An Introduction to Tensors for the Abstract-Minded

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Abstract

This article provides an introduction to the concepts of tensors and tensor algebra, aimed at an audience with a solid foundation in abstract linear algebra, such as that provided by Axler's "Linear Algebra Done Right". We begin with the coordinate-free definition of a tensor as a multilinear map, connect this to familiar objects like vectors and linear operators, and then introduce tensor products. Finally, we bridge this abstract view with the component-based perspective common in physics and engineering by discussing transformation laws.

1 Introduction: Beyond \Multi-dimensional Arrays"

You may have heard a tensor described simply as a "multi-dimensional array of numbers". While this is how we often represent a tensor for computation, it misses the essence of what a tensor *is*. A tensor is a geometric or algebraic object that exists independently of any coordinate system. Its numerical components are merely a representation of that object with respect to a chosen basis, much like a vector is an arrow in space, and its column of numbers (v_1, v_2, v_3) is just its representation in a particular basis.

Our approach will mirror the philosophy of LADR: define the object first by its intrinsic properties (its "what"), and only then explore its representation in coordinates (its "how").

2 Revisiting Familiar Concepts as Tensors

Let's start by re-framing concepts you already know. Let $\mathcal{V}$ be a finite-dimensional vector space over a field F (typically $F = \mathbb{R}$). Let $\mathcal{V}^*$ be its dual space, the space of all linear functionals (linear maps) from $\mathcal{V}$ to F.

- **Scalars (Type (0,0) Tensors):** An element of the field F is a scalar. It takes zero vectors and zero covectors and gives a number (itself). It is the simplest tensor.
- **Vectors (Type (1,0) Tensors):** An element $v \in \mathcal{V}$ can be thought of as a linear map that takes a covector $\omega \in \mathcal{V}^*$ and produces a scalar. This map is simply evaluation: $v(\omega) := \omega(v)$. From this perspective, a vector is a linear map $\mathcal{V}^* \to F$. This identifies $\mathcal{V}$ with its double dual, $(\mathcal{V}^*)^*$, which is a natural isomorphism for finite-dimensional spaces.
- **Covectors (Type (0,1) Tensors):** An element $\omega \in \mathcal{V}^*$ is, by definition, a linear map from $\mathcal{V}$ to F. It takes one vector and produces a scalar. This is our first non-trivial example of a tensor.
- **Linear Operators (Type (1,1) Tensors):** A linear operator $T \in \mathcal{L}(\mathcal{V}, \mathcal{V})$ (a map from $\mathcal{V}$ to $\mathcal{V}$) can be viewed as an object that takes a covector $\omega \in \mathcal{V}^*$ and a vector $v \in \mathcal{V}$ and produces a scalar. We define this action as:

  $$T(\omega, v) := \omega(T(v))$$

  Notice that this map is linear in both ω and v. This is the key idea.

3 The Formal De nition: Tensors as Multilinear Maps

The common thread in the examples above is **multilinearity**. This leads to our formal, coordinate-free definition.

De nition. A **tensor of type (or rank)** (p, q) on a vector space $\mathcal{V}$ is a multilinear map T that takes p covectors from $\mathcal{V}^*$ and q vectors from $\mathcal{V}$ and produces a scalar in F.

$$T : \underbrace{\mathcal{V}^* \times \cdots \times \mathcal{V}^*}_{p \text{ times}} \times \underbrace{\mathcal{V} \times \cdots \times \mathcal{V}}_{q \text{ times}} \to F$$

The space of all such tensors is denoted $T^p_q(\mathcal{V})$. The numbers p and q are called the contravariant and covariant ranks, respectively.

Examples Revisited:

- A scalar is in $T^0_0(\mathcal{V})$.
- A vector $v \in \mathcal{V}$ is in $T^1_0(\mathcal{V})$.
- A covector $\omega \in \mathcal{V}^*$ is in $T^0_1(\mathcal{V})$.
- A linear operator $A : \mathcal{V} \to \mathcal{V}$ is in $T^1_1(\mathcal{V})$.
- A **bilinear form** (like an inner product) is a map $B : \mathcal{V} \times \mathcal{V} \to F$, which makes it a type $(0, 2)$ tensor.

4 Building Tensors: The Tensor Product

How do we construct these multilinear maps? The fundamental tool is the **tensor product**, denoted by $\otimes$. The tensor product of two vector spaces, $\mathcal{V} \otimes \mathcal{W}$, is a new, larger vector space. Its key property is that it "linearizes" bilinear maps.

For our purposes, we can define it constructively. Let $v \in \mathcal{V}$ and $w \in \mathcal{W}$. We can form a new object called a **simple tensor** $v \otimes w$. A general element of $\mathcal{V} \otimes \mathcal{W}$ is a linear combination of such simple tensors. The tensor product operation $\otimes$ itself is bilinear:

$$\begin{gathered}
(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w \\
v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2 \\
(cv) \otimes w = v \otimes (cw) = c(v \otimes w) \quad \text{for } c \in F
\end{gathered}$$

If $\{e_i\}_{i=1}^n$ is a basis for $\mathcal{V}$ and $\{f_j\}_{j=1}^m$ is a basis for $\mathcal{W}$, then the set $\{e_i \otimes f_j\}_{i=1,j=1}^{n,m}$ forms a basis for $\mathcal{V} \otimes \mathcal{W}$. This implies $\dim(\mathcal{V} \otimes \mathcal{W}) = \dim(\mathcal{V}) \dim(\mathcal{W})$.

