Linear Algebra Done Right

Notes – Linear Algebra

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Contents

Don’t just read it; fight it!

Ask your own questions.

Look for your own examples.

Discover your own proofs.

Is the hypothesis necessary?

Is the converse true?

What happens in the classical special case?

What about the degenerate cases?

Where does the proof use the hypothesis?

—Paul Holmos

Vector Spaces

Lemma 1 (Linear dependence lemma)**.** *Suppose $v_1, \ldots, v_m$ is a linearly dependent set in V. Then there exists $k \in \{1, 2, \ldots, m\}$ such that*

$$v_k \in \operatorname{span}(v_1, \ldots, v_m).$$

Furthermore, removing the k^{th} term from the list does not change the span.

Theorem 2. *Any two bases of a finite-dimensional vector space have the same length.*[1]

[1] This proposition ensures that the *dimension* of a vector space is well-defined.

Linear Maps

Kernal and Image of Linear Maps

Exercise 3. Suppose U and V are finite-dimensional and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$. Prove that

$$\dim \ker ST \leq \dim \ker S + \dim \ker T.$$

Proof. Restrict to $Z = \ker ST$. By the fundamental theorem of linear maps,

$$\begin{aligned} \dim Z &= \dim T(Z) + \dim \ker T|_Z \\ &\leq \dim T(Z) + \dim \ker T \\ &= \dim ST(Z) + \dim \ker S|_{T(Z)} + \dim \ker T \\ &\leq \dim \ker S + \dim \ker T. \end{aligned}$$ □

Corollary 4 (Sylvester's rank inequality)**.** *Suppose $A \in \mathbb{F}^{m,n}$ and $B \in \mathbb{F}^{n,p}$ are two matrices. Then*

$$\operatorname{rank} A + \operatorname{rank} B - n \leq \operatorname{rank}(AB).$$

Products and Quotients of Vector Spaces

Exercise 5. Suppose $V_1, \ldots, V_m$ are vector spaces. Prove that $\mathcal{L}(V_1 \times \cdots \times V_m, W)$ and $\mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$ are isomorphic vector spaces.

Proof. We construct an isomorphism T between the two vector spaces.

For every $\Gamma \in \mathcal{L}(V_1 \times \cdots \times V_m, W)$, define $\varphi_k : V_k \to W$ for each k by

$$\varphi_k(v_k) = \Gamma(0, \ldots, v_k, \ldots, 0)$$

with v_k in the k^{th} slot and 0 in all other slots. It can be verified that $\varphi_k \in \mathcal{L}(V_k, W)$.

Define T by $T(\Gamma) = (\varphi_1, \ldots, \varphi_m)$. It can be verified that T is a linear map. We prove T is an isomorphism by constructing its inverse linear map S.

For every $(\varphi_1, \ldots, \varphi_m) \in \mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$, let

$$S(\varphi_1, \ldots, \varphi_m)(v_1, \ldots, v_m) = \varphi_1(v_1) + \cdots + \varphi_m(v_m).$$

It can be shown that S is a linear map, and that $S \circ T = I$ and $T \circ S = I$. That proves T is indeed an isomorphism between the two vector spaces. □

Proposition 6. *A nonempty subset A of V is a translate of some subspace of V if and only if $\lambda v + (1 - \lambda) w \in A$ for all $v, w \in A$ and all $\lambda \in \mathbb{F}$.*

Exercise 7. Suppose $A_1 = v + U_1$ and $A_2 = w + U_2$ for some $v, w \in V$ and some subspaces U_1, U_2 of V. Prove that $A_1 \cap A_2$ is either the empty set or a translate of some subspace of V.[2]

[2] Recall Proposition 6.

Proposition 8. *Suppose U is a subspace of V and $v_1 + U, \ldots, v_m + U$ is a basis of V/U and $u_1, \ldots, u_n$ is a basis of U. Then $v_1, \ldots, v_m, u_1, \ldots, u_n$ is a basis of V. In other words, $V = \operatorname{span}(v_1, \ldots, v_m) \oplus U$.*[3]

[3] $V = \operatorname{span}(v_1, \ldots, v_m) \oplus U$ still holds without the hypothesis that U is finite-dimensional.

