COMPETITION MODELS OF TWO SPECIES USING PPLANE

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OUTLINE

- Problem Definition
- Proposed Model by Utah
- pplane Model
- Conclusion

PROBLEM DEFINITION

- Two populations feeding on some constantly available food supply.
- e.g., two kinds of insects feed on fallen fruit
- Competition

PROBLEM DEFINITION (CONT.)

- The following biological assumptions apply to model a two-population competition system:

Verhulst model **1**	Population **1** grows or decays according to the logistic equation: $x`(t) = (a - bx(t))x(t)$ in the absence of population **2.**
Verhulst model **2**	Population **2** grows or decays according to the logistic equation: $y`(t) = (c - dy(t))y(t)$ in the absence of population **1.**
Chance encounters	Population **1** decays at a rate $-pxy$, $p > 0$, due to chance encounters with population **2.** Population **2** decays at a rate $-qxy$, $q > 0$, due to chance encounters with population **1**

PROBLEM DEFINITION (CONT.)

- Adding the Verhulst rates and the chance encounter rates gives the **Volterra competition system:**
- *a*, *c* are growth rates for populations
- constants *b*, *d* measure **inhibition** (due to lack of food or space)
- constants *p*, *q* measure **competition.**

$$
\begin{aligned}
x'(t) &= (a - bx(t) - py(t))x(t), \\
y'(t) &= (c - dy(t) - qx(t))y(t).
\end{aligned} \tag{6}
$$

PROBLEM DEFINITION (CONT.)

- The equilibrium points **x** satisfy $f(\mathbf{x}) = \mathbf{0}$ where f is defined by:

$$f(\mathbf{x}) = \begin{pmatrix} (a - bx - py)x \\ (c - dy - qx)y \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}. \tag{7}$$

- We have 4 equilibria:
  - (0, 0)
  - (a/b, 0)
  - (0, c/d)
  - (x0, y0) It's a unique root of the system $\begin{pmatrix} b & p \\ q & d \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$

And according to Cramer's rule

$$x_0 = \frac{ad - pc}{bd - qp}, \quad y_0 = \frac{bc - qa}{bd - qp}.$$

PROBLEM DEFINITION (CONT.)

- The Jacobian matrix is given explicitly by:

$$J(x,y) = \begin{pmatrix} a - 2bx - py & -px \\ -qy & c - 2dy - qx \end{pmatrix}. \tag{8}$$

$$\partial_x f(\mathbf{x}) = \begin{pmatrix} a - 2bx - py \\ -qy \end{pmatrix}, \quad \partial_y f(\mathbf{x}) = \begin{pmatrix} -px \\ c - 2dy - qx \end{pmatrix}.$$

PROBLEM DEFINITION (CONT.)

Equilibrium $(0,0)$
Unstable node or spiral

$$\mathbf{x}'(t) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \mathbf{x}(t)$$

Equilibrium $(a/b, 0)$
Saddle or nodal sink

$$\mathbf{x}'(t) = \begin{pmatrix} -a & -ap/b \\ 0 & c - qa/b \end{pmatrix} \mathbf{x}(t)$$

Equilibrium $(0, c/d)$
Saddle or nodal sink

$$\mathbf{x}'(t) = \begin{pmatrix} a - cp/d & 0 \\ -qc/d & -c \end{pmatrix} \mathbf{x}(t)$$

Equilibrium (x_0, y_0)
Saddle or nodal sink

$$\mathbf{x}'(t) = \begin{pmatrix} -bx_0 & -px_0 \\ -qy_0 & -dy_0 \end{pmatrix} \mathbf{x}(t)$$

PROBLEM DEFINITION (CONT.)

- If $bd - qp > 0$, then equilibria $(a/b, 0)$, $(0, c/d)$, $(x0, y0)$ are respectively a **saddle, saddle, nodal sink.**
- If $bd - qp < 0$, then equilibria $(a/b, 0)$, $(0, c/d)$, $(x0, y0)$ are respectively a **nodal sink, nodal sink, saddle.**

PROBLEM DEFINITION (CONT.)

