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On the Two Notions of the Tensor Product of Operators

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Abstract

In linear algebra and its applications, the notation $S \otimes T$ for two linear operators S and T can signify two distinct mathematical objects: a type-(2,2) tensor built from the outer product of two type-(1,1) tensors, or a single linear operator on the tensor product space $\mathcal{V} \otimes \mathcal{W}$. This article clarifies the relationship between these two concepts. We demonstrate that while they reside in different vector spaces, a canonical isomorphism connects them, justifying the use of the same notation and revealing a deeper consistency within the formalism of tensor algebra.

1 The Apparent Ambiguity

Let $\mathcal{V}$ and $\mathcal{W}$ be finite-dimensional vector spaces over a field $\mathbb{F}$. Let $S \in \mathcal{L}(\mathcal{V})$ and $T \in \mathcal{L}(\mathcal{W})$ be linear two c dee 1 ni1(oiions)-3 1 for the deeotae

1.2 Definition 2: The Tensor Product of Operators

Drawing from an exercise in Axler's *Linear Algebra Done Right*, the tensor product of two operators $S \in \mathcal{L}(\mathcal{V})$ and $T \in \mathcal{L}(\mathcal{W})$ is defined differently.

Definition 1.1 (Tensor Product of Operators)**.** *There exists a unique linear operator on the tensor product space $\mathcal{V} \otimes \mathcal{W}$, which we shall temporarily denote by Φ, that satisfies*

$$\Phi(v \otimes w) = S(v) \otimes T(w)$$

for all simple tensors $v \otimes w \in \mathcal{V} \otimes \mathcal{W}$. This unique operator is denoted by $S \otimes T$.

So, in this context, $S \otimes T$ is an element of $\mathcal{L}(\mathcal{V} \otimes \mathcal{W})$.

The Core Question: We have two objects, both denoted $S \otimes T$:

1. An object $A \in T^1_1(\mathcal{V}) \otimes T^1_1(\mathcal{W})$, a multilinear map.
2. An operator $\Phi \in \mathcal{L}(\mathcal{V} \otimes \mathcal{W})$.

Are these the same? If not, how are they related?

2 The Canonical Isomorphism

The resolution lies in a canonical (i.e., basis-independent) isomorphism between the spaces where these two objects live. We know that the space of linear operators on a vector space $\mathcal{X}$, $\mathcal{L}(\mathcal{X})$, is naturally isomorphic to the space of type-(1,1) tensors on $\mathcal{X}$, $T^1_1(\mathcal{X})$. Therefore, our operator Φ lives in a space isomorphic to $T^1_1(\mathcal{V} \otimes \mathcal{W})$.

The key is to show that there is a natural isomorphism between the space where A lives and the space where Φ lives.

Proposition 2.1. *There is a canonical isomorphism of vector spaces:*

$$\Psi : T^1_1(\mathcal{V}) \otimes T^1_1(\mathcal{W}) \xrightarrow{\cong} T^1_1(\mathcal{V} \otimes \mathcal{W})$$

Sketch of Proof. We use the fundamental isomorphisms of tensor algebra. First, recall the identification of tensor spaces with tensor products of the underlying spaces and their duals:

$$\begin{aligned} T^1_1(\mathcal{V}) &\cong \mathcal{V} \otimes \mathcal{V}^* \\ T^1_1(\mathcal{W}) &\cong \mathcal{W} \otimes \mathcal{W}^* \\ T^1_1(\mathcal{V} \otimes \mathcal{W}) &\cong (\mathcal{V} \otimes \mathcal{W}) \otimes (\mathcal{V} \otimes \mathcal{W})^* \end{aligned}$$

Next, we use the canonical isomorphism for the dual of a tensor product space:

$$(\mathcal{V} \otimes \mathcal{W})^* \cong \mathcal{V}^* \otimes \mathcal{W}^*$$

An element $\omega \otimes \eta \in \mathcal{V}^* \otimes \mathcal{W}^*$ acts on an element $v \otimes w \in \mathcal{V} \otimes \mathcal{W}$ by $(\omega \otimes \eta)(v \otimes w) := \omega(v)\eta(w)$.

