Math 131C Notes

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This set of notes is very informal and tries its best to simplify often hard to digest ideas. Hopefully it's useful in your learning of the material.

FINISH: put figures in right places

Contents

Lecture 1—March 29, 2021

We're starting by reviewing metric spaces! It's kind of old now (third time I'm writing this for notes lmao) but it's really cool nonetheless. :)

Definition 1.1. A *metric space* is a nonempty set X and a function $d\colon X \times X \to [0, \infty)$ such that for all $x, y, z \in X$,

(i) $d(x, y) = d(y, x)$,

(ii) $d(x, y) = 0$ if and only if $x = y$,

(iii) and $d(x, z) \leq d(x, y) + d(y, z)$.

(This function is called a *metric*.)

My professor does these in class questions to make sure we're following along to check our understanding it seems, so I've tried ot make them pretty.

Question 1.2. Which of the following is not a metric on $\mathbb{R}^2$?

(a) $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$

(b) $d(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$

(c)
$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise} \end{cases}$$

(d) $d(x, y) = \min\{|x_1 - y_1|, |x_2 - y_2|\}$

Answer: (d) as (ii) from Definition 1.1 fails.

Example 1.3. $X \coloneqq \mathbb{R}^2$ with Euclidean metric

$$d(x, y) = \sqrt{\sum_{j=1}^{n} |x_j - y_j|^2}.$$

Definition 1.4. If (X, d) is a metric space and $Y \subseteq X$ is non-empty, then the metric space $(Y, d|_{Y \times Y})$ is called a *subspace* of (X, d).

Definition 1.5. We say that a sequence (x_n) in a metric space (X, d) *converges* if and only if there exists an $x \in X$ such that $d(x, x_n) \to 0$ as $n \to \infty$.

Question 1.6. **TRUE OR FALSE?**
If $x_n \to x$, then every subsequence $x_{n_k} \to x$.
Answer: TRUE. As $n_k \geq k$: for all $\varepsilon > 0$, there exists an integer N such that for all $n \geq N$, $d(x, x_n) < \varepsilon$ (by definition of convergence), so for all $k \geq N$, $d(x, x_{n_k}) < \varepsilon$.

Definition 1.7. We say a sequence (x_n) is *Cauchy* if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

Question 1.8. TRUE OR FALSE? Every Cauchy sequence converges.

Answer: FALSE. Take $(0,1]$ equipped with the Euclidean metric. Consider the sequence $x_n = \frac{1}{n}$. This is Cauchy but does not converge in our metric space (since 0 is not included).

Definition 1.9. We say a metric space is *complete* if and only if every Cauchy sequence converges.

Definition 1.10. Let (X, d) be a metric space. Denote

$$B(x; r) \coloneqq \{y \in X : d(x, y) < r\}.$$

We say a set $U \subseteq X$ is open if and only if for all $x \in U$, there exists an r such that $B(x; r) \subseteq U$. We say a set $F \subseteq X$ is closed if and only if $X \setminus F$ is open.

This last definition seems a bit odd and to me feels not very analysis-ee, so thankfully we have another sequence based definition coming up in this question right now.

Question 1.11. TRUE OR FALSE? A set F is close if whenever $(x_n) \subseteq F$ such that $x_n \to x$ in X, then $x \in F$.

Answer: TRUE. **FINISH: prove this for practice**

Question 1.12. Which of the following sets is not relatively open in $(0, 2]$?

(a) $(0, 1)$

(b) $(1, 2]$

(c) $[0, 1]$

(d) $(0, 2]$

Answer: (c) is relatively closed, the rest are relatively open.

Definition 1.13. Let (X, d_X) and (Y, d_Y) be two metric spaces. A function $f : X \to Y$ is *continuous at a point* $x \in X$ if and only if $d_Y(f(x), f(y)) \to 0$ as $y \to x$.

We say that f is *continuous on* X if and only if it is continuous at every $x \in X$.

