Math 114C Notes

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This set of notes is very informal and tries its best to simplify often hard to digest ideas. Hopefully it's useful in your learning of the material.

FINISH: put figures in right places

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1 Models of computation

test

2 Effective enumerations and undecidability

2.1 Recursive partial functions

In the previous section we talked about different models of computation (hence its title) and generally different ways to think about computability in terms of, well, computing (with computers). Now, we come across a new term: recursive. What recursive means is that the more pure-mathematical way to define computability without machines necessarily. Now let's get into our definition.

Definition 2.1. We define the `class` $\mathcal{R}$ `of recursive partial functions` to be the $\subseteq$-smallest collection of partial functions $f\colon \mathbb{N}^n \rightharpoonup \mathbb{N}$ such that

(1) $\mathcal{R}$ contains the zero function $\underline{0}$, the successor function, and the projection functions U^n_k.

(2) $\mathcal{R}$ is closed under definition by substitution, definition by recursion, and definition by minimalization.

We're also going to give a name to the class of functions we were working with before: the URM computable functions.

Definition 2.2. We define the `class` $\mathcal{C}$ `of URM computable partial functions` to be the class of partial functions $g\colon \mathbb{N}^n \rightharpoonup \mathbb{N}$ such that there exists a URM program P with $g = f_P^{(n)}$.

Our first big result of these notes indeed relating $\mathcal{R}$ and $\mathcal{C}$. In particular,

Theorem 2.3. *A partial function $f\colon \mathbb{N}^n \to \mathbb{N}$ is recursive if and only if it is URM-computable (i.e., $\mathcal{R} = \mathcal{C}$).*

We actually have one direction of this already (the proof is very short and follows mostly by definition).

Lemma 2.4. $\mathcal{R} \subseteq \mathcal{C}$.

Proof. Since $\mathcal{C}$ is closed under the same constraints that define $\mathcal{R}$, since $\mathcal{R}$ is the smallest such set, it is necessarily true that $\mathcal{R} \subseteq \mathcal{C}$. ■

We will prove the other direction in due time.

3 Computably enumerable sets

3.1 Partially decidable relations

In this section, we're going to use the Kleene T predicate a lot, so we're going to restate it here.

Theorem 3.1 (Kleene T predicate)**.** *For each positive integer n, there is a recursive $(n+2)$-ary relation $T_n(e, \mathbf{x}, y)$ and a computable totla function $U\colon \mathbb{N} \to \mathbb{N}$ which satisfy the following condition: for every $e \in \mathbb{N}$ and every $\mathbf{x} \in \mathbb{N}^n$,*

$$\varphi_e^{(n)}(\mathbf{x}) = U(\mu y[T_n(e, \mathbf{x}, y)]).$$

Remark. Let recall what the Kleene T predicate does: it takes a program code e, an input for the program $\mathbf{x}$, and a computation history y, and decides whether y is a computation history for the program defined by e with input $\mathbf{x}$ that halts. It basically simulates a register machine which runs a program with code e and input $\mathbf{x}$. Recall that the Kleene T predicate is indeed decidable.

Definition 3.2. An n-ary relation $R \subseteq \mathbb{N}^n$ is `partially decidable` if and only if the partial function $f\colon \mathbb{N}^n \rightharpoonup \mathbb{N}$ defined by

$$f(\mathbf{x}) = \begin{cases} 1 & \text{if } R(\mathbf{x}) \text{ holds,} \\ \uparrow & \text{otherwise,} \end{cases}$$

is computable.

The function f above is called the `partial characteristic function` of R (which is often denoted χ_R^p) and any algorithm computing f is called a `partial decision procedure` for R.

Remark. Partially decidable has other names too; semi-decidable, semi-computable, and partially solvable are other common names. We will try to use partially decidable mainly though. I don't like the other names lmao.

Do we have any examples of partially decidable relations? We do! In fact, we have many, many such relations. Here's a canonical one, though.

Example 3.3. The (unary diagonal) halting problem (i.e., whether $\varphi_x(x)$ halts) is partially decidable.

Proof. We can define its partial characteristic function $f\colon \mathbb{N} \rightharpoonup \mathbb{N}$ as the following:

$$f(x) = 1 + 0 \cdot \mu y[T(x, x, y)].$$

Observe that f is computable since it is the minimalization of a decidable relation. Then, notice that if $\varphi_x(x)\downarrow$, then there does exist a halting computation history y and our minimalization will eventually return a value, so $f(x) = 1$ if $\varphi_x(x)\downarrow$. If $\varphi_x(x)\uparrow$, then there does not exist a halting computation history y and our minimalization never ends, so $f(x)\uparrow$. ■

Remark. This trick of multiplying a minimalization by 0 to encode divergence of a computation is SUPER useful. Be aware of its existence!

Example 3.4. Any decidable relation is partially decidable.

Proof. Suppose $R \subseteq \mathbb{N}^n$ is decidable. Then, $f\colon \mathbb{N}^n \to \mathbb{N}$ defined by

$$f(\mathbf{x}) = 1 + 0 \cdot \mu y[R(\mathbf{x}) \wedge y = y].$$

If $R(\mathbf{x})$ holds, then the minimalization ends and we get $f(\mathbf{x}) = 1$ as desired. IF $R(\mathbf{x})$ does not hold, then our minimalization never ends and the computation diverges, so $f(\mathbf{x})\uparrow$. ■

Remark. I told you the multiplying by 0 trick would be useful!

