Math 131ABH Notes

Mealaud Mokhtarzad

This set of notes is very informal and tries its best to simplify often hard to digest ideas. Hopefully it's useful in your learning of the material. Also, this set of notes assumes knowledge of induction (i.e. I didn't want to talk about it lmao).

FINISH: put figures in right places

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1 Real numbers and the least upper bound property

1.1 Problems with the rationals and the need for the reals

Rational numbers are pretty cool. You can make them as small as you want, you can make them as close as you want to any other number (just add more digits or something. You figure it out lmfao). They seem just perfect! But, sadly, they are not. They definitely aren't. They are incomplete! (We'll explain precisely what this means later, but the idea is that) The set of rational numbers has holes! What do I mean by this? Well, there are *irrational numbers*. Here's an example.

Example 1.1. $\sqrt{2}$ is an irrational number. In other words, it cannot be written as the ratio of two integers.

Proof. Suppose towards a contradiction that $\sqrt{2}$ is rational. Then there exists corpime integers p and q such that $p/q = \sqrt{2}$. Squaring both sides and multiplying by q^2, we get that $p^2 = 2q^2$. Because the square of an odd integer is always odd, we know that p must then be even, so there exists an integer r such that $p = 2r$. It follows that $4r^2 = 2q^2 \implies 2r^2 = q^2$. By the same reasoning, q must then also be even, contradicting our assumed coprimeness of p and q. Thus $\sqrt{2}$ must be rational. ■

We have now uncovered a 'hole' in our set of rationals, so our rationals aren't 'complete' in some sense. Are the real numbers complete, however? By construction yes, but we'll (maybe) get to that eventually.

1.2 Real numbers and the least upper bound property

How do we know that $\mathbb{R}$ doesn't have the same pitfalls as the rationals? Well, it's equipped with the *least upper bound property* (or *axiom of completeness*, depends on what you want to call it).

Axiom 1.2 (The least upper bound property)**.** *Every nonempty set of real numbers that is bounded above has a least upper bound.* ***fix***

What does literally any of this mean? We'll get to that now.

Definition 1.3 (Upper and lower bounds)**.** A set $A \subseteq \mathbb{R}$ is *bounded above* if there exists a number $u \in \mathbb{R}$ such that if $a \in A$, then $a \leq u$. u is called an *upper bound* for A.

Being bounded below and lower bounds are defined similarly.

Now what does *least upper bound* mean?

Definition 1.4 (Least upper and greatest lower bounds)**.** A real number s is the *least upper bound* or *supremum*for a set $A \subseteq \mathbb{R}$ if and only if the following conditions are met:

(i) s is an upper bound for A;

(ii) if u is any upper bound for A, then $s \leq u$.

s is denoted as $\sup A$.

The *greatest lower bound* or *infimum* is defined similarly.

Does this fill our hole from before, however? Well, consider the set $Q = \left\{q \in \mathbb{Q} \mid q^2 < 2\right\}$. Consider Q as a subset of the reals, we get that $\sup Q = \sqrt{2}$, so our hole is filled! Perfect.

Here's another way to define least upper bound that seems almost too clear to be worth proving, but we'll do it for the sake thoroughness ig.

Lemma 1.5. *Assume $s \in \mathbb{R}$ is an upper bound for a set $A \subseteq \mathbb{R}$. Then $s = \sup A$ if and only if for every $\epsilon > 0$, there exists an element $a \in A$ such that $s - \epsilon < a$. In other words, $s = \sup A$ if and only if any number smaller than s is not an upper bound of A.*

Proof. Refer to Abbott's *Understanding Analysis* Lemma 1.3.8. for proof. **PRove this. Try to keep this self-contained** ■

Now let's move on to some consequences of the least upper bound property.

1.3 Consequences of the least upper bound property

There are two very, very, very, very important consequences of the least upper bound property: the *Nested Interval Property* and the *Archimedean Property*. We'll cover them in that order, also.

Theorem 1.6 (Nested Interval Property)**.** *Assume for each $n = 1, 2, 3, \ldots$, we are given a closed interval $I_n = [\ell_n, r_n]$. Suppose that for all n, $I_n \supseteq I_{n+1}$. Then the intersection of all I_n is nonempty.*

Slogan. The intersection of nested closed intervals is nonempty.

Before we start the proof, looking at Figure 1 may help you digest what the theorem is saying (it also may make following the proof easier).

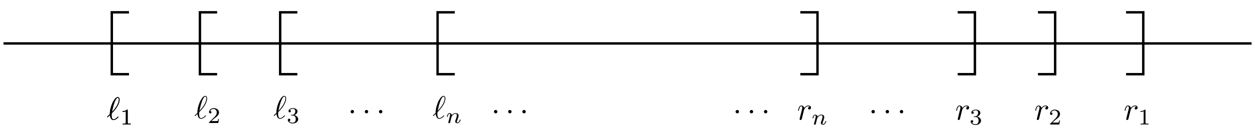

Figure 1: Diagram of the nested intervals for Theorem 1.6

Proof. Let $L = \{\ell_n \mid n = 1, 2, 3, \ldots\}$ and let $R = \{r_n \mid n = 1, 2, 3, \ldots\}$. Let $s = \sup L$. By construction of the intervals, we know that each r_n is an upper bound of L, so, by definition of least upper bound, we know that $s \leq r_n$ for all n. Thus for any n, we have that $\ell_n \leq s \leq r_n$. Thus, $s \in I_n$ for all n and we have that the intersection is nonempty. ■

This may not seem useful now, but it will be used later to do some pretty cool stuff.

Theorem 1.7 (Archimedean Property)**.** *Given any number $r \in \mathbb{R}$, there exists a positive integer n such that $n > r$.*

Slogan. There is always a bigger integer.

Proof. Suppose towards a contradiction that $r > N$ for all positive integers N. Then r is an upper bound for $\mathbb{Z}^+$. Let ℓ be the least upper bound for $\mathbb{Z}^+$ (which exists by the least upper bound property). By Lemma 1.5, $\ell - 1$ is not an upper bound, so there exists an integer $n > \ell - 1$, so $n + 1 > \ell$. Because $n + 1 \in \mathbb{Z}^+$, ℓ is not an upper bound for $\mathbb{Z}^+$ and we have a contradiction. This completes the proof. ■

Corollary 1.8. *Given any real number $r > 0$, there exists a positive integer n such that $1/n < r$.*

Slogan. There's always a smaller reciprocal of a positive integer.

2 Real sequences

2.1 Basics

First we have to define what a sequence is. (It's pretty much what you expect it to be, except it must be infinite in a sense we are about to make precise.)

Definition 2.1 (Sequence). Given a set S, a *sequence* in S is a function from $\mathbb{Z}^+$ to S. A sequence is commonly denoted as (a_n) where n is a positive integer and a_n is the value of the sequence for that integer.

Remark. A sequence in a particular set S can be said to be an S sequence. For example, a sequence in the reals or rationals is called a real sequence or a rational sequence, respectively.

Here are a few examples of sequences.

Example 2.2. (a) $(\text{dog}, \text{cat}, \text{cat}, \text{cat}, \ldots)$ is a sequence on the set of $\{\text{cat}, \text{dog}\}$.

(b) $(n)_{n=1}^{\infty} = (1, 2, 3, \ldots)$ is a sequence on the set of integers.

(c) (a_n) where $a_n = \sqrt{2} - 1/n$ for each positive integer n is a sequence in the real numbers.

(d) (x_n) where $x_1 = \pi$ and $x_n = x_{n-1} + 1$ is a sequence of real numbers.

Now here are two very important inequalities that will come up literally all the time.

Theorem 2.3 (Triangle Inequality). *Let $a, b, c \in \mathbb{R}$. Then, $|a - b| \leq |a - c| + |c - b|$.*

Proof. We aim to show that $|a + b| \leq |a| + |b|$. It suffices to show this because, if this is true, then $|a - c + c - b| \leq |a - c| + |c - b|$, which is our intended triangle inequality.

Because of the definition of absolute value, we must consider the cases where $a + b > 0$ and $a + b \leq 0$.

Suppose $a + b > 0$. Then, $|a + b| = a + b \leq |a| + |b|$.

Suppose $a + b \leq 0$. Then, $|a + b| = -(a + b) = -a - b \leq |a| + |b|$.

This completes the proof. ■

Theorem 2.4 (Reverse Triangle Inequality). *Let $a, b \in \mathbb{R}$. Then, $||a| - |b|| \leq |a - b|$.*

Proof. Consider $|a|$. It follows that

$$|a| = |a - b + b| \leq |a - b| + |b| \implies |a| - |b| \leq |a - b|. \tag{1}$$

Since a was arbitrary, it follows that

$$|b| - |a| \leq |b - a| = |a - b|. \tag{2}$$

Combining (1) and (2) and noting the definition of absolute value, we get that

$$||a| - |b|| \leq |a - b|.$$

This completes the proof. ■

Now we can move on to more cool stuff. B) Note: For the rest of this section, we will only be speaking of real sequences.

2.2 Convergent and divergent sequences

Convergent sequences

Let's get right into it.

Definition 2.5 (Convergence)**.** A sequence (x_n) (in $\mathbb{R}$) converges to a number $x \in \mathbb{R}$ if and only if for every $\epsilon > 0$, there exists an integer N such that $n \geq N$ implies $|x_n - x| < \epsilon$. This is denoted $\lim x_n = x$ or $(x_n) \to x$.

Remark. This is basically saying, a sequence converges to some number if it *eventually* gets arbitrarily close to that number.

Here are some examples of convergent sequences:

Example 2.6. (a) $(1/n)_{n=1}^{\infty}$ (converges to 0);

(b) $(0, 0, 0, \ldots)$ (converges to 0);

(c) (a_n) where $a_n = 10^{99}$ if $n < 50,000$ and $a_n = 1/n$ otherwise (converges to 0).

Remark. (c)'s convergence from the example should really push the point that sequences converge based on their tails.

How do we prove a sequence like (a) converges, however? Well, we'll do it as an example.

Example 2.7. $\lim(1/n) = 0$.

Proof. To prove that the limit of the sequence $(1/n)_{n=1}^{\infty}$ is 0, we have to show that, as per the definition, that for any $\epsilon > 0$, there exists an integer N such that $n \geq N$ implies that $|1/n - 0| < \epsilon$. To this end, recall the Archimedean Property and notice that $|1/n - 0| = 1/n$ since $1/n$ is positive for all positive integers n. It follows then by the Archimedean Property that there exists a positive integer N such that $1/N < \epsilon$. Taking $n \geq N$, we have that $|1/n - 0| = 1/n < \epsilon$. This completes the proof. ■

One question you may (or may not) have is the following: are limits unique? What I mean by this is can you have a sequence that converges to two different numbers? This likely sounds odd because it is impossible! We will show this, but before we do we need a lemma.

Lemma 2.8. *Two real numbers are equal if and only if their difference may be made arbitrarily small. In other words, two real numbers a, b are equal if and only if for every $\epsilon > 0$, $|a - b| < \epsilon$.*

Proof. Let $\epsilon > 0$ be arbitrary.

($\Rightarrow$) Suppose $a = b$. Then $|a - b| = 0 < \epsilon$.

($\Leftarrow$) Suppose $|a - b| < \epsilon$. Suppose towards a contradiction that $a \neq b$. Then, clearly, $a - b \neq 0$ (because otherwise they would be equal), so $|a - b| > 0$. Thus, by hypothesis, we can make $|a - b| < |a - b|$ which is a contradiction. Thus $a = b$. ■

Theorem 2.9 (Uniqueness of limits of convergent sequences)**.** *The limit of a convergent sequence is unique.*

Proof. Let $\epsilon > 0$ be arbitrary. Suppose we have a convergent sequence (x_n) that converges to real numbers a and b. It then follows that there exists integers N_a and N_b such that $n \geq N_a$ implies $|x_n - a| < \epsilon/2$ and

$n \geq N_b$ implies $|x_n - b| < \epsilon/2$, respectively. Taking $n \geq \max\{N_a, N_b\}$ and summing these two inequalities we get that $|x_n - a| + |x_n - b| < \epsilon$. It follows from triangle inequality then that

$$|a - b| \leq |x_n - a| + |x_n - b| < \epsilon \implies |a - b| < \epsilon.$$

It follows by Lemma 2.8 that $a = b$. Thus the limit of a convergent sequence is unique. ■

Divergent sequences

As you might suspect, divergence is the opposite of convergence. More precisely,

Definition 2.10 (Divergence of a sequence)**.** A sequence is *divergent* if and only if it does not converge.

Here are a few examples of divergent sequences:

Example 2.11. Some divergent sequences.

(a) $(0, 1, 0, 1, 0, 1, \ldots)$;

(b) (a_n) where $a_n = 1/n$ for odd n and $a_n = 1$ for even n;

(c) $(\sin(n))_{n=1}^{\infty}$;

(d) $(1, 2, 3, \ldots)$.

2.3 Algebraic and order theorems for convergent sequences

Convergent sequences are very well behaved. Indeed, they behave pretty much exactly how you'd expect them to (i.e. they behave like numbers themselves when it comes to algebra operations). But before we get into the algebra, we first need a theorem that will prove to be helpful, but even before that we need another definition.

Definition 2.12 (Bounded sequence)**.** A sequence (x_n) is bounded if and only if there exists a number $M > 0$ such that $|x_n| \leq M$ for all positive integer n.

To make what this means more explicit, here's an example.

Example 2.13. Here are some examples of bounded sequences.

(a) $(1/n)_{n=1}^{\infty}$ is bounded by 1;

(b) $(1, -1, 1, -1, 1, -1, \ldots)$ is bounded by 1;

(c) $(\sin(n))$ is bounded by 1.

However, not all sequences are bounded! For example, $(1, 2, 3, \ldots)$.
Now we can move on to our useful theorem.

Theorem 2.14. *Every convergent sequence is bounded.*

Slogan. Convergent $\implies$ bounded.

Proof. Suppose (x_n) is a sequence that converges to $x \in \mathbb{R}$. By definition of convergence, we know that there exists an integer N such that $n \geq N$ implies that $|x_n - x| < 1$. It follows by reverse triangle inequality that $|x_n| - |x| < 1$, so $|x_n| < 1 + |x|$. Then, let $M = \max\{|x_n| : n = 1, 2, 3, \ldots, N-1\} \cup \{1 + |x|\}$. Clearly M bounds (x_n) and (x_n) is bounded. This completes the proof. ■

Now we can get into the algebra of limits.

Theorem 2.15 (Algebraic Limit Theorem). *Let* $\lim x_n = x$, $\lim y_n = y$, *and* $c \in \mathbb{R}$. *Then,*

(i) $\lim(cx_n) = cx$;

(ii) $\lim(x_n + y_n) = x + y$;

(iii) $\lim(x_n y_n) = xy$;

(iv) $\lim(x_n/y_n) = x/y$ *if* $y \neq 0$.

Proof. Let $\epsilon > 0$. (i) If $c = 0$, then we have the 0 sequence which clearly converges to 0, so we may assume that $c \neq 0$. Then, because $(x_n) \to x$, there exists an integer N such that $n \geq N$ implies $|x_n - x| < \epsilon/c$. It then follows that

$$|cx_n - cx| = c|x_n - x| < c \cdot \frac{\epsilon}{c} = \epsilon.$$

Thus $(cx_n) \to cx$.

(ii) Because $\lim x_n = x$ and $\lim y_n = y$, there exists integer N_1, N_2 such that $n \geq N_1$ implies $|x_n - x| < \epsilon/2$ and $n \geq N_2$ implies $|y_n - y| < \epsilon/2$, respectively. It then follows that

$$|(x_n + y_n) - (x + y)| \leq |x_n - x| + |y_n - y| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $\lim(x_n + y_n) = x + y$.

(iii) Because (y_n) is convergent, by Theorem 2.14, (y_n) is bounded. Let $Y > 0$ be a bound for (y_n). Moreover, because $\lim x_n = x$ and $\lim y_n = y$, there exists an N_1 and N_2 such that $n \geq N_1$ implies $|x_n - x| < \epsilon/(2Y)$ and $n \geq N_2$ implies $|y_n - y| < \epsilon/(2|x|)$, respectively. It then follows that

$$\begin{aligned}
|x_n y_n - xy| &\leq |x_n y_n - x y_n| + |x y_n - xy| \\
&= |y_n||x_n - x| + |x||y_n - y| \\
&\leq Y|x_n - x| + |x||y_n - y| \\
&< Y \cdot \frac{\epsilon}{2Y} + |x| \cdot \frac{\epsilon}{2|x|} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Thus $\lim(x_n y_n) = xy$.

(iv) By (iii), it suffices to show that $(1/y_n) \to 1/y$ (assuming $y \neq 0$). We will prove this.

