

Notes – Linear Algebra

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Don't just read it; fight it!
Ask your own questions.
Look for your own examples.
Discover your own proofs.
Is the hypothesis necessary?
Is the converse true?
What happens in the classical special case?
What about the degenerate cases?
Where does the proof use the hypothesis?

— Paul Holmos

Linear Maps

Products and Quotients of Vector Spaces

Proposition 1. *A nonempty subset A of V is a translate of some subspace of V if and only if $\lambda v + (1 - \lambda)w \in A$ for all $v, w \in A$ and all $\lambda \in \mathbb{F}$.*

Exercise 2. Suppose U is a subspace of V such that V/U is finite-dimensional.

- (a) Prove that if W is a finite-dimensional subspace of V and $V = U + W$, then $\dim W \geq \dim V/U$.
- (b) Prove that there exists a finite-dimensional subspace W of V such that $V = U \oplus W$ and $\dim W = \dim V/U$.

Proof. Let $\bar{w}_1 + U, \dots, \bar{w}_m + U$ be a basis of V/U . It can be shown that $V = \text{span}(\bar{w}_1, \dots, \bar{w}_m) \oplus U$. Let $W_0 = \text{span}(\bar{w}_1, \dots, \bar{w}_m)$. Then $V = U \oplus W_0$, as desired.

Now we prove that for each subspace W of V such that $V = U + W$, we have $\dim W \geq m = \dim V/U$. For each \bar{w}_k above, by definition we have $\bar{w}_k = u_k + w_k$ for some $u_k \in U$ and $w_k \in W$. It can be shown from the linear independence of $\bar{w}_1 + U, \dots, \bar{w}_m + U$ that $\bar{w}_1 - u_1, \dots, \bar{w}_m - u_m$ are independent vectors in W . Hence $\dim W \geq m$. \square

Duality

Proposition 3. *Suppose V is finite-dimensional and U is a subspace of V . Then¹*

$$U = \left\{ v \in V : \varphi(v) = 0 \text{ for every } \varphi \in U^0 \right\}.$$

¹ Compare this result with $(U^\perp)^\perp = U$ where U is a finite-dimensional subspace of inner product space V .

Exercise 4. Suppose V is finite-dimensional and $\varphi_1, \dots, \varphi_n$ is a basis of V' . Prove that there exists a basis of V whose dual basis is precisely $\varphi_1, \dots, \varphi_n$.

Proof. We start from an arbitrary basis u_1, \dots, u_n of V . Let ψ_1, \dots, ψ_n be its dual basis. In this proof, we take the standard basis e_1, \dots, e_n as the basis of \mathbb{F}^n .

Define $S, T \in \mathcal{L}(V, \mathbb{F}^n)$ by

$$T(v) = (\varphi_1(v), \dots, \varphi_n(v)), \quad S(v) = (\psi_1(v), \dots, \psi_n(v)).$$

Then $\mathcal{M}(S, (u_1, \dots, u_n)) = I$.

Let A be the change of basis matrix from ψ 's to φ 's, i.e.,

$$\begin{bmatrix} \varphi_1 & \cdots & \varphi_n \end{bmatrix} = \begin{bmatrix} \psi_1 & \cdots & \psi_n \end{bmatrix} A.$$

Then by the definition of change of basis matrix, we have

$$\mathcal{M}(T, (u_1, \dots, u_n)) = \begin{bmatrix} \mathcal{M}(\varphi_1, (u_1, \dots, u_n)) \\ \vdots \\ \mathcal{M}(\varphi_n, (u_1, \dots, u_n)) \end{bmatrix} = A^t \begin{bmatrix} \mathcal{M}(\psi_1, (u_1, \dots, u_n)) \\ \vdots \\ \mathcal{M}(\psi_n, (u_1, \dots, u_n)) \end{bmatrix}$$

$$= A^t \cdot \mathcal{M}(S, (u_1, \dots, u_n)) = A^t.$$

Consider basis v_1, \dots, v_n of V such that the change of basis matrix from u 's to v 's is $(A^t)^{-1}$.² Thus

$$\mathcal{M}(T, (v_1, \dots, v_n)) = \mathcal{M}(T, (u_1, \dots, u_n)) \cdot \mathcal{M}(I, (v_1, \dots, v_n), (u_1, \dots, u_n)) = I.$$

