Math 131C Notes

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This set of notes is very informal and tries its best to simplify often hard to digest ideas. Hopefully it's useful in your learning of the material.

FINISH: put figures in right places

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We're starting by reviewing metric spaces! It's kind of old now (third time I'm writing this for notes lmao) but it's really cool nonetheless. :)

Definition 1.1. A *metric space* is a nonempty set X and a function $d\colon X \times X \to [0, \infty)$ such that for all $x, y, z \in X$,

(i) $d(x, y) = d(y, x)$,

(ii) $d(x, y) = 0$ if and only if $x = y$,

(iii) and $d(x, z) \leq d(x, y) + d(y, z)$.

(This function is called a *metric*.)

My professor does these in class questions to make sure we're following along to check our understanding it seems, so I've tried ot make them pretty.

Question 1.2. Which of the following is <u>not</u> a metric on $\mathbb{R}^2$?

(a) $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$

(b) $d(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$

(c)
$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise} \end{cases}$$

(d) $d(x, y) = \min\{|x_1 - y_1|, |x_2 - y_2|\}$

Answer: (d) as (ii) from Definition 1.1 fails.

Example 1.3. $X \coloneqq \mathbb{R}^2$ with Euclidean metric

$$d(x, y) = \sqrt{\sum_{j=1}^{n} |x_j - y_j|^2}.$$

Definition 1.4. If (X, d) is a metric space and $Y \subseteq X$ is non-empty, then the metric space $(Y, d|_{Y \times Y})$ is called a *subspace* of (X, d).

Definition 1.5. We say that a sequence (x_n) in a metric space (X, d) *converges* if and only if there exists an $x \in X$ such that $d(x, x_n) \to 0$ as $n \to \infty$.

Question 1.6. TRUE OR FALSE?

If $x_n \to x$, then every subsequence $x_{n_k} \to x$.

Answer: TRUE. As $n_k \geq k$: for all $\varepsilon > 0$, there exists an integer N such that for all $n \geq N$, $d(x, x_n) < \varepsilon$ (by definition of convergence), so for all $k \geq N$, $d(x, x_{n_k}) < \varepsilon$.

Definition 1.7. We say a sequence (x_n) is *Cauchy* if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

Question 1.8. TRUE OR FALSE?

Every Cauchy sequence converges.

Answer: FALSE. Take $(0, 1]$ equipped with the Euclidean metric. Consider the sequence $x_n = \frac{1}{n}$. This is Cauchy but does not converge in our metric space (since 0 is not included).

Definition 1.9. We say a metric space is *complete* if and only if every Cauchy sequence converges.

Definition 1.10. Let (X, d) be a metric space. Denote

$$B(x; r) \coloneqq \{y \in X : d(x, y) < r\}.$$

We say a set $U \subseteq X$ is open if and only if for all $x \in U$, there exists an r such that $B(x; r) \subseteq U$.
We say a set $F \subseteq X$ is closed if and only if $X \setminus F$ is open.

This last definition seems a bit odd and to me feels not very analysis-ee, so thankfully we have another sequence based definition coming up in this question right now.

Question 1.11. TRUE OR FALSE?

A set F is close if whenever $(x_n) \subseteq F$ such that $x_n \to x$ in X, then $x \in F$.

Answer: TRUE. **FINISH: prove this for practice**

Question 1.12. Which of the following sets is not relatively open in $(0, 2]$?

(a) $(0, 1)$

(b) $(1, 2]$

(c) $[0, 1]$

(d) $(0, 2]$

Answer: (c) is relatively closed, the rest are relatively open.

Definition 1.13. Let (X, d_X) and (Y, d_Y) be two metric spaces. A function $f \colon X \to Y$ is *continuous at a point* $x \in X$ if and only if $d_Y(f(x), f(y)) \to 0$ as $y \to x$.

We say that f is *continuous on* X if and only if it is continuous at every $x \in X$.