Homeworks - Detection Theory - MVA 2021/2022

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1 2.7.1 Birthdays in a Class

Consider a class of 30 students and assume that their birthdays are independent and uniformly distributed variables over the 365 days of the year. We call, for $1 \leq n \leq 30$, C_n the number of n-tuples of students of the class having the same birthday. (This number is computed exhaustively by considering all possible n-tuples. If (for example) students 1, 2, and 3 have the same birthday, then we count three pairs, (1,2), (2,3), (3,1).) We also consider $\mathbb{P}_n = \mathbb{P}(C_n \geq 1)$, the probability that there is at least one n-tuple with the same birthday and p_n, the probability that there is at least one n-tuple and no (n+ 1)-tuple.

1) Prove that $\mathbb{P}_n = 1 - \sum_{i=1}^{n-1} p_i$ and $\mathbb{P}_n = \mathbb{P}_{n-1} - p_{n-1}$.

Proof. Having $\mathbb{P}_n = \mathbb{P}(C_n \geq 1)$, we get:

$1 - \mathbb{P}_n = \mathbb{P}(C_n = 0) = \sum_{i=1}^{n-1} p_i$. Indeed, one must notice that the event $\{C_n = 0\}$ (there is no n-tuples of students of the class having the same birthday) is equal to the disjoint-union of the events A_i="Threre's at least one i-tuple and no (i+1) tuple", for i such that $1 \leq i \leq n-1$.

Therefore

$$\mathbb{P}_n = 1 - \mathbb{P}(C_n = 0) = 1 - \sum_{i=1}^{n-1} \mathbb{P}(A_i) = 1 - \sum_{i=1}^{n-1} p_i.$$

Let's prove that $\mathbb{P}_n = \mathbb{P}_{n-1} - p_{n-1}$:

$\{C_{n-1} \geq 1\} = \{C_n \geq 1\} \cap \{\{C_n = 0\} \cap \{C_{n-1} \geq 1\}\}$

Hence:

$$\begin{aligned}
&\mathbb{P}(\{C_{n-1} \geq 1\}) = \mathbb{P}(\{C_n \geq 1\}) + \mathbb{P}(\{C_n = 0\} \cap \{C_{n-1} \geq 1\}) \\
&\iff \mathbb{P}_{n-1} = \mathbb{P}_n + p_{n-1} \\
&\iff \mathbb{P}_n = \mathbb{P}_{n-1} - p_{n-1}
\end{aligned}$$

□

2) Prove that $\mathbb{E}[C_n] = \frac{1}{365^{n-1}} C_n^{30}$. Check that $\mathbb{E}[C_2] \approx 1.192$, $\mathbb{E}[C_3] \approx 0.03047$ and $\mathbb{E}[C_3] \approx 5.6 \times 10^{-4}$.

Proof. let's take $1 \leq n \leq 30$ students, the probability that all these n students have the same birthday is the probability that $n-1$ of these students have the same birthday as 1 given arbitrary student amongst these n. therefore this probability is $\frac{1}{365^{n-1}}$.

The possible combinations of choosing these n students from 30 is C_{30}^{n}.

Therefore we get

$$\mathbb{E}[C_n] = \frac{1}{365^{n-1}} C_n^{30}$$

Using Python (See the python NOtebook) we get:

$\mathbb{E}[C_2] \approx 1.192$, $\mathbb{E}[C_3] \approx 0.03047$ and $\mathbb{E}[C_3] \approx 5.6 \times 10^{-4}$. □

3) Prove that $\mathbb{P}(C_2 = 0) = \frac{365 \times 364 \times \cdots \times 336}{365^{30}} \approx 0.294$ and Deduce that $\mathbb{P}_2 = 0.706$.

Proof. Given any date among the 365 possible birthdays, the probability that any given student has this birthday is $\frac{1}{365}$. Hence the probability that another student has the same birthday as the previous student is the probability that the second student has a birthday on the same date, this probability is $\frac{1}{365}$.

Therefore, the probability for 2 students to have different birthdays is: $1 - \frac{1}{365} = \frac{364}{365}$.

