Notes – Linear Algebra

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Contents

Don’t just read it; fight it!

Ask your own questions.

Look for your own examples.

Discover your own proofs.

Is the hypothesis necessary?

Is the converse true?

What happens in the classical special case?

What about the degenerate cases?

Where does the proof use the hypothesis?

— Paul Holmos

Vector Spaces

Lemma 1 (Linear dependence lemma)**.** *Suppose $v_1, \ldots, v_m$ is a linearly dependent set in V. Then there exists $k \in \{1, 2, \ldots, m\}$ such that*

$$v_k \in \operatorname{span}(v_1, \ldots, v_m).$$

Furthermore, removing the k^{th} term from the list does not change the span.

Theorem 2. *Any two bases of a finite-dimensional vector space have the same length.*[1]

[1] This proposition ensures that the *dimension* of a vector space is well-defined.

Linear Maps

Kernals and Images

Exercise 3. Suppose U and V are finite-dimensional and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$. Prove that

$$\dim \ker ST \leq \dim \ker S + \dim \ker T.$$

Proof. Restrict to $Z = \ker ST$. By the fundamental theorem of linear maps,

$$\begin{aligned} \dim Z &= \dim T(Z) + \dim \ker T|_Z \\ &\leq \dim T(Z) + \dim \ker T \\ &= \dim ST(Z) + \dim \ker S|_{T(Z)} + \dim \ker T \\ &\leq \dim \ker S + \dim \ker T. \end{aligned}$$ □

Corollary 4 (Sylvester's rank inequality)**.** *Suppose $A \in \mathbb{F}^{m,n}$ and $B \in \mathbb{F}^{n,p}$ are two matrices. Then*

$$\operatorname{rank} A + \operatorname{rank} B - n \leq \operatorname{rank}(AB).$$

Products and Quotients of Vector Spaces

Exercise 5. Suppose $V_1, \ldots, V_m$ are vector spaces. Prove that $\mathcal{L}(V_1 \times \cdots \times V_m, W)$ and $\mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$ are isomorphic vector spaces.

Proof. We construct an isomorphism T between the two vector spaces.

For every $\Gamma \in \mathcal{L}(V_1 \times \cdots \times V_m, W)$, define $\varphi_k : V_k \to W$ for each k by

$$\varphi_k(v_k) = \Gamma(0, \ldots, v_k, \ldots, 0)$$

with v_k in the k^{th} slot and 0 in all other slots. It can be verified that $\varphi_k \in \mathcal{L}(V_k, W)$.

Define T by $T(\Gamma) = (\varphi_1, \ldots, \varphi_m)$. It can be verified that T is a linear map. We prove T is an isomorphism by constructing its inverse linear map S.

For every $(\varphi_1, \ldots, \varphi_m) \in \mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$, let

$$S(\varphi_1, \ldots, \varphi_m)(v_1, \ldots, v_m) = \varphi_1(v_1) + \cdots + \varphi_m(v_m).$$

It can be shown that S is a linear map, and that $S \circ T = I$ and $T \circ S = I$. That proves T is indeed an isomorphism between the two vector spaces. □

Proposition 6. *A nonempty subset A of V is a translate of some subspace of V if and only if $\lambda v + (1-\lambda) w \in A$ for all $v, w \in A$ and all $\lambda \in \mathbb{F}$.*

Proposition 7. *Suppose U is a subspace of V and $v_1 + U, \ldots, v_m + U$ is a basis of V/U and $u_1, \ldots, u_n$ is a basis of U. Then $v_1, \ldots, v_m, u_1, \ldots, u_n$ is a basis of V. In other words, $V = \operatorname{span}(v_1, \ldots, v_m) \oplus U$.*[2]

[2] $V = \operatorname{span}(v_1, \ldots, v_m) \oplus U$ still holds without the hypothesis that U is finite-dimensional.

Exercise 8. Suppose U is a subspace of V such that V/U is finite-dimensional.

(a) Prove that if W is a finite-dimensional subspace of V and $V = U + W$, then $\dim W \geq \dim V/U$.

(b) Prove that there exists a finite-dimensional subspace W of V such that $V = U \oplus W$ and $\dim W = \dim V/U$.

