U = \left\{ v \in V : \varphi(v) = 0 \text{ for every } \varphi \in U^0 \right\}.$$

[1] Compare this result with $(U^\perp)^\perp = U$ where U is a finite-dimensional subspace of inner product space V.

Exercise 5. Suppose V is finite-dimensional and $\varphi_1, \ldots, \varphi_n$ is a basis of V'. Prove that there exists a basis of V whose dual basis is precisely $\varphi_1, \ldots, \varphi_n$.

Proof. We start from an arbitrary basis $u_1, \ldots, u_n$ of V. Let $\psi_1, \ldots, \psi_n$ be its dual basis. In this proof, we take the standard basis $e_1, \ldots, e_n$ as the basis of $\mathbb{F}^n$.

Define $S, T \in \mathcal{L}(V, \mathbb{F}^n)$ by

$$T(v) = (\varphi_1(v), \ldots, \varphi_n(v)), \quad S(v) = (\psi_1(v), \ldots, \psi_n(v)).$$

Then $\mathcal{M}(S, (u_1, \ldots, u_n)) = I$.

Let A be the change of basis matrix from ψ's to φ's, i.e.,

$$\begin{bmatrix} \varphi_1 & \cdots & \varphi_n \end{bmatrix} = \begin{bmatrix} \psi_1 & \cdots & \psi_n \end{bmatrix} A.$$

Then by the definition of change of basis matrix, we have

$$\mathcal{M}(T, (u_1, \ldots, u_n)) = \begin{bmatrix} \mathcal{M}(\varphi_1, (u_1, \ldots, u_n)) \\ \vdots \\ \mathcal{M}(\varphi_n, (u_1, \ldots, u_n)) \end{bmatrix} = A^{\mathrm{t}} \begin{bmatrix} \mathcal{M}(\psi_1, (u_1, \ldots, u_n)) \\ \vdots \\ \mathcal{M}(\psi_n, (u_1, \ldots, u_n)) \end{bmatrix}$$

$$= A^{\mathrm{t}} \cdot \mathcal{M}(S, (u_1, \ldots, u_n)) = A^{\mathrm{t}}.$$

Consider basis $v_1, \ldots, v_n$ of V such that the change of basis matrix from u's to v's is $\left(A^{\mathrm{t}}\right)^{-1}$.[2] Thus

$$\mathcal{M}(T, (v_1, \ldots, v_n)) = \mathcal{M}(T, (u_1, \ldots, u_n)) \cdot \mathcal{M}(I, (v_1, \ldots, v_n), (u_1, \ldots, u_n)) = I.$$

Then the dual basis of $v_1, \ldots, v_n$ is precisely $\varphi_1, \ldots, \varphi_n$, as desired. □

[2] *Proof Idea.* The change of basis for $V' \to V'$ corresponds to the transpose of $V \leftarrow V$, where transpose and inverse both come from duality.

Exercise 6 (A natural isomorphism from primal space onto double dual space)**.** Define $\Lambda : V \to V''$ by

$$(\Lambda v)(\varphi) = \varphi(v)$$

for each $v \in V$ and $\varphi \in V'$.

(a) Prove that if $T \in \mathcal{L}(V)$, then $T'' \circ \Lambda = \Lambda \circ T$.

(b) Prove that if V is finite-dimensional, then Λ is an isomorphism from V onto V''.

Remarks 7. (a) Suppose V is finite-dimensional. Then V and V' are isomorphic, but finding an isomorphism from V onto V' generally requires choosing a basis of V. In contrast, the isomorphism Λ from V onto V'' does not require a choice of basis and thus is more natural.

(b) Another natural isomorphism is $\pi' \in \mathcal{L}((V/U)', V')$ where π is the normal quotient map.

Polynomials

Theorem 8. *Suppose $p \in \mathcal{P}(\mathbb{F})$ is a nonconstant polynomial of degree m. Then p has at most m zeros in $\mathbb{F}$.*[3]

[3] A useful corollary of this theorem: when a polynomial p has too many zeros, $p = 0$.

Remark 9. Theorem 8 implies that the coefficients of a polynomial are uniquely determined. In particular, the *degree* of a polynomial is well-defined.