Hence the probability that a student j has a different birthday than k other students is:

$$\alpha_k = 1 - \sum_{i=1}^{k} \frac{1}{365} = \frac{365-k}{365}$$

Finally,

$$\mathbb{P}(C_2 = 0) = \prod_{k=1}^{29} \alpha_k = \frac{364 \times 363 \times \cdots \times 336}{365^{29}} = \frac{365 \times 364 \times \cdots \times 336}{365^{30}}.$$

Since $\mathbb{P}_2 = \mathbb{P}(C_2 \leq) = 1 - \mathbb{P}(C_2 = 0)$ we get:

$$\mathbb{P}_2 = 0.706$$

□

4) Prove that

$$p_2 = \frac{1}{365^{30}} \sum_{i=1}^{15} \frac{\prod_{j=1}^{i} C_{32-2j}^{2}}{i!} \prod_{k=0}^{29-i} (365-k).$$

Proof. Recall that p_2 is the probability that there is at least one 2-tuple and no 3-tuple. Which means we can't find 3 different students with the same birthday.

$$p_2 = \mathbb{P}(\{C_2 \geq 1\} \cap \{C_3 = 0\})$$

let i be a student and j be a different student with the same birthday as i.

the probability that i and j have the same birthday is $\frac{1}{365}$. The number of 2-tuples possible (students with same birthday) among 30 student is 15.

First let's calculate: $\mathbb{P}(C_2 = i)$ for $1 \leq i \leq 15$.

if there is i 2-tuples of same birthdays, then there i different birthdays to choose and for each birthday, 2 students. Let's call j the j-th possible birthday, to choose 2 students for this birthday we have C_{32-2j}^2 possibilities.

$$
\begin{aligned}
p_2 &= \mathbb{P}(\cup_{i=1}^{15} \Big\{ \{C_2 = i\} \cap \{C_3 = 0\} \Big\}) \\
&= \sum_{i=1}^{15} \mathbb{P}(\{C_2 = i\} \cap \{C_3 = 0\}) \\
&= \sum_{i=1}^{15} \mathbb{P}(\{C_2 = i\}) \times \mathbb{P}(\{C_3 = 0\} | \{C_2 = i\}) \\
&= \sum_{i=1}^{15} \prod_{j=1}^{i} \frac{1}{365} \frac{C_{32-2j}^2}{j} \times \prod_{k=0}^{30-i-1} \frac{365-k}{365} \\
&= \frac{1}{365^{30}} \sum_{i=1}^{15} \frac{\prod_{j=1}^{i} C_{32-2j}^2}{i!} \prod_{k=0}^{29-i} (365-k).
\end{aligned}
$$

□

5) Compute by a small computer program (in Matlab for example): $p_2 \approx 0.678$

Proof. Using Matlab (see Figure 1 and the Matlab file) we get that indeed: $p_2 \approx 0.678$.

```
>> Detection_Theory_UPDATES
E(C2) =  1.191781
E(C3) = 0.030475
E(C4) =  0.000564
P(C2=0) =  0.293684
P2 =  0.706316
p2 =  0.677786
P3 =  0.028531
p3 =  0.024434
P4 =  0.004096
```

Figure 1: Numerical results

□

6) Deduce that $\mathbb{P}_3 \approx 0.0285$.

Proof. Having, $\mathbb{P}_3 = \mathbb{P}_2 - p_2$, $p_2 \approx 0.678$ and $\mathbb{P}_2 \approx 0.706$. We get:

$$\mathbb{P}_3 \approx 0.0285$$

$\square$

7) We denote by [r] the integer part of a real number. Prove that

$$p_3 = \frac{1}{365^{30}} \sum_{i=1}^{10} \frac{\prod_{j=1}^{i} \binom{33-3j}{3}}{i!} \left[\prod_{k=0}^{29-2i} (365-k) + \sum_{l=1}^{[\frac{30-3i}{2}]} \frac{\prod_{m=1}^{l} \binom{30-3i+2-2m}{2}}{l!} \prod_{n=0}^{29-2i-l} (365-n) \right].$$

Figure 2

Proof. $\square$

8) Deduce by a computer program that $p_3 \approx 0.027998$ and $\mathbb{P}_4 \approx 5.410^{-4}$.