Exercise 9. Suppose U is a subspace of V such that V/U is finite-dimensional.

(a) Prove that if W is a finite-dimensional subspace of V and $V = U + W$, then $\dim W \geq \dim V/U$.

(b) Prove that there exists a finite-dimensional subspace W of V such that $V = U \oplus W$ and $\dim W = \dim V/U$.

Proof. Let $\overline{w}_1 + U, \ldots, \overline{w}_m + U$ be a basis of V/U. Then by Proposition 8, we have $V = \operatorname{span}(\overline{w}_1, \ldots, \overline{w}_m) \oplus U$. Let $W_0 = \operatorname{span}(\overline{w}_1, \ldots, \overline{w}_m)$, then $V = U \oplus W_0$, as desired.

Now we prove that for each subspace W of V such that $V = U + W$, we have $\dim W \geq m = \dim V/U$.

For each $\overline{w}_i \in V$ above, by definition we have $\overline{w}_i = u_i + w_i$ for some $u_i \in U$ and $w_i \in W$. It can be shown from the linear independence of $\overline{w}_1 + U, \ldots, \overline{w}_m + U$ that $\overline{w}_1 - u_1, \ldots, \overline{w}_m - u_m$ are independent vectors in W. Hence $\dim W \geq m$. □

Duality

Theorem 10. *Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then*

$$T \text{ is surjective} \iff T' \text{ is injective} \quad \text{and} \quad T \text{ is injective} \iff T' \text{ is surjective.}$$ [4]

[4] This result can be useful because sometimes it is easier to verify that T' is injective (or surjective) than to show directly that T is surjective (or injective).

Proposition 11. *Suppose V is finite-dimensional and U is a subspace of V. Then*

$$U = \left\{ v \in V : \varphi(v) = 0 \text{ for every } \varphi \in U^0 \right\}.$$

Exercise 12. Suppose V is finite-dimensional and U and W are subspaces of V.

(a) Prove that $W^0 \subseteq U^0$ if and only if $U \subseteq W$.

(b) Prove that $W^0 = U^0$ if and only if $U = W$.[5]

[5] Recall Proposition 11.

Exercise 13. Suppose V is finite-dimensional and U and W are subspaces of V.

(a) Prove that $(U + W)^0 = U^0 \cap W^0$.

(b) Prove that $(U \cap W)^0 = U^0 + W^0$.

Proposition 14. *Suppose V is finite-dimensional and $v_1, \ldots, v_n$ is a basis of V. Then $\varphi_1, \ldots, \varphi_n \in V'$ is the dual basis of $v_1, \ldots, v_n$ if and only if*

$$\begin{bmatrix} \mathcal{M}(\varphi_1, (v_1, \ldots, v_n)) \\ \vdots \\ \mathcal{M}(\varphi_n, (v_1, \ldots, v_n)) \end{bmatrix} = I.$$

Exercise 15. Suppose V is finite-dimensional and $\varphi_1, \ldots, \varphi_n$ is a basis of V'. Prove that there exists a basis of V whose dual basis is precisely $\varphi_1, \ldots, \varphi_n$.

Proof. We start from an arbitrary basis $u_1, \ldots, u_n$ of V. Let $\psi_1, \ldots, \psi_n$ be its dual basis. In this proof, we take standard basis $e_1, \ldots, e_n$ as the basis of $\mathbb{F}^n$.

Define $S, T \in \mathcal{L}(V, \mathbb{F}^n)$ by

$$T(v) = (\varphi_1(v), \ldots, \varphi_n(v)), \quad S(v) = (\psi_1(v), \ldots, \psi_n(v)).$$

Then by Proposition 14, $\mathcal{M}(S, (u_1, \ldots, u_n)) = I$.