Discussion 1—March 30, 2021

Example 2.1. Given a metric space (X, d), $x_0 \in X$, $r > 0$, let $B \coloneqq \{x \in X : d(x, x_0) < r\}$, $C \coloneqq \{x \in X : d(x, x_0) \le r\}$. Show that

(a) $\overline{B} \subseteq C$, and that

(b) the inclusion can be strict.

Proof. (a) Recall that for a set E in a metric space X, $\overline{E} \coloneqq$ the set of all points that are limits of sequences in E. Fix $x \in \overline{B}$. By definition of closure, there is a sequence $(x_n) \subseteq B$ such that $\lim x_n = x$. Notice that because (x_n) is a sequence in B,

$$d(x_n, x_0) < r \tag{1}$$

for all n.

By definition of limit, for all $\varepsilon > 0$, there exists an N such that $n \ge N$ implies $d(x_n, x) < \varepsilon$. Using this fact and (1), we have that

$$d(x_0, x) \le d(x_0, x_n) + d(x_n, x) < r + \varepsilon$$

by the triangle inequality. Thus, $d(x_0, x) \le r$ and $x \in C$. Therefore, $\overline{B} \subseteq C$.

(b) Take $\mathbb{R}$ with the discrete metric and concsider $B(0; 1)$. This only contains 0 and is closed, so its closure is itself, while the closed ball is all of $\mathbb{R}$.

■

Example 2.2. Suppose (x_n) is a Cauchy sequence in (X, d) and has a subsequence (x_{n_j}) that converges to $x \in X$. Show $(x_n) \to x$ in X.

Proof. Fix $\varepsilon > 0$. By definition, since (x_n) is Cauchy, there exists an N such that $n, m \ge N$ implies that

$$d(x_n, x_m) < \frac{\varepsilon}{2}. \tag{1}$$

Then, since (x_{n_j}) converges, there exists an M such that $j \ge M$ implies that

$$d(x_{n_j}, x) < \frac{\varepsilon}{2}. \tag{2}$$

Combining (1) and (2) and taking $j \ge \max\{N, M\}$,

$$d(x_j, x) \le d(x_j, x_{n_j}) + d(x_{n_j}, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This completes the proof. ■

Example 2.3. (a) Let $(Y, d|_{Y\times Y})$ be a subspace of a metric space (X, d). Show that $(Y, d|_{Y\times Y})$ completes implies that Y is closed in X.

(b) Suppose (X, d) is complete and $Y \subseteq X$ is closed. Then $(Y, d|_{Y\times Y})$ is complete.

Proof. (a) Suppose (x_n) is a sequence in Y and suppose $(x_n) \to x$ in X.TO show Y is closed, we must show that $x \in Y$. Since (x_n) converges, (x_n) is Cauchy. Since Y is complete with respect to its metric, there exists an $x' \in Y$ such that $(x_n) \to x'$ in Y. Thus, $(x_n) \to x'$ in X and $x' = x$. Thus Y is closed.

■

Example 2.4. Prove that the following are equivalent for $f\colon (X, d_X) \to (Y, d_Y)$:

(1) f is continuous (in the ε-δ sense)

(2) $(x_n) \to x_0$ implies that $(f(x_n)) \to f(x_0)$

(3) for all open $V \subseteq Y$ containing $f(x_0)$, there exists open $U \subseteq X$ containing x_0 such that $f(U) \subseteq V$.

Proof. ((1) $\implies$ (2)) Fix $\varepsilon > 0$. By (1), there exists a $\delta > 0$ such that $d(x, x_0) < \delta$ implies that $d(f(x), f(x_0)) < \varepsilon$. Assume $(x_n) \to x_0$. Then there exists an N such that $d(x_n, x_0) < \delta$ whenever $n \geq N$. Thus, $n \geq N$ implies that $d(f(x_n), f(x_0)) < \varepsilon$. Thus, $\lim f(x_n) = f(x_0)$. ■

Lecture 2—March 31, 2021

Definition 3.1. Let X be a vector space over the reals. A *norm* is a map $\|\cdot\|: X \to [0, \infty)$ such that for all $\lambda \in \mathbb{R}$ and $x, y, z \in X$, then

(1) $\|\lambda x\| = |\lambda| \|x\|$

(2) $\|x\| = 0$ if and only if $x = 0$

(3) $\|x + y\| \leq \|x\| + \|y\|$.