Proof. Using Matlab (see Figure 3 and the Matlab file) we get that indeed: $p_3 \approx 0.027998$ and $\mathbb{P}_4 \approx 5.410^{-4}$. $\square$

```
>> Detection_Theory_UPDATES
E(C2) =  1.191781
E(C3) = 0.030475
E(C4) =  0.000564
P(C2=0) =  0.293684
P2 =  0.706316
p2 =  0.677786
P3 =  0.028531
p3 =  0.024434
P4 =  0.004096
```

Figure 3: Numerical results

9) Be courageous and give a general formula for p_n.

Proof. $\square$

10) Prove that $\mathbb{E}[C_{30}] = \mathbb{P}_{30} = \frac{1}{365^{29}}$, $\mathbb{E}[C_{29}] = \frac{30}{365^{28}}$, $\mathbb{P}_{29} = \frac{30\times 364+1}{365^{29}}$

Proof. Having $\mathbb{E}[C_n] = \frac{1}{365^{n-1}} C_n^{30}$, we get : $\mathbb{E}[C_{30}] = \frac{1}{365^{29}}$ and $\mathbb{E}[C_{29}] = \frac{30}{365^{28}}$.

Recall that $\mathbb{P}_{30} = \mathbb{P}(C_{30} \geq 1) = \mathbb{E}[\mathbb{1}_{\{C_{30} \geq 1\}}] = \mathbb{E}[C_{30}] = \frac{1}{365^{29}}$.

Since $\mathbb{P}_n = \mathbb{P}_{n-1} - p_{n-1}$, we get $\mathbb{P}_{30} = \mathbb{P}_{29} - p_{29}$

p_{29} is the probability that there is at least one 29-tuple but no 30-tuple. we get that $p_{29} = \frac{30\times 364}{365^{29}}$ (because only one (any one of the 30) student should have a different birthday than the other 29. That's 30 possible students in a 364 possible birthday.)

Hence

$$\mathbb{P}_{29} = \frac{30 \times 364 + 1}{365^{29}}$$

$\square$

11) The following table summarizes the comparative results for $\mathbb{E}[C_n]$ and $\mathbb{P}_n$ as well as the relative difference.

Proof. Using Matlab we get (almost the same values):

n	E(C_n)	P_n	relative ratio percentage
2	1.192	0.706	68.73
3	0.030	0.029	6.81
4	0.001	0.004	86.24
...	.....	.....	....
29	30*365/365^29	(30x364+1)/365^29	0.003
30	1/365^29	1/365^29	0

Figure 4: Comparative results

$\square$

12)Explain why $\mathbb{P}_n$ and $\mathbb{E}[C_n]$ are close for $n \geq 3$.

Proof. Notice that $\mathbb{P}_n = \mathbb{E}[1_{\{C_n \geq 1\}}]$ and $\mathbb{E}[C_n] = \sum_{k=1}^{30} \mathbb{P}(C_n \geq k)$, $\square$

2 3.3.2 Hoeffding's Inequality for a Sum of Random Variables

Question 1 *Graph*

To understand the meaning of the inequality, draw the graph of the function $h(p)$.