Theorem 10 (Division algorithm for polynomials)**.** *Suppose that* $p, s \in \mathcal{P}(\mathbb{F})$*, with* $s \neq 0$*. Then there exist unique polynomials* $q, r \in \mathcal{P}(\mathbb{F})$ *such that* $p = sq + r$*.*

Proof. Let $n = \deg p$ and $m = \deg s$. The case where $n < m$ is trivial. Thus we now assume that $n \geq m$.

The list

$$1, z, \ldots, z^{m-1}, s, zs, \ldots, z^{n-m}s$$

is linearly independent in $\mathcal{P}_n(\mathbb{F})$. And it also has length $n+1$. Hence the list is a basis of $\mathcal{P}_n(\mathbb{F})$.

Because $p \in \mathcal{P}_n(\mathbb{F})$, there exist unique constants $a_0, \ldots, a_{m-1}, b_0, \ldots, b_{n-m} \in \mathbb{F}$ such that

$$\begin{aligned} p &= a_0 + a_1 z + \cdots + a_{m-1} z^{m-1} + b_0 s + b_1 zs + \cdots + b_{n-m} z^{n-m} s \\ &= \left(a_0 + a_1 z + \cdots + a_{m-1} z^{m-1}\right) + s\left(b_0 + b_1 z + \cdots + b_{n-m} z^{n-m}\right). \end{aligned}$$ □

Exercise 11. Suppose $p, q \in \mathcal{P}(\mathbb{C})$ are nonconstant polynomials with no zeros in common. Let $m = \deg p$ and $n = \deg q$. Prove that there exist $r \in \mathcal{P}_{n-1}(\mathbb{C})$ and $s \in \mathcal{P}_{m-1}(\mathbb{C})$ such that $rp + sq = 1$.[4]

[4] *Proof Idea.* Define $T : \mathcal{P}_{n-1}(\mathbb{C}) \times \mathcal{P}_{m-1}(\mathbb{C}) \to \mathcal{P}_{m+n-1}(\mathbb{C})$ by $T(r, s) = rp + sq$ and prove the surjectivity of T.

Eigenvalues and Eigenvectors

Invariant Subspaces

Exercise 12. Suppose that $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ are pairwise distinct. Prove that the list $e^{\lambda_1 x}, \ldots, e^{\lambda_n x}$ is linearly independent in $\mathbb{R}^{\mathbb{R}}$.

Proof. Let $V = \operatorname{span}(e^{\lambda_1 x}, \ldots, e^{\lambda_n x})$.[5] Define $D \in \mathcal{L}(V)$ by $Df = f'$. Then $e^{\lambda x}$ is an eigenvector of D corresponding to λ. A list of eigenvectors corresponding to distinct eigenvalues is linearly independent. □

[5] Alternatively we can let $V = D(\mathbb{R})$.

Exercise 13. Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and U is a subspace of V invariant under T. Prove that each eigenvalue of the quotient operator T/U is an eigenvalue of T.

Proof. It suffices to show that $T/U - \lambda I = (T - \lambda I)/U$ is not injective $\Longrightarrow$ $T - \lambda I$ is not injective. We prove that $T - \lambda I$ is invertible $\Longrightarrow$ $(T - \lambda I)/U$ is injective.

Suppose $T - \lambda I$ is invertible. U being invariant under T implies that U is invariant under $T - \lambda I$. Thus $(T - \lambda I)v \in U \iff v \in U$. Suppose $((T - \lambda I)/U)(v + U) = 0$. Then $(T - \lambda I)v \in U$, which implies that $v \in U$, i.e., $v + U = 0$. That proves the injectivity of $(T - \lambda I)/U$. □

Exercise 14. Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T has an eigenvalue if and only if there exists a subspace of V of dimension $\dim V - 1$ that is invariant under T.[6]

[6] *Proof Idea.* Consider the zero entries in $\mathcal{M}(T)$ and its transpose matrix $\mathcal{M}(T')$.

Proof. We first suppose that T has an eigenvalue λ. Then λ is an eigenvalue of T'. There exists $\varphi \in V'$ such that $\varphi \circ T = T'\varphi = \lambda\varphi$. Extend φ to a basis $\varphi, \varphi_2, \dots, \varphi_n$ of V' and let $v, v_2, \dots, v_n$ be the basis of V whose dual basis is $\varphi, \varphi_2, \dots, \varphi_n$. Then $(\varphi \circ T)v_k = \lambda\varphi(v_k) = 0$ for every k. Because $\varphi(Tv_k) = 0$ for every k, we have $Tv_k \in \operatorname{span}(v_2, \dots, v_n)$. That proves that $\operatorname{span}(v_2, \dots, v_n)$ is invariant under T.