- **Biological meaning of *bd-qp* negative or positive:**
  - Reminder: The quantities *bd* and *qp* are measures of inhibition and competition.

Survival-extinction	The inequality ***bd- qp* < 0** means that competition ***qp*** is large compared with inhibition ***bd***. The equilibrium point ($x0$; $y0$) is unstable in this case, which biologically means that the two species cannot coexist: one species **survives** and the other becomes **extinct**.
Co-existence	The inequality ***bd - qp* > 0** means that competition ***qp*** is small compared with inhibition ***bd***. The equilibrium point ($x0$; $y0$) is asymptotically stable in this case, which biologically means the two species **co-exist**.

PROPOSED MODEL BY UTAH (SURVIVAL OF ONE SPECIES)

- Consider populations $x(t)$ and $y(t)$ that satisfy the competition model:

$$
\begin{aligned}
x'(t) &= x(t)(24 - x(t) - 2y(t)), \\
y'(t) &= y(t)(30 - y(t) - 2x(t)).
\end{aligned} \tag{9}
$$

- We apply the general competition theory with $a = 24$, $b = 1$, $p = 2$, $c = 30$, $d = 1$, $q = 2$. The equilibrium points are $(0; 0)$, $(24; 0)$, $(0; 30)$, $(12; 6)$.

PROPOSED MODEL BY UTAH (SURVIVAL OF ONE SPECIES)