Because $(y_n) \to y$, we know that there exists an integer N_1 such that if $n \geq N_1$, $|y_n - y| < |y|/2$. From the reverse triangle inequality it follows that

$$|y| - |y_n| \leq |y_n - y| < \frac{|y|}{2} \implies \frac{|y|}{2} < |y_n|.$$

Now, because $(y_n) \to y$, there exists an integer N_2 such that $n \geq N_2$ implies that $|y_n - y| < \epsilon |y|^2/2$. Taking $n \geq \max\{N_1, N_2\}$, it follows that

$$\left|\frac{1}{y_n} - \frac{1}{y}\right| = \frac{|y - y_n|}{|y||y_n|} < \frac{\epsilon |y|^2}{2} \cdot \frac{1}{|y|\frac{|y|}{2}} = \epsilon.$$

Thus, $\lim(1/y_n) = 1/y$ and the proof is complete. ■

Now let's get to the other part of this section: order theorems.

Theorem 2.16 (Order Limit Theorems). *Let* $\lim x_n = x$ *and* $\lim y_n = y$.

(i) *If* $x_n \geq 0$ *for all positive integer* n*, then* $x \geq 0$.

(ii) *If* $x_n \leq y_n$ *for all positive integer* n*, then* $x \leq y$.

(iii) *If there exists a* $c \in \mathbb{R}$ *for which* $c \leq y_n$ *for all positive integer* n*, then* $c \leq y$*. Similarly, if* $x_n \leq c$ *for all positive integer* n*, then* $x \leq c$.

Proof. (i) Suppose towards a contradiction that $x < 0$. Then, because $(x_n) \to x$, there exists an integer N such that $n \geq N$ implies that $|x_n - x| < |x| = -x$. From the definition of absolute value it follows that $x < x_n - x < -x$, so $x_n < 0$ if $n \geq N$, which is a contradiction as $x_n \geq 0$ for all positive integer n. Thus, $x \geq 0$.

(ii) It follows from the Algebraic Limit Theorem that $\lim(y_n - x_n) = y - x$. Because $y_n - x_n \geq 0$, it follows from part (i) that $y - x \geq 0 \implies y \geq x$.

(iii) Take $x_n = c$ (or $b_n = c$) for all positive integer n, and apply (ii). ■

2.4 Monotone Convergence Theorem

We first have to define a few things about sequences before we can get into the name of this section.

Definition 2.17. A sequence (x_n) is *increasing* if and only if $x_n \leq x_{n+1}$ for all positive integers. A sequence is *decreasing* if and only if $x_n \geq x_{n+1}$ for all positive integers n. A sequence is *monotone* if and only if it is either increasing or decreasing.

With Theorem 2.14, we showed that every convergent sequence is bounded, but does bounded imply convergent? In general, no. Take for example $(0, 1, 0, 1, 0, 1, \ldots)$. This sequence is bounded by 1 but clearly not convergent. But what are some sufficient conditions for convergence given something is bounded? One of them is being monotone. Why? Well, if something is increasing yet bounded (above), then it must slow down or else it will exceed its bound, so then it must converge. This is the idea of the following very important theorem.

Theorem 2.18 (Monotone Convergence Theorem). *Any bounded and monotone sequence converges.*

Slogan. Bounded & monotone $\implies$ convergent

Proof. Let (x_n) be monotone and bounded. In particular, suppose (x_n) is increasing (the decreasing case is handled similarly). Because (x_n) is bounded, $s \coloneqq \sup\{x_n \mid n \in \mathbb{Z}^+\}$ exists. We claim that $(x_n) \to s$.

Now, let $\epsilon > 0$ be arbitrary. By Lemma 1.5, we know that $s - \epsilon$ is not an upper bound, so there exists an x_N such that $s - \epsilon < x_N$. Because (x_n) is increasing, we know that for all $n \geq N$, $s - \epsilon < x_N \leq x_n$. It then follows that

$$s - \epsilon < x_N \leq x_n \leq s < s + \epsilon \implies s - \epsilon < x_n < s + \epsilon \implies |x_n - s| < \epsilon,$$

and $\lim x_n = s$. This completes the proof. ■

Monotone Convergent Theorem will become invaluable once we begin to study series! Very cool stuff. B)

2.5 Subsequences

Basics

A subsequence is exactly what you imagine it to be: it's a subpart of our sequence, but it's a sequence on its own.

Definition 2.19 (Subsequence)**.** Let (x_n) be a sequence of real numbers, and let $n_1 < n_2 < n_3 < \cdots$ be an increasing sequence of positive integers. Then the sequence $(x_{n_1}, x_{n_2}, x_{n_3}, \ldots)$ is called a *subsequence* of (x_n) and is denoted (x_{n_k}) where a positive integer k indexes the subsequence.

Example 2.20. To get you situated, here's an example of a subsequence. If $(x_n) = (1, 2, 3, \ldots)$, then $(1, 3, 5, \ldots)$ and $(2, 4, 6, \ldots)$ are subsequences, but $(2, 1, 3, 4, 5, \ldots)$ is not a subsequence.

Here's a neat fact about subsequences.

Fact 2.21. *Any subsequence of a convergent sequence converges to the same limit as the parent sequence.*

Proof. Let $\epsilon > 0$ be arbitrary. Suppose (x_n) converges to $x \in \mathbb{R}$. Let (x_{n_k}) be a subsequence. Because (x_n) converges to x, there exists an integer N such that $n \geq N$ implies that $|x_n - x| < \epsilon$. Taking $n_k \geq k \geq N$, it follows that $|x_{n_k} - x| < \epsilon$. Thus $(x_{n_k}) \to x$. ■

Now we're prepared to talk about the big boy of subsequences: Bolzano Weierstrass Theorem.

The Bolzano-Weierstrass Theorem

Theorem 2.22 (The Bolzano-Weierstrass Theorem)**.** *Every bounded sequence has a convergent subsequence.*

Slogan. Bounded $\implies \exists$ convergent subsequence

Proof. Let (x_n) be a sequence bounded by some number $M > 0$. Then all points of the sequence are in the interval $[-M, M]$. Bisect this interval and two closed intervals: $[-M, 0]$ and $[0, M]$. Now label the interval with infinitely many points in it I_1. We know that one of these two intervals must have infinitely many points as if neither of them did, the sequence would have only finitely many points, which would be a contradiction. Bisect I_1 once more into two closed intervals (where the midpoint is in both halves), and label the half with infinitely many terms of the sequence I_2. Now, the I_{n+1} interval is the closed half of the interval I_n that contains infinitely many points.

For our subsequence, choose x_{n_1} to be a point in I_1. Then, choose x_{n_2} to be a point in I_2 such that $n_2 > n_1$ (this exists since there are infinitely many points in I_2). In general, choose x_{n_k} to be a point in the I_k interval such that $n_k > n_{k-1}$.

By the Nested Interval Property, we know that there exists at least one point in the intersection of all the $I_1, I_2, I_3, \ldots$. Choose a point from the intersection and call it x. We claim that $(x_{n_k}) \to x$.

Now, let $\epsilon > 0$ be arbitrary. By construction, the length of the interval I_k is $M/2^{k-1}$, which converges to 0. Choose an integer N such that $k \geq N$ implies that $M/2^{k-1} < \epsilon$. Taking $k \geq N$, it follows that

$$|x_{n_k} - x| \leq \frac{M}{2^{k-1}} < \epsilon$$

as x_{n_k} and x are both in I_k. Thus, $(x_{n_k}) \to x$. ■

Figure 2 may be helpful for understanding the proof of Bolzano-Weierstrass Theorem.

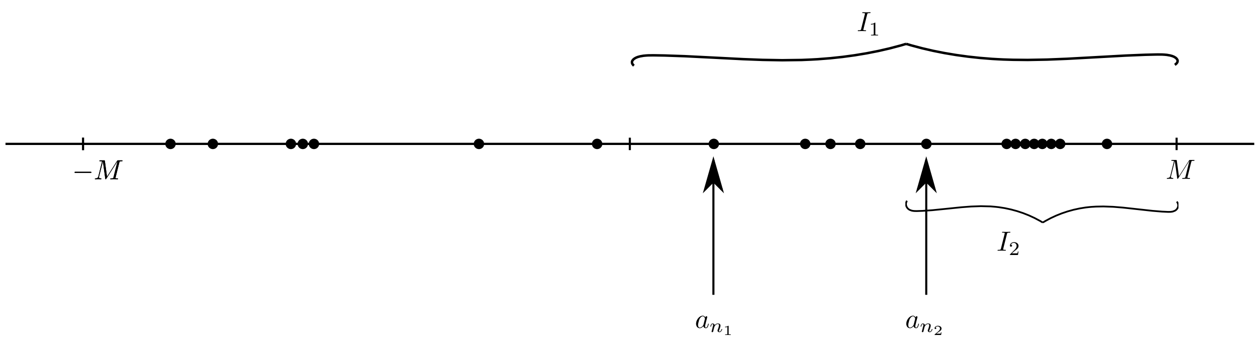

Figure 2: Diagram for the Bolzano-Weierstrass Theorem's proof

2.6 Cauchy sequences

We'll go straight to the definition since the name isn't particularly insightful (it's just the name of some (very important) mathematician).

Definition 2.23 (Cauchy sequence)**.** A sequence (x_n) a *Cauchy sequence* if and only if, for every $\epsilon > 0$, there exists an integer N such that $|x_i - x_j| < \epsilon$ if $i, j \geq N$.

A sequence being Cauchy just means the terms eventually get close together.

Cauchiness, at least to me, seems fairly close to convergence. The terms get close together, so they must get close to a single number, right? In general, no, but in $\mathbb{R}$, yes! We'll work towards showing this. At this moment, we can prove one direction: convergent implies Cauchy.

Theorem 2.24. *Every convergent sequence is a Cauchy sequence*

Proof. Let $\epsilon > 0$ be arbitrary. Suppose (x_n) converges to x. By definition of convergence, we know that there exists an integer N such that $n \geq N$ implies that $|x_n - x| < \epsilon/2$. Taking $n, m \geq N$, it follows by triangle inequality that

$$|x_n - x_m| \leq |x_n - x| + |x - x_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

so (x_n) is Cauchy. ■

How do we prove that Cauchy implies convergent, however? For that, we're going to need a lemma (since we're going to use Bolzano-Weierstrass).

Lemma 2.25. *Cauchy sequences are bounded.*

Proof. Suppose (x_n) is a Cauchy sequence. By definition of Cauchy, we know that there exists an integer N such that $i, j \geq N$ implies that $|x_i - x_j| < 1$. It follows by reverse triangle inequality that

$$|x_i| - |x_j| \leq |x_i - x_j| < 1 \implies |x_i| < 1 + |x_j|.$$

Fixing j to say N, we get that $|x_i| < 1 + |x_N|$. It follows that

$$\max\{|x_1|, \ldots, |x_{N-1}|, 1 + |x_N|\}$$

is a bound for (x_n). ■

Theorem 2.26 (Completeness of $\mathbb{R}$)**.** *A sequence converges if and only if it is a Cauchy sequence.*

Slogan. (In $\mathbb{R}$) Convergent $\iff$ Cauchy

Proof. ($\Rightarrow$) This direction is Theorem 2.24.

($\Leftarrow$) Suppose (x_n) is Cauchy. By Lemma 2.25, we know that (x_n) is then bounded and then by Bolzano-Weierstrass we know that (x_n) has a convergent subsequence (x_{n_k}). Let $(x_{n_k}) \to x$. We claim that $\lim x_n = x$.

Because (x_n) is Cauchy, there exists an integer N_1 such that $i, j \geq N_1$ implies that $|x_i - x_j| < \epsilon/2$. Because (x_{n_k}) converges to x, there exists an N_2 such that $k \geq N_2$ implies that $|x_{n_k} - x| < \epsilon/2$. Taking $n, K \geq \max\{N_1, N_2\}$, we get that

$$|x_n - x| \leq |x_n - x_{n_K}| + |x_{n_K} - x| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

and $\lim x_n = x$. This completes the proof. ■

3 Series

3.1 Basics

What does $\sum_{j=1}^{\infty} a_j = A$ (for some sequence (a_n) and some number A) even mean? Well, in general, $\sum_{j=1}^{N} a_j = a_1 + \cdots + a_N$, so we're saying something like $a_1 + a_2 + a_3 + \cdots = A$. In particular, what we mean is the following: $\lim S_n = A$ where $S_n = \sum_{j=1}^{n} a_j$. So, because of this we can take a lot of my results from sequences and apply them to series!!! Super cool stuff right here! :> But before we do that let's put we just said into an actual definition.

Definition 3.1. Let (a_n) be a sequence. Then,

$$\sum_{j=m}^{n} a_j \coloneqq a_m + a_{m+1} + a_{m+2} + \cdots + a_n$$

for $m \leq n$. With (a_n) we associate the sequence (S_n) where

$$S_n \coloneqq \sum_{j=1}^{n} a_j.$$

For (S_n) we also use the symbolic expression

$$a_1 + a_2 + a_3 + \cdots$$

or

$$\sum_{j=1}^{\infty} a_j. \tag{1}$$

(1) is called an *infinite series* or just *series*. (S_n) is called the *sequence of partial sums of the series*. If $\lim S_n = S$, we say that the series converges and we write

$$\sum_{j=1}^{\infty} a_j = S.$$

S is called the *sum* of the series. If (S_n) diverges, then the sum diverges.

When the bounds are unambiguous we simply write $\sum a_j$.

The following is a direct translation from sequence results to series.

Theorem 3.2 (Algebraic Limit Theorem for Series)**.** *If $\sum_{j=1}^{\infty} a_j = A$ and $\sum_{j=1}^{\infty} b_j = B$, then*

(i) $\sum_{j=1}^{\infty} ca_j = cA$ *for all* $c \in \mathbb{R}$*;*

(ii) $\sum_{j=1}^{\infty}(a_j + b_j) = A + B$.

Remark. We won't talk about products of series just yet. ;)

Remember Cauchy sequences? Well, because we can treat series like sequences (basically), it ends up being pretty improtant.

Theorem 3.3. *The series $\sum_{j=1}^{\infty} a_j$ converges if and only if for all $\epsilon > 0$ there exists an integer N such that whenever $n > m \geq N$ it follows that*

$$|a_{m+1} + a_{m+2} + a_{m+3} + \cdots + a_n| < \epsilon.$$

Proof. Notice that the final line is equivalent to $|S_n - S_m| < \epsilon$. Then also notice that the theorem is saying (S_n) converges if and only if it is Cauchy, which is true by the completeness of $\mathbb{R}$. ∎

Corollary 3.4. *If $\sum_{j=1}^{\infty} a_j$ converges, then $\lim a_n = 0$.*

Proof. Consider the case of $n = m+1$ with regard to Theorem 3.3. We then get that $|a_n| < \epsilon$ and $\lim a_n = 0$. ∎

However, is the converse of this corollary true? No!

Example 3.5 (Harmonic series)**.** The Harmonic Series is $\sum_{n=1}^{\infty} 1/n$. $\lim(1/n) = 0$, but the sum does not converge! (We'll prove this soon.)

Example 3.6 (Geometric Series)**.** A geometric is a series of the form $\sum_{j=0}^{n} ar^j$ for some common ratio (number) r. A geometric series converges if and only if $|r| < 1$. We leave this as an exercise as the identity $\sum_{j=0}^{n} ar^j = \frac{a(1-r^n)}{1-r}$ should be enough to do this. Should be pretty straight forward.

3.2 Tests for convergence

There are a ton of tests. Here are the main ones I know (not all were shown in class).

Theorem 3.7 (Cauchy Condensation Test)**.** *Suppose (a_n) is decreasing and satisfies $a_n \geq 0$ for all n. Then, the series $\sum_{j=1}^{\infty} a_j$. Then the series $\sum_{j=1}^{\infty} a_j$ converges if and only if the series $\sum_{j=1}^{\infty} 2^j a_{2^j}$ converges.*

Proof. finish ∎

Theorem 3.8 (Harmonic Series Test)**.** *The series $\sum_{n=1}^{\infty} 1/n^p$ converges if and only if $p > 1$.*

Proof. FINISH ∎

Theorem 3.9 (Comparison Test)**.** *Suppose (a_n) and (b_n) satisfy $0 \leq a_n \leq b_n$ for all n. Then,*

(i) *If $\sum_{j=1}^{\infty} b_j$ converges, then $\sum_{j=1}^{\infty} a_j$ converges;*

(ii) *If $\sum_{j=1}^{\infty} a_j$ diverges, then $\sum_{j=1}^{\infty} b_j$ diverges.*

Proof. Let $\epsilon > 0$ be arbitrary.

