

*Linear Algebra Done Right*

# Notes – Linear Algebra

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## Prerequisites

We declare several notations below for convenience and clarity.

**Notations 1** (Basic notations).

- $\mathbb{F}$  denotes a number field (often  $\mathbb{R}$  or  $\mathbb{C}$ ).
- $U, V, W$  denotes vector spaces (usually over scalar field  $\mathbb{F}$ ).
- $V^S$  denotes the set of functions from a nonempty set  $S$  to  $V$ .

## Vector Spaces

**Definition 2.** The complexification of  $V$ , denoted by  $V_{\mathbb{C}}$ , equals  $V \times V$  with normal addition and real scalar multiplication for product space. But we write an element  $(u, v)$  of  $V_{\mathbb{C}}$  as  $u + iv$ . Complex scalar multiplication is defined by

$$(a + bi)(u + iv) = (au - bv) + i(av + bu)$$

for all  $a, b \in \mathbb{R}$  and all  $u, v \in V$ .<sup>1</sup>

**Lemma 3** (Linear dependence lemma). Suppose  $v_1, \dots, v_m$  is a linearly dependent set in  $V$ . Then there exists  $k \in \{1, 2, \dots, m\}$  such that

$$v_k \in \text{span}\{v_1, \dots, v_m\}.$$

Further more, removing the  $k^{\text{th}}$  term from the list does not change the span.<sup>2</sup>

**Theorem 4.** Any two bases of a finite-dimensional vector space have the same length.<sup>3</sup>

*Proof.* Suppose  $V$  is finite-dimensional. Let  $B_1$  and  $B_2$  be two bases of  $V$ . Considering  $B_1$  as an independent set and  $B_2$  as a spanning set leads to  $\#B_1 \leq \#B_2$ . Interchanging the roles of  $B_1$  and  $B_2$  and we have  $\#B_2 \leq \#B_1$ . Thus  $\#B_1 = \#B_2$ .  $\square$

## Linear Maps

### Kernal and Image of Linear Maps

**Exercise 5.** Suppose  $U$  and  $V$  are finite-dimensional and  $S \in \mathcal{L}(V, W)$  and  $T \in \mathcal{L}(U, V)$ . Prove that

$$\dim \ker ST \leq \dim \ker S + \dim \ker T.$$

*Proof.* Restrict to  $Z = \ker ST$ . By the fundamental theorem of linear maps,

$$\begin{aligned} \dim Z &= \dim T(Z) + \dim \ker T|_Z \\ &\leq \dim T(Z) + \dim \ker T \\ &= \dim ST(Z) + \dim \ker S|_{T(Z)} + \dim \ker T \\ &\leq \dim \ker S + \dim \ker T. \end{aligned}$$

$\square$

<sup>1</sup> Think of  $V$  as a subset of  $V_{\mathbb{C}}$  by identifying  $u \in V$  with  $u + i0$ . The construction of  $V_{\mathbb{C}}$  from  $V$  can then be thought of as generalizing the construction of  $\mathbb{C}^n$  from  $\mathbb{R}^n$ .

<sup>2</sup> The lemma lays the foundation for a series of basic results for vector spaces.

<sup>3</sup> This proposition ensures that *dimension* is well-defined.

**Corollary 6** (Sylvester's rank inequality). Suppose  $A \in \mathbb{F}^{m,n}$  and  $B \in \mathbb{F}^{n,p}$  are two matrices. Then<sup>4</sup>

$$\text{rank } A + \text{rank } B - n \leq \text{rank}(AB).$$

<sup>4</sup> There is a slicker proof for this inequality using block matrices. But the proof here using linear maps is more informative.

## Products and Quotients of Vector Spaces

**Lemma 7.** Suppose  $V_1, \dots, V_m$  are subspaces of  $V$ .<sup>5</sup> Define a linear map  $\Gamma : V_1 \times \dots \times V_m \rightarrow V_1 + \dots + V_m$  by

$$\Gamma(v_1, \dots, v_m) = v_1 + \dots + v_m.$$

Then  $V_1 + \dots + V_m$  is a direct sum if and only if  $\Gamma$  is injective.

**Theorem 8.** Suppose  $V$  is finite-dimensional and  $V_1, \dots, V_m$  are subspaces of  $V$ . Then  $V_1 + \dots + V_m$  is a direct sum if and only if  $\dim(V_1 + \dots + V_m) = \dim V_1 + \dots + \dim V_m$ .