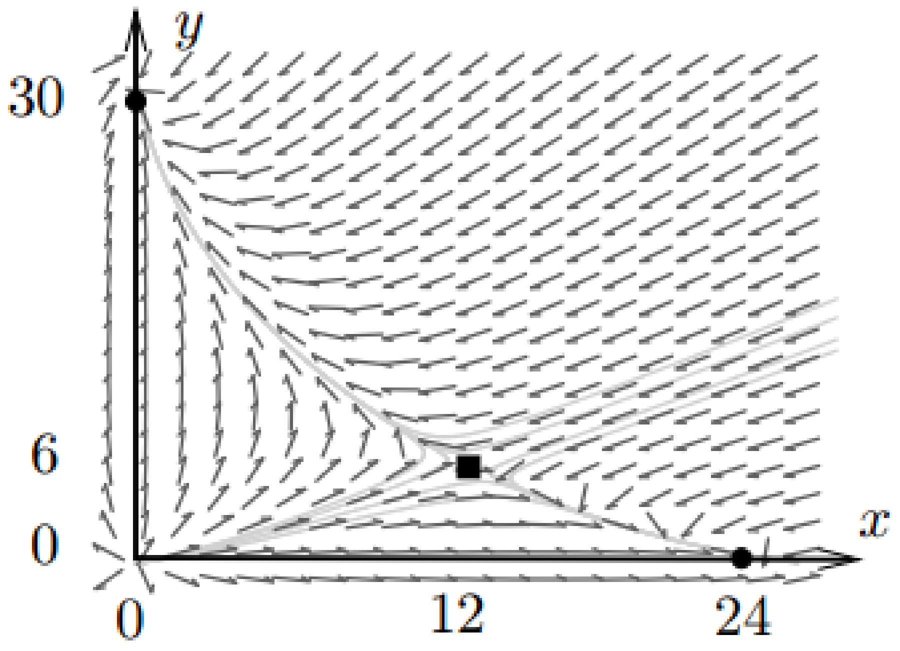
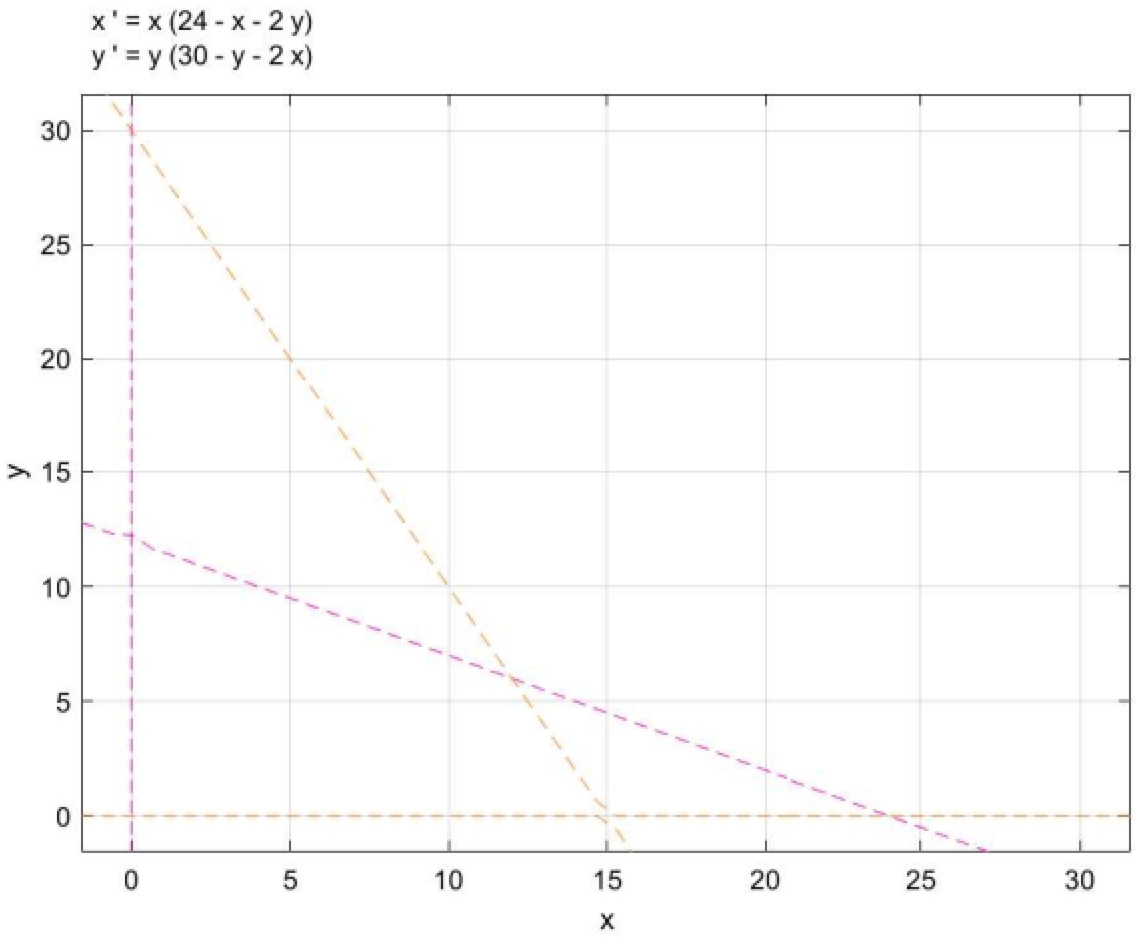