Let A be the change of basis matrix from ψ's to φ's, i.e.,

$$\begin{bmatrix} \varphi_1 & \cdots & \varphi_n \end{bmatrix} = \begin{bmatrix} \psi_1 & \cdots & \psi_n \end{bmatrix} A.$$

Then by the definition of change of basis matrix, we have

$$\mathcal{M}(T, (u_1, \ldots, u_n)) = \begin{bmatrix} \mathcal{M}(\varphi_1, (u_1, \ldots, u_n)) \\ \vdots \\ \mathcal{M}(\varphi_n, (u_1, \ldots, u_n)) \end{bmatrix} = A^t \begin{bmatrix} \mathcal{M}(\psi_1, (u_1, \ldots, u_n)) \\ \vdots \\ \mathcal{M}(\psi_n, (u_1, \ldots, u_n)) \end{bmatrix}$$

$$= A^t \cdot \mathcal{M}(S, (u_1, \ldots, u_n)) = A^t.$$

Consider basis $v_1, \ldots, v_n$ of V such that the change of basis matrix from u's to v's is $\left(A^t\right)^{-1}$.[6] Thus

$$\mathcal{M}(T, (v_1, \ldots, v_n)) = \mathcal{M}(T, (u_1, \ldots, u_n)) \cdot \mathcal{M}(I, (v_1, \ldots, v_n), (u_1, \ldots, u_n)) = I.$$

Then by Proposition 14, the dual basis of $v_1, \ldots, v_n$ is precisely $\varphi_1, \ldots, \varphi_n$, as desired. $\square$

[6] *Proof Idea.* The change of basis for $V' \to V'$ corresponds to the transpose of $V \leftarrow V$, where transpose and inverse both come from duality.

Exercise 16 (A natural isomorphism from primal space onto double dual space)**.** [7,8] Define $\Lambda : V \to V''$ by

$$(\Lambda v)(\varphi) = \varphi(v)$$

for each $v \in V$ and $\varphi \in V'$.

(a) Prove that if $T \in \mathcal{L}(V)$, then $T'' \circ \Lambda = \Lambda \circ T$.

(b) Prove that if V is finite-dimensional, then Λ is an isomorphism from V onto V''.

[7] Suppose V is finite-dimensional. Then V and V' are isomorphic, but finding an isomorphism from V onto V' generally requires choosing a basis of V. In contrast, the isomorphism Λ from V onto V'' does not require a choice of basis and thus is more natural.

[8] Another natural isomorphism is $\pi' \in \mathcal{L}((V/U)', V')$ where π is the normal quotient map.

Polynomials

Theorem 17. *Suppose $p \in \mathcal{P}(\mathbb{F})$ is a nonconstant polynomial of degree m. Then p has at most m zeros in $\mathbb{F}$.*[9,10]

[9] This theorem implies that when a polynomial p has too many zeros, $p = 0$.

[10] This theorem implies that the coefficients of a polynomial are uniquely determined. In particular, the *degree* of a polynomial is well-defined.

Theorem 18 (Division algorithm for polynomials)**.** *Suppose that $p, s \in \mathcal{P}(\mathbb{F})$, with $s \neq 0$. Then there exist unique polynomials $q, r \in \mathcal{P}(\mathbb{F})$ such that $p = sq + r$.*

Proof. Let $n = \deg p$ and $m = \deg s$. The case where $n < m$ is trivial. Thus we now assume that $n \geq m$.

The list

$$1, z, \ldots, z^{m-1}, s, zs, \ldots, z^{n-m}s$$

is linearly independent in $\mathcal{P}_n(\mathbb{F})$. And it also has length $n+1$. Hence the list is a basis of $\mathcal{P}_n(\mathbb{F})$.

Because $p \in \mathcal{P}_n(\mathbb{F})$, there exist unique constants $a_0, \ldots, a_{m-1}, b_0, \ldots, b_{n-m} \in \mathbb{F}$ such that

$$\begin{aligned} p &= a_0 + a_1 z + \cdots + a_{m-1}z^{m-1} + b_0 s + b_1 zs + \cdots + b_{n-m}z^{n-m}s \\ &= \left(a_0 + a_1 z + \cdots + a_{m-1}z^{m-1}\right) + s\left(b_0 + b_1 z + \cdots + b_{n-m}z^{n-m}\right). \end{aligned}$$

□

Theorem 19 (Fundamental theorem of algebra, first version)**.** *Every nonconstant polynomial with complex coefficients has a zero in* $\mathbb{C}$.