Proof. Let $\overline{w}_1 + U, \ldots, \overline{w}_m + U$ be a basis of V/U. Then by Proposition 7, we have $V = \operatorname{span}(\overline{w}_1, \ldots, \overline{w}_m) \oplus U$. Let $W_0 = \operatorname{span}(\overline{w}_1, \ldots, \overline{w}_m)$, then $V = U \oplus W_0$, as desired.

Now we prove that for each subspace W of V such that $V = U + W$, we have $\dim W \geq m = \dim V/U$. For each $\overline{w}_k$ above, by definition we have $\overline{w}_l = u_k + w_k$ for some $u_k \in U$ and $w_k \in W$. It can be shown from the linear independence of $\overline{w}_1 + U, \ldots, \overline{w}_m + U$ that $\overline{w}_1 - u_1, \ldots, \overline{w}_m - u_m$ are independent vectors in W. Hence $\dim W \geq m$. □

Duality

Proposition 9. *Suppose V is finite-dimensional and U is a subspace of V. Then*

$$U = \left\{ v \in V : \varphi(v) = 0 \text{ for every } \varphi \in U^0 \right\}.$$

Exercise 10. Suppose V is finite-dimensional and $\varphi_1, \ldots, \varphi_n$ is a basis of V'. Prove that there exists a basis of V whose dual basis is precisely $\varphi_1, \ldots, \varphi_n$.

Proof. We start from an arbitrary basis $u_1, \ldots, u_n$ of V. Let $\psi_1, \ldots, \psi_n$ be its dual basis. In this proof, we take the standard basis $e_1, \ldots, e_n$ as the basis of $\mathbb{F}^n$.

Define $S, T \in \mathcal{L}(V, \mathbb{F}^n)$ by

$$T(v) = (\varphi_1(v), \ldots, \varphi_n(v)), \quad S(v) = (\psi_1(v), \ldots, \psi_n(v)).$$

Then $\mathcal{M}(S, (u_1, \ldots, u_n)) = I$.

Let A be the change of basis matrix from ψ's to φ's, i.e.,

$$\begin{bmatrix} \varphi_1 & \cdots & \varphi_n \end{bmatrix} = \begin{bmatrix} \psi_1 & \cdots & \psi_n \end{bmatrix} A.$$

Then by the definition of change of basis matrix, we have

$$\mathcal{M}(T, (u_1, \ldots, u_n)) = \begin{bmatrix} \mathcal{M}(\varphi_1, (u_1, \ldots, u_n)) \\ \vdots \\ \mathcal{M}(\varphi_n, (u_1, \ldots, u_n)) \end{bmatrix} = A^{\mathrm{t}} \begin{bmatrix} \mathcal{M}(\psi_1, (u_1, \ldots, u_n)) \\ \vdots \\ \mathcal{M}(\psi_n, (u_1, \ldots, u_n)) \end{bmatrix}$$

$$= A^{\mathrm{t}} \cdot \mathcal{M}(S, (u_1, \ldots, u_n)) = A^{\mathrm{t}}.$$

Consider basis $v_1, \ldots, v_n$ of V such that the change of basis matrix from u's to v's is $\left(A^{\mathrm{t}}\right)^{-1}$.[3] Thus

$$\mathcal{M}(T, (v_1, \ldots, v_n)) = \mathcal{M}(T, (u_1, \ldots, u_n)) \cdot \mathcal{M}(I, (v_1, \ldots, v_n), (u_1, \ldots, u_n)) = I.$$

Then the dual basis of $v_1, \ldots, v_n$ is precisely $\varphi_1, \ldots, \varphi_n$, as desired. $\square$

[3] *Proof Idea.* The change of basis for $V' \to V'$ corresponds to the transpose of $V \leftarrow V$, where transpose and inverse both come from duality.

Exercise 11 (A natural isomorphism from primal space onto double dual space)**.** Define $\Lambda : V \to V''$ by

$$(\Lambda v)(\varphi) = \varphi(v)$$

for each $v \in V$ and $\varphi \in V'$.

(a) Prove that if $T \in \mathcal{L}(V)$, then $T'' \circ \Lambda = \Lambda \circ T$.

(b) Prove that if V is finite-dimensional, then Λ is an isomorphism from V onto V''.