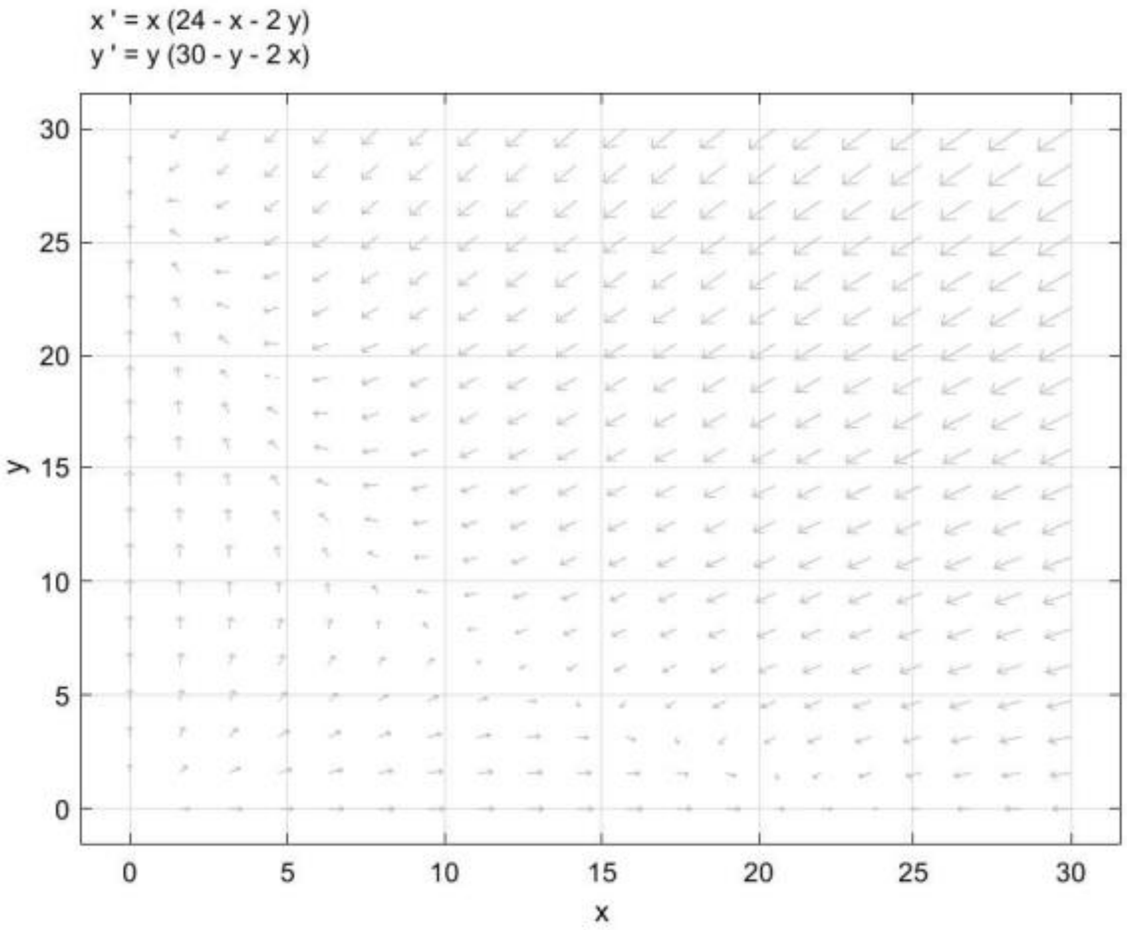