Reversing the steps above leads to an eigenvector of T', completing the proof. □

The Minimal Polynomial

Exercise 15 (Companion matrix of a polynomial)**.** Suppose $a_0, \dots, a_{n-1} \in \mathbb{F}$. Let $T \in \mathcal{L}(\mathbb{F}^n)$ be such that $\mathcal{M}(T)$ (with respect to the standard basis) is

$$\begin{bmatrix} 0 & & & & & -a_0 \\ 1 & 0 & & & & -a_1 \\ & 1 & 0 & & & -a_2 \\ & & \ddots & & & \vdots \\ & & & & 0 & -a_{n-2} \\ & & & & 1 & -a_{n-1} \end{bmatrix}$$

Prove that the minimal polynomial of T is the polynomial

$$a_0 + a_1 z + \dots + a_{n-1}z^{n-1} + z^n.$$

Remark 16. This exercise implies that every monic polynomial is the minimal polynomial of some operator. Hence an algorithm that could produce exact eigenvalues for each operators on each $\mathbb{F}^n$ does not exist.

Exercise 17. Prove that every operator on a finite-dimensional vector space of dimension at least 2 has an invariant subspace of dimension 2.

Proof. Let $T \in \mathcal{L}(V)$ and $n = \dim V$. We use induction on n. The base case $n = 2$ is trivial. Now suppose $n > 2$ and the desired result holds for all smaller positive integers. Let p be the minimal polynomial of T.

If T has an eigenvalue λ, then $p(z) = q(z)(z - \lambda)$ for some monic polynomial q with $\deg q = \deg p - 1$. Because $q(T)|_{\operatorname{im}(T-\lambda I)} = 0$, the desired result holds by induction hypothesis if $\dim \operatorname{im}(T - \lambda I) \geq 2$. If $T - \lambda I = 0$ the desired result trivially holds. If $\dim \operatorname{im}(T - \lambda I) = 1$, then $(T - \lambda I)v$ is a scalar multiple of some fixed $u \in V$ for all $v \in V$. Take $w \in V \setminus \operatorname{span}(u)$ and $\operatorname{span}(u, w)$ will satisfy the desired property.

If T has no eigenvalues, then $\mathbb{F} = \mathbb{R}$ and $p(z) = q(z)(z^2 + bz + c)$ for some $b, c \in \mathbb{R}$ with $b^2 < 4c$ and monic polynomial q with $\deg q = \deg p - 2$. If $\dim \operatorname{im}(T^2 + bT + cI) \geq 2$ the desired result holds by induction hypothesis. If $T^2 + bT + cI = 0$, then let $w \in V$ be such that $w \neq 0$. It can be verified that $\operatorname{span}(w, Tw)$ is invariant under T. If $\dim \operatorname{im}(T^2 + bT + cI) = 1$, because $\dim \ker(T^2 + bT + cI)$ is even, n is odd, which implies that T has an eigenvalue. That completes the proof. □

Commuting Operators

Exercise 18. Suppose $\mathcal{E} \subseteq \mathcal{L}(V)$ and every element of $\mathcal{E}$ is diagonalizable. Prove that there exists a basis of V with respect to which every element of $\mathcal{E}$ has a diagonal matrix if and only if every pair of elements of $\mathcal{E}$ commutes.[7]

Proof. Suppose every pair of elements of $\mathcal{E}$ commutes. We use induction on $n = \dim V$. The base case $n = 1$ is trivial. Now suppose $n > 1$ and the desired result holds for all smaller integers. Without loss of generality, suppose $\mathcal{E} \cap \{\lambda I : \lambda \in \mathbb{F}\} = \emptyset$, or else consider $\mathcal{E} \backslash \{\lambda I : \lambda \in \mathbb{F}\}$.

Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of $T \in \mathcal{E}$. Then $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$. Because $E(\lambda_k, T)$ is invariant under every $S \in \mathcal{E}$, it suffices to show that the desired result holds on $E(\lambda_1, T)$. Because $E(\lambda_1, T) \subsetneqq V$, it holds by induction hypothesis. The other direction is trivial. $\square$

[7] This is an extension of simultaneous diagonalizability to more than 2 (possibly infinitely many) operators.