Then the dual basis of v_1, \dots, v_n is precisely $\varphi_1, \dots, \varphi_n$, as desired. \square

Exercise 5 (A natural isomorphism from primal space onto double dual space). Define $\Lambda : V \rightarrow V''$ by

$$(\Lambda v)(\varphi) = \varphi(v)$$

for each $v \in V$ and $\varphi \in V'$.

- (a) Prove that if $T \in \mathcal{L}(V)$, then $T'' \circ \Lambda = \Lambda \circ T$.
- (b) Prove that if V is finite-dimensional, then Λ is an isomorphism from V onto V'' .

Remarks 6. (a) Suppose V is finite-dimensional. Then V and V' are isomorphic, but finding an isomorphism from V onto V' generally requires choosing a basis of V . In contrast, the isomorphism Λ from V onto V'' does not require a choice of basis and thus is more natural.

- (b) Another natural isomorphism is $\pi' \in \mathcal{L}((V/U)', V')$ where π is the normal quotient map.

Polynomials

Theorem 7. Suppose $p \in \mathcal{P}(\mathbb{F})$ is a nonconstant polynomial of degree m . Then p has at most m zeros in \mathbb{F} .³

Remark 8. Theorem 7 implies that the coefficients of a polynomial are uniquely determined. In particular, the degree of a polynomial is well-defined.

Theorem 9 (Division algorithm for polynomials). Suppose that $p, s \in \mathcal{P}(\mathbb{F})$, with $s \neq 0$. Then there exist unique polynomials $q, r \in \mathcal{P}(\mathbb{F})$ such that $p = sq + r$.

Proof. Let $n = \deg p$ and $m = \deg s$. The case where $n < m$ is trivial. Thus we now assume that $n \geq m$.

The list

$$1, z, \dots, z^{m-1}, s, zs, \dots, z^{n-m}s$$

is linearly independent in $\mathcal{P}_n(\mathbb{F})$. And it also has length $n + 1$. Hence the list is a basis of $\mathcal{P}_n(\mathbb{F})$.

Because $p \in \mathcal{P}_n(\mathbb{F})$, there exist unique constants $a_0, \dots, a_{m-1}, b_0, \dots, b_{n-m} \in \mathbb{F}$ such that

$$\begin{aligned} p &= a_0 + a_1z + \dots + a_{m-1}z^{m-1} + b_0s + b_1zs + \dots + b_{n-m}z^{n-m}s \\ &= (a_0 + a_1z + \dots + a_{m-1}z^{m-1}) + s(b_0 + b_1z + \dots + b_{n-m}z^{n-m}). \end{aligned} \quad \square$$

Exercise 10. Suppose $p, q \in \mathcal{P}(\mathbb{C})$ are nonconstant polynomials with no zeros in common. Let $m = \deg p$ and $n = \deg q$. Prove that there exist $r \in \mathcal{P}_{n-1}(\mathbb{C})$ and $s \in \mathcal{P}_{m-1}(\mathbb{C})$ such that $rp + sq = 1$.⁴

² *Proof Idea.* The change of basis for $V' \rightarrow V'$ corresponds to the transpose of $V \leftarrow V$, where transpose and inverse both come from duality.

³ A useful corollary of this theorem: when a polynomial p has too many zeros, $p = 0$.

⁴ *Proof Idea.* Define $T : \mathcal{P}_{n-1}(\mathbb{C}) \times \mathcal{P}_{m-1}(\mathbb{C}) \rightarrow \mathcal{P}_{m+n-1}(\mathbb{C})$ by $T(r, s) = rp + sq$ and prove the surjectivity of T .

