TD1 Images

January 2021

Exercice 1.4

Question 1

$$
\begin{aligned}
c_k &= \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}\mathrm{d}x \\
&= \frac{1}{2\pi}\int_0^{2\pi} g(x)(e^{ix}-1)e^{-ikx}\mathrm{d}x \\
&= \frac{1}{2\pi}\int_0^{2\pi} g(x)e^{-i(k-1)x}\mathrm{d}x - \frac{1}{2\pi}\int_0^{2\pi} g(x)e^{-ikx}\mathrm{d}x \\
&= \gamma_{k-1}(x) - \gamma_k(x)
\end{aligned}
$$

$$S_{N,M}f(0) = \sum_{k=-M}^{N} c_k = \sum_{k=-M}^{N} \gamma_{k-1} - \gamma_k = \gamma_{-M-1} - \gamma_N$$

Or, comme g est dans $\mathrm{L}^1(0,2\pi)$, d'après le lemme de Riemann-Lebesgue, on a $\gamma_N \underset{|N|\to+\infty}{\to} 0$, d'où :

$$\lim_{N,M\to+\infty} S_{N,M}f = 0$$

Question 2

On fixe $0 < y < 2\pi$.

$$\frac{x-y}{e^{i(x-y)}-1} \underset{x\to y}{\sim} \frac{x-y}{i(x-y)} = -i$$

Ainsi l'expression $x \mapsto \frac{x-y}{e^{i(x-y)}-1}$ se prolonge en une fonction continue sur le segment $[0,2\pi]$.

Question 3

$$g(x) = \frac{f(x)-c}{e^{i(x-y)}-1} = \underbrace{\frac{f(x)-c}{x-y}}_{\mathrm{L}^1(0,2\pi)} \underbrace{\frac{x-y}{e^{i(x-y)}-1}}_{\mathcal{C}^0([0,2\pi])} \in \mathrm{L}^1(0,2\pi)$$

Question 4

$$\begin{aligned}
c_k &= \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}\mathrm{d}x \\
&= \frac{1}{2\pi}\int_0^{2\pi} \big(g(x)(e^{i(x-y)}-1)+c\big)e^{-ikx}\mathrm{d}x \\
&= \frac{e^{-iy}}{2\pi}\int_0^{2\pi} g(x)e^{-i(k-1)x}\mathrm{d}x - \frac{1}{2\pi}\int_0^{2\pi} g(x)e^{-ikx}\mathrm{d}x + \frac{1}{2\pi}\int_0^{2\pi} e^{-ikx}\mathrm{d}x \\
&= e^{-iy}\gamma_{k-1}(x) - \gamma_k(x) + c\delta_{0,k}
\end{aligned}$$

$$\begin{aligned}
S_{N,M}f(y) &= \sum_{k=-M}^{N} c_k e^{iky} \\
&= c + \sum_{k=-M}^{N} \big(\gamma_{k-1}e^{i(k-1)y} - \gamma_k e^{iky}\big) \\
&= c + \gamma_{-M-1}e^{i(-M-1)y} - \gamma_N e^{iNy} \underset{N,M\to+\infty}{\to} c
\end{aligned}$$

Exercice 1.5

Question 1

$$\begin{aligned}
\frac{1}{2\pi}c_k(S(a)*S(b)) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} S(a)*S(b)(x)e^{-ikx}\mathrm{dx} \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} S(a)(y)S(b)(x-y)\mathrm{dy}e^{-ikx}\mathrm{dx} \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} S(a)(y)e^{-iky}S(b)(x-y)e^{-ik(x-y)}\mathrm{dydx} \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} S(a)(y)e^{-iky}dy\int_{-\pi}^{\pi} S(b)(u)e^{-iku}\mathrm{du} \\
&= c_k(S(a))c_k(S(b))
\end{aligned}$$

Où l'avant dernière égalité est vraie par Fubini-Lebesgue: $f : x, y \longrightarrow S(a)(y)S(b)(x-y)e^{-ikx}$ est intégrable car $a, b \in l^2(\mathbb{Z})$.
Par définition de $S(ab)$, on a donc montré que $S(ab) = \frac{1}{2\pi}S(a) * S(b)$