Exercise 19. Suppose V is a finite-dimensional nonzero complex vector space. Suppose that $\mathcal{E} \subseteq \mathcal{L}(V)$ is such that S and T commute for all $S, T \in \mathcal{E}$.

(a) Prove that there is a vector in V that is an eigenvector for every element of $\mathcal{E}$.

(b) Prove that there exists a basis of V with respect to which every element of $\mathcal{E}$ has an upper-triangular matrix.[8]

[8] This is an extension of simultaneous upper triangularizability to more than 2 (possibly infinitely many) operators.

Inner Product Spaces

Inner Products and Norms

Theorems 20 (Polarization identities)**.** *(a) Suppose V is a real inner product space. Then*

$$\langle u, v \rangle = \frac{\|u+v\|^2 - \|u-v\|^2}{4}.$$

(b) Suppose V is a complex inner product space. Then

$$\langle u, v \rangle = \frac{\|u+v\|^2 - \|u-v\|^2 + \|u+iv\|^2 i - \|u-iv\|^2 i}{4}.$$

Exercise 21. Prove that if $\|\cdot\|$ is a norm on U satisfying the parallelogram equality, then there is an inner product $\langle \cdot, \cdot \rangle$ on U such that $\|u\| = \langle u, u \rangle^{1/2}$ for all $u \in U$.[9]

[9] Recall Theorems 20.

Orthonormal Bases

Exercise 22. Suppose $v_1, \ldots, v_m$ is a linearly independent list in V. Prove that the orthonormal list produced by the formulas of the Gram-Schmidt procedure is the only orthonormal list $e_1, \ldots, e_m$ in V such that $\langle v_k, e_k \rangle > 0$ and $\operatorname{span}(v_1, \ldots, v_k) = \operatorname{span}(e_1, \ldots, e_k)$ for each k.

Exercise 23. Suppose V is finite-dimensional. Suppose $\langle \cdot, \cdot \rangle_1, \langle \cdot, \cdot \rangle_2$ are inner products on V with corresponding norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Prove that there exists $b, c > 0$ such that $b\|v\|_2 \leq \|v\|_1 \leq c\|v\|_2$ for every $v \in V$.[10]

[10] Note that *Proof 1* does not use the hypothesis that $\|\cdot\|_1$ and $\|\cdot\|_2$ are associated with an inner product respectively.

Proof. Without loss of generality, suppose $V = \mathbb{F}^n$, $\|x\|_1^2 = x^t Ax$, and $\|x\|_2^2 = x^t Bx$, where A, B are positive-definite and B is diagonal.[11] Note that $\lambda B - A$ is Hermitian and hence diagonalizable. It suffices to show that there exists $\lambda > 0$ such that $\lambda B - A$ is positive-definite. Such a λ exists by the Gershgorin disk theorem, as desired. □

[11] *Proof Idea.* Fixing an orthonormal basis naturally leads to a (simple) diagonal matrix.

Exercise 24. Suppose $\mathbb{F} = \mathbb{C}$ and V is finite-dimensional. Prove that if $T \in \mathcal{L}(V)$ is such that 1 is the only eigenvalue of T and $\|Tv\| \leq \|v\|$ for all $v \in V$, then T is the identity operator.

Proof. By Schur's theorem, there exists an orthonormal basis $e_1, \ldots, e_m$ of V with respect to which T has an upper-triangular matrix. Then all entries on the diagonal of $\mathcal{M}(T)$ is 1. Any $\mathcal{M}(T)_{j,k} \neq 0$ with $j \neq k$ would contradict $\|Te_k\| \leq \|e_k\|$. That proves that $\mathcal{M}(T) = I$, as desired. □

Exercise 25. Suppose $v_1, \ldots, v_n$ is a basis of V. Prove that there exists a basis $u_1, \ldots, u_n$ of V such that[12]

$$\langle u_j, v_k \rangle = \begin{cases} 0 & \text{if } j \neq k, \\ 1 & \text{if } j = k. \end{cases}$$

[12] *Proof Idea.* Linearity in inner product and and patterns of $0, 1$ inspire the use of linear functionals.