Eigenvalues and Eigenvectors

Invariant Subspaces

Exercise 11. Suppose that $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ are pairwise distinct. Prove that the list $e^{\lambda_1 x}, \dots, e^{\lambda_n x}$ is linearly independent in $\mathbb{R}^{\mathbb{R}}$.

Proof. Let $V = \text{span}(e^{\lambda_1 x}, \dots, e^{\lambda_n x})$.⁵ Define $D \in \mathcal{L}(V)$ by $Df = f'$. Then $e^{\lambda x}$ is an eigenvector of D corresponding to λ . A list of eigenvectors corresponding to distinct eigenvalues is linearly independent. \square

⁵ Alternatively we can let $V = D(\mathbb{R})$.

Exercise 12. Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and U is a subspace of V invariant under T . Prove that each eigenvalue of the quotient operator T/U is an eigenvalue of T .

Proof. It suffices to show that $T/U - \lambda I = (T - \lambda I)/U$ is not injective $\implies T - \lambda I$ is not injective. We prove that $T - \lambda I$ is invertible $\implies (T - \lambda I)/U$ is injective.

Suppose $T - \lambda I$ is invertible. U being invariant under T implies that U is invariant under $T - \lambda I$. Thus $(T - \lambda I)v \in U \iff v \in U$. Suppose $((T - \lambda I)/U)(v + U) = 0$. Then $(T - \lambda I)v \in U$, which implies that $v \in U$, i.e., $v + U = 0$. That proves the injectivity of $(T - \lambda I)/U$. \square

Exercise 13. Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T has an eigenvalue if and only if there exists a subspace of V of dimension $\dim V - 1$ that is invariant under T .⁶

⁶ *Proof Idea.* Consider the zero entries in $\mathcal{M}(T)$ and its transpose matrix $\mathcal{M}(T')$.

Proof. We first suppose that T has an eigenvalue λ . Then λ is an eigenvalue of T' . There exists $\varphi \in V'$ such that $\varphi \circ T = T'\varphi = \lambda\varphi$. Extend φ to a basis $\varphi, \varphi_2, \dots, \varphi_n$ of V' and let v, v_2, \dots, v_n be the basis of V whose dual basis is $\varphi, \varphi_2, \dots, \varphi_n$. Then $(\varphi \circ T)v_k = \lambda\varphi(v_k) = 0$ for every k . Because $\varphi(Tv_k) = 0$ for every k , we have $Tv_k \in \text{span}(v_2, \dots, v_n)$. That proves that $\text{span}(v_2, \dots, v_n)$ is invariant under T .

Reversing the steps above leads to an eigenvector of T' , completing the proof. \square

The Minimal Polynomial

Exercise 14 (Companion matrix of a polynomial). Suppose $a_0, \dots, a_{n-1} \in \mathbb{F}$. Let $T \in \mathcal{L}(\mathbb{F}^n)$ be such that $\mathcal{M}(T)$ (with respect to the standard basis) is

$$\begin{bmatrix} 0 & & & & -a_0 \\ 1 & 0 & & & -a_1 \\ & 1 & 0 & & -a_2 \\ & & \ddots & & \vdots \\ & & & 0 & -a_{n-2} \\ & & & 1 & -a_{n-1} \end{bmatrix}$$

Prove that the minimal polynomial of T is the polynomial

$$a_0 + a_1 z + \dots + a_{n-1} z^{n-1} + z^n.$$

Remark 15. This exercise implies that every monic polynomial is the minimal polynomial of some operator. Hence an algorithm that could produce exact eigenvalues for each operators on each \mathbb{F}^n does not exist.

Exercise 16. Prove that every operator on a finite-dimensional vector space of dimension at least 2 has an invariant subspace of dimension 2.

Proof. Let $T \in \mathcal{L}(V)$ and $n = \dim V$. We use induction on n . The base case $n = 2$ is trivial. Now suppose $n > 2$ and the desired result holds for all smaller positive integers. Let p be the minimal polynomial of T .

