

*Linear Algebra Done Right*

# Notes – Linear Algebra

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## Vector Spaces

**Definition 1.** The complexification of  $V$ , denoted by  $V_{\mathbb{C}}$ , equals  $V \times V$  with normal addition and real scalar multiplication for product space. But we write an element  $(u, v)$  of  $V_{\mathbb{C}}$  as  $u + iv$ . Complex scalar multiplication is defined by

$$(a + bi)(u + iv) = (au - bv) + i(av + bu)$$

for all  $a, b \in \mathbb{R}$  and all  $u, v \in V$ .<sup>1</sup>

**Lemma 2** (Linear dependence lemma). Suppose  $v_1, \dots, v_m$  is a linearly dependent set in  $V$ . Then there exists  $k \in \{1, 2, \dots, m\}$  such that

$$v_k \in \text{span}(v_1, \dots, v_m).$$

Furthermore, removing the  $k^{\text{th}}$  term from the list does not change the span.<sup>2</sup>

**Theorem 3.** Any two bases of a finite-dimensional vector space have the same length.<sup>3</sup>

*Proof.* Suppose  $V$  is finite-dimensional. Let  $\mathcal{B}_1$  and  $\mathcal{B}_2$  be two bases of  $V$ . Considering  $\mathcal{B}_1$  as an independent set and  $\mathcal{B}_2$  as a spanning set leads to  $\#\mathcal{B}_1 \leq \#\mathcal{B}_2$ . Interchanging the roles of  $\mathcal{B}_1$  and  $\mathcal{B}_2$  and we have  $\#\mathcal{B}_2 \leq \#\mathcal{B}_1$ . Thus  $\#\mathcal{B}_1 = \#\mathcal{B}_2$ .  $\square$

## Linear Maps

### Kernal and Image of Linear Maps

**Exercise 4.** Suppose  $U$  and  $V$  are finite-dimensional and  $S \in \mathcal{L}(V, W)$  and  $T \in \mathcal{L}(U, V)$ . Prove that

$$\dim \ker ST \leq \dim \ker S + \dim \ker T.$$

*Proof.* Restrict to  $Z = \ker ST$ . By the fundamental theorem of linear maps,

$$\begin{aligned} \dim Z &= \dim T(Z) + \dim \ker T|_Z \\ &\leq \dim T(Z) + \dim \ker T \\ &= \dim ST(Z) + \dim \ker S|_{T(Z)} + \dim \ker T \\ &\leq \dim \ker S + \dim \ker T. \end{aligned} \quad \square$$

**Corollary 5** (Sylvester's rank inequality). Suppose  $A \in \mathbb{F}^{m,n}$  and  $B \in \mathbb{F}^{n,p}$  are two matrices. Then<sup>4</sup>

$$\text{rank } A + \text{rank } B - n \leq \text{rank}(AB).$$

## Products and Quotients of Vector Spaces

**Lemma 6.** Suppose  $V_1, \dots, V_m$  are subspaces of  $V$ .<sup>5</sup> Define a linear map  $\Gamma : V_1 \times \dots \times V_m \rightarrow V_1 + \dots + V_m$  by

$$\Gamma(v_1, \dots, v_m) = v_1 + \dots + v_m.$$

Then  $V_1 + \dots + V_m$  is a direct sum if and only if  $\Gamma$  is injective.

<sup>1</sup> Think of  $V$  as a subset of  $V_{\mathbb{C}}$  by identifying  $u \in V$  with  $u + i0$ . The construction of  $V_{\mathbb{C}}$  from  $V$  can then be thought of as generalizing the construction of  $\mathbb{C}^n$  from  $\mathbb{R}^n$ .

<sup>2</sup> This lemma lays the foundation for a series of basic results for vector spaces.

<sup>3</sup> This proposition ensures that the dimension of a vector space is well-defined.

<sup>4</sup> There is a slicker proof for this inequality using block matrices. But the proof here using linear maps is more informative.

<sup>5</sup> Note that  $V$  does not have to be finite-dimensional. Recall that  $V_1 + \dots + V_m$  is a direct sum if and only if the only way to write 0 as a sum of  $v_1 + \dots + v_m$ , where each  $v_k \in V_k$ , is by taking each  $v_k$  equal to 0.

