The Foundations of Tensor Algebra

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Abstract

Building upon the definition of a tensor as a multilinear map, this article delves into the rich algebraic structure of the space of tensors. We will formalize the primary operations of tensor algebra: the outer product, which combines tensors to create those of higher rank, and contraction, which reduces rank. Each operation will be introduced from a coordinate-free perspective to emphasize its geometric nature, followed by its practical implementation in a component-based framework. We conclude with a discussion of the metric tensor, which endows the algebra with a means to relate covariant and contravariant quantities.

1 Recap: The Landscape of Tensors

Let $\mathcal{V}$ be a finite-dimensional vector space over a field $\mathbb{F}$. We have established that a tensor of type (p, q) is a multilinear map from $(\mathcal{V}^*)^p \times \mathcal{V}^q$ to $\mathbb{F}$. The set of all such tensors forms a vector space, which we denote $T^p_q(\mathcal{V})$.

The foundation of tensor algebra lies in defining meaningful operations on and between these tensor spaces. For any fixed (p, q), the space $T^p_q(\mathcal{V})$ is itself a vector space. This means we can add two tensors of the same type and multiply a tensor by a scalar:

- **Addition**: For $S, T \in T^p_q(\mathcal{V})$, $(S + T)(\omega_1, \ldots, v_q) := S(\omega_1, \ldots, v_q) + T(\omega_1, \ldots, v_q)$.
- **Scalar Multiplication**: For $c \in \mathbb{F}$, $(cT)(\omega_1, \ldots, v_q) := c \cdot T(\omega_1, \ldots, v_q)$.

These operations are fundamental but limited, as they do not allow us to interact with tensors of different types. The power of tensor algebra comes from two additional operations: the outer product and contraction.

2 The Outer Product: Building Higher-Rank Tensors

The outer product, denoted by $\otimes$, is the primary multiplicative operation in tensor algebra. It allows us to combine two tensors to create a new tensor of a higher rank.

Definition 2.1 (Outer Product)**.** *Let $S \in T^p_q(\mathcal{V})$ and $T \in T^r_s(\mathcal{V})$. Their* ***outer product****, denoted $S \otimes T$, is a tensor of type $(p + r, q + s)$ defined by its action on $p + r$ covectors and $q + s$ vectors:*

$$\begin{aligned}(S \otimes T)(\omega_1, \ldots, \omega_{p+r}, v_1, \ldots, v_{q+s}) := {} & S(\omega_1, \ldots, \omega_p, v_1, \ldots, v_q) \\ & \times T(\omega_{p+1}, \ldots, \omega_{p+r}, v_{q+1}, \ldots, v_{q+s})\end{aligned}$$

The result is the product of the scalars generated by S and T acting on their respective arguments.

This definition is inherently coordinate-free. The outer product inherits the multilinearity of its constituent tensors. It is also associative and distributive over tensor addition, making the collection of all tensors a graded algebra.

2.1 Outer Product in Components

The elegance of the outer product becomes apparent in a component representation. Let $\{e_i\}$ be a basis for $\mathcal{V}$ and $\{\epsilon^j\}$ be its dual basis. Let S and T have components

$$S^{i_1 \ldots i_p}_{j_1 \ldots j_q} \quad \text{and} \quad T^{k_1 \ldots k_r}_{l_1 \ldots l_s}$$

The components of the new tensor $U = S \otimes T \in T^{p+r}_{q+s}(\mathcal{V})$ are simply the product of the individual components. The indices are concatenated:

$$U^{i_1 \ldots i_p k_1 \ldots k_r}_{j_1 \ldots j_q l_1 \ldots l_s} = S^{i_1 \ldots i_p}_{j_1 \ldots j_q} T^{k_1 \ldots k_r}_{l_1 \ldots l_s}$$

For example, the outer product of two vectors $u = u^i e_i \in T^1_0(\mathcal{V})$ and $v = v^j e_j \in T^1_0(\mathcal{V})$ is a type-(2,0) tensor $u \otimes v$ with components $(u \otimes v)^{ij} = u^i v^j$.

3 Contraction: Reducing Tensor Rank

Contraction is a uniquely tensorial operation that reduces the rank of a tensor. It "pairs" a contravariant slot (an argument from $\mathcal{V}$) with a covariant slot (an argument from $\mathcal{V}^*$) and sums over the result.

Definition 3.1 (Contraction). *Let $T \in T^p_q(\mathcal{V})$ with $p, q \geq 1$. The* ***contraction*** *of T on the k-th contravariant index and the l-th covariant index (where $1 \leq k \leq p, 1 \leq l \leq q$) is a tensor $C^k_l(T) \in T^{p-1}_{q-1}(\mathcal{V})$. Let $\{e_i\}$ be any basis for $\mathcal{V}$ and $\{\epsilon^i\}$ be its dual basis. The action of the contracted tensor $C^k_l(T)$ is defined as:*

$$\begin{aligned}
&(C^k_l(T))(\omega_1, \ldots, \omega_{p-1}, v_1, \ldots, v_{q-1}) \\
&:= \sum_{i=1}^{\dim(\mathcal{V})} T(\omega_1, \ldots, \omega_{k-1}, \boldsymbol{\epsilon}^{\boldsymbol{i}}, \omega_k, \ldots, \omega_{p-1}, \\
&\qquad\qquad v_1, \ldots, v_{l-1}, \boldsymbol{e}_{\boldsymbol{i}}, v_l, \ldots, v_{q-1})
\end{aligned}$$

where the basis covector $\boldsymbol{\epsilon}^{\boldsymbol{i}}$ is inserted into the k-th contravariant slot of T and the basis vector $\boldsymbol{e}_{\boldsymbol{i}}$ is inserted into the l-th covariant slot.