If T has an eigenvalue λ , then $p(z) = q(z)(z - \lambda)$ for some monic polynomial q with $\deg q = \deg p - 1$. Because $q(T)|_{\text{im}(T - \lambda I)} = 0$, the desired result holds by induction hypothesis if $\dim \text{im}(T - \lambda I) \geq 2$. If $T - \lambda I = 0$ the desired result trivially holds. If $\dim \text{im}(T - \lambda I) = 1$, then $(T - \lambda I)v$ is a scalar multiple of some fixed $u \in V$ for all $v \in V$. Take $w \in V \setminus \text{span}(u)$ and $\text{span}(u, w)$ will satisfy the desired property.

If T has no eigenvalues, then $\mathbb{F} = \mathbb{R}$ and $p(z) = q(z)(z^2 + bz + c)$ for some $b, c \in \mathbb{R}$ with $b^2 < 4c$ and monic polynomial q with $\deg q = \deg p - 2$. If $\dim \text{im}(T^2 + bT + cI) \geq 2$ the desired result holds by induction hypothesis. If $T^2 + bT + cI = 0$, then let $w \in V$ be such that $w \neq 0$. It can be verified that $\text{span}(w, Tw)$ is invariant under T . If $\dim \text{im}(T^2 + bT + cI) = 1$, because $\dim \ker(T^2 + bT + cI)$ is even, n is odd, which implies that T has an eigenvalue. That completes the proof. \square

Commuting Operators

Exercise 17. Suppose $\mathcal{E} \subseteq \mathcal{L}(V)$ and every element of \mathcal{E} is diagonalizable. Prove that there exists a basis of V with respect to which every element of \mathcal{E} has a diagonal matrix if and only if every pair of elements of \mathcal{E} commutes.⁷

Proof. Suppose every pair of elements of \mathcal{E} commutes. We use induction on $n = \dim V$. The base case $n = 1$ is trivial. Now suppose $n > 1$ and the desired result holds for all smaller integers. Without loss of generality, suppose $\mathcal{E} \cap \{\lambda I : \lambda \in \mathbb{F}\} = \emptyset$, or else consider $\mathcal{E} \setminus \{\lambda I : \lambda \in \mathbb{F}\}$.

Let $\lambda_1, \dots, \lambda_m$ be the distinct eigenvalues of $T \in \mathcal{E}$. Then $V = E(\lambda_1, T) \oplus \dots \oplus E(\lambda_m, T)$. Because $E(\lambda_k, T)$ is invariant under every $S \in \mathcal{E}$, it suffices to show that the desired result holds on $E(\lambda_1, T)$. Because $E(\lambda_1, T) \subsetneq V$, it holds by induction hypothesis. The other direction is trivial. \square

Exercise 18. Suppose V is a finite-dimensional nonzero complex vector space. Suppose that $\mathcal{E} \subseteq \mathcal{L}(V)$ is such that S and T commute for all $S, T \in \mathcal{E}$.

- (a) Prove that there is a vector in V that is an eigenvector for every element of \mathcal{E} .
- (b) Prove that there exists a basis of V with respect to which every element of \mathcal{E} has an upper-triangular matrix.⁸

⁷ This exercise extends simultaneous diagonalizability to more than 2 (possibly infinitely many) operators.

⁸ This exercise extends simultaneous upper triangularizability to more than 2 (possibly infinitely many) operators.

Inner Product Spaces

Inner Products and Norms

Theorems 19 (Polarization identities). (a) Suppose V is a real inner product space. Then

$$\langle u, v \rangle = \frac{\|u + v\|^2 - \|u - v\|^2}{4}.$$

(b) Suppose V is a complex inner product space. Then

$$\langle u, v \rangle = \frac{\|u + v\|^2 - \|u - v\|^2 + \|u + iv\|^2 - \|u - iv\|^2}{4}i.$$

Exercise 20. Prove that if $\|\cdot\|$ is a norm on U satisfying the parallelogram equality, then there is an inner product $\langle \cdot, \cdot \rangle$ on U such that $\|u\| = \langle u, u \rangle^{1/2}$ for all $u \in U$.⁹

⁹ Recall Theorems 19.