*Proof.* Recall Lemma 7. Because  $\Gamma$  is surjective, by the fundamental theorem of linear maps,  $V_1 + \dots + V_m$  is a direct sum if and only if  $\dim(V_1 + \dots + V_m) = \dim(V_1 \times \dots \times V_m) = \dim V_1 + \dots + \dim V_m$ .  $\square$

**Notation 9.** Suppose  $T \in \mathcal{L}(V, W)$ . Define  $\tilde{T} : V + \ker V \rightarrow V$  by

$$\tilde{T}(v + \ker T) = Tv.$$

**Exercise 10.** Suppose  $V_1, \dots, V_m$  are vector spaces. Prove that  $\mathcal{L}(V_1 \times \dots \times V_m, W)$  and  $\mathcal{L}(V_1, W) \times \dots \times \mathcal{L}(V_m, W)$  are isomorphic vector spaces.

*Proof.* We construct an isomorphism  $T$  between the two vector spaces.

For every  $\Gamma \in \mathcal{L}(V_1 \times \dots \times V_m, W)$ , define  $\varphi_i : V_i \rightarrow W$  for each  $i \in \{1, \dots, m\}$  by

$$\varphi_i(v_i) = \Gamma(\mathbf{0}, \dots, v_i, \dots, \mathbf{0})$$

with  $v_i$  in the  $i^{\text{th}}$  slot and  $\mathbf{0}$  in all other slots. It can be verified that  $\varphi_i \in \mathcal{L}(V_i, W)$ .

Let  $T(\Gamma) = (\varphi_1, \dots, \varphi_m)$ . It can be verified that  $T$  is a linear map. We prove  $T$  is an isomorphism by constructing its inverse linear map  $S$ .

For every  $(\varphi_1, \dots, \varphi_m) \in \mathcal{L}(V_1, W) \times \dots \times \mathcal{L}(V_m, W)$ , let

$$S(\varphi_1, \dots, \varphi_m)(v_1, \dots, v_m) = \varphi_1(v_1) + \dots + \varphi_m(v_m).$$

It can be shown that  $S$  is a linear map, and that  $S \circ T = I$  and  $T \circ S = I$ , where  $I$  is the identity operator on the proper vector space. That proves  $T$  is indeed an isomorphism between the two vector spaces, as desired.  $\square$

**Proposition 11.** A nonempty subset  $A$  of  $V$  is a translate of some subspace of  $V$  if and only if  $\lambda v + (1 - \lambda)w \in A$  for all  $v, w \in A$  and all  $\lambda \in \mathbb{F}$ .

**Exercise 12.** Suppose  $A_1 = v + U_1$  and  $A_2 = w + U_2$  for some  $v, w \in V$  and some subspaces  $U_1, U_2$  of  $V$ . Prove that  $A_1 \cap A_2$  is either the empty set or a translate of some subspace of  $V$ .<sup>6</sup>

<sup>5</sup> Note that  $V$  does not have to be finite-dimensional. Recall that  $V_1 + \dots + V_m$  is a direct sum if and only if the only way to write  $\mathbf{0}$  as a sum of  $v_1 + \dots + v_m$ , where each  $v_k \in V_k$ , is by taking each  $v_k$  equal to  $\mathbf{0}$ .

<sup>6</sup> Recall Proposition 11.

**Proposition 13.** Suppose  $\mathbf{U}$  is a subspace of  $\mathbf{V}$  and  $\mathbf{v}_1 + \mathbf{U}, \dots, \mathbf{v}_m + \mathbf{U}$  is a basis of  $\mathbf{V}/\mathbf{U}$  and  $\mathbf{u}_1, \dots, \mathbf{u}_n$  is a basis of  $\mathbf{U}$ . Then  $\mathbf{v}_1, \dots, \mathbf{v}_m, \mathbf{u}_1, \dots, \mathbf{u}_n$  is a basis of  $\mathbf{V}$ . In other words,  $\mathbf{V} = \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_m\} \oplus \mathbf{U}$ .<sup>7</sup>

<sup>7</sup>  $\mathbf{V} = \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_m\} \oplus \mathbf{U}$  still holds without the hypothesis that  $\mathbf{U}$  is finite-dimensional.

**Exercise 14.** Suppose  $\mathbf{U}$  is a subspace of  $\mathbf{V}$  such that  $\mathbf{V}/\mathbf{U}$  is finite-dimensional.

- (a) Prove that if  $\mathbf{W}$  is a finite-dimensional subspace of  $\mathbf{V}$  and  $\mathbf{V} = \mathbf{U} + \mathbf{W}$ , then  $\dim \mathbf{W} \geq \dim \mathbf{V}/\mathbf{U}$ .
- (b) Prove that there exists a finite-dimensional subspace  $\mathbf{W}$  of  $\mathbf{V}$  such that  $\mathbf{V} = \mathbf{U} \oplus \mathbf{W}$  and  $\dim \mathbf{W} = \dim \mathbf{V}/\mathbf{U}$ .

