Lemma. *For any $A \subseteq \mathbf{R}$ and any bounded open interval I,*

$$|A| = |A \cap I| + |A \cap I^c|.$$

Proof. $A = (A \cap I) \cup (A \cap I^c) \implies$ LHS $\leq$ RHS. Assume that LHS $<$ RHS, i.e., $\exists \{I_k\}$ s.t.

$$A \subseteq \bigcup_{k=1}^{\infty} I_k, \quad \sum_{k=1}^{\infty} \ell(I_k) < |A \cap I| + |A \cap I^c|.$$

Split each I_k into one open interval inside I, collected in a new $\{I_k\}$, and two open intervals outside I, collected in $\{J_k\}$. Then

$$A \backslash \{a, b\} \subseteq \left(\bigcup_{k=1}^{\infty} I_k \right) \cup \left(\bigcup_{k=1}^{\infty} J_k \right), \quad I_k \subseteq I, J_k \subseteq I^c. \tag{1}$$

$$\sum_{k=1}^{\infty} \ell(I_k) + \sum_{k=1}^{\infty} \ell(J_k) < |A \cap I| + |A \cap I^c|. \tag{2}$$

Add $(a - \varepsilon, a + \varepsilon)$ and $(b - \varepsilon, b + \varepsilon)$ to $\{J_k\}$. (1) becomes

$$A \subseteq \left(\bigcup_{k=1}^{\infty} I_k \right) \cup \left(\bigcup_{k=1}^{\infty} J_k \right), \tag{3}$$

and ε is small enough s.t. (2) still holds.

Analyze (1) and (3). We have

$$A \cap I \subseteq \bigcup_{k=1}^{\infty} I_k, \quad A \cap I^c \subseteq \bigcup_{k=1}^{\infty} J_k,$$

contradicting (2). □

Problem. Prove that for all $A \subseteq \mathbf{R}$,

$$|A| = \lim_{t \to \infty} |A \cap (-t, t)|.$$

Proof. The case where $|A| = 0$ is trivial.

Now suppose $|A| \in \mathbf{R}^+$. Let $\{I_k\}$ be s.t.

$$A \subseteq \bigcup_{k=1}^{\infty} I_k, \tag{4}$$

where each I_k is bounded and

$$|A| \leq \sum_{k=1}^{\infty} \ell(I_k) < |A| + \varepsilon.$$

Then $\exists n$ s.t.

$$\sum_{k=1}^{n} \ell(I_k) > |A| - \varepsilon.$$

$\exists t$ s.t. $\bigcup_{k=1}^{n} I_k \subseteq (-t, t)$. Now by (4), we have

$$A \cap (-t, t)^c \subseteq \bigcup_{k=n+1}^{\infty} I_k.$$

$$|A \cap (-t,t)^c| \leq \sum_{k=n+1}^{\infty} \ell(I_k) < (|A| + \varepsilon) - (|A| - \varepsilon) = 2\varepsilon.$$

Now suppose $|A| = \infty$. WLOG, suppose $A \cap \mathbf{Z} = \emptyset$; otherwise consider $A \backslash \mathbf{Z}$ instead of A. By repeatedly using the lemma, we can prove that

$$|A| = \sum_{n \in \mathbf{Z}} |A \cap (n, n+1)| .$$

Because $|A| = \infty$, $\exists m$ s.t.

$$\sum_{n=-m}^{m} |A \cap (n, n+1)| > C.$$

Hence

$$|A \cap (-m, m)| = \sum_{n=-m}^{m-1} |A \cap (n, n+1)| > C.$$

Here the first equality follows from repeatedly using the lemma. □