Orthonormal Bases

Exercise 21. Suppose v_1, \dots, v_m is a linearly independent list in V . Prove that the orthonormal list produced by the formulas of the Gram-Schmidt procedure is the only orthonormal list e_1, \dots, e_m in V such that $\langle v_k, e_k \rangle > 0$ and $\text{span}(v_1, \dots, v_k) = \text{span}(e_1, \dots, e_k)$ for each k .

Exercise 22. Suppose V is finite-dimensional. Suppose $\langle \cdot, \cdot \rangle_1, \langle \cdot, \cdot \rangle_2$ are inner products on V with corresponding norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Prove that there exists $b, c > 0$ such that $b\|v\|_2 \leq \|v\|_1 \leq c\|v\|_2$ for every $v \in V$.¹⁰

Proof. Without loss of generality, suppose $V = \mathbb{F}^n$, $\|x\|_1^2 = x^t A x$, and $\|x\|_2^2 = x^t B x$, where A, B are positive-definite and B is diagonal.¹¹ Note that $\lambda B - A$ is Hermitian and hence diagonalizable. It suffices to show that there exists $\lambda > 0$ such that $\lambda B - A$ is positive-definite. Such a λ exists by the Gershgorin disk theorem, as desired. \square

Exercise 23. Suppose $\mathbb{F} = \mathbb{C}$ and V is finite-dimensional. Prove that if $T \in \mathcal{L}(V)$ is such that 1 is the only eigenvalue of T and $\|Tv\| \leq \|v\|$ for all $v \in V$, then T is the identity operator.

Proof. By Schur's theorem, there exists an orthonormal basis e_1, \dots, e_m of V with respect to which T has an upper-triangular matrix. Then all entries on the diagonal of $\mathcal{M}(T)$ is 1. Any $\mathcal{M}(T)_{j,k} \neq 0$ with $j \neq k$ would contradict $\|Te_k\| \leq \|e_k\|$. That proves that $\mathcal{M}(T) = I$, as desired. \square

Exercise 24. Suppose v_1, \dots, v_n is a basis of V . Prove that there exists a basis u_1, \dots, u_n of V such that¹²

$$\langle u_j, v_k \rangle = \begin{cases} 0 & \text{if } j \neq k, \\ 1 & \text{if } j = k. \end{cases}$$

Proof. Define $\varphi_k \in V'$ for each k by $\varphi_k(u) = \langle u, v_k \rangle$. By the Riesz representation theorem, $\varphi_1, \dots, \varphi_n$ is a spanning list in V' . Thus it is a basis of V' . Let u_1, \dots, u_n be the basis of V whose dual basis is $\varphi_1, \dots, \varphi_n$. Then u_1, \dots, u_n satisfies the desired property. \square

¹⁰ Note that *Proof 1* does not use the hypothesis that $\|\cdot\|_1$ and $\|\cdot\|_2$ are associated with an inner product respectively.

¹¹ *Proof Idea.* Fixing an orthonormal basis naturally leads to a (simple) diagonal matrix.

¹² *Proof Idea.* Linearity in inner product and patterns of 0, 1 inspire the use of linear functionals.

Orthogonal Complements and Minimization Problems

Theorem 25 (Riesz representation theorem). *Suppose V is finite-dimensional. For each $v \in V$, define $\varphi_v \in V'$ by*

$$\varphi_v(u) = \langle u, v \rangle$$

*for each $u \in V$. Then $v \mapsto \varphi_v$ is a one-to-one map from V onto V' .*¹³

Exercise 26. Suppose V is finite-dimensional and $P \in \mathcal{L}(V)$ is such that $P^2 = P$.