Figure 17. Survival of one species.

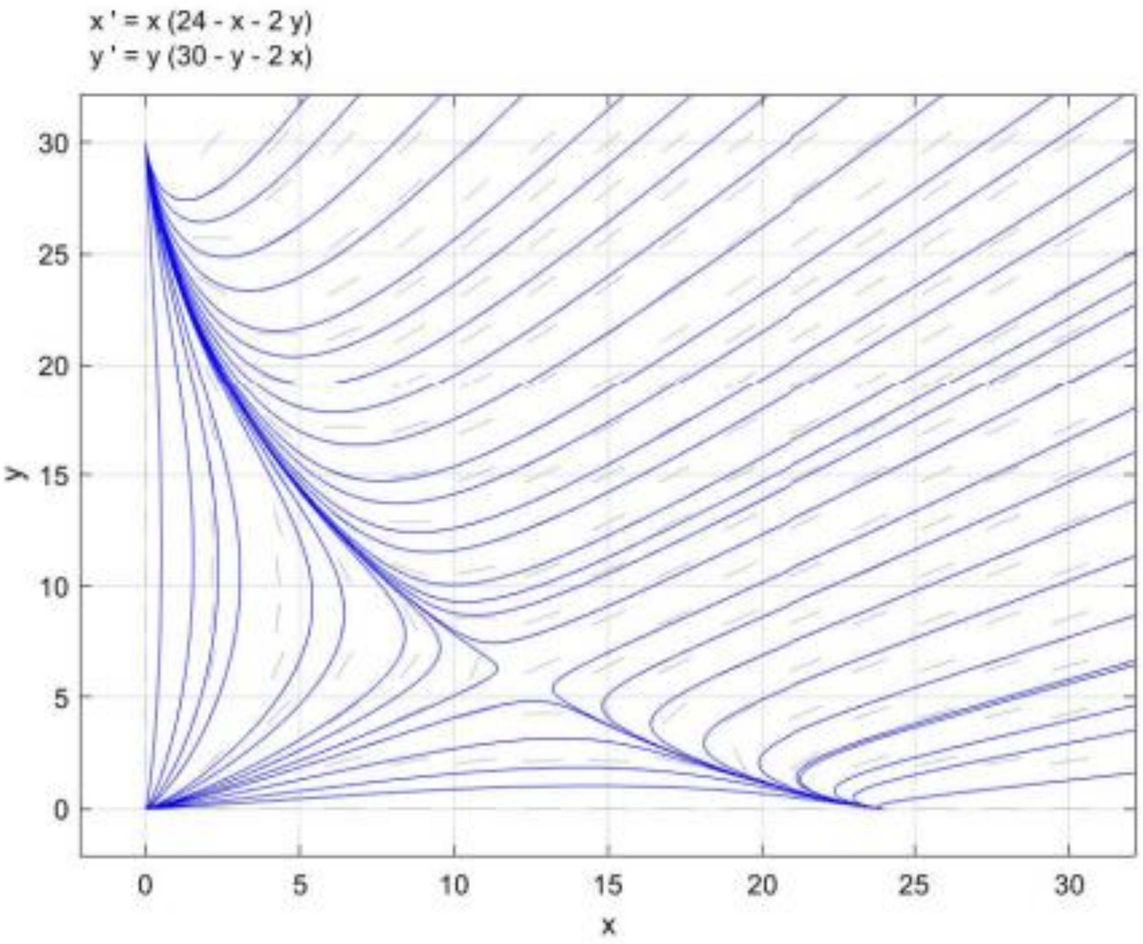
The equilibria are $(0,0)$, $(0,30)$, $(24,0)$ and $(12,6)$. They are classified as node, node, node, saddle, respectively. The population with initial advantage survives, while the other dies out.

PPLANE MODEL





PPLANE MODEL



PROPOSED MODEL BY UTAH (CO-EXISTENCE)

- Consider populations $x(t)$ and $y(t)$ that satisfy the competition model:

$$\begin{aligned} x'(t) &= x(t)(24 - 2x(t) - y(t)), \\ y'(t) &= y(t)(30 - 2y(t) - x(t)). \end{aligned} \tag{10}$$

- We apply the general competition theory with $a = 24$, $b = 2$, $p = 2$, $c = 30$, $d = 2$, $q = 2$. The equilibrium points are $(0; 0)$, $(24; 0)$, $(0; 30)$, $(12; 6)$.

PROPOSED MODEL BY UTAH (CO-EXISTENCE)