(i) Suppose $\sum_{j=1}^{\infty} b_j$ converges. From Theorem 3.3, there exists an integer N such that $n > m \geq N$ such that $|b_{m+1} + \cdots + b_n| < \epsilon$. Because $0 \leq a_n \leq b_n$ for all n, we have that

$$|a_{m+1} + \cdots + a_n| \leq |b_{m+1} + \cdots + b_n| < \epsilon.$$

Thus $\sum_{j=1}^{\infty} a_j$ converges.

(ii) Similar argument to the proof of (i). ∎

Theorem 3.10 (Absolute Convergence Test). *If $\sum_{j=1}^{\infty} |a_j|$ converges, then $\sum_{j=1}^{\infty} a_j$ converges.*

Proof. Let $\epsilon > 0$ be arbitrary. It follows from Theorem 3.3 that there exists an integer N such that $n > m \geq N$ implies that $||a_{m+1}| + \cdots + |a_n|| = |a_{m+1}| + \cdots + |a_n| < \epsilon$. It follows from Triangle Inequality that $|a_{m+1} + \cdots + a_n| \leq |a_{m+1}| + \cdots + |a_n| < \epsilon$, so $\sum_{j=1}^{\infty} a_j$ converges. ■

Converse is not true however! $\sum_{j=1}^{\infty}(-1)^j/j$ converges but $\sum_{j=1}^{\infty} 1/j$ does not. But how do we know that $\sum_{j=1}^{\infty}(-1)^j/j$ converges? With the following test:

Theorem 3.11 (Alternating Series Test). *Let (a_n) be a seuqence satisfying,*

(i) $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots$ *and*

(ii) $\lim a_n = 0$.

Then, the alternating series $\sum_{j=1}^{\infty}(-1)^j a_j$ converges.

Proof. **FINISH** ■

4 Functions on $\mathbb{R}$

4.1 The functional limit

4.2 Continuous functions

Basics

Continuous functions (as they were defined in your calculus class) are functions that can be drawn without lifting your pencil. How do we formalize this, however? We need something that encodes a lack of jumps, a lack of holes, a lack of *discontinuities*, but how do we do this?

Definition 4.1 (Continuity)**.** Let $S \subseteq \mathbb{R}$. A function $f\colon S \to \mathbb{R}$ is *continuous at a point* $s \in S$ if and only if, for all $\epsilon > 0$, there exists a $\delta > 0$ such that $|x - s| < \delta$ (for $x \in S$) implies that $|f(x) - f(s)| < \epsilon$.

If f is continuous at every point in its domain, then we say that f is *continuous*.

Why does represent our primitive idea of continuity? Well, at any given point, we must have our function approach the same value from both sides (left and right) of the point, so our function meets in the middle, so to speak.

We also have an alternate definition of continuity.

Theorem 4.2 (Sequential continuity)**.** *Let $S \subseteq \mathbb{R}$ and let $f\colon S \to \mathbb{R}$ be a function. Suppose (s_n) is a sequence in S such that $\lim s_n = s \in S$. Then, f is continuous at s if and only if $\lim f(s_n) = f(\lim s_n) = f(s)$.*

Slogan. Continuous $\iff$ preserves limits of sequences.

Proof. ($\Rightarrow$) Suppose f is continuous at s. We aim to show that $\lim f(s_n) = f(\lim s_n) = f(s)$. Let $\epsilon > 0$ be arbitrary. By f's continuity, we know that there exists a $\delta > 0$ such that $|s_n - s| < \delta$ implies $|f(s_n) - f(s)| < \epsilon$. Since $\lim s_n = s$, there exists an integer N such that $|s_n - s| < \delta$ if $n \geq N$. Therefore, taking $n \geq N$ once more, we get that $|f(s_n) - f(s)| < \epsilon$. Thus, $\lim f(s_n) = f(\lim s_n) = f(s)$.

($\Leftarrow$) Suppose that f preserves limits of sequences. Suppose towards a contradiction that f is not continuous at $s \in S$. Then, by definition of continuity, there exists an $\epsilon > 0$ such that for all $\delta > 0$, there exists a $t \in S$ such that $|t - s| < \delta$ but $|f(t) - f(s)| < \epsilon$. Now define a sequence (t_n) as follows: $t_n \in S$ is some number such that $|t_n - s| < 1/n$. By the Archimedean Property, we know that $\lim t_n = s$, but $|f(r_n) - f(t)| \geq \epsilon$, so $\lim f(t_n) \neq f(s)$, which contradicts our assumption. Thus f must be continuous. ■

Intermediate Value Theorem

Are you convinced that the definition of continuity represents our intended idea? If you are, the following should seem immediate, but if you are not, the Intermediate Value Theorem should cement the idea that our definition is perfect. The Intermediate Value Theorem is the idea that continuous functions do not have jumps. More precisely:

Theorem 4.3 (Intermediate Value Theorem)**.** *Suppose $f\colon [a, b] \to \mathbb{R}$ is continuous. If r is a real number satisfying $f(a) < r < f(b)$ or $f(a) > r > f(b)$, then there exists a point $c \in (a, b)$ where $f(c) = r$.*

Slogan. Continuous functions don't have jumps.

Figure 3 should help illustrate what is being said. Essentially, if we have a continuous function on an interval of this form, we function attains every value between the minimum and maximum in its range. In

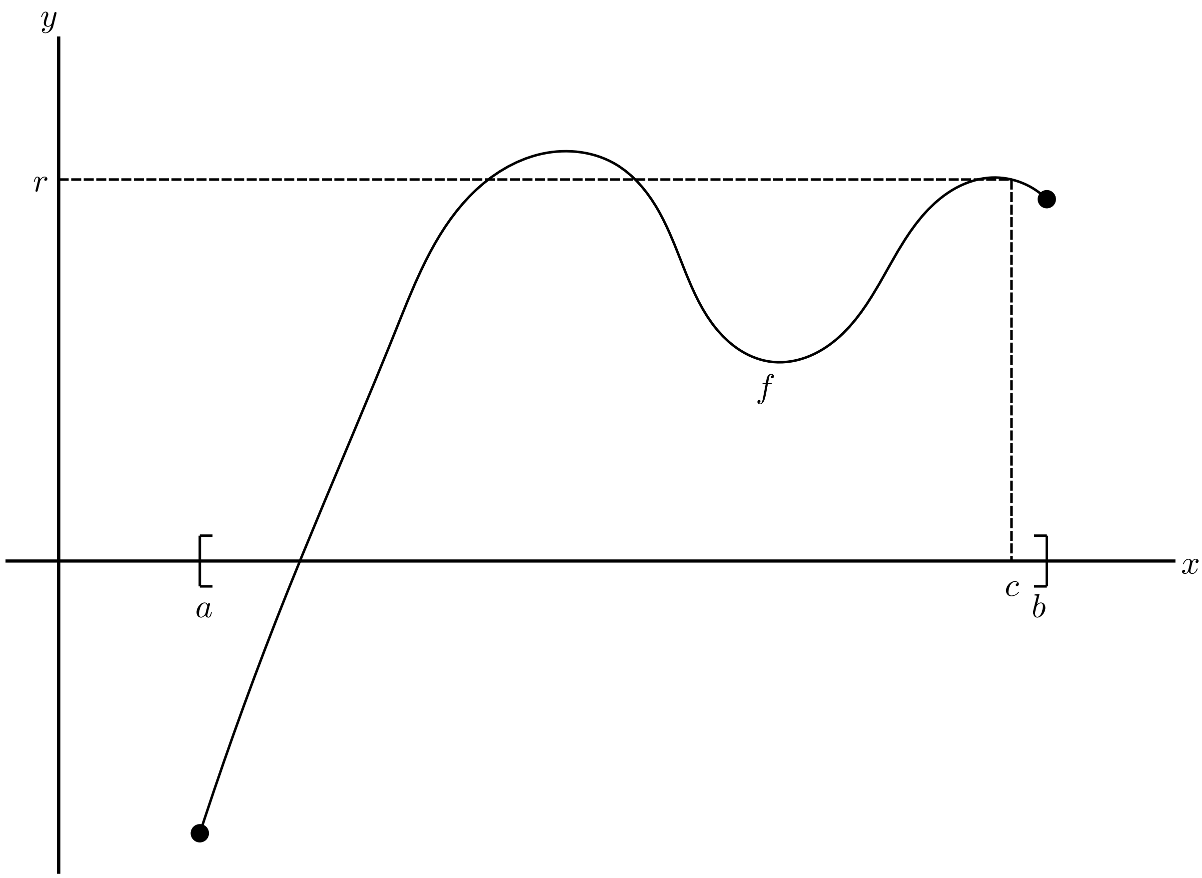

Figure 3: Diagram of Theorem 4.3.

other words, if we draw a line through a continuous function of this sort, we must cross that line, as seen in Figure 3. Now, to the proof.

Proof. We will only prove the case of $f(a) < r < f(b)$ as the other case is similar.

Let $C \coloneqq \{x \in [a,b] : f(x) < r\}$. Define c as the least upper bound of C (we know such a number exists as C is bounded above by b). We claim that $f(c) = r$.

Let $\epsilon > 0$ be arbitrary. Since f is continuous, there exists a $\delta > 0$ such that $|x - c| < \delta$ implies $|f(x) - f(c)| < \epsilon$. In other words, $f(x) - \epsilon < f(c) < f(x) + \epsilon$ for $x \in (c - \delta, c + \delta)$. Now, taking $\alpha \in (c - \delta, c]$, it follows that

$$f(\alpha) - \epsilon < f(c) < f(\alpha) + \epsilon < r + \epsilon \implies f(c) < r + \epsilon. \tag{1}$$

Then, taking $\beta \in (c, c + \delta)$ (notice that $\beta \notin C$), it follows that

$$r - \epsilon \leq f(\beta) - \epsilon < f(c) < f(\beta) + \epsilon \implies r - \epsilon < f(c). \tag{2}$$

(The leftmost sub-inequality of the inequality comes from the fact that since $\beta \notin C$, $r \leq f(\beta)$.) Combining (1) and (2) we get that $r - \epsilon < f(c) < r + \epsilon \implies |f(c) - r| < \epsilon$. It follows then from Lemma 2.8 that $f(c) = r$. This completes the proof. ■

Algebra and composition of continuous functions

Like convergent sequences, continuous functions interact with each other nicely (you can do algebra with them). They behave exactly as you expect. :)

Theorem 4.4 (Algebraic Continuity Theorem)**.** *Let $S \subseteq \mathbb{R}$. Suppose $f\colon S \to \mathbb{R}$ and $g\colon S \to \mathbb{R}$ are continuous at a point $s \in S$. Then,*

(i) *$cf(x)$ is continuous at s for all real c;*

(ii) *$f(x) + g(x)$ is continuous at s;*

(iii) *$f(x)g(x)$ is continuous at s;*

(iv) *$f(x)/g(x)$ is continuous at c if the quotient is defined.*

Proof. Do this after we cover functional limits and stuff. ∎

The following is another case of continuous functions behaving very nicely.

Theorem 4.5 (Composition of continuous functions)**.** *Let $S, T \subseteq \mathbb{R}$. Suppose that $f\colon S \to \mathbb{R}$ and $g\colon T \to \mathbb{R}$, and $f(S) \subseteq T$ (so that $(g \circ f)(x)$ is defined on S).*

If f is continuous at $s \in S$ and g is continuous at $f(s) \in T$, then $g \circ f$ is continuous at s.

Proof. Let $\epsilon > 0$ be arbitrary. Suppose (s_n) is a sequence in S that converges to $s \in S$. By hypothesis, we know that $f(s_n)$ is a sequence in T, so $g(f(s_n))$ is defined. By f's and g's continuity, we know that $\lim g(f(s_n)) = g(\lim f(s_n)) = g(f(s))$, so $g \circ f$ preserves limits of sequences and is continuous by Theorem 4.2. ∎

4.3 Uniform Continuity

Let's get right into it.

Definition 4.6 (Uniform Continuity)**.** Let $S \subseteq \mathbb{R}$. A function $f\colon S \to \mathbb{R}$ is uniformly continuous (on S) if and only if, given an $\epsilon > 0$, there is a $\delta > 0$ such that for all $x, y \in S$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

What does any of this mean? This sounds just like regular continuity, right? It does, yes, but they are not the same–they are different, though subtly. To make the difference more pronounced, we'll restate the the original definition of continuity:

Definition 4.1 (Continuity)**.** Let $S \subseteq \mathbb{R}$. A function $f\colon S \to \mathbb{R}$ is *continuous at a point* $s \in S$ if and only if, for all $\epsilon > 0$, there exists a $\delta > 0$ such that $|x - s| < \delta$ (for $x \in S$) implies that $|f(x) - f(s)| < \epsilon$.

If f is continuous at every point in its domain, then we say that f is *continuous*.

The thing to notice is the following: for continuity, δ can depend on ϵ and the point being considered; for uniform continuity, δ can only depend on ϵ (as the point being considered is not fixed since the δ must work for all x, y in the domain). Do not forget this!

Example 4.7. Some examples of uniformly continuous functions:

(a) $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$;

(b) $f\colon [0, 1] \to \mathbb{R}$ defined by $f(x) = x^2$;

(c) $f\colon [0, \pi] \to \mathbb{R}$ defined by $f(x) = \sin x$.

We will prove that (b) is uniformly continuous.

Example 4.8. $f\colon [0,1] \to \mathbb{R}$ defined by $f(x) = x^2$ is uniformly continuous.

Proof. Let $\epsilon > 0$ be arbitrary. Suppose that $|x-y| < \epsilon/2$. It then follows that

$$\begin{aligned} |x^2 - y^2| &= |x-y||x+y| \\ &\leq |x-y| \cdot 2 \\ &< \frac{\epsilon}{2} \cdot 2 = \epsilon. \end{aligned}$$

Thus f is uniformly continuous. ■

Is there something special about this example, however? Is there a general pattern/idea behind x^2 being continuous over a closed interval? Actually, yes. In fact, a continuous function over a *compact* interval is uniformly continuous, but we'll get to that later in the topology sections. :>

Are all functions uniformly continuous? No. Take $f(x) = 1/x$ on $(0,1)$, for example, but how do we know that $1/x$ is not uniformly continous? What if you're just not clever enough to figure out a δ that only depends on ϵ? Sometimes the latter might be the case, but there are ways to guarantee something is not uniformly continuous. The following is one to do this:

Theorem 4.9 (Sequential Criterion for Absence of Uniform Continuity)**.** *Let $S \subseteq \mathbb{R}$. A function $f\colon S \to \mathbb{R}$ is not uniformly continuous on S if and only if there exists a particular $\epsilon_0 > 0$ and two sequences (x_n) and (y_n) in S satisfying*

$$|x_n - y_n| \to 0 \quad \textit{but} \quad |f(x_n) - f(y_n)| \geq \epsilon_0.$$

Proof. **DO THIS** ■

TEST2

5 Metric space topology

Metric space topology was tough to understand at first (since it was presented fairly abstractly in my class), so I'll be trying to make it as easy to follow as I can. The general structure of the section will be the following: first we consider some idea in $\mathbb{R}$, then we consider it in general. Let's start.

5.1 Metric spaces

Basics

What is a metric space? Its definition comes in two parts: 1) what a metric is; and 2) what a space is. A *metric* is a particular kind of function that measures the distance between two points. For example, a function that measures the distance from one part to another part of your town would be a metric (we'll get to exactly why later). A *space* is just another term for a set, basically. Now let's get into more precise definitions for these (I think we've been handwavey enough). We'll first start with what a metric is exactly.

Definition 5.1 (Metric)**.** Let X be a set. A function $d\colon X \times X \to \mathbb{R}$ is said to be a metric if and only if for all points $x, y, z \in X$ d satisfies the following conditions:

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, y) \leq d(x, z) + d(z, y)$.

The conditions, in order, are called: 'Identity of Indiscernables' (kind of a lame name imo), 'Symmetry', and 'Triangle Inequality.' There's another 'hidden' condition, however: the value of a metric is always nonnegative. Why is this not on our list? Because we can derive this! We'll present this as a fact.

Fact 5.2 (Nonnegativity of metrics)**.** *Let d be a metric on a set X and let $x, y \in X$. Then, $d(x, y) \geq 0$.*

Proof. Consider $d(x, x)$. It follows that

$$\begin{aligned} 0 &= d(x, x) && \text{(by Identity of Indiscernables)} \\ &\leq d(x, y) + d(y, x) && \text{(by Triangle Inequality)} \\ &= d(x, y) + d(x, y) && \text{(by Symmetry)} \\ &= 2d(x, y) \\ \implies 0 &\leq d(x, y). \end{aligned}$$

Thus $d(x, y)$ is nonnegative. ■

Why can't distances be negative, however? Intuitively, I mean. Well, if we're measuring the distance from my house to your house (and our distance does not have a direction associated with it (**REMEMBER**: the output of a metric is a number, **not** a vector)), I can only be say 1 mile away as -1 mile away would have little meaning.