A *normed vector space* is a vector space equipped with a norm.

Lemma 3.2. *If X is a normed vecotr space, it is a metric space with*

$$d(x, y) = \|x - y\|.$$

Example 3.3. If $1 \leq p < \infty$ and $x \in \mathbb{R}^n$, define

$$\|x\|_p := \left(\sum_{j=1}^{n} |x_j|^p \right)^{\frac{1}{p}}.$$

If $p = \infty$ and $x \in \mathbb{R}^n$, define

$$\|x\|_\infty := \sup_{1 \leq j \leq n} |x_j|.$$

If $p = 2$, we get the Euclidean norm.

Lemma 3.4 (Holder's Inequality)**.** *Let $1 \leq p, q \leq \infty$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$ (where $\frac{1}{\infty} = 0$), then if $x, y \in \mathbb{R}^n$*

$$\sum_{j=1}^{n} |x_j||y_j| \leq \|x\|_p \|y\|_q.$$

Lemma 3.5 (Minkowski's Inequality)**.** *If $1 \leq p \leq \infty$ and $x, y \in \mathbb{R}^n$,*

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p.$$

(Look these up in Copson's book. Learn about these inequalities and their proofs.)

Question 3.6. **TRUE OR FALSE?** $(\mathbb{R}^n, \|\cdot\|_p)$ is complete.
Answer: TRUE. All norms on $\mathbb{R}^n$ are equivalent, so it's equivalent to the Euclidean norm which is complete.

Definition 3.7. A complete normed vector space is called a *Banach space*.

Example 3.8. For $p = \infty$, we define ℓ^∞ to be the set of bounded sequences, with norm

$$\|(x_n)\|_\infty = \sup_{n \geq 1} |x_n|.$$

For $1 \leq p < \infty$, we define ℓ^p to be the set of sequences for which

$$\sum_{n=1}^{\infty} |x_n|^p < \infty$$

with norm

$$\|(x_n)\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}}.$$

We can prove the triangle inequality for this stuff using Holder's/Minkowski's Inequality.

Theorem 3.9. *For $1 \leq p \leq \infty$, $(\ell^p, \|\cdot\|_p)$ is a Banach space.*

Proof. Note that we will use function notation for this proof to make the proof easier to read.

Suppose $1 \leq p < \infty$. (We do not handle the $p = \infty$ case as it is very similar.) Moreover, suppose that (f_n) is a Cauchy sequence in ℓ^p. Taking $k \geq 1$, we have that

$$|f_n(k) - f_m(k)| \leq \left(\sum_{j=1}^{\infty} |f_n(j) - f_m(j)|^p\right)^{\frac{1}{p}} = \|f_n - f_m\|_p.$$

Thus, since (f_n) is Cauchy (so we can make $\|f_n - f_m\|$ for large enough n and m), $(f_n(k))_{n\geq 1}$ is Cauchy for each $k \geq 1$. Now, since $\mathbb{R}$ is complete, $(f_n(k))$ must then converge. Call its limit $f(k)$ where, in general, $f(k) := \lim_{n\to\infty} f_n(k)$.

Fix $\varepsilon > 0$. Then there exists an $N \geq 1$ such that for all $n, m \geq N$,

$$\|f_n - f_m\|_p < \frac{\varepsilon}{10}.$$

Then, clearly, for all $J \geq 1$,

$$\left(\sum_{j=1}^{J} |f_n(j) - f_m(j)|^p\right)^{\frac{1}{p}} < \frac{\varepsilon}{10}.$$

Now taking $m \to \infty$ (which we can do since $|\cdot|$ is continuous),

$$\left(\sum_{j=1}^{J} |f_n(j) - f(j)|^p\right)^{\frac{1}{p}} \leq \frac{\varepsilon}{10}$$

