1 (Aside:) Completeness

1.1 Basics

In the very first section (in the first paragraph, in fact!), we said the rationals are 'incomplete.' We will now explain exactly what that meant.

Definition 1.1 (Completeness). A metric space (X, d) is complete if and only if every Cauchy sequence in (X, d) converges in (X, d).

Do we have an example of a complete metric space? Indeed, we do!

Example 1.2. $\mathbb{R}$ with the Euclidean metric is complete.

Proof. This is Theorem 1.26. ■

In fact, we have many more examples of complete metric spaces! Say ℓ_2 or $C([0,1])$ (with the sup-norm metric), or even $\mathbb{R}^n$ (with the Euclidean metric) for any finite n! (We'll prove ℓ_2, $C([0,1])$, and $\mathbb{R}^2$ is complete in the upcoming section.) Now to some examples.

1.2 Examples of cool complete metric spaces (with proof)

Fact 1.3. $\mathbb{R}^2$ *with the Euclidean metric is complete.*

Proof. Let $\epsilon > 0$. Suppose $(\mathbf{a}_n)$ is a Cauchy sequence. We may rewrite $(\mathbf{a}_n)$ as $((x_n, y_n))$. First we will show that both (x_n) and (y_n) are Cauchy in $\mathbb{R}$.

Because $(\mathbf{a}_n)$ is Cauchy, there exists an integer N such that $d(\mathbf{a}_i, \mathbf{a}_j) = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} < \epsilon$ if $i, j \geq N$. It follows that

$$|x_i - x_j| = \sqrt{(x_i - x_j)^2} \leq \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} < \epsilon$$
$$|y_i - y_j| = \sqrt{(y_i - y_j)^2} \leq \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} < \epsilon$$

if $i, j \geq N$. Thus (x_n) and (y_n) are Cauchy in $\mathbb{R}$ (with the Euclidean metric), so they have limits x and y, respectively. We claim that $\lim \mathbf{a}_n = (x, y)$.

Because $\lim x_n = x$, there exists an integer N_1 such that $n \geq N_1$ implies that $|x_n - x| < \epsilon/\sqrt{2}$. Similarly, there exists an N_2 such that $|y_n - y| < \epsilon/\sqrt{2}$ if $n \geq N_2$. Taking $n \geq \max\{N_1, N_2\}$, we get that

$$\begin{aligned} d(\mathbf{a}_n, \mathbf{a}) &= \sqrt{(x_n - x)^2 + (y_n - y)^2} \\ &= \sqrt{|x_n - x|^2 + |y_n - y|^2} \\ &< \sqrt{\left(\frac{\epsilon}{\sqrt{2}}\right)^2 + \left(\frac{\epsilon}{\sqrt{2}}\right)^2} \\ &= \sqrt{\frac{\epsilon^2}{2} + \frac{\epsilon^2}{2}} \\ &= \sqrt{\epsilon^2} \\ &= \epsilon. \end{aligned}$$

Hence $\lim \mathbf{a}_n = (x, y)$ and $\mathbb{R}^2$ (with the Euclidean metric is complete.) ■

Corollary 1.4. $\mathbb{R}^n$ *with the Euclidean metric is complete.*

Proof. Mimic the proof for the completeness of $\mathbb{R}^2$. ■

Fact 1.5. $\mathbb{R}^2$ *with the Taxicab metric is complete.*

Proof. Mimic the proof of the completeness of $\mathbb{R}^2$. ■

Fact 1.6. ℓ_2 *with the metric* $d\colon \ell_2 \times \ell_2 \to \mathbb{R}$ *defined by*

$$d((x_n),(y_n)) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^2\right)^{\frac{1}{2}}$$

is complete.

Proof. FINISH ■

Fact 1.7. $C([0,1])$ *with the metric* $d\colon C([0,1]) \times C([0,1]) \to \mathbb{R}$ *defined by*

$$d(f,g) = \sup\{|f(x) - g(x)| : x \in [0,1]\}$$

is complete.

Proof. Let $\epsilon > 0$ be arbitrary. Let (f_n) be a Cauchy sequence. Then there exists an integer N such that $d(f_i, f_j) < \epsilon/2$ if $i, j \geq N$. By the definition d and supremum, we know that $|f_i(x) - f_j(x)| < \epsilon/2 < \epsilon$ and $(f_n(x))$ is a Cauchy sequence in $\mathbb{R}$ with the Euclidean metric for all $x \in [0,1]$.

Taking $i, j \geq N$ again and then taking $j \to \infty$, we get

$$\sup_{y \in [0,1]} |f_i(y) - f_j(y)| < \frac{\epsilon}{2} \to \sup_{y \in [0,1]} |f_i(y) - f_0(y)| \leq \frac{\epsilon}{2} < \epsilon.$$

Thus, $d(f_i, f_0) < \epsilon$ if $i \geq N$ and $\lim f_n = f_0$.

We must still show that $f_0 \in C([0,1])$ (i.e. we must show that f_0 is continuous). Let $c \in [0,1]$. Since $\lim f_n = f_0$, there exists an integer N' such that for $|f_n(x) - f_0(x)| \leq d(f_n, f_0) < \epsilon/3$ if $n \geq N'$ for all $x \in [0,1]$. Recall also that because $f_{N'}$ is continuous, there exists a $\delta > 0$ such that $|x - c| < \delta$ implies that $|f_{N'} - f_{N'}(c)| < \epsilon/3$. Now taking $|x - c| < \delta$, it follows that

$$|f_0(x) - f_0(c)| \leq |f_0(x) - f_{N'}(x)| + |f_{N'}(x) - f_{N'}(c)| + |f_{N'}(c) - f_0(c)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Thus f_0 is continuous.

This completes the proof. ■

Remark. The final inequality in the proof is called the "Three Term Estimate." It's very important stuff. Don't forget it! We might make a section/bit on it. idk

1.3 Complete subspaces

When is the subset of a metric space complete? When it's closed!

Fact 1.8. *Let* (X, d) *be a complete metric space and suppose* $Y \subseteq X$*. If* Y *is closed in* X*, then* $(Y, d|_{Y \times Y})$ *is complete.*

Proof. Suppose (x_n) is a Cauchy sequence in Y. Then (x_n) converges to x in X. Y contains all its limit points because it is closed, so $x \in Y$. Thus Y is complete. ■

Now back to topology!