1 Series

1.1 Basics

What does $\sum_{j=1}^{\infty} a_j = A$ (for some sequence (a_n) and some number A) even mean? Well, in general, $\sum_{j=1}^{N} a_j = a_1 + \cdots + a_N$, so we're saying something like $a_1 + a_2 + a_3 + \cdots = A$. In particular, what we mean is the following: $\lim S_n = A$ where $S_n = \sum_{j=1}^{n} a_j$. So, because of this we can take a lot of my results from sequences and apply them to series!!! Super cool stuff right here! :> But before we do that let's put we just said into an actual definition.

Definition 1.1. Let (a_n) be a sequence. Then,

$$\sum_{j=m}^{n} a_j := a_m + a_{m+1} + a_{m+2} + \cdots + a_n$$

for $m \leq n$. With (a_n) we associate the sequence (S_n) where

$$S_n := \sum_{j=1}^{n} a_j.$$

For (S_n) we also use the symbolic expression

$$a_1 + a_2 + a_3 + \cdots$$

or

$$\sum_{j=1}^{\infty} a_j. \tag{1}$$

(1) is called an *infinite series* or just *series*. (S_n) is called the *sequence of partial sums of the series*. If $\lim S_n = S$, we say that the series converges and we write

$$\sum_{j=1}^{\infty} a_j = S.$$

S is called the *sum* of the series. If (S_n) diverges, then the sum diverges.

When the bounds are unambiguous we simply write $\sum a_j$.

The following is a direct translation from sequence results to series.

Theorem 1.2 (Algebraic Limit Theorem for Series)**.** *If $\sum_{j=1}^{\infty} a_j = A$ and $\sum_{j=1}^{\infty} b_j = B$, then*

(i) $\sum_{j=1}^{\infty} ca_j = cA$ *for all* $c \in \mathbb{R}$*;*

(ii) $\sum_{j=1}^{\infty} (a_j + b_j) = A + B$.

Remark. We won't talk about products of series just yet. ;)

Remember Cauchy sequences? Well, because we can treat series like sequences (basically), it ends up being pretty improtant.

Theorem 1.3. *The series $\sum_{j=1}^{\infty} a_j$ converges if and only if for all $\epsilon > 0$ there exists an integer N such that whenever $n > m \geq N$ it follows that*

$$|a_{m+1} + a_{m+2} + a_{m+3} + \cdots + a_n| < \epsilon.$$

Proof. Notice that the final line is equivalent to $|S_n - S_m| < \epsilon$. Then also notice that the theorem is saying (S_n) converges if and only if it is Cauchy, which is true by the completeness of $\mathbb{R}$. ■

Corollary 1.4. *If $\sum_{j=1}^{\infty} a_j$ converges, then $\lim a_n = 0$.*

Proof. Consider the case of $n = m+1$ with regard to Theorem 1.3. We then get that $|a_n| < \epsilon$ and $\lim a_n = 0$. ■

However, is the converse of this corollary true? No!

Example 1.5 (Harmonic series)**.** The Harmonic Series is $\sum_{n=1}^{\infty} 1/n$. $\lim(1/n) = 0$, but the sum does not converge! (We'll prove this soon.)

Example 1.6 (Geometric Series)**.** A geometric is a series of the form $\sum_{j=0}^{n} ar^j$ for some common ratio (number) r. A geometric series converges if and only if $|r| < 1$. We leave this as an exercise as the identity $\sum_{j=0}^{n} ar^j = \frac{a(1-r^n)}{1-r}$ should be enough to do this. Should be pretty straight forward.

1.2 Tests for convergence

There are a ton of tests. Here are the main ones I know (not all were shown in class).

Theorem 1.7 (Cauchy Condensation Test)**.** *Suppose (a_n) is decreasing and satisfies $a_n \geq 0$ for all n. Then, the series $\sum_{j=1}^{\infty} a_j$. Then the series $\sum_{j=1}^{\infty} a_j$ converges if and only if the series $\sum_{j=1}^{\infty} 2^j a_{2^j}$ converges.*

Proof. finish ■

Theorem 1.8 (Harmonic Series Test)**.** *The series $\sum_{n=1}^{\infty} 1/n^p$ converges if and only if $p > 1$.*

Proof. FINISH ■

Theorem 1.9 (Comparison Test)**.** *Suppose (a_n) and (b_n) satisfy $0 \leq a_n \leq b_n$ for all n. Then,*

(i) *If $\sum_{j=1}^{\infty} b_j$ converges, then $\sum_{j=1}^{\infty} a_j$ converges;*

(ii) *If $\sum_{j=1}^{\infty} a_j$ diverges, then $\sum_{j=1}^{\infty} b_j$ diverges.*

Proof. Let $\epsilon > 0$ be arbitrary.

(i) Suppose $\sum_{j=1}^{\infty} b_j$ converges. From Theorem 1.3, there exists an integer N such that $n > m \geq N$ such that $|b_{m+1} + \cdots + b_n| < \epsilon$. Because $0 \leq a_n \leq b_n$ for all n, we have that

$$|a_{m+1} + \cdots + a_n| \leq |b_{m+1} + \cdots + b_n| < \epsilon.$$

Thus $\sum_{j=1}^{\infty} a_j$ converges.

(ii) Similar argument to the proof of (i). ■

Theorem 1.10 (Absolute Convergence Test). *If $\sum_{j=1}^{\infty} |a_j|$ converges, then $\sum_{j=1}^{\infty} a_j$ converges.*

Proof. Let $\epsilon > 0$ be arbitrary. It follows from Theorem 1.3 that there exists an integer N such that $n > m \geq N$ implies that $||a_{m+1}| + \cdots + |a_n|| = |a_{m+1}| + \cdots + |a_n| < \epsilon$. It follows from Triangle Inequality that $|a_{m+1} + \cdots + a_n| \leq |a_{m+1}| + \cdots + |a_n| < \epsilon$, so $\sum_{j=1}^{\infty} a_j$ converges. ■

Converse is not true however! $\sum_{j=1}^{\infty}(-1)^j/j$ converges but $\sum_{j=1}^{\infty} 1/j$ does not. But how do we know that $\sum_{j=1}^{\infty}(-1)^j/j$ converges? With the following test:

Theorem 1.11 (Alternating Series Test). *Let (a_n) be a seuqence satisfying,*

(i) $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots$ *and*

(ii) $\lim a_n = 0$.

Then, the alternating series $\sum_{j=1}^{\infty}(-1)^j a_j$ converges.

Proof. **FINISH** ■