Do we have an example of a metric? Yes!

Example 5.3 (Euclidean Metric on $\mathbb{R}$)**.** Let $x, y \in \mathbb{R}$. Then $d\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $d(x, y) = |x - y|$ is a metric.

We have a ton more, also. For example, the Euclidean Metric on $\mathbb{R}^n$ is also a metric.

Example 5.4 (Euclidean Metric on $\mathbb{R}^n$). Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then $d \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{j=1}^{n} (x_j - y_j)^2}$$

is a metric.

Before we prove that this is a metric, we will need to introduce an inequality that will be of great use to us: the Cauchy-Schwarz Inequality.

Theorem 5.5 (Cauchy-Schwarz Inequality). *Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then, $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|$ where $\cdot$ is the usual dot product on $\mathbb{R}^n$.*

Proof. Let $\lambda = \frac{\mathbf{x} \cdot \mathbf{y}}{\mathbf{y} \cdot \mathbf{y}}$. Consider $(\mathbf{x} - \lambda\mathbf{y}) \cdot (\mathbf{x} - \lambda\mathbf{y})$. It follows that

$$\begin{aligned} 0 \leq (\mathbf{x} - \lambda\mathbf{y}) \cdot (\mathbf{x} - \lambda\mathbf{y}) = \|\mathbf{x}\|^2 - 2\lambda(\mathbf{x} \cdot \mathbf{y}) + \lambda^2\|\mathbf{y}\|^2 \\ 0 \leq \|\mathbf{x}\|^2 - 2\frac{\mathbf{x} \cdot \mathbf{y}}{\mathbf{y} \cdot \mathbf{y}}(\mathbf{x} \cdot \mathbf{y}) + \left(\frac{\mathbf{x} \cdot \mathbf{y}}{\mathbf{y} \cdot \mathbf{y}}\right)^2 \|\mathbf{y}\|^2 \\ 0 \leq \|\mathbf{x}\|^2 - \frac{(\mathbf{x} \cdot \mathbf{y})^2}{\|\mathbf{y}\|^2} \\ \frac{(\mathbf{x} \cdot \mathbf{y})^2}{\|\mathbf{y}\|^2} \leq \|\mathbf{x}\|^2 \\ (\mathbf{x} \cdot \mathbf{y})^2 \leq \|\mathbf{x}\|^2\|\mathbf{y}\|^2 \\ |\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|. \end{aligned}$$

This completes the proof. ■

Now we can get to the actual proof of the Euclidean metric being a metric.

Proof. Recall from the definition of a metric that we must verify that the Euclidean metric satisfies the Identity of Indiscernables, symmetry, and the triangle inequality.

First the indiscernables. ($\Rightarrow$) Suppose $d(\mathbf{x}, \mathbf{y}) = 0$. It follows that each term of the sum must then also be 0, meaning that each component must be equal and hence $\mathbf{x} = \mathbf{y}$. ($\Leftarrow$) Suppose $\mathbf{x} = \mathbf{y}$. It is then clear that $d(\mathbf{x}, \mathbf{y}) = 0$.

Now symmetry. By the properties of squares, we know that $(x_i - y_i)^2 = (y_i - x_i)^2$, so $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$.

Finally, the triangle inequality. Consider the Cauchy-Schwarz Inequality. It follows that

$$\begin{aligned} \mathbf{x} \cdot \mathbf{y} \leq \|\mathbf{x}\|\|\mathbf{y}\| \\ \implies \|\mathbf{x}\|^2 + 2(\mathbf{x} \cdot \mathbf{y}) + \|\mathbf{y}\|^2 \leq \|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\| + \|\mathbf{y}\|^2 \\ \implies \|\mathbf{x} + \mathbf{y}\|^2 \leq (\|\mathbf{x}\| + \|\mathbf{y}\|)^2 \\ \implies \|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|. \end{aligned}$$

Thus the Euclidean metric is indeed a metric. ■

Here's another neat metric on $\mathbb{R}^2$.

Example 5.6 (Taxicab/Manhattan Metric). Define $d \colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|.$$

Figure 4 should illustrate the difference between Euclidean and Taxicab metrics.

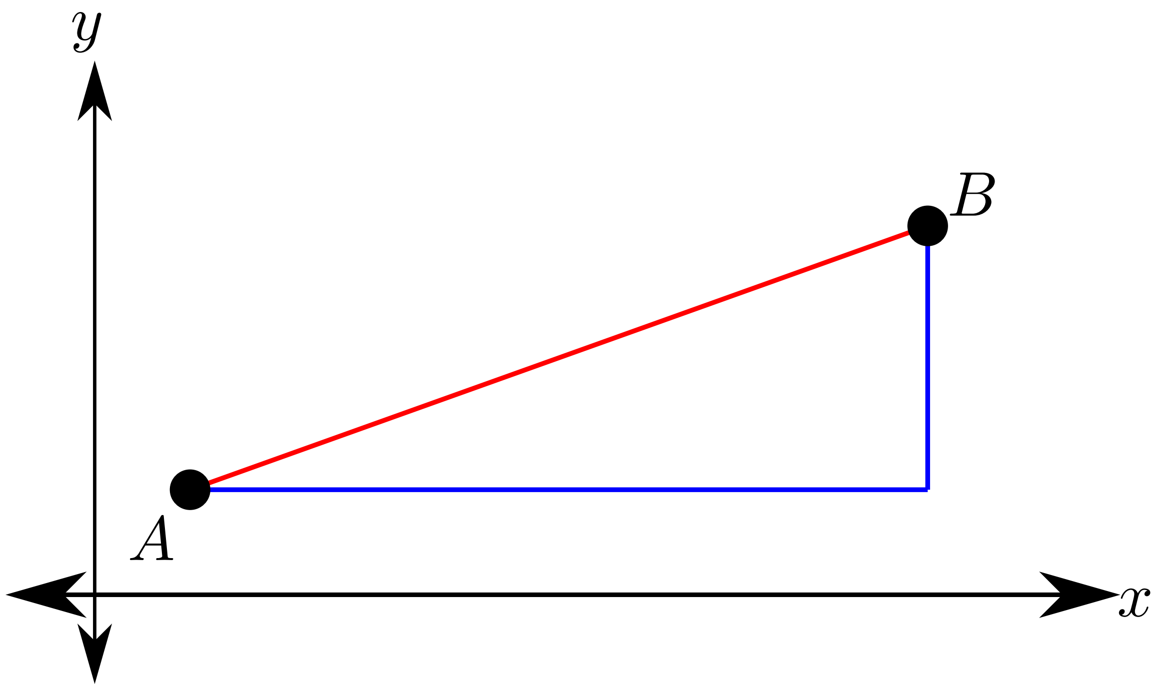

Figure 4: Comparison of the Euclidean and Taxicab metrics. The red path is the Euclidean distance and the blue path is the taxicab distance. **FINISH this figure**

Now that we know what metrics are, we can talk about *metric spaces* pretty easily.

Definition 5.7 (Metric space)**.** A *metric space* is a set with a metric defined on it.

Notation 5.8. A metric space X with the metric d is denoted (X, d) or simply X if the metric is unambiguous.

We've actually already been talking about metric spaces! Whenever we've mentioned a metric, we've always had a corresponding set (out of necessity). So, here are some examples of metric spaces:

Example 5.9. Examples of metric spaces.

(a) $\mathbb{R}$ with the Euclidean metric;

(b) $\mathbb{R}^n$ with the Euclidean metric;

(c) $\mathbb{R}^2$ (or $\mathbb{R}^n$) with the Taxicab metric.

*

Less basic examples of metric spaces

Following from the very general definition of metric spaces, there are tons of examples of metric spaces! We're going to go through a few of them.

Example 5.10. ℓ_2 is the set of real sequences (x_n) such that $\sum_{j=1}^{\infty} |x_j|^2$ is finite. When equipped with the metric $d\colon \ell_2 \times \ell_2 \to \mathbb{R}$ defined by

$$d((x_n),(y_n)) = \sqrt{\sum_{j=1}^{\infty} |x_j - y_j|^2},$$

(ℓ_2, d) is a metric space.

Example 5.11. $C([0,1])$ is the set of all continuous functions from $[0,1]$ to $\mathbb{R}$. When combined with the metric $d\colon C([0,1]) \times C([0,1]) \to \mathbb{R}$ defined by

$$d(f,g) = \sup\{|f(x) - g(x)| : x \in [0,1]\}$$

$(C([0,1]), d)$ is a metric space.

Example 5.12 (Sup-norm metric)**.** $\mathbb{R}^2$ with $d(\mathbf{x}, \mathbf{y}) \coloneqq \sup\{|x_i - y_i| : i = 1, 2\}$ is a metric space.

Some translating

We've developed quite of a lot of machinery so far in the previous sections of these notes. In particular, we've talked about convergence, Cauchy sequences, continuity, and uniform continuity, but we've only defined it for the case of $\mathbb{R}$. In this section we will define them for general metric spaces. :)

Will defining things in more generality change any of the ideas, though? Not at all! In fact, our reference to $\mathbb{R}$ is indeed a special case of the general definition, so everything should seem fairly similar and need little explanation, so let's get right into it.

Definition 5.13 (Convergence in general metric spaces)**.** Let (x_n) be a sequence in a metric space (X,d). (x_n) converges to $x_0 \in X$ if and only if for all $\epsilon > 0$ there exists an integer N such that $n \geq N$ implies $d(x_n, x) < \epsilon$. This is denoted $\lim x_n = x$ or $(x_n) \to x$.

Comparing this to Definition 2.5 and we say that they are almost the same! The only difference is what we write for the metric. Very cool. All other definitions we will introduce will follow this same pattern.

Definition 5.14 (Cauchiness in general metric spaces)**.** Let (x_n) be a sequence in a metric space (X,d). (x_n) is a Cauchy sequence if and only if for all $\epsilon > 0$ there exists an integer N such that $i, j \geq N$ implies $d(x_i, x_j) < \epsilon$.

Definition 5.15 (Continuity in general metric spaces)**.** Let (X, d_x) and (Y, d_y) be metric spaces and let $f\colon (X, d_x) \to (Y, d_y)$ be a function. Then, f is continuous at a point $c \in X$ if and only if for all $\epsilon > 0$ there exists a $\delta > 0$ such that $d_y(f(x), f(c)) < \epsilon$ if $d_x(x, c) < \delta$.

Definition 5.16 (Uniform Continuity in general metric spaces)**.** Let (X, d_x) and (Y, d_y) be metric spaces and let $f\colon (X, d_x) \to (Y, d_y)$ be a function. f is uniformly continuous (on X) if and only if, given an $\epsilon > 0$, there is a $\delta > 0$ such that for all $x_1, x_2 \in X$, if $d_x(x_1, x_2) < \delta$, then $d_y(f(x_1), f(x_2)) < \epsilon$.

5.2 Open sets

Basics

Before we get into the title of this section, we have to introduce a very important idea: the open ball (or neighborhood).

Definition 5.17 (Open ball)**.** Let (X, d) be a metric space and suppose $x \in X$. Then, an *open ball* around x of radius r is the set $\{y \in X \mid d(x, y) < r\}$ and is denoted $B_r^X(x)$ or $B_r(x)$ if the metric space the ball is in is unambiguous.

Why do we call this an open ball? We'll get to the 'open' part later, but we can answer the 'ball' part now; in $\mathbb{R}^n$, it literally defines a ball lmao.

There's one more definition before we get to the definition of open set: the interior point.

Definition 5.18 (Interior point)**.** A point $x \in E \subseteq X$ is an interior point of E if and only if there exists an open ball around x that is completely contained in E.

We'll get to use this more later.

Now we can get to the titular object of this section: the open set!

Definition 5.19 (Open set)**.** Let (X, d) be a metric space. $U \subseteq X$ is open in if and only if for every $u \in U$, there exists an $r_u > 0$ such that $B_{r_u}(u) \subseteq U$.

Slogan. Open $\iff$ around every point there is an open ball that is completely contained

Now let's do our first proof regarding open sets. Let's give reason to our name 'open ball' from earlier. :)

Theorem 5.20 (Open balls are open)**.** *Let (X, d) be a metric space, $x \in X$, and $r > 0$. Then, $B_r(x)$ is an open set.*

While looking through the following proof, please refer to Figure 5 to get a better idea of what we're doing (as the proof will likely be difficult to follow at first glance).

Proof. Consider some $y \in B_r(x)$. Then consider the ball of radius $r - d(x, y)$ around y. Choose some z in that ball. It follows that

$$d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + r - d(x, y) = r,$$

so $z \in B_r(x)$ and $B_{r-d(x,y)}(y) \subseteq B_r(x)$. Thus $B_r(x)$ is open. This completes the proof. ■

Open sets seem kind of abstract right? Or at least they did to me at first, so let's try to help with that. To that end, consider the following example of some more familiar open sets.

Example 5.21. Some open sets in Euclidean space.

(a) $(0, 1) \subseteq \mathbb{R}$;

(b) $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\} \subseteq \mathbb{R}^2$;

(c) $(0, 1) \cup (2, 3) \subseteq \mathbb{R}$.

Remark. I will not prove these but they should be similar to the proof of Theorem 5.20.

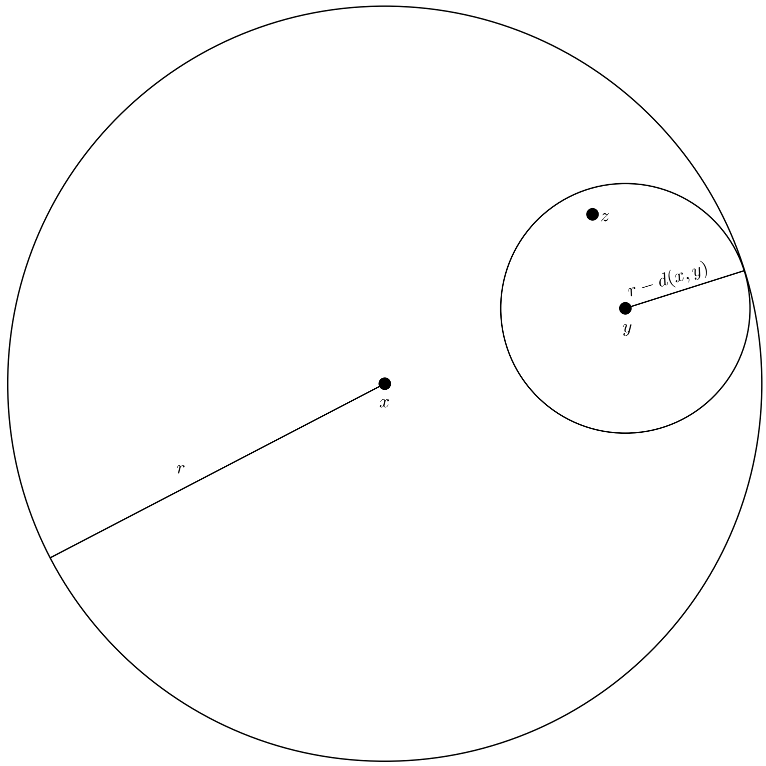

Figure 5: Diagram for the proof Theorem 5.20

Now let's move on to some interesting properties of open sets.

Relativity (of openness)

This will be a short section but an important one nonetheless. It's just a really important idea.

Fact 5.22 (Relativity of openness)**.** *A set's openness is relative to the metric space it is in.*

Here an example with proof.

Example 5.23. $(0,1)$ is open as a subset of $\mathbb{R}$ but not open as a subset of $\mathbb{R}^2$.

Proof. $(0,1)$ is clearly open in $\mathbb{R}$ as it is the ball of radius $1/2$ centered at $1/2$.

$(0,1)$ is not open as a subset of $\mathbb{R}^2$ (we treat $(0,1) \subseteq \mathbb{R}^2$ as $(0,1) \times \{0\} \subseteq \mathbb{R}^2$) as there exists a point of $(0,1) \subseteq \mathbb{R}^2$ that is not an interior point (take $(1/2,0)$ for example).

See Figure 6 for further explanation. ∎

There's a little bit more to say, however.