Remarks 12. (a) Suppose V is finite-dimensional. Then V and V' are isomorphic, but finding an isomorphism from V onto V' generally requires choosing a basis of V. In contrast, the isomorphism Λ from V onto V'' does not require a choice of basis and thus is more natural.

(b) Another natural isomorphism is $\pi' \in \mathcal{L}((V/U)', V')$ where π is the normal quotient map.

Polynomials

Theorem 13. *Suppose $p \in \mathcal{P}(\mathbb{F})$ is a nonconstant polynomial of degree m. Then p has at most m zeros in $\mathbb{F}$.*[4]

[4] A useful corollary of this theorem: when a polynomial p has too many zeros, $p = 0$.

Remark 14. Theorem 13 implies that the coefficients of a polynomial are uniquely determined. In particular, the *degree* of a polynomial is well-defined.

Theorem 15 (Division algorithm for polynomials)**.** *Suppose that $p, s \in \mathcal{P}(\mathbb{F})$, with $s \neq 0$. Then there exist unique polynomials $q, r \in \mathcal{P}(\mathbb{F})$ such that $p = sq + r$.*

Proof. Let $n = \deg p$ and $m = \deg s$. The case where $n < m$ is trivial. Thus we now assume that $n \geq m$.

The list

$$1, z, \ldots, z^{m-1}, s, zs, \ldots, z^{n-m}s$$

is linearly independent in $\mathcal{P}_n(\mathbb{F})$. And it also has length $n+1$. Hence the list is a basis of $\mathcal{P}_n(\mathbb{F})$.

Because $p \in \mathcal{P}_n(\mathbb{F})$, there exist unique constants $a_0, \ldots, a_{m-1}, b_0, \ldots, b_{n-m} \in \mathbb{F}$ such that

$$\begin{aligned} p &= a_0 + a_1 z + \cdots + a_{m-1} z^{m-1} + b_0 s + b_1 zs + \cdots + b_{n-m} z^{n-m} s \\ &= \left(a_0 + a_1 z + \cdots + a_{m-1} z^{m-1}\right) + s\left(b_0 + b_1 z + \cdots + b_{n-m} z^{n-m}\right). \end{aligned}$$

$\square$

Exercise 16. Suppose $p, q \in \mathcal{P}(\mathbb{C})$ are nonconstant polynomials with no zeros in common. Let $m = \deg p$ and $n = \deg q$. Prove that there exist $r \in \mathcal{P}_{n-1}(\mathbb{C})$ and $s \in \mathcal{P}_{m-1}(\mathbb{C})$ such that $rp + sq = 1$.

Proof. Define $T : \mathcal{P}_{n-1}(\mathbb{C}) \times \mathcal{P}_{m-1}(\mathbb{C}) \to \mathcal{P}_{m+n-1}(\mathbb{C})$ by $T(r, s) = rp + sq$. It can be shown that T is an injective linear map. Because the domain space and target space have the same dimension, T is surjective, completing the proof. □

Eigenvalues and Eigenvectors

Invariant Subspaces

Exercise 17. Suppose that $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ are pairwise distinct. Prove that the list $e^{\lambda_1 x}, \ldots, e^{\lambda_n x}$ is linearly independent in $\mathbb{R}^{\mathbb{R}}$.

Proof. Let $V = \operatorname{span}(e^{\lambda_1 x}, \ldots, e^{\lambda_n x})$.[5] Define $D \in \mathcal{L}(V)$ by $Df = f'$. Then $e^{\lambda x}$ is an eigenvector of D corresponding to λ. A list of eigenvectors corresponding to distinct eigenvalues is linearly independent. □

[5] Alternatively we can let $V = D(\mathbb{R})$.

Exercise 18. Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and U is a subspace of V invariant under T. Prove that each eigenvalue of the quotient operator T/U is an eigenvalue of T.

Proof. It suffices to show that $T/U - \lambda I = (T - \lambda I)/U$ is not injective $\implies T - \lambda I$ is not injective. We prove that $T - \lambda I$ is invertible $\implies (T - \lambda I)/U$ is injective.