Answer 1

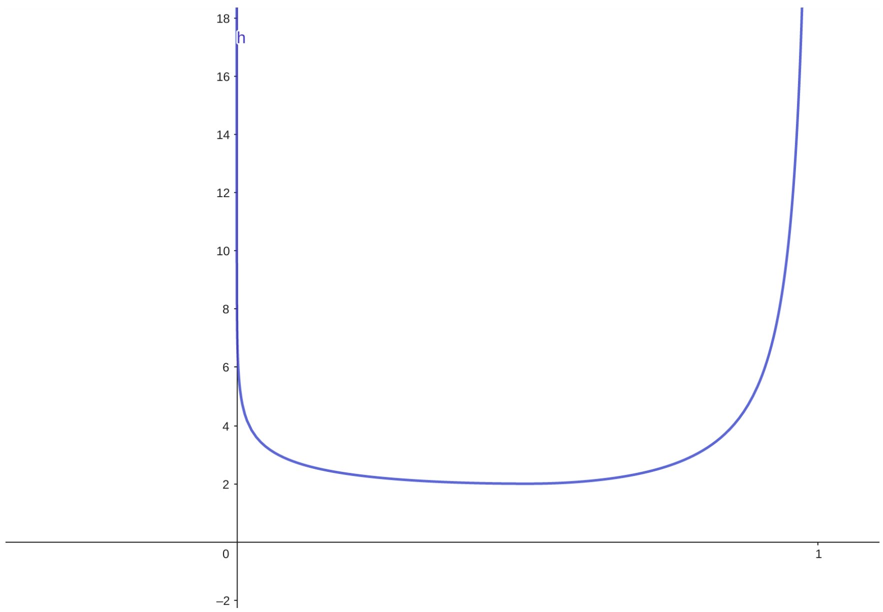

Figure 5: The graph of the function $h(p)$

Question 2 *Inequality*

Let X be a random variable such that $a \leq X \leq b$. Let α be a positive real number. Prove that

$$\mathbb{E}[\exp(\lambda X)] \leq \frac{b - \mathbb{E}[X]}{b-a}\exp(\lambda a) + \frac{\mathbb{E}[X] - a}{b-a}\exp(\lambda b)$$

Answer 2

The function $x \to \exp(x)$ is convex (it's second order derivative is positive). Therefore for x such that $a \leq x \leq b$ we can write:

$\lambda x = \frac{b-x}{b-a}\lambda a + \frac{x-a}{b-a}\lambda b$, with $0 \leq \frac{b-x}{b-a}, \frac{x-a}{b-a} \leq 1$.

Using the convexity of the exponential function we get :

$$\exp(\lambda x) \leq \frac{b-x}{b-a}\exp(\lambda a) + \frac{x-a}{b-a}\exp(\lambda b).$$

Since the expected value of a positive random variable is a positive value we get:

$$\mathbb{E}[\exp(\lambda X)] \leq \frac{b - \mathbb{E}[X]}{b-a}\exp(\lambda a) + \frac{\mathbb{E}[X] - a}{b-a}\exp(\lambda b).$$

Question 3

The main trick of large deviation estimates is to use the very simple inequality $\mathbb{1}_{\{x \geq 0\}} \leq \exp(\lambda x)$, true for $\lambda > 0$. Prove this inequality. Then apply it to $\mathbb{1}_{\{S_l - \mathbb{E}[S_l] - lt \geq 0\}}$ to deduce that

$$\mathbb{P}(S_l \geq (p+t)l) \leq \exp(-\lambda(p+t)l)\prod_{i=1}^{l}\mathbb{E}[\exp(\lambda X_i)]$$

.

Answer 3

For any $\lambda > 0$:

if $x < 0$, we get:

$$\mathbb{1}_{\{x \geq 0\}} = 0 \leq \exp(\lambda x)$$

else:

$$\mathbb{1}_{\{x \geq 0\}} = 1 \leq \exp(\lambda x)$$

Therefore:

$$\mathbb{1}_{\{x \geq 0\}} \leq \exp(\lambda x).$$

- Take $X = S_l - \mathbb{E}[S_l] - lt = S_l - (p+t)l$, (recall that $p = \frac{\mathbb{E}[S_l]}{l}$). We get:

$$\begin{aligned}
\mathbb{P}(S_l \geq (p+t)l) &= \mathbb{E}[\mathbb{1}_{\{S_l - \mathbb{E}[S_l] - lt \geq 0\}}] \\
&= \mathbb{E}[\mathbb{1}_{\{X \geq 0\}}] \\
&\leq \mathbb{E}[\exp(\lambda X)] \\
&= \mathbb{E}[\exp(-\lambda(p+t)l + \lambda(\sum_{i=1}^{l} X_i)] \\
&= \exp(-\lambda(p+t)l)\prod_{i=1}^{l}\mathbb{E}[\exp(\lambda X_i)]. \text{ Since } X_1, \cdot, X_l \text{ are independent.}
\end{aligned}$$

Therefore: $\mathbb{P}(S_l \geq (p+t)l) \leq \exp(-\lambda(p+t)l)\prod_{i=1}^{l}\mathbb{E}[\exp(\lambda X_i)]$.

Question 4

Set $p_i = \mathbb{E}[X_i]$. Applying question 2 with $a = 0$ and $b = 1$, deduce that

$$\prod_{i=1}^{l} \mathbb{E}[e^{\lambda X_i}] \leq \prod_{i=1}^{l} (1 - p_i + p_i e^{\lambda}).$$

Be sure to check that this inequality becomes an identity when the X_i's are Bernoulli random variables.