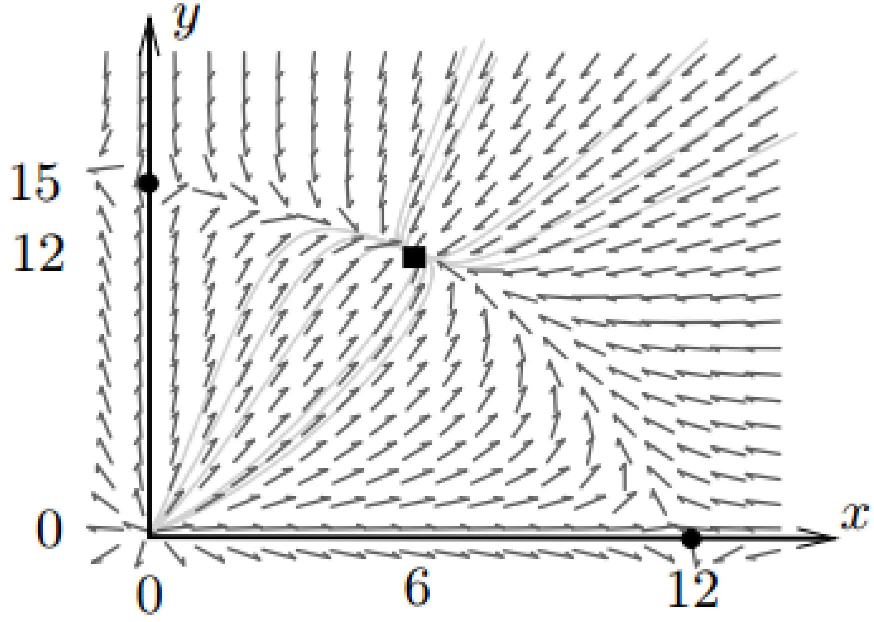
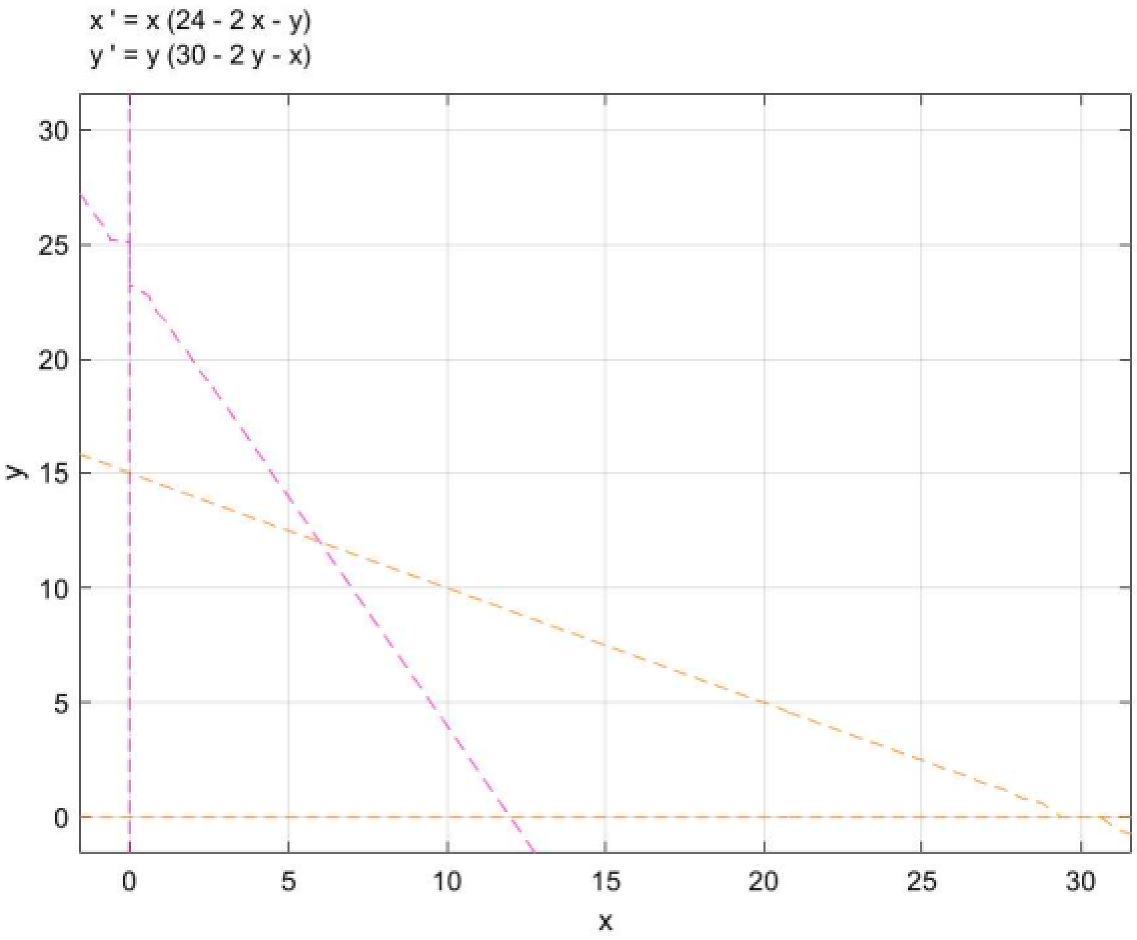