Proof. Suppose $p \in \mathcal{P}(\mathbb{C})$ is a nonconstant polynomial with highest-order nonzero term $c_m z^m$. Then $|p(z)| \to \infty$ as $|z| \to \infty$. Thus the continuous function $z \mapsto |p(z)|$ has a global minimum at some $\zeta \in \mathbb{C}$. Assume that $p(\zeta) \neq 0$.

Consider polynomial $q(z) = p(z+\zeta)/p(\zeta)$. The function $z \mapsto |q(z)|$ has a global minimum at $z = 0$. Write

$$q(z) = 1 + a_k z^k + \cdots + a_m z^m$$

where k is the smallest positive integer such that the coefficient of z_k is nonzero.

Let β be a k^{th} root of $-1/a_k$. There is a constant $c > 1$ such that if $t \in (0,1)$, then

$$|q(t\beta)| \leq \left|1 + a_k t^k \beta^k\right| + ct^{k+1} = 1 - t^k(1 - tc).$$

Thus taking t to be $1/(2c)$ leads to $|q(t\beta)| < 1$.[11] The contradiction implies that $p(\zeta) = 0$, as desired. □

[11] *Proof Idea.* z^k is the leading term in this infinitesimal. We hope its coefficient is negative real, and thus use de Moivre's theorem.

Exercise 20. Suppose $p, q \in \mathcal{P}(\mathbb{C})$ are nonconstant polynomials with no zeros in common. Let $m = \deg p$ and $n = \deg q$. Prove that there exist $r \in \mathcal{P}_{n-1}(\mathbb{C})$ and $s \in \mathcal{P}_{m-1}(\mathbb{C})$ such that $rp + sq = 1$.

Proof. Define $T : \mathcal{P}_{n-1}(\mathbb{C}) \times \mathcal{P}_{m-1}(\mathbb{C}) \to \mathcal{P}_{m+n-1}(\mathbb{C})$ by $T(r,s) = rp + sq$. It can be shown that T is an injective linear map. Because the domain space and target space have the same dimension, T is surjective, completing the proof. □

Eigenvalues and Eigenvectors

Invariant Subspaces

Exercise 21. Suppose that $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ are pairwise distinct. Prove that the list $e^{\lambda_1 x}, \ldots, e^{\lambda_n x}$ is linearly independent in $\mathbb{R}^{\mathbb{R}}$.

Proof. Let $V = \operatorname{span}(e^{\lambda_1 x}, \ldots, e^{\lambda_n x})$.[12] Define $D \in \mathcal{L}(V)$ by $Df = f'$. Then $e^{\lambda x}$ is an eigenvector of D corresponding to λ. A list of eigenvectors corresponding to distinct eigenvalues is linearly independent. □

[12] Alternatively we can let $V = D(\mathbb{R})$.

Exercise 22. Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and U is a subspace of V invariant under T. Prove that each eigenvalue of the quotient operator T/U is an eigenvalue of T.

Proof. It suffices to show that $T/U - \lambda I = (T - \lambda I)/U$ is not injective $\implies T - \lambda I$ is not injective. We prove that $T - \lambda I$ is invertible $\implies (T - \lambda I)/U$ is injective.

Suppose $T - \lambda I$ is invertible. U being invariant under T implies that U is invariant under $T - \lambda I$. Thus $(T - \lambda I)v \in U \iff v \in U$. Suppose $((T - \lambda I)/U)(v + U) = 0$. Then $(T - \lambda I)v \in U$, which implies that $v \in U$, i.e., $v + U = 0$. That proves the injectivity of $(T - \lambda I)/U$. $\square$

Exercise 23. Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T has an eigenvalue if and only if there exists a subspace of V of dimension $\dim V - 1$ that is invariant under T.[13]

[13] *Proof Idea.* Consider the zero entries in $\mathcal{M}(T)$ and its transpose matrix $\mathcal{M}(T')$.