Proof. Define $\varphi_k \in V'$ for each k by $\varphi_k(u) = \langle u, v_k \rangle$. By the Riesz representation theorem, $\varphi_1, \ldots, \varphi_n$ is a spanning list in V'. Thus it is a basis of V'. Let $u_1, \ldots, u_n$ be the basis of V whose dual basis is $\varphi_1, \ldots, \varphi_n$. Then $u_1, \ldots, u_n$ satisfies the desired property. □

Orthogonal Complements and Minimization Problems

Theorem 26 (Riesz representation theorem)**.** *Suppose V is finite-dimensional. For each $v \in V$, define $\varphi_v \in V'$ by*

$$\varphi_v(u) = \langle u, v \rangle$$

for each $u \in V$. Then $v \mapsto \varphi_v$ is a one-to-one map from V onto V'.[13]

[13] *Proof Idea.* If $\varphi(u) = \langle u, v \rangle$ holds for all $u \in V$, then $v \in (\ker \varphi)^\perp$. However, $(\ker \varphi)^\perp$ has dimension 1 (except when $\varphi = 0$). Hence we can obtain the right v by choosing an arbitrary nonzero $w \in (\ker \varphi)^\perp$ and then multiplying by an appropriate scalar.

Proof. The injectivity is trivial. We prove the surjectivity. Suppose $\varphi \in V'$. The case where $\varphi = 0$ is trivial. Thus assume $\varphi \neq 0$. Hence $\ker \varphi \neq V$, which implies that $(\ker \varphi)^\perp \neq \{0\}$. Let $w \in (\ker \varphi)^\perp$ be such that $w \neq 0$. Let[14]

[14] *Proof Idea.* Apply w to u to find the supposedly right scalar c in $\varphi(u) = \langle u, cw \rangle$.

$$v = \frac{\overline{\varphi(w)}}{\|w\|^2} w. \tag{26.1}$$

Then $v \in (\ker \varphi)^\perp$ and $v \neq 0$. Now we prove that $\varphi(u) = \langle u, v \rangle$ for each $u \in V$.

Let $u \in V$. The orthogonal decomposition leads to

$$u = \frac{\langle u, v \rangle}{\|v\|^2} v + \left(u - \frac{\langle u, v \rangle}{\|v\|^2} v \right),$$

where the second term is orthogonal to v and thus in $\ker \varphi$. Applying φ to both sides, we have

$$\varphi(u) = \frac{\langle u, v \rangle}{\|v\|^2} \varphi(v). \tag{26.2}$$

By (26.1), we have

$$\|v\| = \frac{|\varphi(w)|}{\|w\|}, \qquad \varphi(v) = \frac{|\varphi(w)|^2}{\|w\|^2}.$$

Applying them to (26.2) leads to $\varphi(u) = \langle u, v\rangle$, as desired. □

Exercise 27. Suppose V is finite-dimensional and $P \in \mathcal{L}(V)$ is such that $P^2 = P$.

(a) Suppose every vector in $\ker P$ is orthogonal to every vector in $\operatorname{im} P$. Prove that there exists a subspace U of V such that $P = P_U$.[15]

(b) Suppose $\|Pv\| \le \|v\|$ for every $v \in V$. Prove that there exists a subspace U of V such that $P = P_U$.[16]

[15] *Proof Idea.* Observe that $\ker P = (\operatorname{im} P)^\perp$. Thus we prove that P and $P_{\operatorname{im} P}$ agree on $\ker P$ and $\operatorname{im} P$.

[16] *Proof Idea.* Observe that among all v's with the same Pv, the one in $(\ker P)^\perp$ is the shortest. Applying this v makes the best use of the inequality. Thus we prove that P and $P_{(\ker P)^\perp}$ agree on $\ker P$ and $(\ker P)^\perp$.

Propositions 28 (Algebraic properties of the pseudoinverse)**.** *Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$.*

(a) The pseudoinverse of an orthogonal projection is the operator itself.

(b) $\ker T^\dagger = (\operatorname{im} T)^\perp$ *and* $\operatorname{im} T^\dagger = (\ker T)^\perp$.

(c) $TT^\dagger T = T$ *and* $T^\dagger T T^\dagger = T^\dagger$.

(d) $\left(T^\dagger\right)^\dagger = T$.

Operators on Inner Product Spaces

From now on, we suppose that V and W are nonzero finite-dimensional inner product spaces over $\mathbb{F}$.