Since this statement above holds for all $J \geq 1$, we may take $J \to \infty$ to get

$$\|f_n - f\|_p \leq \frac{\varepsilon}{10}.$$

We then have by the triangle inequality that

$$\|f\|_p \leq \|f_n - f\|_p + \|f_n\|_p < \infty$$

(so that $f \in \ell^p$) AND $(f_n) \to f$ in ℓ^p. ■

Theorem 3.10. *Let (X, d) be a metric space. The space $C_b(X)$ (b for bounded) of bounded continous functions from $X \to \mathbb{R}$ endowed with the norm*

$$\|f\| := \sup_{x\in X} |f(x)|$$

is a Banach space.

Remark. The supremum exists since the function is bounded.

Proof. Can easily check $\|\cdot\|$ is a norm. Remains to prove completeness.

Let (f_n) be a Cauchy sequence in $C_b(X)$ (with respect to our aforementioned norm). If $x \in X$,then $(f_n(x))$ is a Cauchy sequence in $\mathbb{R}$ (since we're basically evaluating at x), so it converges to some limit $f(x)$.

Given $\varepsilon > 0$, choose $N \geq 1$ such that

$$\|f_n - f_m\| < \frac{\varepsilon}{6}$$

for $n, m \geq N$. Then, sending $m \to \infty$,

$$\|f_n(x) - f_m\| \leq \|f_n - f_m\| < \frac{\varepsilon}{6}$$

so $\|f_n(x) - f(x)\| \leq \frac{\varepsilon}{6}$.

Then

$$|f(x)| \leq \frac{\varepsilon}{6} + |f_n(x)| \leq \frac{\varepsilon}{6} + \|f_n\| < \infty$$

uniformly for $x \in X$. So f is bounded. Also,

$$\|f_n - f\| \leq \frac{\varepsilon}{6}.$$

So $(f_n) \to f$ uniformly on X. Then f is continuous, so $(f_n) \to f$ in $C_b(X)$. ■

Definition 3.11. A subset K of a metric space (X, d) is said to be *compact* if every open cover of K has a finite subcover.

Definition 3.12. A subset K of a metric space (X, d) is said to be *sequentially compact* if every sequence (x_n) in K has a convergent subsequence, with limit lying in K.

Lemma 3.13. *In a metric space, K is compact if and only if K is sequentially compact.*

Question 3.14. TRUE OR FALSE? In a metric space, a set is compact if and only if it is closed and bounded.

Answer: FALSE. Compact implies closed and bounded, yes, but the converse is not true in general, but it is true in $\mathbb{R}^n$: this is known as the Heine-Borel Theorem.

Question 3.15. TRUE OR FALSE? If (X, d) is compact, then $C(X) = C_b(X)$.

Answer: TRUE. Certainly, $C_b(X) \subseteq C(X)$. If $f \in C(X)$, then $f(X) \subseteq \mathbb{R}$ is compact, so by our previous statement it must be bounded. Thus, $f \in C_b(X)$.

Definition 3.16. An *algebra* is a vector space $\mathcal{A}$ over $\mathbb{R}$ with an operator $\times\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ such that for any $x, y, z \in \mathcal{A}$ and $\lambda \in \mathbb{R}$,

(i) $x \times (y \times z) = (x \times y) \times z$

(ii) $(x + y) \times z = x \times z + y \times z$

(iii) $x \times (y + z) = x \times y + x \times z$

(iv) $\lambda(x \times y) = (\lambda x) \times y = x \times (\lambda y)$.

Example 3.17. $C(x)$ and $C_b(X)$ are algebras. (Multiplication is just pointwise multiplication.)

Definition 3.18. A set $S \subseteq C(X)$ is said to *separate the points of* X if for all $x \neq y$, there exists some function $f \in S$ such that $f(x) \neq f(y)$.

Theorem 3.19 (The Stone-Weierstrass Theorem)**.** *Let (X, d) be a compact metric space. Let $\mathcal{A}$ be a close subalgebra that separates the points of x and contains the constant functions. Then $\mathcal{A} = C(X)$.*