Answer 4

Using the inequality seen in question 2 with $a = 0$ and $b = 1$ we get for i such that, $1 \leq i \leq l$:

$$\begin{aligned}
\mathbb{E}[\exp(\lambda X_i)] &\leq \frac{b - \mathbb{E}[X_i]}{b - a} \exp(\lambda a) + \frac{\mathbb{E}[X_i] - a}{b - a} \exp(\lambda b) \\
&= \frac{1 - p_i}{1 - 0} \exp(\lambda \times 0) + \frac{p_i - 0}{1 - 0} \exp(\lambda \times 1) \\
&= (1 - p_i + p_i e^{\lambda}).
\end{aligned}$$

Therefore, $0 \leq \mathbb{E}[e^{\lambda X_i}] \leq (1 - p_i + p_i e^{\lambda})$. Which gives us :

$$\prod_{i=1}^{l} \mathbb{E}[e^{\lambda X_i}] \leq \prod_{i=1}^{l} (1 - p_i + p_i e^{\lambda}).$$

Assume that the $X_i \sim \mathcal{B}(p_i)$'s are Bernoulli random variables, we get for any i such that $1 \leq i \leq l$:

$$\mathbb{E}[e^{\lambda X_i}] = \mathbb{P}(X_i = 0) e^{\lambda \times 0} + \mathbb{P}(X_i = 1) e^{\lambda \times 1} = 1 - p_i + p_i e^{\lambda}.$$

which leads to:

$$\prod_{i=1}^{l} \mathbb{E}[e^{\lambda X_i}] = \prod_{i=1}^{l} (1 - p_i + p_i e^{\lambda}).$$

Question 5

Prove the geometric-arithmetic mean inequality: if $a_1, \cdot, a_l$ are positive real numbers, then

$$\left(\prod_{i=1}^{l} a_i \right)^{1/l} \leq \frac{1}{l} \sum_{i=1}^{l} a_i.$$

Answer 5

When the $a_1, \cdots, a_l$ are positive real numbers, using the identity, $a_i = e^{\ln(a_i)}$ we get:

$$\begin{aligned}
\Big(\prod_{i=1}^{l} a_i\Big)^{1/l} &= e^{\left(\sum_{i=1}^{l} \frac{\ln(a_i)}{l}\right)} \\
&\leq \sum_{i=1}^{l} \frac{e^{\ln(a_i)}}{l} \quad \text{, (by Jensen Inequality applied to the convex function } x \to e^x.) \\
&= \frac{1}{l}\sum_{i=1}^{l} a_i.
\end{aligned}$$

Therefore :

$$\Big(\prod_{i=1}^{l} a_i\Big)^{1/l} \leq \frac{1}{l}\sum_{i=1}^{l} a_i.$$

Question 6

Deduce that

$$\prod_{i=1}^{l} \mathbb{E}[e^{\lambda X_i}] \leq (1 - p + pe)^l.$$

Answer 6

Recall that we proved $\prod_{i=1}^{l} \mathbb{E}[e^{\lambda X_i}] \leq \prod_{i=1}^{l}(1 - p_i + p_i e^{\lambda})$.

Take $a_i = 1 - p_i + p_i e^{\lambda}$, positive real numbers, using the the geometric-arithmetic mean inequality, we get:

$$\begin{aligned}
\prod_{i=1}^{l} \mathbb{E}[e^{\lambda X_i}] &\leq \prod_{i=1}^{l}(1 - p_i + p_i e^{\lambda}) \\
&= \Big(\prod_{i=1}^{l} a_i\Big) \\
&\leq \Big(\frac{1}{l}\sum_{i=1}^{l} a_i\Big)^l \\
&= \Big(\frac{1}{l}\sum_{i=1}^{l} 1 - p_i + p_i e^{\lambda}\Big)^l \\
&= \Big(1 - p + pe^{\lambda}\Big)^l. \quad (\text{since} \frac{1}{l}\sum_{i=1}^{l} p_i = \mathbb{E}[\frac{S_l}{l}] = p)
\end{aligned}$$

Hence

$$\prod_{i=1}^{l} \mathbb{E}[e^{\lambda X_i}] \leq (1 - p + pe^{\lambda})^l.$$

Question 7

Combine questions 3 and 6 and get an inequality. Prove that the right-hand side of this inequality is minimal for $\lambda = \log(\frac{(1-p)(p+t)}{(1-p-t)p})$. Check that this number is positive when $0 < t < 1-p$ and obtain the first Hoeffding inequality.