*Proof.* Let  $\bar{\mathbf{w}}_1 + \mathbf{U}, \dots, \bar{\mathbf{w}}_m + \mathbf{U}$  be a basis of  $\mathbf{V}/\mathbf{U}$ . Then by Proposition 13, we have  $\mathbf{V} = \text{span}\{\bar{\mathbf{w}}_1, \dots, \bar{\mathbf{w}}_m\} \oplus \mathbf{U}$ . Let  $\mathbf{W}_0 = \text{span}\{\bar{\mathbf{w}}_1, \dots, \bar{\mathbf{w}}_m\}$ , then  $\mathbf{V} = \mathbf{U} \oplus \mathbf{W}_0$ , as desired.

Now we prove that for each subspace  $\mathbf{W}$  of  $\mathbf{V}$  such that  $\mathbf{V} = \mathbf{U} + \mathbf{W}$ , we have  $\dim \mathbf{W} \geq m = \dim \mathbf{V}/\mathbf{U}$ .

For each  $\bar{\mathbf{w}}_i \in \mathbf{V}$  above, by definition we have  $\bar{\mathbf{w}}_i = \mathbf{u}_i + \mathbf{w}_i$  for some  $\mathbf{u}_i \in \mathbf{U}$  and  $\mathbf{w}_i \in \mathbf{W}$ . It can be shown from the linear independence of  $\bar{\mathbf{w}}_1 + \mathbf{U}, \dots, \bar{\mathbf{w}}_m + \mathbf{U}$  that  $\bar{\mathbf{w}}_1 - \mathbf{u}_1, \dots, \bar{\mathbf{w}}_m - \mathbf{u}_m$  are independent vectors in  $\mathbf{W}$ . Hence  $\dim \mathbf{W} \geq m$ .  $\square$

## Duality

**Theorem 15.** Suppose  $\mathbf{V}$  and  $\mathbf{W}$  are finite-dimensional and  $T \in \mathcal{L}(\mathbf{V}, \mathbf{W})$ . Then

$$T \text{ is surjective} \iff T' \text{ is injective} \quad \text{and} \quad T \text{ is injective} \iff T' \text{ is surjective.}^8$$

**Proposition 16.** Suppose  $\mathbf{V}$  is finite-dimensional and  $\mathbf{U}$  is a subspace of  $\mathbf{V}$ . Then<sup>9</sup>

$$\mathbf{U} = \left\{ \mathbf{v} \in \mathbf{V} : \varphi(\mathbf{v}) = \mathbf{0} \text{ for every } \varphi \in \mathbf{U}^0 \right\}.$$

<sup>8</sup> This result can be useful because sometimes it is easier to verify that  $T'$  is injective (surjective) than to show directly that  $T$  is surjective (injective).

<sup>9</sup> The proposition can be easily verified from definition. But it can be useful.

**Exercise 17.** Suppose  $\mathbf{V}$  is finite-dimensional and  $\mathbf{U}$  and  $\mathbf{W}$  are subspaces of  $\mathbf{V}$ .

- (a) Prove that  $\mathbf{W}^0 \subseteq \mathbf{U}^0$  if and only if  $\mathbf{U} \subseteq \mathbf{W}$ .
- (b) Prove that  $\mathbf{W}^0 = \mathbf{U}^0$  if and only if  $\mathbf{U} = \mathbf{W}$ .<sup>10</sup>

<sup>10</sup> Recall Proposition 16.

**Exercise 18.** Suppose  $\mathbf{V}$  is finite-dimensional and  $\mathbf{U}$  and  $\mathbf{W}$  are subspaces of  $\mathbf{V}$ .

- (a) Prove that  $(\mathbf{U} + \mathbf{W})^0 = \mathbf{U}^0 \cap \mathbf{W}^0$ .
- (b) Prove that  $(\mathbf{U} \cap \mathbf{W})^0 = \mathbf{U}^0 + \mathbf{W}^0$ .