Proof. We first suppose that T has an eigenvalue λ. Then λ is an eigenvalue of T'. There exists $\varphi \in V'$ such that $\varphi \circ T = T'\varphi = \lambda\varphi$. Extend φ to a basis $\varphi, \varphi_2, \ldots, \varphi_n$ of V' and let $v, v_2, \ldots, v_n$ be the basis of V whose dual basis is $\varphi, \varphi_2, \ldots, \varphi_n$. Then $(\varphi \circ T)v_k = \lambda\varphi(v_k) = 0$ for every k. Because $\varphi(Tv_k) = 0$ for every k, we have $Tv_k \in \operatorname{span}(v_2, \ldots, v_n)$. That proves that $\operatorname{span}(v_2, \ldots, v_n)$ is invariant under T.

Reversing the steps above leads to an eigenvector of T', completing the proof. $\square$

Minimal Polynomials

Exercise 24 (Companion matrix of a polynomial)**.** Suppose $a_0, \ldots, a_{n-1} \in \mathbb{F}$. Let $T \in \mathcal{L}(\mathbb{F}^n)$ be such that $\mathcal{M}(T)$ (with respect to the standard basis) is

$$\begin{bmatrix} 0 & & & & & -a_0 \\ 1 & 0 & & & & -a_1 \\ & 1 & 0 & & & -a_2 \\ & & \ddots & & & \vdots \\ & & & & 0 & -a_{n-2} \\ & & & & 1 & -a_{n-1} \end{bmatrix}$$

Prove that the minimal polynomial of T is the polynomial[14]

$$a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n.$$

[14] This exercise implies that every monic polynomial is the minimal polynomial of some operator. Hence an algorithm that could produce exact eigenvalues for each operators on each $\mathbb{F}^n$ does not exist.

Exercise 25. Prove that every operator on a finite-dimensional vector space of dimension at least 2 has an invariant subspace of dimension 2.

Proof. Let $T \in \mathcal{L}(V)$ and $n = \dim V$. We use induction on n. The base case $n = 2$ is trivial. Now suppose $n > 2$ and the desired result holds for all smaller positive integers. Let p be the minimal polynomial of T.

If T has an eigenvalue λ, then $p(z) = q(z)(z - \lambda)$ for some monic polynomial q with $\deg q = \deg p - 1$. Because $q(T)|_{\operatorname{im}(T - \lambda I)} = 0$, the desired result holds by induction hypothesis if $\dim \operatorname{im}(T - \lambda I) \geq 2$. If $T - \lambda I = 0$ the desired result trivially holds. If $\dim \operatorname{im}(T - \lambda I) = 1$, then $(T - \lambda I)v$ is a scalar multiple of some fixed $u \in V$ for all $v \in V$. Take $w \in V \setminus \operatorname{span}(u)$ and $\operatorname{span}(u, w)$ will satisfy the desired property.

If T has no eigenvalues, then $\mathbb{F} = \mathbb{R}$ and $p(z) = q(z)(z^2 + bz + c)$ for some $b, c \in \mathbb{R}$ with $b^2 < 4c$ and monic polynomial q with $\deg q = \deg p - 2$. If $\dim \operatorname{im}(T^2 + bT + cI) \geq 2$ the desired result holds by induction hypothesis. If $T^2 + bT + cI = 0$, then let $w \in V$ be such that $w \neq 0$. It can be verified that $\operatorname{span}(w, Tw)$ is invariant under T. If $\dim \operatorname{im}(T^2 + bT + cI) = 1$, because $\dim \ker(T^2 + bT + cI)$ is even, n is odd, which implies that T has an eigenvalue. That completes the proof. □

Commuting Operators

Exercise 26. Suppose $\mathcal{E} \subseteq \mathcal{L}(V)$ and every element of $\mathcal{E}$ is diagonalizable. Prove that there exists a basis of V with respect to which every element of $\mathcal{E}$ has a diagonal matrix if and only if every pair of elements of $\mathcal{E}$ commutes.[15]

Proof. Suppose every pair of elements of $\mathcal{E}$ commutes. The other direction is trivial. We use induction on $n = \dim V$. The base case $n = 1$ is trivial. Now suppose $n > 1$ and the desired result holds for all smaller integers. Without loss of generality, suppose $\mathcal{E} \cap \{\lambda I : \lambda \in \mathbb{F}\} = \emptyset$, or else consider $\mathcal{E} \setminus \{\lambda I : \lambda \in \mathbb{F}\}$.

Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of $T \in \mathcal{E}$. Then $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$. Because $E(\lambda_k, T)$ is invariant under every $S \in \mathcal{E}$, it suffices to show that the desired result holds on $E(\lambda_1, T)$. Because $E(\lambda_1, T) \subsetneq V$, it holds by induction hypothesis, completing the proof. □

Exercise 27. Suppose V is a finite-dimensional nonzero complex vector space. Suppose that $\mathcal{E} \subseteq \mathcal{L}(V)$ is such that S and T commute for all $S, T \in \mathcal{E}$.

(a) Prove that there is a vector in V that is an eigenvector for every element of $\mathcal{E}$.

(b) Prove that there exists a basis of V with respect to which every element of $\mathcal{E}$ has an upper-triangular matrix.[16]

Inner Product Spaces

Inner Products and Norms

Theorem 28 (Polarization identities)**.** *(a) Suppose V is a real inner product space. Then*

$$\langle u, v \rangle = \frac{\|u + v\|^2 - \|u - v\|^2}{4}.$$

(b) Suppose V is a complex inner product space. Then

$$\langle u, v \rangle = \frac{\|u + v\|^2 - \|u - v\|^2 + \|u + iv\|^2 i - \|u - iv\|^2 i}{4}.$$

Exercise 29. Prove that if $\|\cdot\|$ is a norm on U satisfying the parallelogram equality, then there is an inner product $\langle \cdot, \cdot \rangle$ on U such that $\|u\| = \langle u, u \rangle^{1/2}$ for all $u \in U$.[17]

[15] This is an extension of simultaneous diagonalizability to more than 2 (possibly infinitely many) operators.

[16] This is an extension of simultaneous upper triangularizability to more than 2 (possibly infinitely many) operators.

[17] Recall Theorems 28.

Exercise 30. Suppose f, g are differentiable functions from $\mathbb{R}$ to $\mathbb{R}^n$.

(a) Prove that
$$\langle f(t), g(t)\rangle' = \langle f'(t), g(t)\rangle + \langle f(t), g'(t)\rangle .$$

(b) Suppose $\|f(t)\| \equiv c > 0$ for every $t \in \mathbb{R}$. Prove that $\langle f'(t), f(t)\rangle = 0$ for every $t \in \mathbb{R}$.[18]

[18] This result has a nice geometric interpretation.

Orthonormal Bases

Exercise 31. Suppose $v_1, \ldots, v_m$ is a linearly independent list in V. Prove that the orthonormal list produced by the formulas of the Gram-Schmidt procedure is the only orthonormal list $e_1, \ldots, e_m$ in V such that $\langle v_k, e_k\rangle > 0$ for each k.

Exercise 32. Suppose V is finite-dimensional. Suppose $\langle\cdot,\cdot\rangle_1, \langle\cdot,\cdot\rangle_2$ are inner products on V with corresponding norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Prove that there exists $b, c > 0$ such that $b\|v\|_2 \le \|v\|_1 \le c\|v\|_2$ for every $v \in V$.[19]

[19] Note that *Proof 1* does not use the hypothesis that $\|\cdot\|_1$ and $\|\cdot\|_2$ are associated with an inner product respectively.

Proof 1. Let $u_1, \ldots, u_n$ be a basis of V. Without loss of generality, suppose that[20]
$$\|c_1u_1 + \cdots + c_nu_n\|_2 = \max\{|c_1|, \ldots, |c_n|\}.$$

[20] As an alternative to infinity norm, we might also use the 2-norm, which naturally induces an inner product by Exercise 29.

We interpret $\|\cdot\|_1$ as a function $(c_1, \ldots, c_n) \mapsto \|c_1u_1 + \cdots + c_nu_n\|_1$ from $\mathbb{F}^n$ to $\mathbb{R}$. It suffices to show that the desired result holds on the sphere
$$S = \{(c_1, \ldots, c_n) \in \mathbb{F}^n : \|c_1u_1 + \cdots + c_nu_n\|_2 = 1\},$$
i.e. that $\|\cdot\|_1$ is bounded on S.[21] Because S is a closed and bounded set, it suffices to show that the function $\|\cdot\|_1$ is continuous on $\mathbb{F}^n$.