Question 2

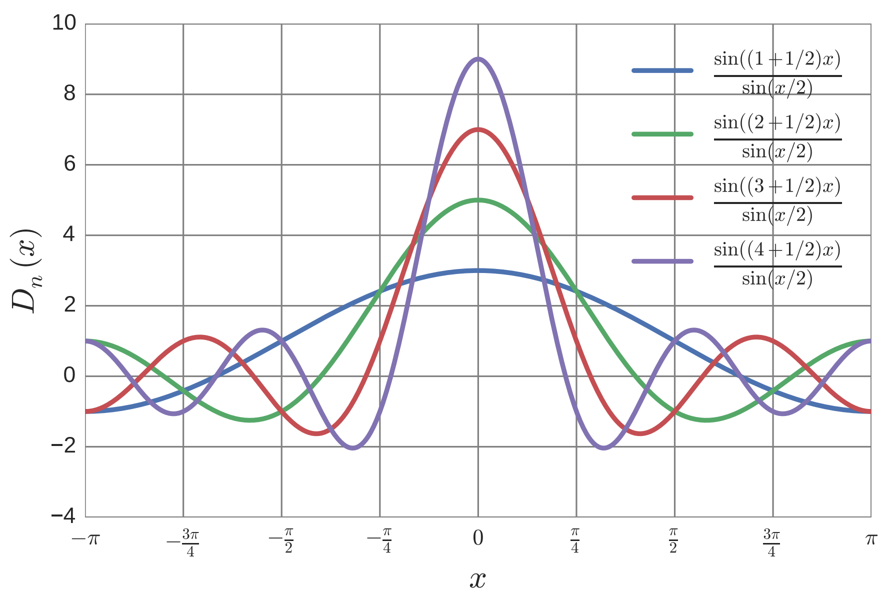
$$S(b^N) = \sum_{k \in \mathbb{Z}} b_k^N e^{-ikx}$$
$$= \sum_{-N \leq k \leq N} e^{-ikx}$$
$$= \begin{cases} 2N+1 & \text{si } x = 2k\pi. \\ \frac{sin((N+\frac{1}{2})x)}{sin(\frac{x}{2})} & \text{Sinon.} \end{cases} \qquad = \frac{sin((N+\frac{1}{2})x)}{sin(\frac{x}{2})}$$

Car $x \in [-\pi, \pi]$.

Question 3

D'après la question 2, $S(b^N) = 2\pi h_N$.
D'après la question 1, $S(b^N c(f)) = \frac{1}{2\pi} S(b^N) * S(c(f))$.
Donc, $S_N f = S(b^N c(f)) = h_N * f$.

Question 4



1 Exercice 2.7

$$u(x,y) = \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} c_{m,n}(u) e^{\frac{2i\pi mx}{a}} e^{\frac{2i\pi ny}{a}}$$

En posant $\omega_N = e^{\frac{2i\pi}{N}}$, on a :

$$\begin{aligned}
\tilde{u}_{m,n} &= \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{\ell=0}^{N-1} u(\frac{ka}{N}, \frac{\ell a}{N}) \omega_N^{-mk} \omega_N^{-n\ell} \\
&= \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{\ell=0}^{N-1} \sum_{p,q=-\infty}^{+\infty} c_{p,q}(u) e^{\frac{2i\pi pk}{N}} e^{\frac{2i\pi q\ell}{N}} \omega_N^{-mk} \omega_N^{-n\ell} \\
&= \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{\ell=0}^{N-1} \sum_{p,q=-\infty}^{+\infty} c_{p,q}(u) \omega_N^{(p-m)k} \omega_N^{(q-n)\ell} \\
&= \frac{1}{N^2} \sum_{p,q=-\infty}^{+\infty} c_{p,q}(u) \sum_{k=0}^{N-1} \omega_N^{(p-m)k} \sum_{\ell=0}^{N-1} \omega_N^{(q-n)\ell}
\end{aligned}$$

Or :

$$\sum_{k=0}^{N-1} \omega_N^{(p-m)k} = \begin{cases} N & \text{si } p-m \text{ est multiple de } N \\ 0 & \text{sinon} \end{cases}$$

$$\sum_{\ell=0}^{N-1} \omega_N^{(q-n)\ell} = \begin{cases} N & \text{si } q-n \text{ est multiple de } N \\ 0 & \text{sinon} \end{cases}$$

En effet, si $l = 0 \bmod N$, $\omega_N^l = 1$.
Donc, $\forall k \in \{0 \cdots N\}$, $\omega_N^{lk} = (\omega_N^l)^k = 1$.
Sinon, $l \neq 0 \bmod N$ et $\omega_N^l \neq 1$. Ainsi,

$$1 - \omega_N^{lN} = (1 - \omega_N^l) \sum_{k=0}^{N-1} \omega_N^{lk} = 0.$$

On en conclut que $\sum_{k=0}^{N-1} \omega_N^{lk} = 0$.

On conserve donc uniquement les termes d'indices $p = m + Np'$ et $q = m + Nq'$ avec $(p', q') \in \mathbb{Z}^2$, d'où :

$$\tilde{u}_{m,n} = \sum_{p,q=-\infty}^{+\infty} c_{m+Np,n+Nq}(u)$$