Suppose $T - \lambda I$ is invertible. U being invariant under T implies that U is invariant under $T - \lambda I$. Thus $(T - \lambda I)v \in U \iff v \in U$. Suppose $((T - \lambda I)/U)(v + U) = 0$. Then $(T - \lambda I)v \in U$, which implies that $v \in U$, i.e., $v + U = 0$. That proves the injectivity of $(T - \lambda I)/U$. □

Exercise 19. Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T has an eigenvalue if and only if there exists a subspace of V of dimension $\dim V - 1$ that is invariant under T.[6]

[6] *Proof Idea.* Consider the zero entries in $\mathcal{M}(T)$ and its transpose matrix $\mathcal{M}(T')$.

Proof. We first suppose that T has an eigenvalue λ. Then λ is an eigenvalue of T'. There exists $\varphi \in V'$ such that $\varphi \circ T = T'\varphi = \lambda\varphi$. Extend φ to a basis $\varphi, \varphi_2, \ldots, \varphi_n$ of V' and let $v, v_2, \ldots, v_n$ be the basis of V whose dual basis is $\varphi, \varphi_2, \ldots, \varphi_n$. Then $(\varphi \circ T)v_k = \lambda\varphi(v_k) = 0$ for every k. Because $\varphi(Tv_k) = 0$ for every k, we have $Tv_k \in \operatorname{span}(v_2, \ldots, v_n)$. That proves that $\operatorname{span}(v_2, \ldots, v_n)$ is invariant under T.

Reversing the steps above leads to an eigenvector of T', completing the proof. □

The Minimal Polynomial

Exercise 20 (Companion matrix of a polynomial)**.** Suppose $a_0, \ldots, a_{n-1} \in \mathbb{F}$. Let $T \in \mathcal{L}(\mathbb{F}^n)$ be such that $\mathcal{M}(T)$ (with respect to the standard basis) is

$$\begin{bmatrix} 0 & & & & & -a_0 \\ 1 & 0 & & & & -a_1 \\ & 1 & 0 & & & -a_2 \\ & & & \ddots & & \vdots \\ & & & & 0 & -a_{n-2} \\ & & & & 1 & -a_{n-1} \end{bmatrix}$$

Prove that the minimal polynomial of T is the polynomial

$$a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n.$$

Remark 21. This exercise implies that every monic polynomial is the minimal polynomial of some operator. Hence an algorithm that could produce exact eigenvalues for each operators on each $\mathbb{F}^n$ does not exist.

Exercise 22. Prove that every operator on a finite-dimensional vector space of dimension at least 2 has an invariant subspace of dimension 2.

Proof. Let $T \in \mathcal{L}(V)$ and $n = \dim V$. We use induction on n. The base case $n = 2$ is trivial. Now suppose $n > 2$ and the desired result holds for all smaller positive integers. Let p be the minimal polynomial of T.

If T has an eigenvalue λ, then $p(z) = q(z)(z - \lambda)$ for some monic polynomial q with $\deg q = \deg p - 1$. Because $q(T)|_{\operatorname{im}(T-\lambda I)} = 0$, the desired result holds by induction hypothesis if $\dim \operatorname{im}(T - \lambda I) \geq 2$. If $T - \lambda I = 0$ the desired result trivially holds. If $\dim \operatorname{im}(T - \lambda I) = 1$, then $(T - \lambda I)v$ is a scalar multiple of some fixed $u \in V$ for all $v \in V$. Take $w \in V \setminus \operatorname{span}(u)$ and $\operatorname{span}(u, w)$ will satisfy the desired property.

If T has no eigenvalues, then $\mathbb{F} = \mathbb{R}$ and $p(z) = q(z)(z^2 + bz + c)$ for some $b, c \in \mathbb{R}$ with $b^2 < 4c$ and monic polynomial q with $\deg q = \deg p - 2$. If $\dim \operatorname{im}(T^2 + bT + cI) \geq 2$ the desired result holds by induction hypothesis. If $T^2 + bT + cI = 0$, then let $w \in V$ be such that $w \neq 0$. It can be verified that $\operatorname{span}(w, Tw)$ is invariant under T. If $\dim \operatorname{im}(T^2 + bT + cI) = 1$, because $\dim \ker(T^2 + bT + cI)$ is even, n is odd, which implies that T has an eigenvalue. That completes the proof. $\square$

Commuting Operators

Exercise 23. Suppose $\mathcal{E} \subseteq \mathcal{L}(V)$ and every element of $\mathcal{E}$ is diagonalizable. Prove that there exists a basis of V with respect to which every element of $\mathcal{E}$ has a diagonal matrix if and only if every pair of elements of $\mathcal{E}$ commutes.[7]

[7] This is an extension of simultaneous diagonalizability to more than 2 (possibly infinitely many) operators.