Now, we can construct the chain of isomorphisms.

$$\begin{aligned} T^1_1(\mathcal{V}) \otimes T^1_1(\mathcal{W}) &\cong (\mathcal{V} \otimes \mathcal{V}^*) \otimes (\mathcal{W} \otimes \mathcal{W}^*) \\ &\cong \mathcal{V} \otimes \mathcal{W} \otimes \mathcal{V}^* \otimes \mathcal{W}^* \quad \text{(rearranging terms)} \\ &\cong (\mathcal{V} \otimes \mathcal{W}) \otimes (\mathcal{V}^* \otimes \mathcal{W}^*) \\ &\cong (\mathcal{V} \otimes \mathcal{W}) \otimes (\mathcal{V} \otimes \mathcal{W})^* \quad \text{(using the dual space isomorphism)} \\ &\cong T^1_1(\mathcal{V} \otimes \mathcal{W}) \end{aligned}$$

Since all isomorphisms used are canonical, the resulting isomorphism between the start and end spaces is also canonical. $\square$

3 Verifying the Correspondence

The existence of an isomorphism is good, but we must show that it maps our specific tensor A (from Def. 1) to our specific operator Φ (from Def. 2). We will do this by showing they represent the same underlying mapping when we interpret Φ as a tensor.

Let's take the operator $\Phi = S \otimes T \in \mathcal{L}(\mathcal{V} \otimes \mathcal{W})$ and view it as a type-(1,1) tensor on the space $\mathcal{V} \otimes \mathcal{W}$. This means Φ is a bilinear map

$$\Phi : (\mathcal{V} \otimes \mathcal{W})^* \times (\mathcal{V} \otimes \mathcal{W}) \to \mathbb{F}$$

defined by $\Phi(\alpha, z) := \alpha(\Phi(z))$ for $\alpha \in (\mathcal{V} \otimes \mathcal{W})^*$ and $z \in \mathcal{V} \otimes \mathcal{W}$.

To check if this corresponds to the tensor A from Definition 1, we evaluate it on simple (elementary) inputs. Let $\alpha = \omega \otimes \eta \in \mathcal{V}^* \otimes \mathcal{W}^* \cong (\mathcal{V} \otimes \mathcal{W})^*$ and let $z = v \otimes w \in \mathcal{V} \otimes \mathcal{W}$.

$$\begin{aligned}
\Phi(\omega \otimes \eta, v \otimes w) &= (\omega \otimes \eta)(\Phi(v \otimes w)) && \text{(by definition of } \Phi \text{ as a tensor)} \\
&= (\omega \otimes \eta)(S(v) \otimes T(w)) && \text{(by definition of the operator } \Phi\text{)} \\
&= \omega(S(v))\eta(T(w)) && \text{(by definition of action of } \mathcal{V}^* \otimes \mathcal{W}^*\text{)}
\end{aligned}$$

Now, let's recall the definition of our other object, the tensor $A = S \otimes T$ from Definition 1. It is a multilinear map whose action is:

$$A(\omega, \eta, v, w) = \omega(S(v))\eta(T(w))$$

The results are identical. The evaluation of the operator Φ (interpreted as a (1,1)-tensor on $\mathcal{V} \otimes \mathcal{W}$) on a pair of simple tensors $(\omega \otimes \eta, v \otimes w)$ yields the same scalar as the evaluation of the tensor A on the collection of individual vectors and covectors (ω, η, v, w).

This demonstrates that the canonical isomorphism maps the object from Definition 1 to the object from Definition 2.

4 Conclusion

The use of the notation $S \otimes T$ for both an outer product of tensors and an operator on a tensor product space is not an abuse of notation. It is a deliberate and elegant identification of two objects that are canonically equivalent.

- $S \otimes T \in T^1_1(\mathcal{V}) \otimes T^1_1(\mathcal{W})$ is a multilinear map taking four arguments.
- $S \otimes T \in \mathcal{L}(\mathcal{V} \otimes \mathcal{W})$ is a linear map taking one argument from $\mathcal{V} \otimes \mathcal{W}$.

The isomorphism $T^1_1(\mathcal{V}) \otimes T^1_1(\mathcal{W}) \cong \mathcal{L}(\mathcal{V} \otimes \mathcal{W})$ provides the precise dictionary for translating between these two viewpoints. This consistency is a hallmark of the power of the tensor formalism, allowing for flexibility in perspective without mathematical contradiction.