Theorem 3.1. *The definition of contraction is independent of the choice of basis.*

Proof Sketch. Let $\{\bar{e}_j\}$ be another basis with $\bar{e}_j = \Lambda^i_j e_i$. Then the dual basis transforms as $\bar{\epsilon}^j = (\Lambda^{-1})^j_k \epsilon^k$. Substituting these into the sum, we get

$$\sum_j T(\ldots, \bar{\epsilon}^j, \ldots, \bar{e}_j, \ldots) = \sum_j T(\ldots, (\Lambda^{-1})^j_k \epsilon^k, \ldots, \Lambda^i_j e_i, \ldots)$$

By multilinearity of T, we can pull out the transformation matrices:

$$\sum_j (\Lambda^{-1})^j_k \Lambda^i_j \sum_{i,k} T(\ldots, \epsilon^k, \ldots, e_i, \ldots) = \sum_{i,k} \delta^i_k T(\ldots, \epsilon^k, \ldots, e_i, \ldots)$$

Since $\sum_j (\Lambda^{-1})^j_k \Lambda^i_j = \delta^i_k$ (the Kronecker delta), the sum reduces to $\sum_i T(\ldots, \epsilon^i, \ldots, e_i, \ldots)$, proving the result is basis-independent. $\square$

3.1 Contraction in Components

The coordinate-free definition, while rigorous, is cumbersome. In components, contraction is beautifully simple: it corresponds to setting an upper index equal to a lower index and summing over it, per the Einstein summation convention.

Let T have components $T^{i_1:::i_k:::i_p}_{j_1:::j_l:::j_q}$. The components of the contracted tensor $S = C^k_l(T)$ are:

$$S^{i_1:::i_{k-1}i_{k+1}:::i_p}_{j_1:::j_{l-1}j_{l+1}:::j_q} = T^{i_1:::m:::i_p}_{j_1:::m:::j_q}$$

where the index m replaces both i_k and j_l and is summed over from 1 to $\dim(\mathcal{V})$.

Example: The Trace. The most famous example of contraction is the trace of a linear operator. An operator $A \in \mathcal{L}(\mathcal{V})$ is a tensor in $T^1_1(\mathcal{V})$. Its contraction $C^1_1(A)$ is a tensor in $T^0_0(\mathcal{V})$, which is a scalar. In components, this is:

$$\mathrm{Tr}(A) = A^i_i = A^1_1 + A^2_2 + \cdots + A^n_n$$

4 The Metric Tensor and Index Manipulation

Tensor algebra becomes particularly powerful in vector spaces equipped with additional structure, such as an inner product.

Definition 4.1 (Metric Tensor)**.** *A* ***metric tensor*** g *on* $\mathcal{V}$ *is a type-(0,2) tensor that is:*

1. ***Symmetric:*** $g(u; v) = g(v; u)$ *for all* $u; v \in \mathcal{V}$.
2. ***Non-degenerate:*** *If* $g(u; v) = 0$ *for all* $v \in \mathcal{V}$*, then* $u = 0$.

A metric tensor is simply a choice of an inner product for $\mathcal{V}$.

The non-degeneracy of g guarantees that it establishes a canonical isomorphism between the vector space $\mathcal{V}$ and its dual $\mathcal{V}^*$. This is often called the **musical isomorphism**.

- The **flat** map $[: \mathcal{V} \to \mathcal{V}^*$ sends a vector v to a covector $v^{[}$. This covector is defined by its action on any vector u:
$$v^{[}(u) := g(v; u)$$
- The **sharp** map $] : \mathcal{V}^* \to \mathcal{V}$ is the inverse map.

In components, if g has components $g_{ij} = g(e_i; e_j)$, the flat map corresponds to **lowering an index**:

$$v_j = g_{ji} v^i$$

The inverse metric tensor g^{-1}, with components g^{ij} satisfying $g^{ik} g_{kj} = \delta^i_j$, is used to **raise an index**:

$$v^i = g^{ij} v_j$$

The existence of a metric allows us to freely convert between contravariant and covariant indices. This simplifies many formulas in physics and differential geometry, as we can place indices in the vertical position that is most convenient. For instance, the inner product of two vectors u and v can be written in four equivalent ways:

$$g(u; v) = g_{ij} u^i v^j = u_j v^j = u^i v_i = g^{ij} u_i v_j$$

5 Conclusion

Tensor algebra provides a complete and consistent framework for manipulating multilinear objects. The operations of addition, scalar multiplication, outer product, and contraction form a rich structure. The outer product allows for the construction of complexity by increasing tensor rank, while contraction allows for the extraction of simpler, invariant quantities (often scalars) by reducing rank. When supplemented with a metric tensor, the algebra gains a powerful tool for relating vectors to their duals, unifying the concepts of contravariance and covariance. This algebraic machinery is the essential prerequisite for moving on to tensor analysis, where these objects are allowed to vary smoothly over a space, forming the language of modern geometry and physics.