[21] *Proof Idea.* We cannot manipulate two norms simultaneously. Hence we restrict to S and fix an orthonormal basis. Then V behaves like $\mathbb{F}^n$. Nice properties of S lead to proof of the continuity of $\|\cdot\|_1$.

Suppose $a \in \mathbb{F}^n$ and $e_1, \ldots, e_n$ is the standard basis of $\mathbb{F}^n$. As $\|x - a\| \to 0$,
$$\begin{aligned}
|\|x\|_1 - \|a\|_1| \le \|x - a\|_1 &= \left\|\sum_{k=1}^n (x_k - a_k)e_k\right\|_1 \\
&\le \sum_{k=1}^n |x_k - a_k| \, \|e_k\|_1 \\
&\le \left(\sum_{k=1}^n \|e_k\|_1^2\right)^{1/2} \|x - a\| \to 0,
\end{aligned}$$
where the last inequality follows from the Cauchy-Schwarz inequality. Because every continuous function on a closed and bounded set is bounded, the proof is completed. □

Proof 2. Without loss of generality, suppose $V = \mathbb{F}^n$, $\|x\|_1^2 = x^tAx$, and $\|x\|_2^2 = x^tBx$, where A, B are positive-definite and B is diagonal.[22] Note that $\lambda B - A$ is Hermitian and hence diagonalizable. It suffices to show that there exists $\lambda > 0$ such that $\lambda B - A$ is positive-definite. Such a λ exists by the Gershgorin disk theorem, as desired. □

[22] *Proof Idea.* Fixing an orthonormal basis naturally leads to a (simple) diagonal matrix.

Exercise 33. Suppose $\mathbb{F} = \mathbb{C}$ and V is finite-dimensional. Prove that if $T \in \mathcal{L}(V)$ is such that 1 is the only eigenvalue of T and $\|Tv\| \leq \|v\|$ for all $v \in V$, then T is the identity operator.

Proof. By Schur's theorem, there exists an orthonormal basis $e_1, \ldots, e_m$ of V with respect to which T has an upper-triangular matrix. Then all entries on the diagonal of $\mathcal{M}(T)$ is 1. Any $\mathcal{M}(T)_{j,k} \neq 0$ with $j \neq k$ would contradict $\|Te_k\| \leq \|e_k\|$. That proves that $\mathcal{M}(T) = I$, as desired. □

Exercise 34. Suppose $v_1, \ldots, v_n$ is a basis of V. Prove that there exists a basis $u_1, \ldots, u_n$ of V such that[23]

$$\langle u_j, v_k \rangle = \begin{cases} 0 & \text{if } j \neq k, \\ 1 & \text{if } j = k. \end{cases}$$

Proof. Define $\varphi_k \in V'$ for each k by $\varphi_k(u) = \langle u, v_k \rangle$. By the Riesz representation theorem, $\varphi_1, \ldots, \varphi_n$ is a spanning list in V'. Thus it is a basis of V'. Let $u_1, \ldots, u_n$ be the basis of V whose dual basis is $\varphi_1, \ldots, \varphi_n$. Then $u_1, \ldots, u_n$ satisfies the desired property. □

[23] *Proof Idea.* Linearity in inner product and and patterns of $0, 1$ inspire the use of linear functionals.

Orthogonal Complements and Minimization Problems

Theorem 35 (Riesz representation theorem)**.** *Suppose V is finite-dimensional. For each $v \in V$, define $\varphi_v \in V'$ by*

$$\varphi_v(u) = \langle u, v \rangle$$

for each $u \in V$. Then $v \mapsto \varphi_v$ is a one-to-one map from V onto V'.[24]

Proof. The injectivity is trivial. We prove the surjectivity. Suppose $\varphi \in V'$. The case where $\varphi = 0$ is trivial. Thus assume $\varphi \neq 0$. Hence $\ker \varphi \neq V$, which implies that $(\ker \varphi)^\perp \neq \{0\}$. Let $w \in (\ker \varphi)^\perp$ be such that $w \neq 0$. Let[25]

$$v = \frac{\overline{\varphi(w)}}{\|w\|^2} w. \tag{35.1}$$

Then $v \in (\ker \varphi)^\perp$ and $v \neq 0$. Now we prove that $\varphi(u) = \langle u, v \rangle$ for each $u \in V$.