Theorem 5.24. *Let (X,d) be a metric space and let $Y \subseteq X$. $E \subseteq Y$ is open relative to Y if and only if $E = Y \cap G$ for some open $G \subseteq X$.*

Proof. ($\Rightarrow$) Suppose E is open relative to Y. Let $p \in E$. Let $r_p > 0$ be such that $B^Y_{r_p}(p) \subseteq E$. Define $G = \bigcup_{p \in E} B^Y_{r_p}(p)$. Then G is open in X. ($\subseteq$) It is clear that $E \subseteq G \cap Y$. ($\supseteq$) By construction, $B^Y_{r_p}(p) \cap Y \subseteq E$ for all $p \in E$, so $G \cap Y \subseteq E$. It follows that $G \cap Y = E$.

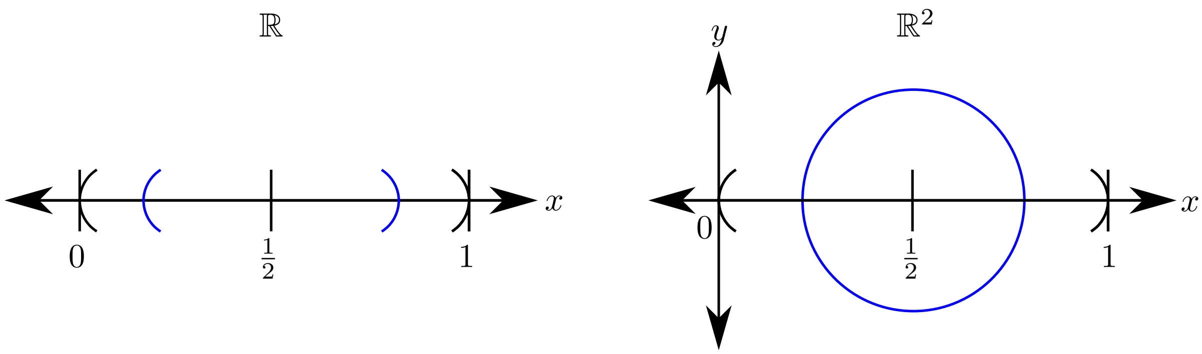

Figure 6: The blue lines are the open balls in $\mathbb{R}$ and $\mathbb{R}^2$, respectively. Notice how the open ball in $\mathbb{R}^2$ is not completely contained in $\mathbb{R}$. This hopefully makes clear why $(0, 1)$ is open in $\mathbb{R}$ but not $\mathbb{R}^2$.

($\Leftarrow$) Suppose $E = Y \cap G$ for some open $G \subseteq X$. Since G is open, for every $p \in E$, there exists a B_p such that $B_p \subseteq G$. Then, $B_p \cap Y \subseteq E$ and E is open relative to Y. ■

Properties of open sets

Are there any sets that are always open? Yes, in fact! The empty set and the entire metric space are open.

Fact 5.25. *Let (X, d) be a metric space. Then X and $\varnothing$ are open.*

Proof. X is open as it is universe in which our (sub)sets are taken from. Hence, treating X as a subset of itself, any ball around any point of X must be contained in X (since there are nothing 'outside' of X).
$\varnothing$ is vacuously open. ■

The section will mainly be out the unions and intersections of open sets. We'll start with unions.

Theorem 5.26. *For any collection $\{U_\alpha\}_{\alpha \in A}$ of open sets, $\bigcup_{\alpha \in A} U_\alpha$ is open.*

Slogan. Any union of open sets is open.

Proof. Suppose $u \in \bigcup_{\alpha \in A} U_\alpha$. Then, by definition of union, there exists some $\beta \in A$ such that $u \in U_\beta$. Then there exists an open ball B around u that is completely contained in $U_\beta \subseteq \bigcup_{\alpha \in A} U_\alpha$. Thus, $\bigcup_{\alpha \in A} U_\alpha$ is open. ■

Now to intersections.

Theorem 5.27. *For any finite collection $U_1, \ldots, U_n$ of open sets, $\bigcap_{j=1}^n U_j$ is open.*

Slogan. The finite intersection of open sets is open.

Proof. Suppose $\bigcap_{j=1}^n U_j$ is empty. Then by Fact 5.25, we know that this is open, so we may assume that the intersection is nonempty.

Suppose $u \in \bigcap_{j=1}^n U_j$. Then, for each $j = 1, 2, 3, \ldots, n$, we have a corresponding radius r_j such that $B_{r_j}(u) \subseteq U_j$ since $u \in U_j$ and U_j is open. Taking $r = \min\{r_j \mid j = 1, 2, 3, \ldots, n\}$, it is clear that $B_r(u) \subseteq U_j$ for all $j = 1, 2, 3, \ldots, n$, so $B_r(u) \subseteq \bigcap_{j=1}^n U_j$ and the intersection is open. ■

Why isn't the infinite intersection open, however? Because of examples like this: $\bigcap_{n=1}^{\infty}(-1/n, 1/n)$. The sets composing the intersection are open, but the intersection is equal to the singleton set 0, so infinite intersections are not, in general, open.

Relation to continuous functions

Our idea of continuity actually has an equivalent formulation using open sets!

Theorem 5.28 (Topological definition of continuity). *Let (X, d_x) and (Y, d_y) be metric spaces and let $f\colon (X, d_x) \to (Y, d_y)$ be a function. f is continuous if and only if for every open set $V \subseteq Y$, $f^{-1}(V) \subseteq X$ is open (where $f^{-1}(V)$ is the preimage of V).*

Slogan. Continuous $\iff$ preimages preserve openness

Before we start the proof of this theorem, here's a little notational thing so we avoid ambiguity.

Proof. ($\Rightarrow$) Suppose f is continuous. We aim to show that $f^{-1}(V)$ is open. Suppose $f^{-1}(V)$ is empty. Then $f^{-1}(V)$ is open by Fact 5.25, so we may assume that $f^{-1}(V)$ is nonempty. Choose a point $c \in f^{-1}(V)$. Notice that by the definition of preimages, $f(c) \in V$. By V's openness, there exists an $\epsilon > 0$ such that $B_{\epsilon}^{Y}(f(c)) \subseteq V$. **finish**

We aim to find a $r > 0$ such that $B_r^X(c) \subseteq f^{-1}(V)$ (where $B_r^X(c)$ is the r-ball around c in (X, d_x)). Because $f(c) \in V$ (since $c \in f^{-1}(V)$) and by V's openness, there exists an $\epsilon > 0$ such that $B_{\epsilon}^{Y}(f(c)) \subseteq V$. By f's continuity, there exists a $\delta > 0$ such that $B_{\delta}^{X}(c) \subseteq f^{-1}(V)$ as $f(B_{\delta}^{X}(c)) \subseteq B_{\epsilon}^{Y}(f(c))$.

($\Leftarrow$) Suppose $f^{-1}(V)$ is open. ■

5.3 Closed sets

Basics

Closed sets, as you might expect, are the opposites of open sets (in some very precise sense), but this isn't immediately obvious from the definition I'm going to present to you. (We'll get to the more definition after). On that note, let's get into the preliminary ideas behind a closed set. **Rework. Finish**

Definition 5.29 (Limit point). Let (X, d) be a metric space. A point $x \in E \subseteq X$ is a limit point if and only if every ball around x contains a point $y \neq x$ such that $y \in E$.

Basically, a point is a limit point if and only if it is the limit of an eventually non-constant sequence. There's also the 'opposite' of a limit point: an isolated point.

Definition 5.30 (Isolated point). Let (X, d) be a metric space. $x \in E \subseteq X$ is an isolated point if and only if it is not a limit point.

Remark. In other words, a point is an isolated point if and only if there exists a ball around the point that only contains itself (considering only the subset the point is in).

Figure 7 should hopefully make the difference between limit and isolated point more clear.

Now we can get into what a closed set is.

Definition 5.31 (Closed set). Let (X, d) be a metric space. A set $E \subseteq X$ is closed if and only if E contains all its limit points.

Slogan. Closed $\iff$ contains all limit points

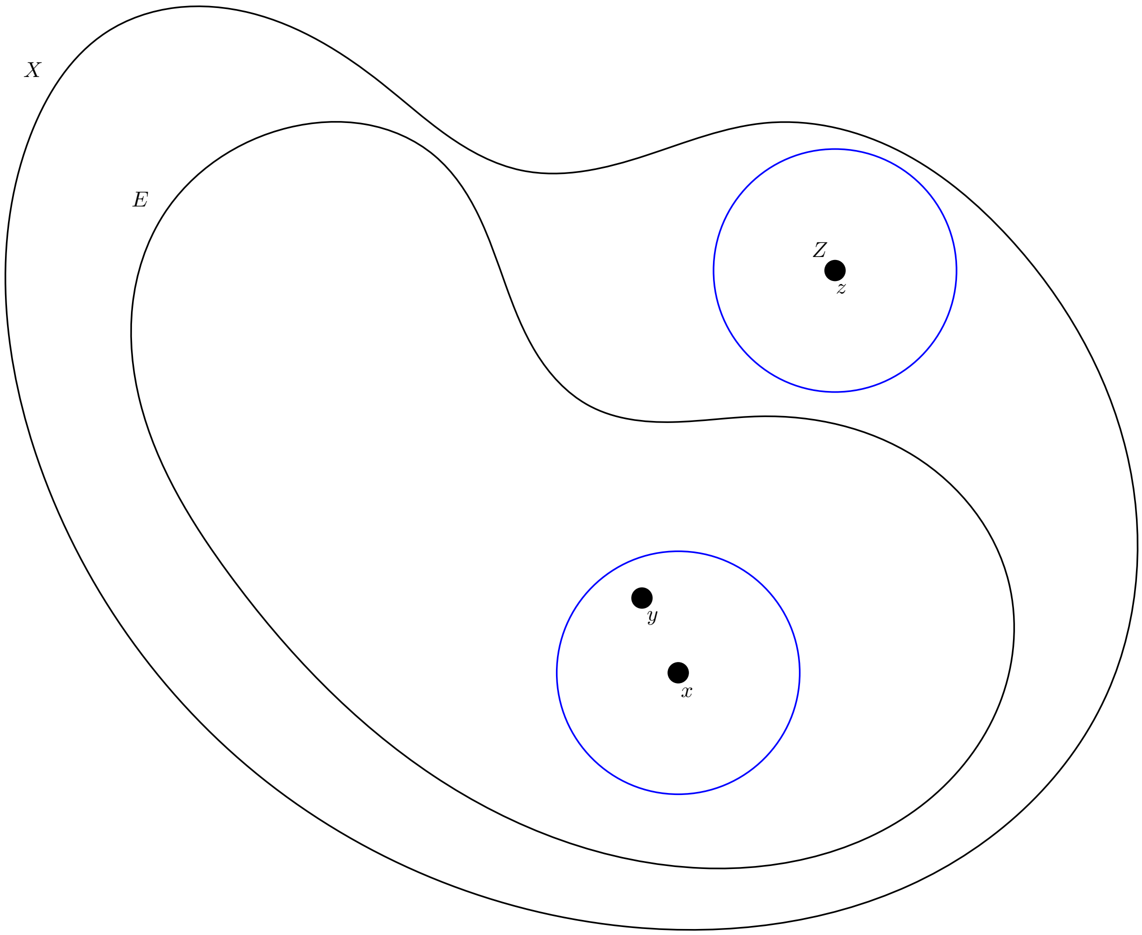

Figure 7: Illustration of the difference between a limit point and an isolated point. x is a limit point of E and z is an isolated point (and the only point) of Z. The blue circles are the balls around x and z, respectively.

Relationship to open sets

There's another way to define closed sets using open sets! (We'll see this type of idea a lot because these are called *topological properties*–topological properties are properties that can be defined using open sets.)

Theorem 5.32 (Alternate definition of closed set)**.** *Let (X, d) be a metric space. A set $E \subseteq X$ is closed if and only if its complement is open.*

Slogan. Closed $\iff$ complement is open

This is the very specific sense in which closed sets are the opposites of open sets! :) Very cool.

Proof. ($\Rightarrow$) Suppose E is closed and consider some $x \in X - E$ (where $X - E = X \setminus E = E^c$). Because E contains all its limit points (as E is closed), x is not a limit point of E, so there exists a ball B centered at x such that $B \cap E = \varnothing$. Hence, by definition of complement (and because B is nonempty since it contains x), $B \subseteq X - E$ and x is an interior point of $X - E$. Since x was arbitrary we now know that $X - E$ is open.

($\Leftarrow$) Suppose $X - E$ is open. Suppose x is a limit point of E. We aim to show that $x \in E$. Because x is a limit point of E, every neighborhood of x contains a point $y \neq x$ in E, so that x is not an interior point of $X - E$. Since $X - E$ is open, we have that $x \in E$. ■

Corollary 5.33 (Alternate definition of open set)**.** *Let (X, d) be a metric space. A set $E \subseteq X$ is open if and only if its complement is closed.*

Slogan. Open $\iff$ complement is closed.

Now we can more easily talk about some cool properties of closed sets!

Relativity (again) (of closed sets)

Like the relativity of openness section, this section will be incredibly short. In fact, it's even shorter (since I won't prove my example).

Fact 5.34 (Relativity of closedness)**.** *A set's closedness is relative to the metric space it is in.*

Example 5.35. $(0, 1)$ is closed as a subset of $(0, 1)$ but not closed as a subset of $\mathbb{R}$.

Cool properties

Similar to how we had some sets (the whole metric space and the empty set) that were always open, we have some sets that are always closed!

Fact 5.36. *Let (X, d) be a metric space. Then X and $\varnothing$ are closed.*

Proof. This is clear from Theorem 5.32 since $X - X = \varnothing$ and $X - \varnothing = X$. ■

The rest of this section will be about unions and intersections of closed sets.

Theorem 5.37. *For any collection $\{C_\alpha\}_{\alpha \in A}$ of closed sets, $\bigcap_{\alpha \in A} C_\alpha$ is closed.*

Slogan. Any intersection of closed sets is closed.

Proof. Suppose that the intersection is empty. Then it is closed by Fact 5.36, so we may assume the intersection is nonempty. To show that the intersection of our closed sets is closed, it suffices to show that its complement is open. Using DeMorgan's Laws, we have that

$$X - \bigcap_{\alpha \in A} C_\alpha = \bigcup_{\alpha \in A} (X - C_\alpha).$$

Since each C_α is closed, we know that $X - C_\alpha$ is open, and hence the union of $X - C_\alpha$ is open. Thus, $\bigcap_{\alpha \in A} C_\alpha$ is closed. ■

Theorem 5.38. *For any finite collection $C_1, \ldots, C_n$ of closed sets, $\bigcup_{j=1}^{n} C_j$ is closed.*

Slogan. The finite union of closed sets is closed.

Proof. Similar to proof of Theorem 5.37. ■

Why isn't any union of closed sets closed? Consider $\bigcup_{n=1}^{\infty} [-1 + 1/n, 1 - 1/n]$. The union is equal to $(-1, 1)$ which is not closed.

5.4 Interiors of sets

The interior of a set is just the set of interior points of a set. More formally:

Definition 5.39 (Interior of a set)**.** Let (X, d) be a metric space and let $E \subseteq X$. The interior of E, denoted E°, is the set of all interior points of E.

There's actually another definition of this using the unions of open sets.

Theorem 5.40 (Interior of a set (alternate))**.** *Let (X, d) be a metric space and let $E \subseteq X$. E° is equal to the union of all open sets contained in E. Symbolically this is,*

$$E^\circ = \bigcup_{\substack{U \subseteq E \\ U \text{ open}}} U.$$

Proof. ($\subseteq$) Suppose $x \in E^\circ$. Then x is an interior point and thus there exists an open ball B completely contained in E. Hence,

$$B \in \bigcup_{\substack{U \subseteq E \\ U \text{ open}}} U.$$

($\supseteq$) Suppose

$$x \in \bigcup_{\substack{U \subseteq E \\ U \text{ open}}} U.$$

Then x is in some open set V contained E. Hence, there exists an open ball B contained in V that contains x. Hence, x is an interior point of E and thus $x \in E^\circ$.

This completes the proof. ∎

FINISH

5.5 Closures

FINISH

5.6 Homeomorphism

Homeomorphism means the spaces are the same in some specific sense. That specific sense is the following:

Definition 5.41 (Homeomorphism)**.** A metric space (X, d_x) is *homeomorphic* to a metric space (Y, d_y) if and only if there exists a bijective bicontinuous function $f\colon X \to Y$ (i.e. both f and f^{-1} are continuous). Then the function f is said to be a *homeomorphism* between X and Y.

Remember how continuous function means 'preimages preserve open sets?' Well, if both the function and its inverse are continuous, then that means both images and preimages preserve open sets. So if two metric spaces are homeomorphic, then there exists function under which the two metric spaces have the same open sets!; and if they have the same open sets, they are fundamentally the same as open sets define all topological properties! Very neat stuff.