Self-Adjoint and Normal Operators

Exercise 29. Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. Prove that

$$\lambda \text{ is an eigenvalue of } T \iff \overline{\lambda} \text{ is an eigenvalue of } T^*.$$

Exercise 30. Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V. Prove that

$$U \text{ is invariant under } T \iff U^\perp \text{ is invariant under } T^*.$$

Exercise 31. Suppose $T \in \mathcal{L}(V)$ is normal. Prove that

$$\ker T^k = \ker T \quad \text{and} \quad \operatorname{im} T^k = \operatorname{im} T.$$

Exercise 32. Suppose $T \in \mathcal{L}(V, W)$. Prove that under the standard identification of V with V' (by Riesz representation theorem) and W with W', the adjoint map T^* corresponds to the dual map T'. More precisely, prove that

$$T'(\varphi_w) = \varphi_{T^* w}$$

for all $w \in W$.

Remark 33. Furthermore, under this identification of V with V', the orthogonal complement corresponds to the annihilator; the formulas for $\ker T^*$ and $\operatorname{im} T^*$ become identical to the formulas for $\ker T'$ and $\operatorname{im} T'$. Note that orthogonal complements and adjoints are easier to deal with than annihilators and dual maps.

Spectral Theorem

Theorems 34. *Suppose $T \in \mathcal{L}(V)$.*

(a) *Suppose $\mathbb{F} = \mathbb{R}$. Then T is self-adjoint if and only if all pairs of eigenvectors corresponding to distinct eigenvalues of T are orthogonal and $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$, where $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of T.*

(b) *Suppose $\mathbb{F} = \mathbb{C}$. Then T is normal if and only if all pairs of eigenvectors corresponding to distinct eigenvalues of T are orthogonal and $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$, where $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of T.*

Positive Operators

Complexifcation

Definition 35. The complexification of V, denoted by $V_{\mathbb{C}}$, equals $V \times V$ with normal addition and real scalar multiplication for product space. But we write an element (u, v) of $V_{\mathbb{C}}$ as $u + iv$. Complex scalar multiplication is defined by

$$(a + bi)(u + iv) = (au - bv) + i(av + bu)$$

for all $a, b \in \mathbb{R}$ and all $u, v \in V$.[17]

[17] Think of V as a subset of $V_{\mathbb{C}}$ by identifying $u \in V$ with $u + i0$. The construction of $V_{\mathbb{C}}$ from V can then be thought of as generalizing the construction of $\mathbb{C}^n$ from $\mathbb{R}^n$.

Propositions 36 (Properties of complexification)**.** (a) *$\lambda \in \mathbb{R}$ is an eigenvalue of T if and only if λ is an eigenvalue of $T_{\mathbb{C}}$.*

(b) *$\lambda \in \mathbb{C}$ is an eigenvalue of $T_{\mathbb{C}}$ if and only if $\overline{\lambda}$ is an eigenvalue of $T_{\mathbb{C}}$.*

(c) *The minimal polynomial of $T_{\mathbb{C}}$ equals the minimal polynomial of T.*

(d) *Suppose V is a real inner product space. For $u, v, w, x \in V$, define*

$$\langle u + iv, w + ix \rangle_{\mathbb{C}} = \langle u, w \rangle + \langle v, x \rangle + (\langle v, w \rangle - \langle u, x \rangle)i.$$

Then $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ makes $V_{\mathbb{C}}$ into a complex inner product space. If $u, v \in V$, then

$$\langle u, v \rangle_{\mathbb{C}} = \langle u, v \rangle \quad \text{and} \quad \|u + iv\|_{\mathbb{C}}^2 = \|u\|^2 + \|v\|^2.$$

Proposition 37. *Every operator on a finite-dimensional nonzero vector space has an invariant subspace of dimension* 1 *or* 2*.*

Proof. The case where $\mathbb{F} = \mathbb{C}$ is trivial. Now assume V is a real vector space and $T \in \mathcal{L}(V)$. Then $T_{\mathbb{C}}$[18] has an eigenvalue $a + bi$ with $a, b \in \mathbb{R}$. Thus there exist $u, v \in V$, not both 0, such that

[18] *Proof Idea.* Field extension.

$$Tu + iTv = (au - bv) + (av + bu)i.$$

Hence $\operatorname{span}(u, v)$ is invariant under T, as desired. □