- (a) Suppose every vector in $\ker P$ is orthogonal to every vector in $\operatorname{im} P$. Prove that there exists a subspace U of V such that $P = P_U$.¹⁴
- (b) Suppose $\|Pv\| \leq \|v\|$ for every $v \in V$. Prove that there exists a subspace U of V such that $P = P_U$.¹⁵

Propositions 27 (Algebraic properties of the pseudoinverse). *Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$.*

- (a) *The pseudoinverse of an orthogonal projection is the operator itself.*
- (b) $\ker T^\dagger = (\operatorname{im} T)^\perp$ and $\operatorname{im} T^\dagger = (\ker T)^\perp$.
- (c) $TT^\dagger T = T$ and $T^\dagger TT^\dagger = T^\dagger$.
- (d) $(T^\dagger)^\dagger = T$.

¹³ *Proof Idea.* If $\varphi(u) = \langle u, v \rangle$ holds for all $u \in V$, then $v \in (\ker \varphi)^\perp$. However, $(\ker \varphi)^\perp$ has dimension 1 (except when $\varphi = 0$). Hence we can obtain the right v by choosing an arbitrary nonzero $w \in (\ker \varphi)^\perp$ and then multiplying by an appropriate scalar.

¹⁴ *Proof Idea.* Observe that $\ker P = (\operatorname{im} P)^\perp$. Thus we prove that P and $P_{\operatorname{im} P}$ agree on $\ker P$ and $\operatorname{im} P$.

¹⁵ *Proof Idea.* Observe that among all v 's with the same Pv , the one in $(\ker P)^\perp$ is the shortest. Applying this v makes the best use of the inequality. Thus we prove that P and $P_{(\ker P)^\perp}$ agree on $\ker P$ and $(\ker P)^\perp$.

Operators on Inner Product Spaces

From now on, we suppose that V and W are nonzero finite-dimensional inner product spaces over \mathbb{F} .

Self-Adjoint and Normal Operators

Exercise 28. Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. Prove that

$$\lambda \text{ is an eigenvalue of } T \iff \bar{\lambda} \text{ is an eigenvalue of } T^*.$$

Exercise 29. Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V . Prove that

$$U \text{ is invariant under } T \iff U^\perp \text{ is invariant under } T^*.$$

Exercise 30. Suppose $T \in \mathcal{L}(V)$ is normal. Prove that

$$\ker T^k = \ker T \quad \text{and} \quad \operatorname{im} T^k = \operatorname{im} T.$$

Exercise 31. Suppose $T \in \mathcal{L}(V, W)$. Prove that under the standard identification of V with V' (by Riesz representation theorem) and W with W' , the adjoint map T^* corresponds to the dual map T' . More precisely, prove that

$$T'(\varphi_w) = \varphi_{T^*w}$$

for all $w \in W$.

Remark 32. Furthermore, under this identification of V with V' , the orthogonal complement corresponds to the annihilator; the formulas for $\ker T^*$ and $\operatorname{im} T^*$ become identical to the formulas for $\ker T'$ and $\operatorname{im} T'$. Note that orthogonal complements and adjoints are easier to deal with than annihilators and dual maps.

Spectral Theorem

Theorems 33. (a) Suppose $T \in \mathcal{L}(V)$. Suppose $\mathbb{F} = \mathbb{R}$. Then T is self-adjoint if and only if all pairs of eigenvectors corresponding to distinct eigenvalues of T are orthogonal and $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$, where $\lambda_1, \dots, \lambda_m$ denote the distinct eigenvalues of T .

(b) Suppose $T \in \mathcal{L}(V)$. Suppose $\mathbb{F} = \mathbb{C}$. Then T is normal if and only if all pairs of eigenvectors corresponding to distinct eigenvalues of T are orthogonal and $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$, where $\lambda_1, \dots, \lambda_m$ denote the distinct eigenvalues of T .

Exercise 34. (a) Suppose $\mathbb{F} = \mathbb{C}$ and $\mathcal{E} \subseteq \mathcal{L}(V)$. Prove that there is an orthonormal basis of V with respect to which every element of \mathcal{E} has a diagonal matrix if and only if S and T are commuting normal operators for all $S, T \in \mathcal{E}$.