With this tool, we can identify the space of (p, q) tensors with a tensor product of vector spaces:

$$T^p_q(\mathcal{V}) \cong \underbrace{\mathcal{V} \otimes \ldots \otimes \mathcal{V}}_{p \text{ times}} \otimes \underbrace{\mathcal{V}^* \otimes \ldots \otimes \mathcal{V}^*}_{q \text{ times}}$$

5 Tensors in Coordinates: The Physicist's View

Now, let's connect this abstract picture to the "multi-dimensional array" view. This requires choosing a basis.

Let $\{e_i\}_{i=1}^n$ be a basis for $\mathcal{V}$. Let $\{\epsilon^j\}_{j=1}^n$ be the corresponding **dual basis** for $\mathcal{V}^*$, defined by the property $\epsilon^j(e_i) = \delta^j_i$ (the Kronecker delta).

Any vector $v \in \mathcal{V}$ can be written as $v = v^i e_i$. (We now adopt the **Einstein summation convention**: repeated upper and lower indices are summed over). The numbers v^i are the components of v. Similarly, any covector $\omega \in \mathcal{V}^*$ can be written as $\omega = \omega_j \epsilon^j$. The numbers ω_j are its components.

Now consider a tensor $T \in T^p_q(\mathcal{V})$. We can find its components by feeding it the basis vectors and covectors:

$$T^{i_1 \dots i_p}_{j_1 \dots j_q} := T(\epsilon^{i_1}, \dots, \epsilon^{i_p}, e_{j_1}, \dots, e_{j_q})$$

This gives us a set of n^{p+q} numbers, which can be arranged in a $(p+q)$-dimensional array. The tensor T can be reconstructed from its components and the basis tensors:

$$T = T^{i_1 \dots i_p}_{j_1 \dots j_q} \, (e_{i_1} \otimes \dots \otimes e_{i_p} \otimes \epsilon^{j_1} \otimes \dots \otimes \epsilon^{j_q})$$

5.1 The All-Important Transformation Law

The statement "a tensor is an object whose components transform in a certain way" arises from asking what happens to the components when we change the basis.

Let's move from our old basis $\{e_i\}$ to a new basis $\{\bar{e}_k\}$. The relationship is given by a change-of-basis matrix Λ:

$$\bar{e}_k = \Lambda^j_k e_j$$

The inverse relationship is $e_j = (\Lambda^{-1})^m_j \bar{e}_m$.

One can show that the components of a vector $v = v^i e_i = \bar{v}^k \bar{e}_k$ transform as:

$$\bar{v}^k = (\Lambda^{-1})^k_i v^i \quad \text{(Contravariant transformation)}$$

And the components of a covector $\omega = \omega_j \epsilon^j = \bar{\omega}_k \bar{\epsilon}^k$ transform as:

$$\bar{\omega}_k = \Lambda^j_k \omega_j \quad \text{(Covariant transformation)}$$

This generalizes to any tensor $T \in T^p_q(\mathcal{V})$. Its components transform according to the rule:

$$\bar{T}^{k_1 \dots k_p}_{l_1 \dots l_q} = (\Lambda^{-1})^{k_1}_{i_1} \dots (\Lambda^{-1})^{k_p}_{i_p} \, \Lambda^{j_1}_{l_1} \dots \Lambda^{j_q}_{l_q} \, T^{i_1 \dots i_p}_{j_1 \dots j_q}$$

Physicists and engineers often take this transformation law as the *definition* of a tensor. For us, it is a *consequence* of the more fundamental definition of a tensor as an invariant multilinear map.

6 A Glimpse of Tensor Algebra and Analysis

With tensors defined, we can perform operations on them.

- **Addition**: Tensors of the same type (p, q) can be added component-wise (in a given basis). This corresponds to the standard vector space addition in $T^p_q(\mathcal{V})$.
- **Outer Product**: The tensor product can be used to combine tensors of different types. If $T \in T^p_q(\mathcal{V})$ and $S \in T^r_s(\mathcal{V})$, their outer product $T \otimes S$ is a tensor of type $(p + r, q + s)$. In components, this is simply:

  $$(T \otimes S)^{i_1 \dots p, k_1 \dots r}_{j_1 \dots q, l_1 \dots s} = T^{i_1 \dots p}_{j_1 \dots q} S^{k_1 \dots r}_{l_1 \dots s}$$

- **Contraction**: This is a crucial operation that reduces the rank of a tensor. It involves "pairing" a contravariant (upper) index with a covariant (lower) index and summing over them. For a type $(1, 1)$ tensor T, the contraction $C(T)$ is:

  $$C(T) = T^i_i = T^1_1 + T^2_2 + \dots + T^n_n$$

  This is precisely the **trace** of the linear operator corresponding to T. Contraction is the component representation of applying a covector slot to a vector slot.

Tensor Analysis (or tensor calculus) extends these ideas to manifolds (curved spaces). Here, the vector space $\mathcal{V}$ becomes the tangent space at each point on the manifold. A tensor is no longer a single object but a **tensor field**—a smooth assignment of a tensor to each point. Operations like the covariant derivative are introduced to differentiate these fields in a way that respects the geometry of the space. This is the mathematical language of General Relativity and differential geometry.

7 Conclusion

A tensor is a fundamental mathematical object that generalizes scalars, vectors, and linear operators. By defining it as a multilinear map, we capture its intrinsic, coordinate-independent nature. This abstract viewpoint, familiar from modern linear algebra, clarifies that the component representations and their transformation laws are consequences of this deeper structure. Understanding tensors from this perspective provides a robust foundation for their application in virtually every field of science and engineering, from continuum mechanics to quantum field theory.