**Lemma 19.** Suppose  $\mathbf{V}$  is finite-dimensional.  $\mathbf{v}_1, \dots, \mathbf{v}_n$  is a basis of  $\mathbf{V}$ . Then  $\varphi_1, \dots, \varphi_n \in \mathbf{V}'$  is the dual basis of  $\mathbf{v}_1, \dots, \mathbf{v}_n$  if and only if

$$\begin{bmatrix} \mathcal{M}(\varphi_1) \\ \vdots \\ \mathcal{M}(\varphi_n) \end{bmatrix} = I$$

where  $\mathcal{M}(\varphi_i)$  is the  $1 \times n$  matrix of  $\varphi_i$  with respect to basis  $\mathbf{v}_1, \dots, \mathbf{v}_n$  of  $\mathbf{V}$  for each  $i \in \{1, \dots, n\}$ .

**Exercise 20.** Suppose  $V$  is finite-dimensional and  $\varphi_1, \dots, \varphi_n$  is a basis of  $V'$ . Prove that there exists a basis of  $V$  whose dual basis is  $\varphi_1, \dots, \varphi_n$ .

*Proof.* We start from an arbitrary basis  $u_1, \dots, u_n$  of  $V$ . Let  $\psi_1, \dots, \psi_n$  be its dual basis. In this proof, we take standard basis  $e_1, \dots, e_n$  as the basis of  $\mathbb{F}^n$ .

Define  $S, T \in \mathcal{L}(V, \mathbb{F}^n)$  by

$$T(v) = (\varphi_1(v), \dots, \varphi_n(v)), S(v) = (\psi_1(v), \dots, \psi_n(v)).$$

Then by Lemma 19,  $\mathcal{M}(S, (u_1, \dots, u_n)) = I$ .

Let  $A$  be the change of basis matrix from  $\psi$ 's to  $\varphi$ 's, i.e.,

$$A = \mathcal{M}(I, (\psi_1, \dots, \psi_n), (\varphi_1, \dots, \varphi_n)).$$

Then by the definition of change of basis matrix, we have

$$\begin{aligned} \mathcal{M}(T, (u_1, \dots, u_n)) &= \begin{bmatrix} \mathcal{M}(\varphi_1, (u_1, \dots, u_n)) \\ \vdots \\ \mathcal{M}(\varphi_n, (u_1, \dots, u_n)) \end{bmatrix} \\ &= A^t \begin{bmatrix} \mathcal{M}(\psi_1, (u_1, \dots, u_n)) \\ \vdots \\ \mathcal{M}(\psi_n, (u_1, \dots, u_n)) \end{bmatrix} = A^t \mathcal{M}(S, (u_1, \dots, u_n)) = A^t. \end{aligned}$$

Consider basis  $\bar{v}_1, \dots, \bar{v}_n$  of  $V$  such that the change of basis matrix from  $v$ 's to  $\bar{v}$ 's is  $A^t$ . Thus

$$\mathcal{M}(T, (\bar{v}_1, \dots, \bar{v}_n)) = \mathcal{M}(T, (v_1, \dots, v_n)) \mathcal{M}(I, (\bar{v}_1, \dots, \bar{v}_n), (v_1, \dots, v_n)) = I.$$

Then by Lemma 19, the dual basis of  $\bar{v}_1, \dots, \bar{v}_n$  is precisely  $\varphi_1, \dots, \varphi_n$ , as desired.  $\square$

## Polynomials

**Theorem 21** (Division algorithm for polynomials). Suppose that  $p, s \in \mathcal{P}(\mathbb{F})$ , with  $s \neq 0$ . Then there exist unique polynomials  $q, r \in \mathcal{P}(\mathbb{F})$  such that  $p = sq + r$ .<sup>11</sup>

*Proof.* Let  $n = \deg p$  and  $m = \deg s$ . The case where  $n < m$  is trivial. Thus we now assume that  $n \geq m$ .

The list

$$1, z, \dots, z^{m-1}, s, zs, \dots, z^{n-m}s$$

is linearly independent in  $\mathcal{P}_n(\mathbb{F})$ . And it also has length  $n + 1$ . Hence the list is a basis of  $\mathcal{P}_n(\mathbb{F})$ .

Because  $p \in \mathcal{P}_n(\mathbb{F})$ , there exist unique constants  $a_0, \dots, a_{m-1}, b_0, \dots, b_{n-m} \in \mathbb{F}$  such that

$$\begin{aligned} p &= a_0 + a_1 z + \dots + a_{m-1} z^{m-1} + b_0 s + b_1 zs + \dots + b_{n-m} z^{n-m} s \\ &= (a_0 + a_1 z + \dots + a_{m-1} z^{m-1}) + s(b_0 + b_1 z + \dots + b_{n-m} z^{n-m}). \end{aligned} \quad \square$$

<sup>11</sup> The division algorithm for polynomials can be proved without using any linear algebra. This proof makes a nice use of a basis of  $\mathcal{P}_n(\mathbb{F})$ .