5.7 Translating convergence

Now that we have an idea of how metric space topology works, we can translate convergence into a topological context.

Theorem 5.42 (Topological definition of convergence)**.** *Let (X, d) be a metric space. Let (p_n) be a sequence in X and let $p \in X$. $\lim p_n = p$ if and only if $B_\epsilon(p)$ contains all but finitely many points (p_n) for all $\epsilon > 0$.*

Proof. ($\Rightarrow$) Let $\epsilon > 0$ be arbitrary. Suppose $\lim p_n = p$. Then, there exists a positive integer N such that $n \geq N$ implies that $d(p_n, p) < \epsilon$. This means that the ϵ-ball around p contains all terms of (p_n) with $n \geq N$. In other words, it contains all but $N - 1$ terms of (p_n).

($\Leftarrow$) We will now prove the converse. Let $\epsilon > 0$ be arbitrary. Suppose $B_\epsilon(p)$ contains all but finitely many points of (p_n). In other words, there exists some positive integer N such that $n \geq N$ implies that $p_n \in B_\epsilon(p)$, which means that $d(p_n, p) < \epsilon$. This is precisely the definition of a sequence converging to a value. This completes the proof. ■

6 (Aside:) Completeness

6.1 Basics

In the very first section (in the first paragraph, in fact!), we said the rationals are 'incomplete.' We will now explain exactly what that meant.

Definition 6.1 (Completeness). A metric space (X, d) is complete if and only if every Cauchy sequence in (X, d) converges in (X, d).

Do we have an example of a complete metric space? Indeed, we do!

Example 6.2. $\mathbb{R}$ with the Euclidean metric is complete.

Proof. This is Theorem 2.26. ■

In fact, we have many more examples of complete metric spaces! Say ℓ_2 or $C([0,1])$ (with the sup-norm metric), or even $\mathbb{R}^n$ (with the Euclidean metric) for any finite n! (We'll prove ℓ_2, $C([0,1])$, and $\mathbb{R}^2$ is complete in the upcoming section.) Now to some examples.

6.2 Examples of cool complete metric spaces (with proof)

Fact 6.3. *$\mathbb{R}^2$ with the Euclidean metric is complete.*

Proof. Let $\epsilon > 0$. Suppose $(\mathbf{a}_n)$ is a Cauchy sequence. We may rewrite $(\mathbf{a}_n)$ as $((x_n, y_n))$. First we will show that both (x_n) and (y_n) are Cauchy in $\mathbb{R}$.

Because $(\mathbf{a}_n)$ is Cauchy, there exists an integer N such that $d(\mathbf{a}_i, \mathbf{a}_j) = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} < \epsilon$ if $i, j \geq N$. It follows that

$$\begin{aligned} |x_i - x_j| &= \sqrt{(x_i - x_j)^2} \leq \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} < \epsilon \\ |y_i - y_j| &= \sqrt{(y_i - y_j)^2} \leq \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} < \epsilon \end{aligned}$$

if $i, j \geq N$. Thus (x_n) and (y_n) are Cauchy in $\mathbb{R}$ (with the Euclidean metric), so they have limits x and y, respectively. We claim that $\lim \mathbf{a}_n = (x, y)$.

Because $\lim x_n = x$, there exists an integer N_1 such that $n \geq N_1$ implies that $|x_n - x| < \epsilon/\sqrt{2}$. Similarly, there exists an N_2 such that $|y_n - y| < \epsilon/\sqrt{2}$ if $n \geq N_2$. Taking $n \geq \max\{N_1, N_2\}$, we get that

$$\begin{aligned} d(\mathbf{a}_n, \mathbf{a}) &= \sqrt{(x_n - x)^2 + (y_n - y)^2} \\ &= \sqrt{|x_n - x|^2 + |y_n - y|^2} \\ &< \sqrt{\left(\frac{\epsilon}{\sqrt{2}}\right)^2 + \left(\frac{\epsilon}{\sqrt{2}}\right)^2} \\ &= \sqrt{\frac{\epsilon^2}{2} + \frac{\epsilon^2}{2}} \\ &= \sqrt{\epsilon^2} \\ &= \epsilon. \end{aligned}$$

Hence $\lim \mathbf{a}_n = (x, y)$ and $\mathbb{R}^2$ (with the Euclidean metric is complete.) ■

Corollary 6.4. *$\mathbb{R}^n$ with the Euclidean metric is complete.*

Proof. Mimic the proof for the completeness of $\mathbb{R}^2$. ■

Fact 6.5. $\mathbb{R}^2$ *with the Taxicab metric is complete.*

Proof. Mimic the proof of the completeness of $\mathbb{R}^2$. ■

Fact 6.6. ℓ_2 *with the metric* $d\colon \ell_2 \times \ell_2 \to \mathbb{R}$ *defined by*

$$d((x_n),(y_n)) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^2\right)^{\frac{1}{2}}$$

is complete.

To make the following proof easier, we will denote the elements of ℓ_2 in a new yet familiar way: as functions! So we will define ℓ_2 as

$$\ell_2 := \left\{ f\colon \mathbb{Z}^+ \to \mathbb{R} \mid \sum_{k=1}^{\infty} |f(k)|^2 < \infty \right\}.$$

Why can we do this? Well, sequences are just functions from the positive integers into the the reals, so writing them this way is perfectly fine. Now we may begin with the proof.

Proof. das ■

Fact 6.7. $C([0,1])$ *with the metric* $d\colon C([0,1]) \times C([0,1]) \to \mathbb{R}$ *defined by*

$$d(f,g) = \sup\{|f(x) - g(x)| : x \in [0,1]\}$$

is complete.

Proof. Let $\epsilon > 0$ be arbitrary. Let (f_n) be a Cauchy sequence. Then there exists an integer N such that $d(f_i, f_j) < \epsilon/2$ if $i, j \geq N$. By the definition d and supremum, we know that $|f_i(x) - f_j(x)| < \epsilon/2 < \epsilon$ and $(f_n(x))$ is a Cauchy sequence in $\mathbb{R}$ with the Euclidean metric for all $x \in [0,1]$.

Taking $i, j \geq N$ again and then taking $j \to \infty$, we get

$$\sup_{y\in[0,1]} |f_i(y) - f_j(y)| < \frac{\epsilon}{2} \to \sup_{y\in[0,1]} |f_i(y) - f_0(y)| \leq \frac{\epsilon}{2} < \epsilon.$$

Thus, $d(f_i, f_0) < \epsilon$ if $i \geq N$ and $\lim f_n = f_0$.

We must still show that $f_0 \in C([0,1])$ (i.e. we must show that f_0 is continuous). Let $c \in [0,1]$. Since $\lim f_n = f_0$, there exists an integer N' such that for $|f_n(x) - f_0(x)| \leq d(f_n, f_0) < \epsilon/3$ if $n \geq N'$ for all $x \in [0,1]$. Recall also that because $f_{N'}$ is continuous, there exists a $\delta > 0$ such that $|x - c| < \delta$ implies that $|f_{N'} - f_{N'}(c)| < \epsilon/3$. Now taking $|x - c| < \delta$, it follows that

$$|f_0(x) - f_0(c)| \leq |f_0(x) - f_{N'}(x)| + |f_{N'}(x) - f_{N'}(c)| + |f_{N'}(c) - f_0(c)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Thus f_0 is continuous.

This completes the proof. ■

Remark. The final inequality in the proof is called the "Three Term Estimate." It's very important stuff. Don't forget it! We might make a section/bit on it. idk

6.3 Complete subspaces

When is the subset of a metric space complete? When it's closed!

Fact 6.8. *Let (X, d) be a complete metric space and suppose $Y \subseteq X$. If Y is closed in X, then $(Y, d|_{Y \times Y})$ is complete.*

Proof. Suppose (x_n) is a Cauchy sequence in Y. Then (x_n) converges to x in X. Y contains all its limit points because it is closed, so $x \in Y$. Thus Y is complete. ■

Now back to topology!

7 Metric space topology (cont'd)

7.1 Preliminary comment on compactness

There are two types of compactness that we will (spoiler!!!) show to be equivalent. Both are incredibly important and hopefully these sections help you get a better idea of what they are. We start with the one that was first shown to me in class.

7.2 Sequential compactness

Basics

We'll get right into it since the name isn't particularly insightful (besides it having to do with sequences) at this point.

Definition 7.1 (Sequential compactness)**.** Let (X, d) be a metric space. $Y \subseteq X$ is *sequentially comapct* if and only if every sequence in Y has a subsequence that converges in Y.

These seems kinda wild to me–this seems like a super strong property! Well, it actually is! But it does sound weird, for sure. However, we do have some examples (and we'll explain why).

Example 7.2. $[0, 1]$ (with the Euclidean metric) is sequentially compact.

Proof. Let (x_n) be a sequence in $[0, 1]$. Because $[0, 1]$ is bounded, (x_n) must also be bounded, so by Bolzano-Weierstrass we know that (x_n) has a convergent subsequence (x_{n_k}). Because $[0, 1]$ is closed, the limit of this subsequence must be in $[0, 1]$. Thus $[0, 1]$ is sequentially compact. ■

Remark. This is easily generalizable to intervals $[a, b]$.

The remark for Example 7.2 hints that a whole class of subsets of $\mathbb{R}$ that are sequentially compact. What ties all of these subsets together? Closed and boundedness! In fact, this is known as *Heine-Borel Theorem*, which we'll eventually.

Consider this example, also.

Example 7.3. $[0, 1] \subseteq \mathbb{R}$ (with the Euclidean mertic) is sequentially compact.

Proof. Do it yourself. ■

Huh???????? Everything else we've talked about has been relative. Is this not also? In fact, no! Sequential compactness is not relative!!!! If something is sequentially compact with regard to some superset, then its sequentially compact relative to any superset (that contains it, obviously). This leads us to the following fact (which we will not prove yet).

Fact 7.4. *(X, d) be a metric space and let $K \subset Y \subset X$. K is sequentially compact in Y if and only if K is sequentially compact in X.*

Neat properties

Let's start off with some basic sounding stuff: what subsets of sequentiall compact spaces are sequentially compact?

Theorem 7.5. *Let (X, d) be a sequentially compact metric space and let $Y \subseteq X$. If Y is closed, then Y is sequentially compact.*

Proof. Let (y_n) be a sequence in Y. Because (y_n) is also a sequence in X, there exists a subsequence (y_{n_k}) that converges to $y \in X$. Because Y is closed, $y \in Y$. Thus Y is sequentially compact. ■

Here's a less basic result: sequential compactness implies closed and bounded. We'll start off with closed first, however.

Theorem 7.6. *Let (X, d) be a metric space and suppose $Y \subseteq X$. If Y is sequentially compact, then Y is closed.*

For the following proof, referring to Figure 8 could aid in understanding.

Proof. Suppose towards a contradiction that that Y is not closed (i.e. $X - Y$ is not open). Then there exists some point $a \in X - Y$ such that for all $\epsilon > 0$, $B_\epsilon(a) \nsubseteq X - Y$. Now, let $\delta > 0$. Then there exists a point $b \in Y$ such that $b \in B_\delta(a)$ (otherwise $B_\delta(a)$ would be a subset of $X - Y$). This property tells us that there are infinitely many points of Y inside any δ-ball around a. This will be shown.

Suppose towards a contradiction that there are only finitely many points of Y, in particular N-many points, in $B_\delta(a)$. Then, choose $\delta' = \frac{1}{2}\min\{d(p_1, a), \ldots, d(p_N, a)\}$ (where $p_1, \ldots, p_N$ are the points in $B_\delta(a)$). Then, because any ball of any radius around a must contain a point of Y within it (otherwise the ball would be completely contained in $X - Y$ which would contradict the property of our chosen a), there exists a point $p_{N+1} \in B_{\delta'}(a) \subsetneq B_\delta(a)$, meaning that there are $N + 1$ points in $B_\delta(a)$ which is a contradiction. Hence, any δ-ball around a contains infinitely many points of Y.

Now we may define a sequence (y_n). Define it by the following:

(i) y_1 is some point in Y;

(ii) y_n $(n > 1)$ in some point in $B_{d(y_{n-1}, a)}(a)$ also in Y.

Thus, for any $\gamma > 0$, $B_\gamma(a)$ contains all but finitely many terms of (y_n). Then, by Theorem 5.42, $\lim y_n = a$ and hence any subsequence of (y_n) converges to a but $a \notin Y$, contradicting the sequential compactness of Y. Thus Y is closed. ■

Theorem 7.7. *Let (X, d) be a metric space and suppose $Y \subseteq X$. If Y is sequentially compact, then Y is bounded.*

Proof. Suppose towards a contradiction that Y is not bounded. Define a sequence (y_n) as follows:

(i) y_1 is some point in Y;

(ii) y_n $(n > 1)$ is some point in Y with the property that $d(y_n, y_m) > 1$ for all integers $1 \leq m < n$.

First we must show that the terms of (y_n) $(n > 1)$ exists in Y. Because Y is unbounded, we may get arbitrarily far from each y_i $(1 \leq i < n)$. Now take the open ball around each of these y_i with radius 1. Then, there exists points outside of the union of these balls because otherwise Y would be bounded (as we could take a ball that contains all these balls, bounding Y). Thus the terms of the sequence (y_n) exist.

We know that the terms are always a distance 1 apart from one another, so this property holds for subsequences also. Thus no subsequence is Cauchy, so no subsequence converges, contradicting sequential compactness. Thus Y is bounded. ■

Functions on sequentially compact sets

There are some cool properties of functions on compact sets! Let's start with functions *preserving* sequential compactness. Very cool stuff. B)

Theorem 7.8 (Preservation of sequential compactness)**.** *Let (X, d_x) and (Y, d_y) be metric spaces and suppose $f\colon X \to Y$ is a continuous function. If (X, d_x) is compact, then $f(X)$ is compact.*

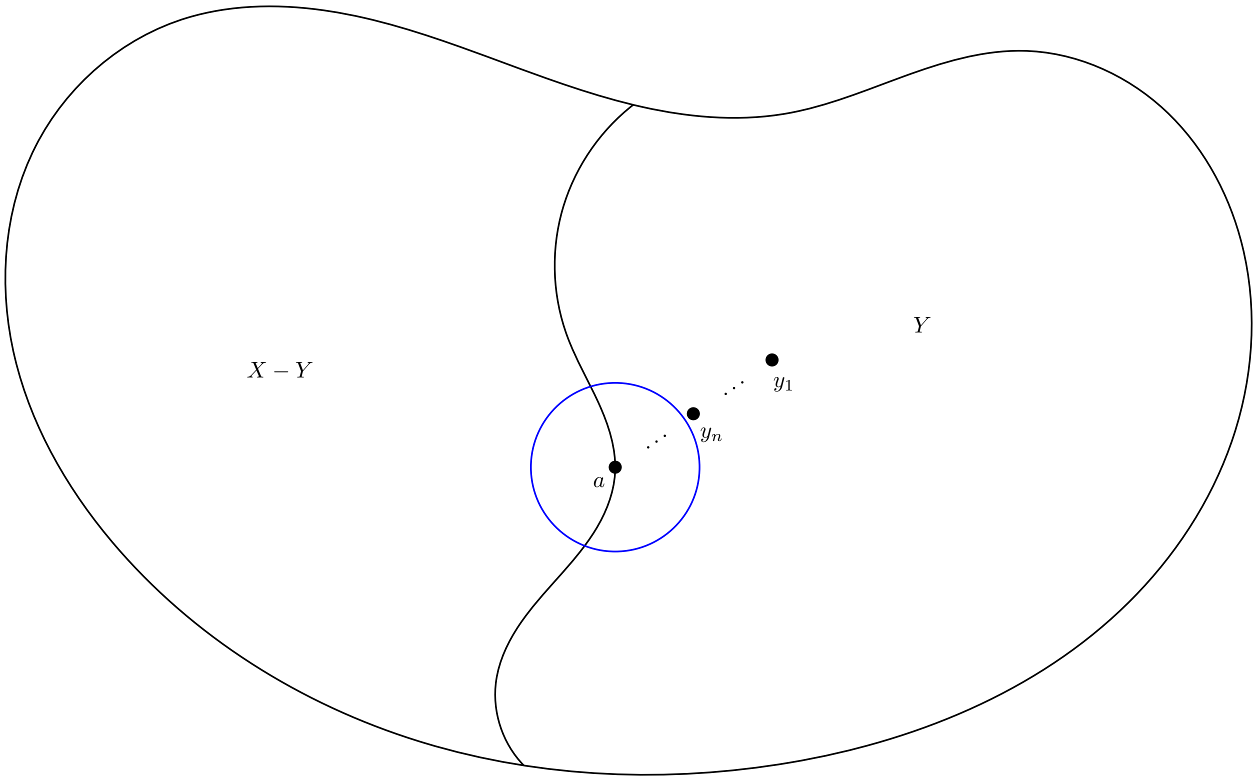

Figure 8: Diagram for the proof of Theorem 7.6.

