Math 191 Notes

Mealaud Mokhtarzad

This set of notes is very informal and tries its best to simplify often hard to digest ideas. Hopefully it's useful in your learning of the material.

FINISH: put figures in right places

Contents

Lecture 1—March 29, 2021

Categories

History: Eilenberg and MacLane needed to make sense of "naturality." Here are a bunch of examples of natural maps!

Example 1.1. (a) Let X be a set and let $\mathcal{P}(X)$ be the powerset of X. Then there is a natural function from $X \to \mathcal{P}(X)$ defined by $x \mapsto \{x\}$.

(b) For any sets X and Y, let $Y^X := \{f : X \to Y\}$. Then there exists a natural bijection from $\mathcal{P}(X)$ to $\{0,1\}^X$ by $A \mapsto \chi_A$ where χ_A is the characteristic function of A.

(c) For any sets X and Y, $X \times Y := \{(x, y) : x \in X \wedge y \in Y\}$. Then there is a natural bijection from $X \times Y$ to $Y \times X$: $(x, y) \mapsto (y, x)$.

How do we make this "naturality" precise, however? Well, there are a few problems that Eilenberg and MacLane had to face. Here they are:

- We're talking about "natural maps"—they need domains and codomains (these are called functors)
  - Functor is a "construction"
  - Functors also have inputs and outputs $\rightsquigarrow$ some kind of mapping, so they need domain and codomain, also.
  - The domain and codomain of functors are *categories*.

Now let's define what a category is. Spoiler: it's actually really long lmao.

Definition 1.2. A *category* $\mathcal{C}$ consists of

(1) a collection of objects $A, B, C, \ldots,$

(2) and a collection of morphisms (arrows) $f, g, h, \ldots$

such that

(i) each morphism has a domain and a codomain object. We write $f : A \to B$ as a shorthand for "f is a morphism with domain A and codomain B," and we write $\mathcal{C}(A, B)$ for the collection of all morphisms $f : A \to B$.

(ii) Each object A has an idntity morphism $1_A : A \to A$.

(iii) For any pair of "composable morphisms" g and f with $\mathrm{Dom}(g) = \mathrm{Cod}(f)$, there is a composite morphism $g \circ f$ with $\mathrm{Dom}(g \circ f) = \mathrm{Dom}(f)$ and $\mathrm{Cod}(g \circ f) = \mathrm{Cod}(g)$. This is exemplified in the following diagram:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

These data are subject to two axioms:

(C1) (associativity) for any $f : A \to B$, $g : B \to C$, and $h : C \to D$, $(h \circ g) \circ f = h \circ (g \circ f)$; and

(C2) (unitality) for any $f : A \to B$, $f \circ 1_A = f = 1_B \circ f$.

Finally that definition is done. It's the longest I've ever seen (so far). Now to some examples of categories.

Example 1.3. (a) The category of sets consists of all sets and all functions between sets. We write Set.

(b) A *pointed set* is a pair (X, x) where $x \in X$ is a distinguished element. A morphism $f\colon (X, x) \to (Y, y)$ is a function $f\colon X \to Y$ such that $f(x) = y$. We denote this Set_*.

(c) A *monoid* is a triple $(M, \cdot, e)$ such that

(i) M is a set,

(ii) $\cdot\colon M \times M \to M$ is a binary operation, and

(iii) $e \in M$ is a distinguished element

such that

(M1) (associativity) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in M$ and

(M2) (unitality) $e \cdot x = x = x \cdot e$ for all $x \in M$.

A *monoid homomorphism* $f\colon (M, \cdot_M, e_M) \to (N, \cdot_N, e_N)$ is a function $f\colon M \to N$ such that

(i) $f(e_M) = e_N$ and

(ii) $f(x \cdot_M y) = f(x) \cdot_N f(y)$.

These data assemble into the category Mon.

(d) A group is a quadruple $(G, \cdot, e, (-)^{-1})$ such that

(i) G is a set,

(ii) $\cdot\colon G \times G \to G$ is a binary operation,

(iii) e is a distinguished element, and

(iv) $(-)^{-1}\colon G \to G$ is a unary operation

such that

(G1) $(G, \cdot, e)$ is a monoid and

(G2) $x^{-1} \cdot x = e = x \cdot x^{-1}$ for all $x \in G$.

A *group homomorphism* $f\colon (G, \cdot_G, e_G, (-)^{-1}_G) \to (H, \cdot_H, e_H, (-)^{-1}_H)$ is a function $f\colon G \to H$ such that

(i) $f(xy) = f(x)f(y)$ for all $x, y \in G$,

(ii) $f(e_G) = e_H$, and

(iii) $f(x^{-1}) = f(x)^{-1}$ for all $x \in G$.

We denote this category Grp.

(e) A *preorder* is a pair $(P, \leq)$ such that P is a set and $\leq$ is a binary relation on P such that

(P1) (reflexivity) $x \leq x$ for all $x \in P$ and

(P2) (transivitiy) $x \leq y$ and $y \leq z$ implies $x \leq z$ for all $x, y, z \in P$.

A morphism of preorders $f\colon P \to Q$ is a function such that $x \leq y$ implies $f(x) \leq f(y)$ (*order-preserving functions*).

We denote this category Preord.

Definition 1.4. Let $\mathcal{C}$ be a category. A *subcategory* $\mathcal{D}$ of $\mathcal{C}$ consists of a collection of objects of $\mathcal{C}$ and a collection of morphisms of $\mathcal{C}$ such that

(1) (closed under domain/codomain) if $f\colon A \to B$ is in $\mathcal{D}$, then so are A and B;

(2) (closed under composition) if $f\colon A \to B$, $g\colon B \to C$ are in $\mathcal{D}$, then so is $g \circ f$; and

(3) (contains identities) if A is an object of $\mathcal{D}$, then so is 1_A.

Now to examples, again.

Example 1.5. (a) The collection of all finite sets and all the maps between them is a subcategory of Set.

We denote this category FinSet.

(b) A *commutative monoid* is a monoid $(M, \cdot, e)$ such that $x \cdot y = y \cdot x$ for all $x, y \in M$. The collection of all commutative monoids and monoid homomorphisms between tehm for a subcategory of Mon.

We denote this category CMon.

(c) An *abelian group* is a group $(G, \ldots)$ such that $\cdot$ is commutative. This is a subcategory of Grp.

We denote this category Ab.

(d) A *poset* $(P, \leq)$ is a preorder $(P, \leq)$ such that $\leq$ is antisymmetric, i.e. $x \leq y$ and $y \leq x$ implies that $x = y$. The collection of all posets and order-preserving maps between them form a subcategory of Preord.

We denote this category Pos.

Definition 1.6. A subcategory $\mathcal{D}$ of $\mathcal{C}$ is *full* if and only if for any objects A and B of $\mathcal{D}$, every morphism $f\colon A \to B$ in $\mathcal{C}$ is also in $\mathcal{D}$.