Answer 7

Recall that we proved in question 3 and 6 that:

$\mathbb{P}(S_l \geq (p+t)l) \leq e^{(-\lambda(p+t)l)} \prod_{i=1}^{l} \mathbb{E}[e^{\lambda X_i}]$, and $\prod_{i=1}^{l} \mathbb{E}[e^{\lambda X_i}] \leq (1-p+pe^{\lambda})^l$.

Combining the two inequalities above we get:

$$\mathbb{P}\Big(S_l \geq (p+t)l\Big) \leq \Big((1-p+pe^{\lambda})e^{-\lambda(p+t)}\Big)^l.$$

The right-hand side of the inequality above is minimal when $\lambda \to (1-p+pe^{\lambda})e^{-\lambda(p+t)}$ reaches its minima.

Consider $f : \lambda \to \lambda \to (1-p+pe^{\lambda})e^{-\lambda(p+t)}$. we get :

$f'(\lambda) = [pe^{\lambda}(1-p-t) - (1-p)(p+t)]e^{-\lambda(p+t)}$, $f'(\lambda) = 0$ iff $\lambda = \log(\frac{(1-p)(p+t)}{(1-p-t)p})$. and since $f'(\lambda) \leq 0$ for $\lambda \leq \log(\frac{(1-p)(p+t)}{(1-p-t)p})$, and $f'(\lambda) \geq 0$ for $\lambda \geq \log(\frac{(1-p)(p+t)}{(1-p-t)p})$.

the function f, and therefore the right-hand side of this inequality is minimal for

$$\lambda = \log(\frac{(1-p)(p+t)}{(1-p-t)p}).$$

For t such that $0 < t < 1-p$, consider $g : t \to (1-p)(p+t) - (1-p-t)p$. we get:

$g'(t) = (1-p) + p = 1 > 0$ then the function g is strictly non-decreasing,
Hence $g(t) = (1-p)(p+t) - (1-p-t)p > g(0) = 0$. Therefore $\frac{(1-p)(p+t)}{(1-p-t)p} > 1$.

Which proves that $\lambda = \log(\frac{(1-p)(p+t)}{(1-p-t)p})$ is positive when $0 < t < 1-p$.

- **The first Hoeffding inequality**:

By taking $\lambda = \log(\frac{(1-p)(p+t)}{(1-p-t)p})$. we get that:

$$\mathbb{P}\Big(S_l \geq (p+t)l\Big) \leq \Big(\frac{p}{p+t}\Big)^{l(p+t)} \Big(\frac{1-p}{1-p-t}\Big)^{l(1-p-t)}.$$

Question 8

To prove the second inequality, one can remark that the first proved upper bound has a form $e^{-lt^2G(t,p}$ where $G(t,p)$ is defined by

$$G(t,p) = \frac{p+t}{t^2}\log(\frac{p+t}{p}) + \frac{1-p-t}{t^2}\log(\frac{1-p-t}{1-p}).$$

Answer 8

$$\begin{aligned}
\frac{\partial G}{\partial t}(t,p) &= \Big(\frac{1-\frac{2(p+t)t}{t^2}}{t^2}\Big)\log(\frac{p+t}{p}) + \frac{p+t}{t^2}\times\frac{1}{p+t} \\
&+ \Big(\frac{-1-\frac{2(1-p-t)t}{t^2}}{t^2}\Big)\log(\frac{1-p-t}{1-p}) + \frac{1-p-t}{t^2}\times\frac{-1}{1-p-t} \\
&= \frac{1}{t^2}\Big[\Big(1-2\frac{1-p}{t}\Big)\log(1-\frac{t}{1-p}) - \Big(1-2\frac{p+t}{t}\Big)\log(1-\frac{t}{t+p})\Big]
\end{aligned}$$

Which leads us to the equality:

$$\begin{aligned}
t^2\frac{\partial G}{\partial t}(t,p) &= \Big(1-2\frac{1-p}{t}\Big)\log(1-\frac{t}{1-p}) - \Big(1-2\frac{p+t}{t}\Big)\log(1-\frac{t}{t+p}) \\
&= H(\frac{t}{1-p}) - H(\frac{t}{t+p})
\end{aligned}$$

Where $H(x) = (1-\frac{2}{x})\log(1-x)$, for $0 < x < 1$.