**Theorem 7.** Suppose  $V$  is finite-dimensional and  $V_1, \dots, V_m$  are subspaces of  $V$ . Then  $V_1 + \dots + V_m$  is a direct sum if and only if<sup>6</sup>

<sup>6</sup> Recall Lemma 6.

$$\dim(V_1 + \dots + V_m) = \dim V_1 + \dots + \dim V_m.$$

**Notation 8.** Suppose  $T \in \mathcal{L}(V, W)$ . Define  $\tilde{T} : V + \ker T \rightarrow V$  by

$$\tilde{T}(v + \ker T) = Tv.$$

**Exercise 9.** Suppose  $V_1, \dots, V_m$  are vector spaces. Prove that  $\mathcal{L}(V_1 \times \dots \times V_m, W)$  and  $\mathcal{L}(V_1, W) \times \dots \times \mathcal{L}(V_m, W)$  are isomorphic vector spaces.

*Proof.* We construct an isomorphism  $T$  between the two vector spaces.

For every  $\Gamma \in \mathcal{L}(V_1 \times \dots \times V_m, W)$ , define  $\varphi_i : V_i \rightarrow W$  for each  $i \in \{1, \dots, m\}$  by

$$\varphi_i(v_i) = \Gamma(0, \dots, v_i, \dots, 0)$$

with  $v_i$  in the  $i^{\text{th}}$  slot and 0 in all other slots. It can be verified that  $\varphi_i \in \mathcal{L}(V_i, W)$ .

Define  $T(\Gamma) = (\varphi_1, \dots, \varphi_m)$ . It can be verified that  $T$  is a linear map. We prove  $T$  is an isomorphism by constructing its inverse linear map  $S$ .

For every  $(\varphi_1, \dots, \varphi_m) \in \mathcal{L}(V_1, W) \times \dots \times \mathcal{L}(V_m, W)$ , let

$$S(\varphi_1, \dots, \varphi_m)(v_1, \dots, v_m) = \varphi_1(v_1) + \dots + \varphi_m(v_m).$$

It can be shown that  $S$  is a linear map, and that  $S \circ T = I$  and  $T \circ S = I$ . That proves  $T$  is indeed an isomorphism between the two vector spaces.  $\square$

**Proposition 10.** A nonempty subset  $A$  of  $V$  is a translate of some subspace of  $V$  if and only if  $\lambda v + (1 - \lambda)w \in A$  for all  $v, w \in A$  and all  $\lambda \in \mathbb{F}$ .

**Exercise 11.** Suppose  $A_1 = v + U_1$  and  $A_2 = w + U_2$  for some  $v, w \in V$  and some subspaces  $U_1, U_2$  of  $V$ . Prove that  $A_1 \cap A_2$  is either the empty set or a translate of some subspace of  $V$ .<sup>7</sup>

<sup>7</sup> Recall Proposition 10.

**Proposition 12.** Suppose  $U$  is a subspace of  $V$  and  $v_1 + U, \dots, v_m + U$  is a basis of  $V/U$  and  $u_1, \dots, u_n$  is a basis of  $U$ . Then  $v_1, \dots, v_m, u_1, \dots, u_n$  is a basis of  $V$ . In other words,  $V = \text{span}(v_1, \dots, v_m) \oplus U$ .<sup>8</sup>

<sup>8</sup>  $V = \text{span}(v_1, \dots, v_m) \oplus U$  still holds without the hypothesis that  $U$  is finite-dimensional.

**Exercise 13.** Suppose  $U$  is a subspace of  $V$  such that  $V/U$  is finite-dimensional.

- Prove that if  $W$  is a finite-dimensional subspace of  $V$  and  $V = U + W$ , then  $\dim W \geq \dim V/U$ .
- Prove that there exists a finite-dimensional subspace  $W$  of  $V$  such that  $V = U \oplus W$  and  $\dim W = \dim V/U$ .