Slogan. Continuous image of a sequentially compact metric space is compact.

Proof. Suppose (y_n) is a sequence in $f(X)$. Thus, for each (y_n), there exists a $x_n \in X$ such that $f(x_n) = y_n$. Because X is sequentially compact, there then exists a convergent subsequence of (x_n), (x_{n_k}), that converges to $x \in X$. Then, by f's continuity and definition of (x_n), we get that $\lim f(x_{n_k}) = \lim y_{n_k} = f(x) \in f(X)$. We have now exhibited a convergent subsequence of (y_n) and $f(X)$ is sequentially compact. ■

We can now talk about the *Extreme Value Theorem*! Very, very cool stuff. But before we do, we have to talk about a compact subset of $\mathbb{R}$ containing its supremum and infimum.

Lemma 7.9. *Let E be a sequentially compact subset of $\mathbb{R}$. Then E contains its supremum and infimum.*

Proof. We will only prove this for supremums as the proof for infimums is similar.

Suppose E is finite. Then the supremum is contained in E, so we may assume that E is infinite. By Theorems 7.6 and 7.7, we know that E is both closed and bounded. Thus $\sup E = s$ exists. By Theorem 1.5, we can construct a sequence (x_n) such that $s - 1/n < x_n$ for $x_n \in E$. Thus, $s - x_n < 1/n$ and $|s - x_n| < 1/n$ and (x_n) converges to s by the Archimedean Property. Thus s is a limit of E and by E's closedness is contained in E. ■

Theorem 7.10 (Extreme Value Theorem (EVT))**.** *Let (X, d) be a metric space and let $f\colon (X, d) \to \mathbb{R}$ (with the Euclidean metric) be continuous. Then, f attains its maximum and minimum.*

Slogan. Continuous function into $\mathbb{R} \implies$ attains max and min

Proof. We know by Theorem 7.8 that $f(X) \subseteq \mathbb{R}$ is sequentially compact. Then by Lemma 7.9, we know that $M = \inf f(X) \in f(X)$, so there exists an $x_M \in X$ such that $f(x_M) = M$. Thus f attains its maximum value.

The proof for minimum is very similar. ■

Now let's move on to the other type of compactness: covering compactness.

7.3 Covering compactness

Basics

Definition 7.11 (Covering compactness)**.** Let (X, d) be a metric space and let $\{U_\alpha\}_{\alpha \in A}$ be a collection of open sets in (X, d). (X, d) is covering compact if and only if, if $\bigcup_{\alpha \in A} U_\alpha \supseteq X$, then there exists a finite subcollection of $\{U_\alpha\}_{\alpha \in A}$, $U_{\alpha_1}, \ldots, U_{\alpha_n}$ such that $\bigcup_{j=1}^{n} U_{\alpha_j} \supseteq X$.

Slogan. Covering compact $\iff$ every open cover has a finite subcover

Here's an example of a covering compact set.

Example 7.12. The set $A = \{0\} \cup \{1/n \mid n = 1, 2, 3, \ldots\}$ (with the Euclidean metric) is covering compact.

Proof. First notice that 0 is the only limit point of A. Now, consider any open cover of A. Choose a cover that contains 0. By the topological definition of convergence, this contains all but finitely many $1/n$ elements of the set. Say $\{0\} \cup \{1/n \mid n = m, m+1, m+2, \ldots\}$ are covered by this open set. Choose an open set that contains $1/n$ for each $n = 1, 2, 3, \ldots, m-1$. We have exhibited a finite subcover and A is compact. ■

Example 7.13. The set $A = \{0\} \cup \{1/n \mid n = 1, 2, 3, \ldots\} \subseteq \mathbb{R}$ (with the Euclidean metric) is covering compact.

Proof. Again, I leave it to you. ■

Like with sequential compactness, this covering compactness seems to be independent of the set the set we're focused on is nestled in. Indeed, covering compactness is also independent of the superset its contained in.

Theorem 7.14. *Let (X, d) be a metric space and let $K \subset Y \subset X$. K is covering compact in Y if and only if K is compact covering in X.*

Proof. **FINISH** ■

Remark. From this, it actually makes sense to talk about compact sets without invoking the metric space containing it! Very interesting stuff.

Cool properties

Theorem 7.15. *If $\{K_\alpha\}_{\alpha \in A}$ is a collection of covering compact subsets of a metric space (X, d) such that the intersection of every finite subcollection of $\{K_\alpha\}_{\alpha \in A}$, then $\bigcap_{\alpha \in A} K_\alpha$ is nonempty.*

Proof. Suppose towards a contradiction that $\bigcap_{\alpha \in A} K_\alpha$ is empty. Fix $K \in \{K_\alpha\}_{\alpha \in A}$ and define $G_\alpha = X - K_\alpha$. Since the intersection is empty, no point of K is in every $\{K_\alpha\}_{\alpha \in A} - \{K\}$. Then $\bigcup_{\alpha \in A} G_\alpha$ is an open cover of K. Since K is compact, there exists a finite subcollection $G_{\alpha_1}, \ldots, G_{\alpha_n}$ that covers K (i.e. $K \subseteq G_{\alpha_1} \cup \cdots \cup G_{\alpha_n} = X - (K_{\alpha_1} \cap \cdots \cap K_{\alpha_n})$). However, then $K \cap K_{\alpha_1} \cap \cdots \cap K_{\alpha_n}$ is empty, contradicting our hypothesis. This completes the proof. ■

Corollary 7.16 (Generalized Nested Interval Property)**.** *If $\{K_n\}_{n=1}^{\infty}$ is a collection of covering compact sets with the property that $K_n \supseteq K_{n+1}$ (for all $n = 1, 2, 3, \ldots$), then $\bigcap_{n=1}^{\infty} K_n$ is nonempty.*

Slogan. The intersection nonempty nested compact sets is nonempty

Theorem 7.17. *Let (X, d) be a metric space and let K be a covering compact subset of X. Then K is bounded.*

Slogan. Any covering compact set is bounded.

Proof. It is clear that the set $\{B_1(x)\}_{x \in K}$ is an open cover of K. Since K is covering compact, there exists a finite subcollection of 1-balls (say n many balls that covers K. Then, clearly, the $n+1$ ball centered around any point of K covers K and K is bounded. ■

Theorem 7.18. *Let (X, d) be a metric space and let K be a covering compact subset of X. Then K is closed.*

Slogan. Any covering compact set is closed.

Proof. Suppose towards the contraposition that K is not closed. We aim to show that K is then not compact.

Since K is not closed, there exists a point $p \in \overline{K} - K$. Let $U_n := \{x \in X \mid d(x, p) > 1/n\}$. Then, $\bigcup_{n=1}^{\infty} U_n$ is an open cover of K, but any finite subcollection with largest index m would not include $\{x \in X \mid d(x, p) \leq 1/m\}$ whose intersection with K is nonempty by definition of limit point (since p is a limit point), so no finite subcollection can cover K. Thus K is not compact and the proof is complete. ■

Now let's move on to an idea that shouldn't feel so crazy anymore: the equivalence of covering and sequential compactness.

7.4 Equivalence of covering and sequential compactness, and Heine-Borel Theorem

Theorem 7.19 (Equivalence of covering and sequential compactness)**.** *Let (X, d) be a metric space and let $K \subseteq X$. Then, K is covering compact if and only if K is sequentially compact.*

Slogan. Covering compact $\iff$ sequentially compact

Proof. ($\Rightarrow$) Let (a_n) be a sequence in K. Suppose (a_n) is eventually constant (i.e. there are only finitely many distinct terms), then (a_n) converges to that constant term and is necessarily in K. We may now assume that (a_n) has infinitely many distinct terms.

Consider $A \coloneqq \{a_n \mid n = 1, 2, 3, \ldots\}$. Since we have assumed that (a_n) has infinitely many distinct terms, A is infinite. It suffices to show that A has a limit point in K. Suppose towards the contraposition that this is not the case. Then each $p \in K$ would have an open ball B_p around p that contains at most one point of A (one point if $p \in A$). Then $\{B_p \mid p \in K\}$ forms an open cover of A but no finite subcollection covers A as if it did, A would have a limit point in K which would contradict our assumption. Thus this cover cannot have a finite subcover of K. Thus K is not compact. This completes this direction.

($\Leftarrow$) Suppose K is sequentially compact. **FINISH** ■

Because we have shown that sequential compactness and covering compactness are the same thing, we will now refer to them as just *compactness*.

Now let's get to Heine-Borel Theorem.

Theorem 7.20 (Heine-Borel Theorem)**.** *Let $K \subseteq \mathbb{R}^n$ (with the Euclidean metric). K is sequentially compact if and only if K is closed (in $\mathbb{R}^n$) and bounded.*

Slogan. (In $\mathbb{R}^n$) Sequentially compact $\iff$ closed and bounded

Before we do the proof, we need a lemma.

Theorem 7.21 (Bolzano-Weierstrass Theorem in Euclidean space)**.** *Let $(\mathbf{a}_n)$ be a sequence in $\mathbb{R}^n$ (with the Euclidean metric). Then, if $(\mathbf{a}_n)$ is bounded, $(\mathbf{a}_n)$ has a convergent subsequence.*

Proof. We will proceed by induction on the dimension of $\mathbb{R}$. We know that the base case holds by Bolzano-Weierstrass Theorem. Suppose the statement holds for R^n. We will show that the property holds for $\mathbb{R}^{n+1}$. Since $(\mathbf{a}_n)$ is a sequence in $\mathbb{R}^{n+1}$, we may rewrite this as $(\mathbf{x}_n, y_n)$ where $(\mathbf{x}_n)$ is a sequence in $\mathbb{R}^n$. Because this sequence is bounded in $\mathbb{R}^n$, it must be bounded in each component sequence as otherwise the sequence would not be bounded. Now, by hypothesis, we know that there is a convergent subsequence $(\mathbf{x}_{n_k})$. Now, consider $(\mathbf{a}_{n_k})$. Because $(\mathbf{a}_{n_k})$ is still bounded, and we know there is a convergent subsequence of (y_{n_k}), $(y_{n_{k_\ell}})$. It follows that $(\mathbf{a}_{n_{k_\ell}})$ converges. This completes the proof. ■

Now let's move on to the proof of Heine-Borel.

Proof. ($\Leftarrow$) Let $(\mathbf{a}_n)$ be a sequence in K. Because K is bounded, it follows from Lemma 7.21 that $(\mathbf{a}_n)$ has a convergent subsequence $(\mathbf{a}_{n_k})$. Because K is closed, this limit must be in K. Thus K is sequentially compact.

($\Rightarrow$) Suppose Y is (sequentially) compact. It follows from Theorems 7.6 and 7.7 that Y is closed and bounded.

This completes the proof. ■

7.5 Closing remark on compactness

In general, when people say compactness, they mean covering compactness, so there is often no need to specify what type of compactness you're talking about, but if you're going to use sequential compactness, put 'sequential' in parentheses or something. Also, a big takeaway from the idea of compactness (particularly covering compactness) is that the sets are, in fact, 'compact' in some specific sense. The set is small in that even open cover has a finite subcover. Compactness is sort of like the best thing after finiteness. Now let's move on to other cool stuff.

7.6 Dense sets, Separability, and Bases

Dense set

Definition 7.22 (Dense set)**.** Let (X, d) be a metric space. A set Y is *dense* in X if and only if $\overline{Y} = X$. In other words, Y is dense in X if and only if every point of X is a point of Y or a limit point of Y. In other words, $Y \subseteq X$ is dense in X if and only if if $G \subseteq X$ is open, then $G \cap Y$ is nonempty.

Do we have any examples of dense sets? Yup!

Example 7.23. $\mathbb{Q}$ is dense in $\mathbb{R}$.

Now let's move on to the more interesting idea of separability.

Separability

Definition 7.24 (Separable)**.** Let (X, d) be a metric space. X is separable if and only if there exists a countable dense subset of X.

Our example from density actually works here!

Example 7.25. $\mathbb{R}$ is separable as $\overline{\mathbb{Q}} = \mathbb{R}$ and $\mathbb{Q}$ is countable.

This also gives us the following example.

Example 7.26. $\mathbb{R}^n$ is separable with countable dense subset $\mathbb{Q}^n$.

There is also a relation to compactness here.

Theorem 7.27. *If a metric space (X, d) is compact, then (X, d) is separable.*

Proof. Consider $\mathcal{B}_n := \left\{B_{1/n}(x) \mid x \in X\right\}$. Each B_n is an open cover of X, so it admits a finite subcover B_n'. Name the set of the centers of the balls in B_n' C_n. Then $\bigcup_{n=1}^{\infty} C_n$ is a countable dense subset of X. ■

There is another.

Theorem 7.28. *Suppose (X, d) is separable and $X = \bigcup_{\lambda \in \Lambda} U_\lambda$, U_λ open. Show*

$$X = \bigcup_{j=1}^{\infty} U_{\lambda_j}$$

for some $\lambda_1, \lambda_2, \ldots \in \Lambda$.

Slogan. Separable $\implies$ every cover has a countable subcover

Proof. Let S be a countable dense subset of X. It follows from the construction in Lemma 7.2. that $\mathcal{B} := \{B_q(s) \mid s \in S \wedge q \in \mathbb{Q}^+\}$ is a countable base for X. Then because $\bigcup_{\lambda \in \Lambda} U_\lambda$ is open, it is the union of a countable subcollection of $\mathcal{B}$, say $\mathcal{B}' := \{B_1, B_2, B_3, \ldots\}$.

Now, for each $B_i \in \mathcal{B}'$, choose a $U_{\lambda_i} \in \{U_\lambda\}_{\lambda\in\Lambda}$ such that $B_i \subseteq U_{\lambda_i}$. Then our subcollection of $\{U_\lambda\}_{\lambda\in\Lambda}$ is clearly countable and $\bigcup_{j=1}^{\infty} U_{\lambda_j} \supseteq \bigcup_{j=1}^{\infty} B_j = \bigcup_{\lambda\in\Lambda} U_\lambda = X$, so $\left\{U_{\lambda_j}\right\}_{j\geq 1}$ is a countable subcover of X. ■

Bases

FINISH write something about vector spaces or smth

Definition 7.29 (Base of a metric space)**.** A collection $\{V_\alpha\}$ of open subsets of X is said to be a *base* for X if and only if the following is true: For every $x \in X$ and every open set $G \subseteq X$ such that $x \in G$, we have $x \in V_\alpha \subseteq G$ for some α. In other words, every open set in X is the union of a subcollection of $\{V_\alpha\}$.

There is a very cool connection between countable bases and separability: they are equivalent!

Theorem 7.30. *Let (X, d) be a metric space. X is separable if and only if it has a countable base.*

Slogan. Separable $\iff$ has a countable base

Proof. Let (X, d) be a metric space.

($\Rightarrow$) Suppose (X, d) is separable with a countable dense subset S. Let $\mathcal{B} := \{B_q(s) \mid s \in S \wedge q \in \mathbb{Q}^+\}$. We will show that $\mathcal{B}$ is a countable base for X. To that end, suppose $x \in X$ and let G be an open subset that contains x. Because G is open, there exists an $r > 0$ such that $B_r(x) \subseteq G$. Because S is dense in X, $B_{r/2}(x)$ intersects nontrivially with S, so there exists a $t \in S$ such that $t \in B_{r/2}(x)$. Now, choose a rational $d(x, t) < p < r/2$. Notice that then $x \in B_p(t)$. Now, choose some $u \in B_p(t)$. It follows that

$$d(x, u) \leq d(x, t) + d(t, u) < \frac{r}{2} + p < \frac{r}{2} + \frac{r}{2} = r.$$

Thus, $u \in B_r(x)$. We now have that $x \in B_p(t) \subseteq B_r(x) \subseteq G$. Since $B_p(t) \in \mathcal{B}$, $\mathcal{B}$ is a base of (X, d).

We will now show that $\mathcal{B}$ is a countable. Let $f \colon \mathcal{B} \to \mathbb{Q} \times S$ be defined by $B_q(s) \mapsto (q, s)$. f is clearly a bijection, so $\#\mathcal{B} = \#(\mathbb{Q} \times S)$. Now, since S is countable, there exists a bijection g from S to $\mathbb{Q}$. Thus, define $h \colon \mathbb{Q} \times S \to \mathbb{Q} \times \mathbb{Q}$ by $(q, s) \mapsto (q, g(s))$. This is clearly bijective, so we have that $\#\mathcal{B} = \#(\mathbb{Q} \times S) = \#(\mathbb{Q} \times \mathbb{Q})$, so $\mathcal{B}$ is countable. This completes the proof.