Proof. Suppose every pair of elements of $\mathcal{E}$ commutes. The other direction is trivial. We use induction on $n = \dim V$. The base case $n = 1$ is trivial. Now suppose $n > 1$

and the desired result holds for all smaller integers. Without loss of generality, suppose $\mathcal{E} \cap \{\lambda I : \lambda \in \mathbb{F}\} = \varnothing$, or else consider $\mathcal{E} \backslash \{\lambda I : \lambda \in \mathbb{F}\}$.

Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of $T \in \mathcal{E}$. Then $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$. Because $E(\lambda_k, T)$ is invariant under every $S \in \mathcal{E}$, it suffices to show that the desired result holds on $E(\lambda_1, T)$. Because $E(\lambda_1, T) \subsetneq V$, it holds by induction hypothesis, completing the proof. □

Exercise 24. Suppose V is a finite-dimensional nonzero complex vector space. Suppose that $\mathcal{E} \subseteq \mathcal{L}(V)$ is such that S and T commute for all $S, T \in \mathcal{E}$.

(a) Prove that there is a vector in V that is an eigenvector for every element of $\mathcal{E}$.

(b) Prove that there exists a basis of V with respect to which every element of $\mathcal{E}$ has an upper-triangular matrix.[8]

[8] This is an extension of simultaneous upper triangularizability to more than 2 (possibly infinitely many) operators.

Inner Product Spaces

Inner Products and Norms

Theorems 25 (Polarization identities). *(a) Suppose V is a real inner product space. Then*

$$\langle u, v \rangle = \frac{\|u+v\|^2 - \|u-v\|^2}{4}.$$

(b) Suppose V is a complex inner product space. Then

$$\langle u, v \rangle = \frac{\|u+v\|^2 - \|u-v\|^2 + \|u+iv\|^2 i - \|u-iv\|^2 i}{4}.$$

Exercise 26. Prove that if $\|\cdot\|$ is a norm on U satisfying the parallelogram equality, then there is an inner product $\langle \cdot, \cdot \rangle$ on U such that $\|u\| = \langle u, u \rangle^{1/2}$ for all $u \in U$.[9]

[9] Recall Theorems 25.

Orthonormal Bases

Exercise 27. Suppose $v_1, \ldots, v_m$ is a linearly independent list in V. Prove that the orthonormal list produced by the formulas of the Gram-Schmidt procedure is the only orthonormal list $e_1, \ldots, e_m$ in V such that $\langle v_k, e_k \rangle > 0$ for each k.

Exercise 28. Suppose V is finite-dimensional. Suppose $\langle \cdot, \cdot \rangle_1, \langle \cdot, \cdot \rangle_2$ are inner products on V with corresponding norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Prove that there exists $b, c > 0$ such that $b\|v\|_2 \le \|v\|_1 \le c\|v\|_2$ for every $v \in V$.[10]

[10] Note that *Proof 1* does not use the hypothesis that $\|\cdot\|_1$ and $\|\cdot\|_2$ are associated with an inner product respectively.

Proof. Without loss of generality, suppose $V = \mathbb{F}^n$, $\|x\|_1^2 = x^{\mathrm{t}} A x$, and $\|x\|_2^2 = x^{\mathrm{t}} B x$, where A, B are positive-definite and B is diagonal.[11] Note that $\lambda B - A$ is Hermitian and hence diagonalizable. It suffices to show that there exists $\lambda > 0$ such that $\lambda B - A$ is positive-definite. Such a λ exists by the Gershgorin disk theorem, as desired. □

[11] *Proof Idea.* Fixing an orthonormal basis naturally leads to a (simple) diagonal matrix.

Exercise 29. Suppose $\mathbb{F} = \mathbb{C}$ and V is finite-dimensional. Prove that if $T \in \mathcal{L}(V)$ is such that 1 is the only eigenvalue of T and $\|Tv\| \leq \|v\|$ for all $v \in V$, then T is the identity operator.