Using Taylor's formula we get:

$\log(1-x) = -\sum_{n\geq 0}\frac{(x)^{n+1}}{n+1}$

Hence,

$$\begin{aligned}
H(x) &= -(1-\frac{2}{x})\sum_{n\geq 0}\frac{(x)^{n+1}}{n+1} \\
&= -\sum_{n\geq 0}\frac{(x)^{n+1}}{n+1} + 2\sum_{n\geq 0}\frac{(x)^n}{n+1} \\
&= -\sum_{n\geq 0}\frac{(x)^{n+1}}{n+1} + 2 + 2\sum_{n\geq 1}\frac{(x)^n}{n+1} \\
&= -\sum_{n\geq 0}\frac{(x)^{n+1}}{n+1} + 2 + 2\sum_{n\geq 0}\frac{(x)^{n+1}}{n+2} \\
&= 2 + \sum_{n\geq 0}(\frac{2}{n+2} - \frac{1}{n+1})x^{n+1} \\
&= 2 + (\frac{2}{3} - \frac{1}{2})x^2 + (\frac{2}{4} - \frac{1}{3})x^3 + (\frac{2}{5} - \frac{1}{4})x^4 + \cdots
\end{aligned}$$

H is $\mathcal{C}^\infty$ on $]0,1[$, therefore using $H(x) = 2 + \sum_{n\geq 0}(\frac{2}{n+2} - \frac{1}{n+1})x^{n+1}$ seen above, we get:

$$\begin{aligned}
H'(x) &= 0 + \sum_{n\geq 0}(\frac{2}{n+2} - \frac{1}{n+1})x^n \\
&= \sum_{n\geq 0}\frac{n}{(n+2)(n+1)}x^n \\
&> 0.
\end{aligned}$$

Hence $H(x)$ is strictly increasing for $0 < x < 1$.

For $\frac{t}{1-p} > \frac{t}{p+t}$ i.e $t > 1-2p$, we get:

$$\begin{aligned}
\frac{\partial G}{\partial t}(t,p) &= \frac{1}{t^2}\Big[H(\frac{t}{1-p}) - H(\frac{t}{t+p})\Big] \\
&> 0.
\end{aligned}$$

Therefore $t \to \frac{\partial G}{\partial t}(t,p) > 0$ iff $\frac{t}{1-p} > \frac{t}{p+t}$ i.e $t > 1-2p$.

When $1-2p > 0$, $G(.,p)$ is defined and continuous in $1-2p$ and since for $t > 1-2p$, $t \to G(t,p)$ is increasing, therefore $G(t,p) \geq G(1-2p,p)$.

G, therefore attains its minimum for $t = 1-2p$. and

$$\begin{aligned}
G(1-2p,p) &= \frac{p+(1-2p)}{(1-2p)^2}\log(\frac{p+(1-2p)}{p}) + \frac{1-p-(1-2p)}{(1-2p)^2}\log(\frac{1-p-(1-2p)}{1-p}). \\
&= \frac{1-p}{(1-2p)^2}\log(\frac{1-p}{p}) + \frac{p}{(1-2p)^2}\log(\frac{p}{1-p}) \\
&= \frac{1-2p}{(1-2p)^2}\log(\frac{1-p}{p}) \\
&= \frac{1}{1-2p}\log(\frac{1-p}{p}) \\
&= h(p). \;\; (0 < 1-2p \text{ and } p > 0, \text{ implies } 0 < p < \frac{1}{2}.)
\end{aligned}$$

Question 9

To prove the second inequality, one can remark that the first proved upper bound has a form $e^{-lt^2G(t,p}$ where $G(t,p)$ is defined by

$$G(t,p) = \frac{p+t}{t^2}\log(\frac{p+t}{p}) + \frac{1-p-t}{t^2}\log(\frac{1-p-t}{1-p}).$$