*Proof.* Let  $\bar{w}_1 + U, \dots, \bar{w}_m + U$  be a basis of  $V/U$ . Then by Proposition 12, we have  $V = \text{span}(\bar{w}_1, \dots, \bar{w}_m) \oplus U$ . Let  $W_0 = \text{span}(\bar{w}_1, \dots, \bar{w}_m)$ , then  $V = U \oplus W_0$ , as desired.

Now we prove that for each subspace  $W$  of  $V$  such that  $V = U + W$ , we have  $\dim W \geq m = \dim V/U$ .

For each  $\bar{w}_i \in V$  above, by definition we have  $\bar{w}_i = u_i + w_i$  for some  $u_i \in U$  and  $w_i \in W$ . It can be shown from the linear independence of  $\bar{w}_1 + U, \dots, \bar{w}_m + U$  that  $\bar{w}_1 - u_1, \dots, \bar{w}_m - u_m$  are independent vectors in  $W$ . Hence  $\dim W \geq m$ .  $\square$

## Duality

**Theorem 14.** Suppose  $V$  and  $W$  are finite-dimensional and  $T \in \mathcal{L}(V, W)$ . Then

$$T \text{ is surjective} \iff T' \text{ is injective} \quad \text{and} \quad T \text{ is injective} \iff T' \text{ is surjective.}^9$$

**Proposition 15.** Suppose  $V$  is finite-dimensional and  $U$  is a subspace of  $V$ . Then

$$U = \left\{ v \in V : \varphi(v) = 0 \text{ for every } \varphi \in U^0 \right\}.$$

**Exercise 16.** Suppose  $V$  is finite-dimensional and  $U$  and  $W$  are subspaces of  $V$ .

- (a) Prove that  $W^0 \subseteq U^0$  if and only if  $U \subseteq W$ .
- (b) Prove that  $W^0 = U^0$  if and only if  $U = W$ .<sup>10</sup>

<sup>9</sup> This result can be useful because sometimes it is easier to verify that  $T'$  is injective (surjective) than to show directly that  $T$  is surjective (injective).

<sup>10</sup> Recall Proposition 15.

**Exercise 17.** Suppose  $V$  is finite-dimensional and  $U$  and  $W$  are subspaces of  $V$ .

- (a) Prove that  $(U + W)^0 = U^0 \cap W^0$ .
- (b) Prove that  $(U \cap W)^0 = U^0 + W^0$ .

**Proposition 18.** Suppose  $V$  is finite-dimensional and  $v_1, \dots, v_n$  is a basis of  $V$ . Then  $\varphi_1, \dots, \varphi_n \in V'$  is the dual basis of  $v_1, \dots, v_n$  if and only if

$$\begin{bmatrix} \mathcal{M}(\varphi_1, (v_1, \dots, v_n)) \\ \vdots \\ \mathcal{M}(\varphi_n, (v_1, \dots, v_n)) \end{bmatrix} = I.$$

**Exercise 19.** Suppose  $V$  is finite-dimensional and  $\varphi_1, \dots, \varphi_n$  is a basis of  $V'$ . Prove that there exists a basis of  $V$  whose dual basis is  $\varphi_1, \dots, \varphi_n$ .

*Proof.* We start from an arbitrary basis  $u_1, \dots, u_n$  of  $V$ . Let  $\psi_1, \dots, \psi_n$  be its dual basis. In this proof, we take standard basis  $e_1, \dots, e_n$  as the basis of  $\mathbb{F}^n$ .

Define  $S, T \in \mathcal{L}(V, \mathbb{F}^n)$  by

$$T(v) = (\varphi_1(v), \dots, \varphi_n(v)), \quad S(v) = (\psi_1(v), \dots, \psi_n(v)).$$

Then by Proposition 18,  $\mathcal{M}(S, (u_1, \dots, u_n)) = I$ .