Proof. By Schur's theorem, there exists an orthonormal basis $e_1, \ldots, e_m$ of V with respect to which T has an upper-triangular matrix. Then all entries on the diagonal of $\mathcal{M}(T)$ is 1. Any $\mathcal{M}(T)_{j,k} \neq 0$ with $j \neq k$ would contradict $\|Te_k\| \leq \|e_k\|$. That proves that $\mathcal{M}(T) = I$, as desired. □

Exercise 30. Suppose $v_1, \ldots, v_n$ is a basis of V. Prove that there exists a basis $u_1, \ldots, u_n$ of V such that[12]

$$\langle u_j, v_k \rangle = \begin{cases} 0 & \text{if } j \neq k, \\ 1 & \text{if } j = k. \end{cases}$$

[12] *Proof Idea.* Linearity in inner product and and patterns of $0, 1$ inspire the use of linear functionals.

Proof. Define $\varphi_k \in V'$ for each k by $\varphi_k(u) = \langle u, v_k \rangle$. By the Riesz representation theorem, $\varphi_1, \ldots, \varphi_n$ is a spanning list in V'. Thus it is a basis of V'. Let $u_1, \ldots, u_n$ be the basis of V whose dual basis is $\varphi_1, \ldots, \varphi_n$. Then $u_1, \ldots, u_n$ satisfies the desired property. □

Orthogonal Complements and Minimization Problems

Theorem 31 (Riesz representation theorem)**.** *Suppose V is finite-dimensional. For each $v \in V$, define $\varphi_v \in V'$ by*

$$\varphi_v(u) = \langle u, v \rangle$$

for each $u \in V$. Then $v \mapsto \varphi_v$ is a one-to-one map from V onto V'.[13]

[13] *Proof Idea.* If $\varphi(u) = \langle u, v \rangle$ holds for all $u \in V$, then $v \in (\ker \varphi)^{\perp}$. However, $(\ker \varphi)^{\perp}$ has dimension 1 (except when $\varphi = 0$). Hence we can obtain the right v by choosing an arbitrary nonzero $w \in (\ker \varphi)^{\perp}$ and then multiplying by an appropriate scalar.

Proof. The injectivity is trivial. We prove the surjectivity. Suppose $\varphi \in V'$. The case where $\varphi = 0$ is trivial. Thus assume $\varphi \neq 0$. Hence $\ker \varphi \neq V$, which implies that $(\ker \varphi)^{\perp} \neq \{0\}$. Let $w \in (\ker \varphi)^{\perp}$ be such that $w \neq 0$. Let[14]

[14] *Proof Idea.* Apply w to u to find the supposedly right scalar c in $\varphi(u) = \langle u, cw \rangle$.

$$v = \frac{\overline{\varphi(w)}}{\|w\|^2} w. \tag{31.1}$$

Then $v \in (\ker \varphi)^{\perp}$ and $v \neq 0$. Now we prove that $\varphi(u) = \langle u, v \rangle$ for each $u \in V$.

Let $u \in V$. The orthogonal decomposition leads to

$$u = \frac{\langle u, v \rangle}{\|v\|^2} v + \left(u - \frac{\langle u, v \rangle}{\|v\|^2} v \right),$$

where the second term is orthogonal to v and thus in $\ker \varphi$. Applying φ to both sides, we have

$$\varphi(u) = \frac{\langle u, v \rangle}{\|v\|^2} \varphi(v). \tag{31.2}$$

By (31.1), we have

$$\|v\| = \frac{|\varphi(w)|}{\|w\|}, \qquad \varphi(v) = \frac{|\varphi(w)|^2}{\|w\|^2}.$$

Applying them to (31.2) leads to $\varphi(u) = \langle u, v \rangle$, as desired. □

Exercise 32. Suppose V is finite-dimensional and $P \in \mathcal{L}(V)$ is such that $P^2 = P$.

(a) Suppose every vector in $\ker P$ is orthogonal to every vector in $\operatorname{im} P$. Prove that there exists a subspace U of V such that $P = P_U$.[15]

(b) Suppose $\|Pv\| \leq \|v\|$ for every $v \in V$. Prove that there exists a subspace U of V such that $P = P_U$.[16]

[15] *Proof Idea.* Observe that $\ker P = (\operatorname{im} P)^{\perp}$. Thus we prove that P and $P_{\operatorname{im} P}$ agree on $\ker P$ and $\operatorname{im} P$.