($\Leftarrow$) Suppose (X, d) has a countable base $\{V_n\}_{n\geq 1}$. Now, let $x \in X$. Choose p_n to be some point in V_n. Name this set of points P. P is clearly countable. We will now show that P is dense in X. Suppose U is an open subset of X. Then there exists a subcollection $\left\{V_{n_j}\right\}$ such that $\bigcup_{j=1}^{\infty} V_{n_j} = U$. Since $P \cap \left(\bigcup_{j=1}^{\infty} V_{n_j}\right) \neq \varnothing$, $P \cap U \neq \varnothing$, so P is dense in X. ■

We can also use bases to prove that the subset of a separable metric space is indeed separable!

Theorem 7.31. *Let (X, d) be a separable metric space. If $S \subseteq X$, then S is also separable.*

Slogan. Separable $\implies$ subsets are separable

Proof. Let Y be a countable dense subset of X. Then $\mathcal{B} := \{B_q(y) \mid y \in Y \wedge q \in \mathbb{Q}^+\}$ forms a countable base of X by the forward direction of Lemma 7.30. Let $\mathcal{C} := \{B \in \mathcal{B} \mid B \cap S \neq \varnothing\}$. $\mathcal{C}$ is clearly countable since $\mathcal{B}$ is countable. What remains to be shown is that $\mathcal{C}$ is a base for S.

Suppose $T \subseteq S$ is open. Suppose $t \in T$. Because T is open, there exists a number $r > 0$ such that $B_r(t) \subseteq T$. Choose a rational number $0 < s < r$ (this exists since $\mathbb{Q}$ is dense in $\mathbb{R}$). Let (y_n) be a sequence of points in Y that have limit t (this exists since $t \in X$ and Y is dense in X). There then exists a point of the sequence $y_m \in B_{s/2}(t) \subseteq B_r(t)$. Then draw an $s/2$-ball around y_m and choose a point w within it. It then suffices to show that $w \in B_r(t)$ since then we would have $w \in B_{s/2}(y_m) \subseteq B_r(t) \subseteq T$ which would then satisfy the definition of a base (since $B_{s/2}(y_m) \in \mathcal{C}$ by construction). Using the triangle inequality, we have that

$$d(w,t) \leq d(w,y_m) + d(y_m,t) < s/2 + s/2 = s < r.$$

Thus $w \in B_r(t)$ and the proof is complete. ■

7.7 Totally boundedness

Before we define totally bounded, we need to define something else first.

Definition 7.32. Let (X,d) be a metric space. A set S is said to be *ϵ-dense in X* if and only if for all $x \in X$, there exists an $s \in S$ such that $d(x,s) < \epsilon$.

Slogan. ϵ-dense $\iff$ there exists finite collection of ϵ-balls in X whose union contains X

Now we can talk about totally boundedness! :)

Definition 7.33 (Totally boundedness)**.** A metric space (X,d) is *totally bounded* if and only if for all $\epsilon > 0$, there exists an ϵ-dense subset of X.

There are some interesting connections to compactness here.

Theorem 7.34. *Let (X,d) be a metric space. If (X,d) is compact, then (X,d) is totally bounded.*

Slogan. Compact $\implies$ totally bounded

Proof. Let $\epsilon > 0$. Consider the open cover $\{B_\epsilon(x)\}_{x\in X}$. Because X is compact, there is a finite subcover. Name the set of centers of these balls in the finite subcover S. Then S is ϵ-dense in X by construction. ■

When does totally boundedness mean compact, however? When the space is also complete! But before we do that we need a lemma/theorem.

Lemma 7.35. *Let (X,d) be a metric space and let $S \subseteq X$. If (X,d) is totally bounded, then S is totally bounded.*

Slogan. Totally bounded $\implies$ subsets are totally bounded

Proof. Let $\epsilon > 0$. Let $Y = \{y_1, \ldots, y_n\}$ be finite, $\epsilon/2$-dense subset of X (this exists because X is totally bounded). Define $W = \{w_1, \ldots, w_k\} \subseteq Y$ to be the set of y_j's such that $B_{\epsilon/2}(y_j) \cap S \neq \varnothing$. We will show that $\bigcup_{j=1}^k B_\epsilon(w_j) \supseteq S$. Suppose $s \in S$. Then there exists a $y \in Y$ such that $s \in B_{\epsilon/2}(y)$. Then there $w \in W$ such that $w \in B_{\epsilon/2}(y)$. It then follows from the triangle inequality that

$$d(w,s) \leq d(w,y) + d(y,s) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $s \in B_\epsilon(w) \subseteq \bigcup_{j=1}^k B_\epsilon(w_j)$. ■

Now we can move on to the main event! B)

Theorem 7.36. *Let (X, d) be a metric space. If (X, d) is complete and totally bounded, then (X, d) is compact.*

Proof. Suppose (x_n) is a sequence in (X, d). By Theorem 7.35, $S = \{x_n \mid n \in \mathbb{Z}^+\} \subseteq X$ is totally bounded. Hence there exists a finite 1-dense subset Y_1 in S. Then, there exists a $y_1 \in Y_1$ such that closed 1-ball around y_1 has infinitely many points of S (because if none of the closed 1-balls around points of Y_1 had infinitely many points, S would be finite which would be a contradiction). Then by Theorem 7.35, there exists a 1/2-dense subset Y_2 of Y_1 (since Y_1 is totally bounded). By a similar argument, there exists a $y_2 \in Y_2$ such that the closed 1/2-ball around y_2 contains infinitely many points of $Y_1 \subseteq S$. Similarly, there exists a 1/3-dense subset Y_3 of Y_2 and there exists a $y_3 \in Y_3$ such that the closed 1/3-ball around y_3 contains infinitely many points of $Y_2 \subseteq Y_1 \subseteq S$. Continuing this process, we get a subsequence (y_n) of (x_n).

We will now show that (y_n) is Cauchy. Let $\epsilon > 0$. Then there exists a positive integer N such that $1/N < \epsilon$ by the Archimedean Property. Take $i, j \geq N + 1$. Suppose without loss of generality that $j \geq i$. Then by construction of (y_n), we know that

$$y_j \in \overline{B}_{\frac{1}{i}}(y_i) \implies d(y_i, y_j) \leq \frac{1}{i} \leq \frac{1}{N+1} < \frac{1}{N} < \epsilon.$$

Thus (y_n) is Cauchy. Now, since $(y_n) \subseteq Y_1$ and Y_1 is closed (since any finite set is closed in any metric space), Y_1 is complete and so (y_n) converges. Thus (x_n) has a convergent subsequence and (X, d) is (sequentially) compact. ■

7.8 -Connectedness

We call this section '-connectedness' because we actually have two types of connectedness to discuss: connectedness and path/arcwise-connectedness. We'll get to the former now.

Connectedness

Connectedness means exactly what you think it means: (colloquially) a set is connected if it's, well, connected. What I mean by this is that if you think something looks connected, it probably is. But how do we turn this into a mathematical definition? It actually turns out we don't do this directly. Indeed, it is actually easier to define *disconnectedness* in order to define *connectedness*.

Definition 7.37. A metric space (X, d) is *disconnected* if and only if there exists $A, B \subseteq X$ such that A, B are nonempty, open, disjoint, and $A \cup B = X$.

A metric space (X, d) is *connected* if and only if it is not disconnected.

A subset Y of a metric space (X, d) is *connected* if and only if $(Y, d|_{Y \times Y})$ is connected.

Here's an example of a connected set.

Example 7.38. $[0, 1]$ is connected.

By virtue of the definition of connectedness, we actually have to do proof by contradiction most of the time.

Proof. Suppose towards a contradiction that $[0, 1]$ is disconnected. Then there exists $A, B \subseteq [0, 1]$ such that A, B are open, nonempty, disjoint, and $A \cup B = [0, 1]$.

FINISH ■

Here's an example of a disconnected set.

Example 7.39. $(0,1) \cup (2,3)$ is disconnected.

Proof. Clear. ■

Path-connectedness

Two points being path-connected means exaclty what you imagine it to mean (like with connectedness). More precisely this is:

Definition 7.40 (Path connectedness)**.** A metric space (X, d) is *path-connected* if and only if for all $x, y \in X$, there exists a continuous function $\gamma\colon [0,1] \to X$ such that $\gamma(0) = x_1$ and $\gamma(1) = y$.

It seems (at least to me) clear that path-connectedness is a stronger condition than connectedness. This hunch turns out to be correct, however 'proof by inspection' is not enough, so we'll actually go through this.

Theorem 7.41. *If a metric space (X, d) is path-connected, then (X, d) is connected.*

Proof. Suppose towards a contradiction that (X, d) is not connected. Then there exists $A, B \subseteq X$ such that A, B are open, nonempty, disjoint, and have union equal to X. Choose $a \in A$ and $b \in B$. We will show that there does not exist a path γ from a to b, contradicting X's path-connectedness.

$\gamma^{-1}(A)$ and $\gamma^{-1}(B)$ are open since γ is continuous. Regarding disjointness, we have that $\gamma^{-1}(A) \cap \gamma^{-1}(B) = \gamma^{-1}(A \cap B) = \gamma^{-1}(\varnothing) = \varnothing$. We also have that $\gamma^{-1}(A) \cup \gamma^{-1}(B) = \gamma^{-1}(A \cup B) = \gamma^{-1}(X) = [0,1]$. Thus $[0,1]$ is disconnected which is a contradiction. This completes the proof. ■

However, this does not go both ways! The topologists sine curve: $\{\sin 1/x \mid x \in (0,1]\} \cup \{0\}$ is connected but not path connected. However, for open subsets in $\mathbb{R}^n$, this is true. We'll get to that now.

Connectedness and Path-connectedness in $\mathbb{R}^n$

Theorem 7.42. *Let $U \subseteq \mathbb{R}^n$ be open. Then, U is connected if and only if it is path-connected.*

For the following proof, referring to Figure 9

Proof. $(\Rightarrow)$ Suppose $p \in U$. Let $S_p \coloneqq \{u \in U \mid \exists \gamma\colon [0,1] \to U : \gamma(0) = p, \gamma(1) = u\}$.

We will first show that S_p is open. Suppose $q \in S_p$. Then there exists a number $r > 0$ such that $B_r(q) \subseteq U$. Let $s \in B_r(q)$. Let $\gamma'\colon [0,1] \to U$ be the straight line from q to s. Now, let $\Gamma\colon [0,1] \to U$ be defined by

$$t \mapsto \begin{cases} \gamma(2t) & t \in \left[0, \frac{1}{2}\right); \\ \gamma'(2t) & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Now we have a continuous path Γ from p to s. Thus $B_r(q) \subseteq S_p$ and S_p is open.

We will now show that $S_p = U$. Suppose towards a contradiction that $U - S_p$ is nonempty. Then there exists a point $a \in U - S_p$. Let $r' > 0$. Consider a point $b \in B_{r'}(a)$. Suppose towards a contradiction that $b \notin B_{r'(a)}$. Then, $b \in S_p$. Then a would be reachable from b with a continuous path from the same argument in the previous paragraph, which would be a contradiction, so $b \in B_{r'}(a)$. Thus, $U - S_p$ is open. Then, $U = S_p \cup (U - S_p)$ which means that U is disconnected which is a contradiction. Therefore, $U - S_p = \varnothing$ and $S_p = U$.

Hence, U is path-connected.

$(\Leftarrow)$ This direction follows from 7.41.

This completes the proof. ■

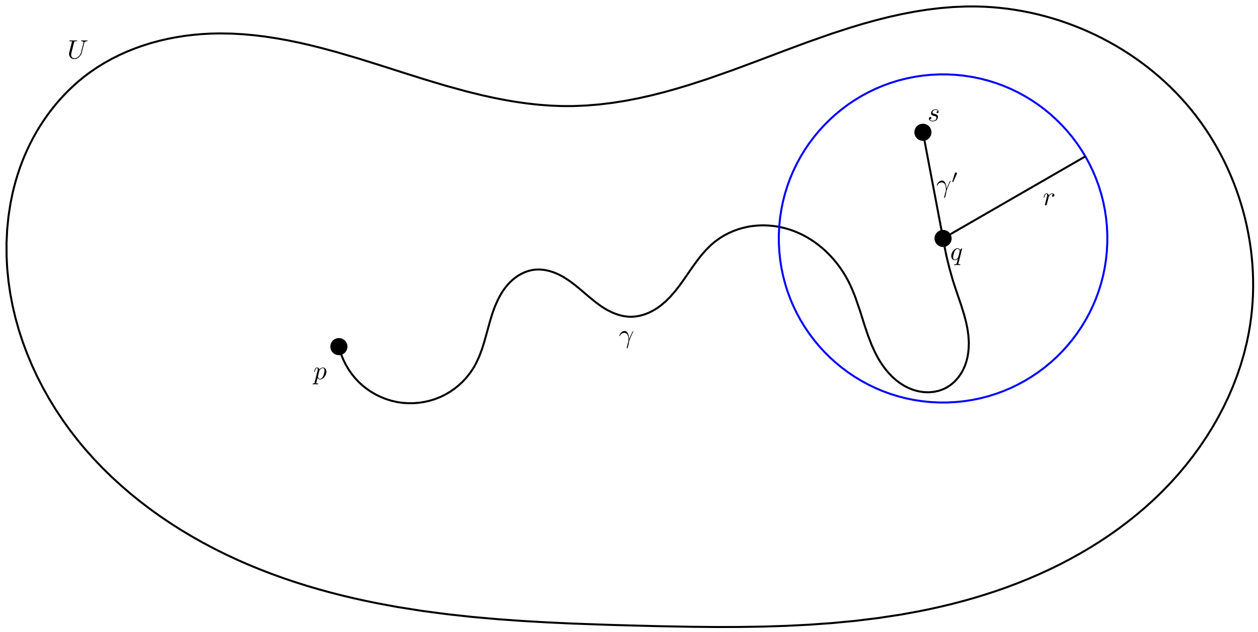

Figure 9: Diagram for the first part of the proof of Theorem 7.42.

8 Baire Category Theorem

The name of Baire Category Theorem (BCT) isn't particularly enlightening, so we'll just get into the statement of the theorem, though there are several statements so we'll go through them individually.

8.1 BCT I

Theorem 8.1 (BCT I). *Let* $\{U_n\}_{n=1}^{\infty}$ *be a sequence of dense open subsets of a complete metric space* (X, d)*. Then,* $\bigcap_{n=1}^{\infty} U_n$ *is also dense in* X*.*

Proof. finish ■

I don't have any good examples of this particular thing being useful.

8.2 BCT II

We first need to define what a *nowhere dense set* is before we get to the statement of BCT II.

Definition 8.2 (Nowhere dense set). Let (X, d) be a metric space. A subset $Y \subseteq X$ is *nowhere dense* if and only if it has no interior points (i.e. $Y^\circ = \varnothing$).

Theorem 8.3 (BCT II). *Let* $\{E_n\}_{n=1}^{\infty}$ *be a sequence of nowhere dense subsets of a complete metric space* (X, d)*. Then,* $\bigcup_{n=1}^{\infty} E_n$ *has empty interior.*

Proof. Apply BCT I to the dense open sets $U_n = X - \overline{E}_n$. ■

8.3 BCT III

Theorem 8.4 (BCT III). *Any complete metric space is not the countable union of closed sets with empty interior.*

Corollary 8.5 (BCT IIIa). *Any complete metric space no isolated points is uncountable.*

Here's an application of BCT III:

Example 8.6. $\mathbb{R}^3$ is not the countable union of planes.

Proof. By Baire Category Theorem, we know that because $\mathbb{R}^3$ is complete it cannot be the countable union of closed sets with empty interior. It suffices to show that each plane in $\mathbb{R}^3$ is closed and has an empty interior. Let P be an arbitrary plane in $\mathbb{R}^3$.

First we will show that P is closed. It suffices to show that $\mathbb{R}^3 - P$ is open. Suppose we have a point $(x, y, z) \in \mathbb{R}^3$. Let s be the minimal distance from (x, y, z) to P. Then the ball $B_s((x, y, z)) \subseteq \mathbb{R}^3 - P$. Thus $\mathbb{R}^3 - P$ is open and P is closed.

Now we will show that P has empty interior. Suppose P° is nonempty. Then there exists a point $p \in P^\circ$ such that there exists a number $r > 0$ such that $B_r(p) \subseteq P^\circ$. However, a ball in $\mathbb{R}^3$ cannot be contained in P since the ball expands in the direction normal to the plane. Thus $p \notin P^\circ$. Thus $P^\circ = \varnothing$.

This completes the proof. ■