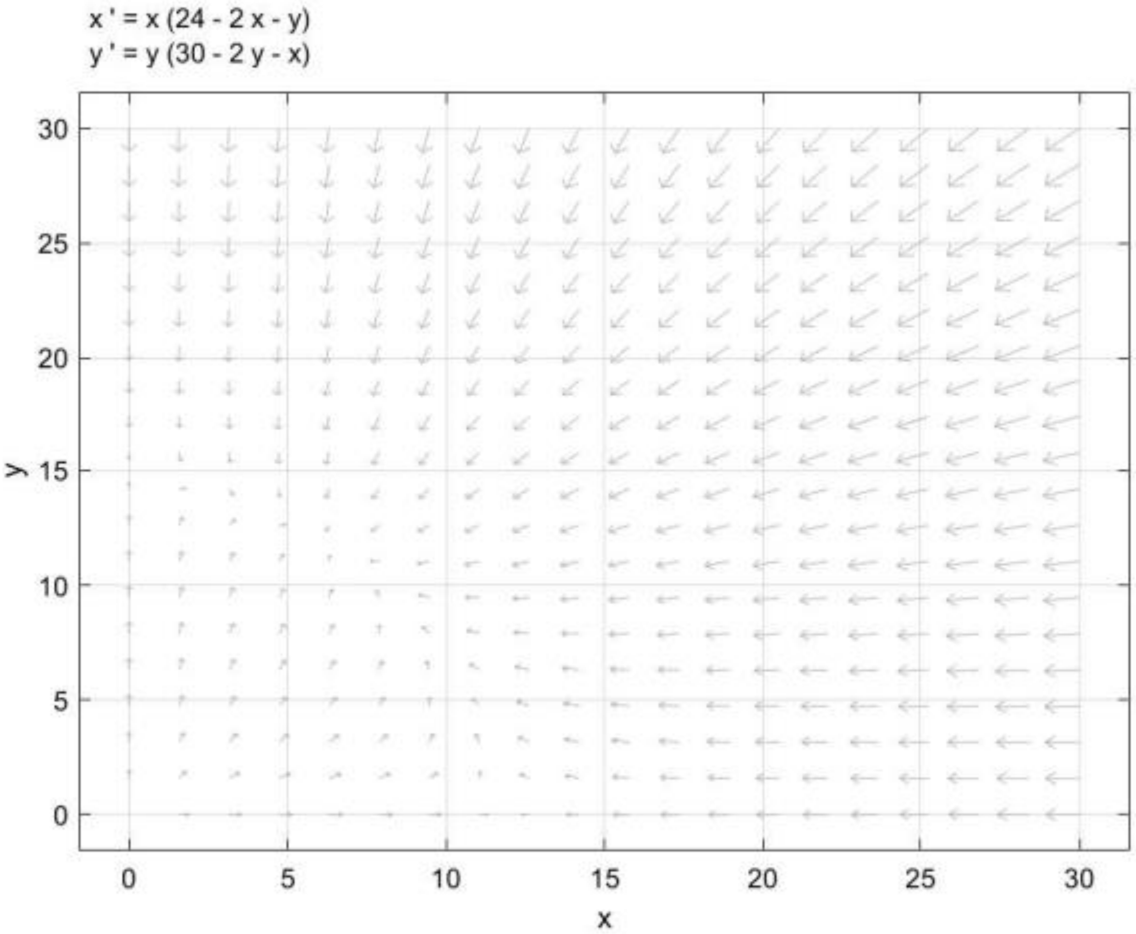