Let  $A$  be the change of basis matrix from  $\psi$ 's to  $\varphi$ 's, i.e.,

$$\begin{bmatrix} \varphi_1 & \cdots & \varphi_n \end{bmatrix} = \begin{bmatrix} \psi_1 & \cdots & \psi_n \end{bmatrix} A.$$

Then by the definition of change of basis matrix, we have

$$\begin{aligned} \mathcal{M}(T, (u_1, \dots, u_n)) &= \begin{bmatrix} \mathcal{M}(\varphi_1, (u_1, \dots, u_n)) \\ \vdots \\ \mathcal{M}(\varphi_n, (u_1, \dots, u_n)) \end{bmatrix} = A^t \begin{bmatrix} \mathcal{M}(\psi_1, (u_1, \dots, u_n)) \\ \vdots \\ \mathcal{M}(\psi_n, (u_1, \dots, u_n)) \end{bmatrix} \\ &= A^t \cdot \mathcal{M}(S, (u_1, \dots, u_n)) = A^t. \end{aligned}$$

Consider basis  $v_1, \dots, v_n$  of  $V$  such that the change of basis matrix from  $u$ 's to  $v$ 's is  $(A^t)^{-1}$ .<sup>11</sup> Thus

$$\mathcal{M}(T, (v_1, \dots, v_n)) = \mathcal{M}(T, (u_1, \dots, u_n)) \cdot \mathcal{M}(I, (v_1, \dots, v_n), (u_1, \dots, u_n)) = I.$$

Then by Proposition 18, the dual basis of  $v_1, \dots, v_n$  is precisely  $\varphi_1, \dots, \varphi_n$ , as desired.  $\square$

**Exercise 20** (A natural isomorphism from primal space onto double dual space). Define  $\Lambda : V \rightarrow V''$  by

$$(\Lambda v)(\varphi) = \varphi(v)$$

for each  $v \in V$  and  $\varphi \in V'$ .

- (a) Prove that if  $T \in \mathcal{L}(V)$ , then  $T'' \circ \Lambda = \Lambda \circ T$ .
- (b) Prove that if  $V$  is finite-dimensional, then  $\Lambda$  is an isomorphism from  $V$  to  $V''$ .<sup>12</sup>

## Polynomials

**Theorem 21.** Suppose  $p \in \mathcal{P}(\mathbb{F})$  is a nonconstant polynomial of degree  $m$ . Then  $\lambda \in \mathbb{F}$  is a zero of  $p$  if and only if there exists a polynomial  $q \in \mathcal{P}(\mathbb{F})$  of degree  $m - 1$  such that  $p(z) = (z - \lambda)q(z)$  for every  $z \in \mathbb{F}$ .

**Theorem 22.** Suppose  $p \in \mathcal{P}(\mathbb{F})$  is a nonconstant polynomial of degree  $m$ . Then  $p$  has at most  $m$  zeros in  $\mathbb{F}$ .<sup>13,14</sup>

**Theorem 23** (Division algorithm for polynomials). Suppose that  $p, s \in \mathcal{P}(\mathbb{F})$ , with  $s \neq 0$ . Then there exist unique polynomials  $q, r \in \mathcal{P}(\mathbb{F})$  such that  $p = sq + r$ .

*Proof.* Let  $n = \deg p$  and  $m = \deg s$ . The case where  $n < m$  is trivial. Thus we now assume that  $n \geq m$ .

The list

$$1, z, \dots, z^{m-1}, s, zs, \dots, z^{n-m}s$$

is linearly independent in  $\mathcal{P}_n(\mathbb{F})$ . And it also has length  $n + 1$ . Hence the list is a basis of  $\mathcal{P}_n(\mathbb{F})$ .

Because  $p \in \mathcal{P}_n(\mathbb{F})$ , there exist unique constants  $a_0, \dots, a_{m-1}, b_0, \dots, b_{n-m} \in \mathbb{F}$  such that

$$\begin{aligned} p &= a_0 + a_1z + \dots + a_{m-1}z^{m-1} + b_0s + b_1zs + \dots + b_{n-m}z^{n-m}s \\ &= (a_0 + a_1z + \dots + a_{m-1}z^{m-1}) + s(b_0 + b_1z + \dots + b_{n-m}z^{n-m}). \end{aligned} \quad \square$$

**Theorem 24** (Fundamental theorem of algebra, first version). Every nonconstant polynomial with complex coefficients has a zero in  $\mathbb{C}$ .