[16] *Proof Idea.* Observe that among all v's with the same Pv, the one in $(\ker P)^{\perp}$ is the shortest. Applying this v makes the best use of the inequality. Thus we prove that P and $P_{(\ker P)^{\perp}}$ agree on $\ker P$ and $(\ker P)^{\perp}$.

Propositions 33 (Algebraic properties of the pseudoinverse)**.** *Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$.*

(a) The pseudoinverse of an orthogonal projection is the operator itself.

(b) $\ker T^{\dagger} = (\operatorname{im} T)^{\perp}$ *and* $\operatorname{im} T^{\dagger} = (\ker T)^{\perp}$.

(c) $TT^{\dagger}T = T$ *and* $T^{\dagger}TT^{\dagger} = T^{\dagger}$.

(d) $\left(T^{\dagger}\right)^{\dagger} = T$.

Operators on Inner Product Spaces

From now on, we suppose that V and W are nonzero finite-dimensional inner product spaces over $\mathbb{F}$.

Self-Adjoint and Normal Operators

Exercise 34. Suppose $T \in \mathcal{L}(V, W)$. Prove that under the standard identification of V with V' (by Riesz representation theorem) and W with W', the adjoint map T^* corresponds to the dual map T'. More precisely, prove that

$$T'(\varphi_w) = \varphi_{T^*w}$$

for all $w \in W$.

Remark 35. Furthermore, under this identification of V with V', the orthogonal complement corresponds to the annihilator; the formulas for $\ker T^*$ and $\operatorname{im} T^*$ become identical to the formulas for $\ker T'$ and $\operatorname{im} T'$. Note that orthogonal complements and adjoints are easier to deal with than annihilators and dual maps.

Complexifcation

Definition 36. The complexification of V, denoted by $V_{\mathbb{C}}$, equals $V \times V$ with normal addition and real scalar multiplication for product space. But we write an element (u, v) of $V_{\mathbb{C}}$ as $u + iv$. Complex scalar multiplication is defined by

$$(a + bi)(u + iv) = (au - bv) + i(av + bu)$$

for all $a, b \in \mathbb{R}$ and all $u, v \in V$.[17]

[17] Think of V as a subset of $V_{\mathbb{C}}$ by identifying $u \in V$ with $u + i0$. The construction of $V_{\mathbb{C}}$ from V can then be thought of as generalizing the construction of $\mathbb{C}^n$ from $\mathbb{R}^n$.

Propositions 37 (Properties of complexification)**.** *(a) $\lambda \in \mathbb{R}$ is an eigenvalue of T if and only if λ is an eigenvalue of $T_{\mathbb{C}}$.*

(b) $\lambda \in \mathbb{C}$ *is an eigenvalue of* $T_{\mathbb{C}}$ *if and only if* $\overline{\lambda}$ *is an eigenvalue of* $T_{\mathbb{C}}$.

(c) *The minimal polynomial of* $T_{\mathbb{C}}$ *equals the minimal polynomial of* T.

(d) *Suppose* V *is a real inner product space. For* $u, v, w, x \in V$, *define*

$$\langle u + iv, w + ix \rangle_{\mathbb{C}} = \langle u, w \rangle + \langle v, x \rangle + (\langle v, w \rangle - \langle u, x \rangle)i.$$

Then $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ *makes* $V_{\mathbb{C}}$ *into a complex inner product space. If* $u, v \in V$, *then*

$$\langle u, v \rangle_{\mathbb{C}} = \langle u, v \rangle \quad \text{and} \quad \|u + iv\|_{\mathbb{C}}^2 = \|u\|^2 + \|v\|^2.$$

Proposition 38. *Every operator on a finite-dimensional nonzero vector space has an invariant subspace of dimension* 1 *or* 2.

Proof. The case where $\mathbb{F} = \mathbb{C}$ is trivial. Now assume V is a real vector space and $T \in \mathcal{L}(V)$. Then $T_{\mathbb{C}}$[18] has an eigenvalue $a + bi$ with $a, b \in \mathbb{R}$. Thus there exist $u, v \in V$, not both 0, such that

$$Tu + iTv = (au - bv) + (av + bu)i.$$

Hence $\operatorname{span}(u, v)$ is invariant under T, as desired. $\square$

[18] *Proof Idea.* Field extension.