Let $u \in V$. The orthogonal decomposition leads to

$$u = \frac{\langle u, v \rangle}{\|v\|^2} v + \left(u - \frac{\langle u, v \rangle}{\|v\|^2} v \right),$$

where the second term is orthogonal to v and thus in $\ker \varphi$. Applying φ to both sides, we have

$$\varphi(u) = \frac{\langle u, v \rangle}{\|v\|^2} \varphi(v). \tag{35.2}$$

By (35.1), we have

$$\|v\| = \frac{|\varphi(w)|}{\|w\|}, \qquad \varphi(v) = \frac{|\varphi(w)|^2}{\|w\|^2}.$$

Applying them to (35.2) leads to $\varphi(u) = \langle u, v \rangle$, as desired. □

[24] *Proof Idea.* If $\varphi(u) = \langle u, v \rangle$ holds for all $u \in V$, then $v \in (\ker \varphi)^\perp$. However, $(\ker \varphi)^\perp$ has dimension 1 (except when $\varphi = 0$). Hence we can obtain the right v by choosing an arbitrary nonzero $w \in (\ker \varphi)^\perp$ and then multiplying by an appropriate scalar.

[25] *Proof Idea.* Apply w to u to find the supposedly right scalar c in $\varphi(u) = \langle u, cw \rangle$.

Complexifcation

Definition 36. The complexification of V, denoted by $V_{\mathbb{C}}$, equals $V \times V$ with normal addition and real scalar multiplication for product space. But we write an element (u, v) of $V_{\mathbb{C}}$ as $u + iv$. Complex scalar multiplication is defined by

$$(a + bi)(u + iv) = (au - bv) + i(av + bu)$$

for all $a, b \in \mathbb{R}$ and all $u, v \in V$.[26]

[26] Think of V as a subset of $V_{\mathbb{C}}$ by identifying $u \in V$ with $u + i0$. The construction of $V_{\mathbb{C}}$ from V can then be thought of as generalizing the construction of $\mathbb{C}^n$ from $\mathbb{R}^n$.

Propositions 37 (Properties of complexification)**.** *(a)* *$\lambda \in \mathbb{R}$ is an eigenvalue of T if and only if λ is an eigenvalue of $T_{\mathbb{C}}$.*

(b) *$\lambda \in \mathbb{C}$ is an eigenvalue of $T_{\mathbb{C}}$ if and only if $\overline{\lambda}$ is an eigenvalue of $T_{\mathbb{C}}$.*

(c) *The minimal polynomial of $T_{\mathbb{C}}$ equals the minimal polynomial of T.*

(d) *Suppose V is a real inner product space. For $u, v, w, x \in V$, define*

$$\langle u + iv, w + ix \rangle_{\mathbb{C}} = \langle u, w \rangle + \langle v, x \rangle + (\langle v, w \rangle - \langle u, x \rangle)i.$$

Then $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ makes $V_{\mathbb{C}}$ into a complex inner product space. If $u, v \in V$, then

$$\langle u, v \rangle_{\mathbb{C}} = \langle u, v \rangle \quad \text{and} \quad \|u + iv\|_{\mathbb{C}}^2 = \|u\|^2 + \|v\|^2.$$

Proposition 38. *Every operator on a finite-dimensional nonzero vector space has an invariant subspace of dimension 1 or 2.*

Proof. The case where $\mathbb{F} = \mathbb{C}$ is trivial. Now assume V is a real vector space and $T \in \mathcal{L}(V)$. Then $T_{\mathbb{C}}$[27] has an eigenvalue $a + bi$ with $a, b \in \mathbb{R}$. Thus there exist $u, v \in V$, not both 0, such that

[27] *Proof Idea.* Field extension.

$$Tu + iTv = (au - bv) + (av + bu)i.$$

Hence span(u, v) is invariant under T, as desired. □