*Proof.* Suppose  $p \in \mathcal{P}(\mathbb{C})$  is a nonconstant polynomial with highest-order nonzero term  $c_m z^m$ . Then  $|p(z)| \rightarrow \infty$  as  $|z| \rightarrow \infty$ . Thus the continuous function  $z \mapsto |p(z)|$  has a global minimum at some  $\zeta \in \mathbb{C}$ . Assume that  $p(\zeta) \neq 0$ .

<sup>11</sup> The change of basis for  $V' \rightarrow V'$  corresponds to the transpose of  $V \leftarrow V$ , where transpose and inverse both come from duality. That gives the idea of considering  $(A^t)^{-1}$ .

<sup>12</sup> Suppose  $V$  is finite-dimensional. Then  $V$  and  $V'$  are isomorphic, but finding an isomorphism from  $V$  onto  $V'$  generally requires choosing a basis of  $V$ . In contrast, the isomorphism  $\Lambda$  from  $V$  onto  $V''$  does not require a choice of basis and thus is more natural.

<sup>13</sup> This theorem indicates that when a polynomial  $p$  has too many zeros,  $p = 0$ .

<sup>14</sup> This theorem implies that the coefficients of a polynomial are uniquely determined. In particular, the degree of a polynomial is well-defined.

Consider polynomial  $q(z) = p(z + \zeta)/p(\zeta)$ . The function  $z \mapsto |q(z)|$  has a global minimum at  $z = 0$ . Write

$$q(z) = 1 + a_k z^k + \cdots + a_m z^m$$

where  $k$  is the smallest positive integer such that the coefficient of  $z_k$  is nonzero.

Let  $\beta$  be a  $k^{\text{th}}$  root of  $-1/a_k$ . There is a constant  $c > 1$  such that if  $t \in (0, 1)$ , then

$$|q(t\beta)| \leq \left| 1 + a_k t^k \beta^k \right| + c t^{k+1} = 1 - t^k(1 - tc).$$

Thus taking  $t$  to be  $1/(2c)$  leads to  $|q(t\beta)| < 1$ . The contradiction implies that  $p(\zeta) = 0$ , as desired.  $\square$

**Theorem 25** (Fundamental theorem of algebra, second version). *If  $p \in \mathcal{P}(\mathbb{C})$  is a nonconstant polynomial, then  $p$  has a unique factorization of the form*

$$p(z) = c(z - \lambda_1) \cdots (z - \lambda_m),$$

where  $c, \lambda_1, \dots, \lambda_m \in \mathbb{C}$ .

**Theorem 26** (Factorization of a polynomial over  $\mathbb{R}$ ). *If  $p \in \mathcal{P}(\mathbb{R})$  is a nonconstant polynomial, then  $p$  has a unique factorization of the form*

$$p(x) = c(x - \lambda_1) \cdots (x - \lambda_m)(x^2 + b_1x + c_1) \cdots (x^2 + b_Mx + c_M),$$

where  $m, M \in \mathbb{N}$  and  $c, \lambda_1, \dots, \lambda_m, b_1, \dots, b_M, c_1, \dots, c_M \in \mathbb{R}$ , with  $b_k^2 < 4c_k$  for each  $k$ .

**Exercise 27.** Suppose  $p, q \in \mathcal{P}(\mathbb{C})$  are nonconstant polynomials with no zeros in common. Let  $m = \deg p$  and  $n = \deg q$ . Prove that there exist  $r \in \mathcal{P}_{n-1}(\mathbb{C})$  and  $s \in \mathcal{P}_{m-1}(\mathbb{C})$  such that  $rp + sq = 1$ .

*Proof.* Define  $T : \mathcal{P}_{n-1}(\mathbb{C}) \times \mathcal{P}_{m-1}(\mathbb{C}) \rightarrow \mathcal{P}_{m+n-1}(\mathbb{C})$  by  $T(r, s) = rp + sq$ . It can be shown that  $T$  is an injective linear map. Because the domain space and target space have the same dimension,  $T$  is surjective, completing the proof.  $\square$

## Eigenvalues and Eigenvectors