(b) Suppose $\mathbb{F} = \mathbb{R}$ and $\mathcal{E} \subseteq \mathcal{L}(V)$. Prove that there is an orthonormal basis of V with respect to which every element of \mathcal{E} has a diagonal matrix if and only if S and T are commuting self-adjoint operators for all $S, T \in \mathcal{E}$.¹⁶

¹⁶ This exercise extends the spectral theorems to the context of a collection of commuting operators.

Positive Operators

Exercise 35. For $T \in \mathcal{L}(V)$ and $u, v \in V$, define $\langle u, v \rangle_T$ by $\langle u, v \rangle_T = \langle Tu, v \rangle$.

- (a) Suppose $T \in \mathcal{L}(V)$. Prove that $\langle \cdot, \cdot \rangle_T$ is an inner product on V if and only if T is an invertible positive operator (with respect to the original inner product).
- (b) Prove that every inner product on V is of the form $\langle \cdot, \cdot \rangle_T$ for some invertible positive operator $T \in \mathcal{L}(V)$.¹⁷

¹⁷ **Proof Idea.** Let $\langle \cdot, \cdot \rangle_1$ be an inner product on V . We need to find T such that $\langle u, v \rangle_1 = \langle Tu, v \rangle$. Fixing v is not sufficient to find the unique Tu . Thus we fix u and consider $\langle v, u \rangle_1 = \langle v, Tu \rangle$. The map $v \mapsto \langle v, u \rangle_1$ is in V' . Hence we use the Riesz representation theorem to define Tu .

Complexification

Definition 36. The complexification of V , denoted by $V_{\mathbb{C}}$, equals $V \times V$ with normal addition and real scalar multiplication for product space. But we write an element (u, v) of $V_{\mathbb{C}}$ as $u + iv$. Complex scalar multiplication is defined by

$$(a + bi)(u + iv) = (au - bv) + i(av + bu)$$

for all $a, b \in \mathbb{R}$ and all $u, v \in V$.¹⁸

Propositions 37 (Properties of complexification). (a) $\lambda \in \mathbb{R}$ is an eigenvalue of T if and only if λ is an eigenvalue of $T_{\mathbb{C}}$.

(b) $\lambda \in \mathbb{C}$ is an eigenvalue of $T_{\mathbb{C}}$ if and only if $\bar{\lambda}$ is an eigenvalue of $T_{\mathbb{C}}$.

¹⁸ Think of V as a subset of $V_{\mathbb{C}}$ by identifying $u \in V$ with $u + i0$. The construction of $V_{\mathbb{C}}$ from V can then be thought of as generalizing the construction of \mathbb{C}^n from \mathbb{R}^n .

- (c) The minimal polynomial of $T_{\mathbb{C}}$ equals the minimal polynomial of T .
- (d) Suppose V is a real inner product space. For $u, v, w, x \in V$, define

$$\langle u + iv, w + ix \rangle_{\mathbb{C}} = \langle u, w \rangle + \langle v, x \rangle + (\langle v, w \rangle - \langle u, x \rangle)i.$$

Then $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ makes $V_{\mathbb{C}}$ into a complex inner product space. If $u, v \in V$, then

$$\langle u, v \rangle_{\mathbb{C}} = \langle u, v \rangle \quad \text{and} \quad \|u + iv\|_{\mathbb{C}}^2 = \|u\|^2 + \|v\|^2.$$

Proposition 38. Every operator on a finite-dimensional nonzero vector space has an invariant subspace of dimension 1 or 2.

Proof. The case where $\mathbb{F} = \mathbb{C}$ is trivial. Now assume V is a real vector space and $T \in \mathcal{L}(V)$. Then $T_{\mathbb{C}}$ ¹⁹ has an eigenvalue $a + bi$ with $a, b \in \mathbb{R}$. Thus there exist $u, v \in V$, not both 0, such that

$$Tu + iTv = (au - bv) + (av + bu)i.$$

Hence $\text{span}(u, v)$ is invariant under T , as desired. □

¹⁹ *Proof Idea.* Field extension.