Figure 18. Coexistence.

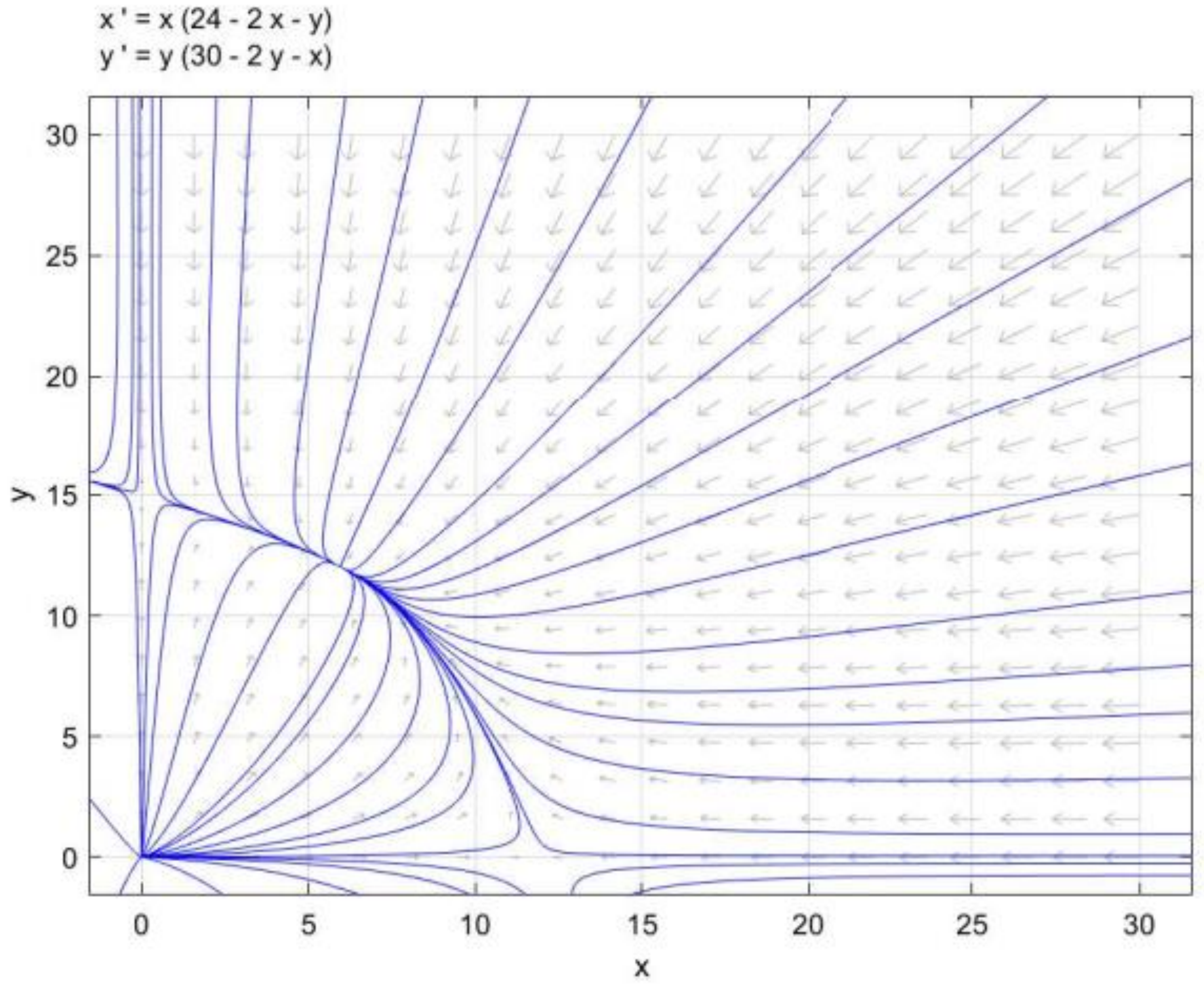
The equilibria are $(0,0)$, $(0,15)$, $(12,0)$ and $(6,12)$. They are classified as node, saddle, saddle, node, respectively. A solution with $x(0) > 0$, $y(0) > 0$ limits to $(6,12)$ at $t = \infty$.

PPLANE MODEL





PPLANE MODEL



CONCLUSION

- Interactions of populations in biology can be described in the language of mathematics.
- Before considering interacting populations, it is necessary to consider the growth of a single population.
- I focused only on 2 models here under the competition category: survival of one species and co-existence.
- pplane showed the same results Utah gave using their code.

Thank you