Math 132H Notes

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This set of notes is very informal and tries its best to simplify often hard to digest ideas. Hopefully it's useful in your learning of the material.

FINISH: put figures in right places

Contents

1 Lecture 1 - March 29, 2021

Today felt more like an overview than anything/basic introduction, so we didn't do anything too crazy today, but it was still pretty cool.

We first define the complex numbers.

Definition 1.1. We define $\mathbb{C} \coloneqq \left\{x + iy : x, y \in \mathbb{R} \wedge i^2 = -1\right\}$.

Notice that $\mathbb{C}$ is a 2D basically $\mathbb{R}^2$ and is a field, but the i gives us some cool stuff. It's also a field since we can add, subtract, multiply, and divide.

Here are some basic operations/facts with $z \coloneqq x + iy$:

(a) $\overline{z} \coloneqq x - iy$;

(b) $z \cdot \overline{z} = |z|^2 = x^2 + y^2$;

(c) $\Re(z) \coloneqq x$ and $\Im(z) \coloneqq y$.

We also have that $\mathbb{C}$ is a metric space with respect to $|z|^2$. Thus it also satisfies the triangle inequality: $|z + w| \leq |z| + |w|$.

We then talked about Euler's Identity/Formula for a bit but it culminated with this fact:

Fact 1.2. *We may represent complex numbers as $re^{i\theta}$ for suitable r and θ.*

Now we're talking about sequences in $\mathbb{C}$.

Definition 1.3. Let (z_n) be a sequence in $\mathbb{C}$. We say that $\lim_{n\to\infty} z_n = w$, or (z_n) *converges to* w if and only if $|z_n - w| \to 0$ as $n \to \infty$. Equivalently, for all $\varepsilon > 0$ there exists an N such that $n \geq N$ implies that $|z_n - w| < \varepsilon$.

Definition 1.4. Let (z_n) be a sequence in $\mathbb{C}$. We say that (z_n) is a *Cauchy sequence* if and only if for all $\varepsilon > 0$, there exists an N such that $|z_n - z_m| < \varepsilon$ whenever $n, m \geq N$.

Now onto our first theorem!

Theorem 1.5. *The set $\mathbb{C}$ is complete (i.e., all Cauchy sequences in $\mathbb{C}$ converge in $\mathbb{C}$).*

Topology of the $\mathbb{C}$

Then we talked a bit about topology and how we're lucky we have a metric because that gives us a lot of nice properties. The first thing we really talked about regarding topology is disc in $\mathbb{C}$.

Definition 1.6. We define the following sets for $r > 0$ and $z_0 \in \mathbb{C}$:

(a) $D_r(z_0) \coloneqq \{z \in \mathbb{C} : |z - z_0| < r\}$ (*open disc of radius r in $\mathbb{C}$*);

(b) $\overline{D}_r(z_0) \coloneqq \{z \in \mathbb{C} : |z - z_0| \leq r\}$ (*closed disc of radius r in $\mathbb{C}$*);

(c) $C_r(z_0) \coloneqq \{z \in \mathbb{C} : |z - z_0| = r\}$ (*circle of radius r in $\mathbb{C}$*).

Definition 1.7. Given a set $\Omega \subseteq \mathbb{C}$ and a point $z_0 \in \Omega$, z_0 is an *interior point of* Ω if and only if there exists an open disc $D_r(z_0) \subseteq \Omega$.

The *interior of* Ω is the set of all interior points of Ω. This is denoted $\operatorname{int}(\Omega)$.

The set Ω is open if and only if $\Omega = \operatorname{int}(\Omega)$.

Definition 1.8. A set $\Omega \subseteq \mathbb{C}$ is closed if and only if $\mathbb{C} \setminus \Omega$ is open.

Remark. Equivalently, Ω is closed if and only if it contains all of its limit points.

This leads us into int's counterpart: closure.

Definition 1.9. The closure of Ω is the union of Ω with all of its limit points. This is denoted $\overline{\Omega}$.

We're just going to keep loading up on definitions for now.

Definition 1.10. The boundary of Ω is $\overline{\Omega} \setminus \operatorname{int}(\Omega)$.

Because we are in a metric space, we have a sense of being bounded. This means that the distance between points is bounded. The actual definition for this that we use in this class is this:

Definition 1.11. A set $\Omega \subseteq \mathbb{C}$ is bounded if and only if it is contained in some $D_r(z_0)$ for some finite r.

Now to compactness! This guy is pretty important. The professor said Heine-Borel is good exercise. Pretty good. b)

Definition 1.12. A set $\Omega \subseteq \mathbb{C}$ is *compact* if and only if Ω is closed and bounded.

Theorem 1.13. *Let Ω be a subset of $\mathbb{C}$. Then, Ω is compact if and only if every sequence has a subsequence that converges in Ω.*

Remark. This is known as *sequential compactness*. Very important property.

Now we can get to the definition of *covering compactness*.

Definition 1.14. Let Ω be a subset of $\mathbb{C}$. An *open cover of* Ω is a family of open sets $\{U_\alpha\}$ (not necessarily countable) such that

$$\Omega \subseteq \bigcup_\alpha U_\alpha.$$

Theorem 1.15. *A subset Ω of $\mathbb{C}$ is compact if and only if every open cover has a finite subcover.*

Now to our first proposition/proof (though it is said to be trivial haha). However, before this we need a quick definition.

Definition 1.16. The diameter of a set $\Omega \subseteq \mathbb{C}$ is defined to be $\sup S$ where $S := \{|z - w| : z, w \in \mathbb{C}\}$.

Proposition 1.17. *Suppose $\Omega_1 \supseteq \Omega_2 \supseteq \cdots \supseteq \Omega_n \supseteq \cdots$ is a sequence of nonempty compact sets such that $\operatorname{diam}(\Omega_n) \to 0$ as $n \to \infty$. Then there is a unique $w \in \mathbb{C}$ such that $w \in \Omega_n$ for all n.*

Proof. Choose $z_n \in \Omega_n$ for each n. Then (z_n) is a Cauchy sequence. Let $w \coloneqq \lim_{n\to\infty} z_n$ (this exists since $\mathbb{C}$ is complete). Then w is clearly in Ω_n for all n and it also trivially unique (since a convergent sequence can only have 1 limit).

My question: how did we get that there is only one sequence? Couldn't we get a bunch? ■

Now to connectedness.

Definition 1.18. A set $\Omega \subseteq \mathbb{C}$ is *connected* if and only if it is not possible to express Ω as the disjoint union of nonempty sets Ω_1 and Ω_2 such that $\overline{\Omega}_1 \cap \Omega_2 = \Omega_1 \cap \overline{\Omega}_2 = \varnothing$.

Remark. In the case of an open Ω, this definition reduces to the following: Ω is connected if and only if you cannot express Ω has the union of disjoint nonempty open sets.

Definition 1.19. A *region* is an open and connected set.

Functions on regions

Definition 1.20. Let Ω be a subset of $\mathbb{C}$ and let $f\colon \Omega \to \mathbb{C}$. The function f is *continuous at a point* $z_0 \in \Omega$ if and only if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $z \in \Omega$ and $|z - z_0| < \delta$, $|f(z) - f(z_0)| < \varepsilon$.

The function f is *continuous on* Ω if and only if it is continuous at every $z_0 \in \Omega$.

Theorem 1.21. *A continuous f on a compact Ω is bounded and attains a maximum and a minimum.*

We talk about holomorphic functions (complex differentiable functions) next!