We know that there exist decidable relations, but do there exist not partially decidable relations? Yes, there do! We actually have another 'simple' example: the negation of the (unary diagonal) Halting Problem.

Example 3.5. Let $Q(x)$ hold if and only if $\varphi_x(x)\uparrow$. The relation Q is not partially decidable.

Proof. Suppose towards a contradiction that Q is partially decidable. In particular, suppose that the the partial characteristic function of Q, $f_Q\colon \mathbb{N} \rightharpoonup \mathbb{N}$, is computable. Morever, let $f_H\colon \mathbb{N} \rightharpoonup \mathbb{N}$ be the (computable) partial characteristic function of the (unary diagonal) Halting Problem. Now, by running $f_Q(x)$ and $f_H(x)$ at the same time, we may decide whether $\varphi_x(x)$ halts (as we will be able to decide if it does or if it does not based on the ouput of $f_H(x)$ and $f_Q(x)$, respectively). Since the Halting Problem is undecidable we have a contradiction and thus Q must not have been partially decidable. ■

The next two theorems will provide some neat equivalent characterizations of partially decidable relations.

Theorem 3.6. *An n-nary relation $R \subseteq \mathbb{N}^n$ is partially decidable if and only if R is the domain of a computable partial function, i.e., there is a computable partial function $g\colon \mathbb{N}^n \rightharpoonup \mathbb{N}$ such that for all $\mathbf{x} \in \mathbb{N}^n$,*

$$R(\mathbf{x}) \iff g(\mathbf{x})\downarrow.$$

Proof. $(\Rightarrow)$ Suppose R is partially decidable. Then the domain of R's partial characteristic function clearly equals R.

$(\Leftarrow)$ Suppose $R = \mathrm{Dom}(\varphi_e^{(n)})$. Then the partial characteristic function of R, $\chi_R^p(\mathbf{x}) = 1 + 0 \cdot \mu y[T_n(e, \mathbf{x}, y)]$ is computable. ■

Remark. It follows that $W_0^{(n)}, W_1^{(n)}, W_2^{(n)}, \ldots$ is an (effective) enumeration of all partially deicdable n-ary relations.

Theorem 3.7. *An n-ary relation $R \subseteq \mathbb{N}^n$ is partially decidable if and only if there is a $(n+1)$-ary decidable relation $Q \subseteq \mathbb{N}^{n+1}$ such that for any $\mathbf{x} \in \mathbb{N}^n$,*

$$R(\mathbf{x}) \iff \exists y[Q(\mathbf{x}, y)].$$

Proof. $(\Rightarrow)$ Suppose R is partially decidable. Then $R = W_e^{(n)}$ for some code $e \in \mathbb{N}$. We then have that

$$R(\mathbf{x}) = \varphi_e^{(n)}(\mathbf{x})\downarrow \iff \exists y[T_n(e, \mathbf{x}, y)]$$

where T_n is the Kleene T predicate. (To be more explicit, deifne $Q(\mathbf{x}, y)$ to hold if and only if $T_n(e, \mathbf{x}, y)$.)

$(\Leftarrow)$ Suppose R is a rleation satisfying

$$R(\mathbf{x}) \iff \exists y[Q(\mathbf{x}, y)]$$

for a decidable $(n+1)$-ary relation $Q \subseteq \mathbb{N}^{n+1}$. Define $f(\mathbf{x}) = \mu y[Q(\mathbf{x}, y)]$. Then f is computable and $R = \mathrm{Dom}(f)$, so R is partially decidable. ■

The next theorem gives closure properties of the class of partially decidable relations.

Theorem 3.8. *The class of partially decidable relations is closed under $\wedge$, $\vee$, $\exists^{<}$, $\forall^{<}$, and $\exists$. It is not closed under negation $\neg$. More formally:*

(a) If R is partially decidable n-ary relations, then the relations $(R \wedge Q)$ and $(R \vee Q)$ are also partially decidable n-ary relations.

(b) If R is a partially decidable $(n+1)$-ary relation, then the relations

$$\begin{aligned}(\exists^{<}R)(\mathbf{x},y) &\iff_{\text{df}} (\exists z<y)[R(\mathbf{x},z)],\\ (\forall^{<}R)(\mathbf{x},y) &\iff_{\text{df}} (\forall z<y)[R(\mathbf{x},z)],\\ (\exists R)(\mathbf{x}) &\iff_{\text{df}} \exists y[R(\mathbf{x},y)],\end{aligned}$$

are also partially decidable.

(c) There are partially decidable relations R such that $\neg R$ is not partially decidable.

Proof. This is annoying. Maybe later lmao. ■

Corollary 3.9. *If R is an $(n+k)$-ary partially decidable relation, then the n-ary relation*

$$Q(\mathbf{x}) \iff_{\text{df}} (\exists y_1)\cdots(\exists y_k)[R(\mathbf{x},y_1,\ldots,y_k)]$$

is also partially decidable.

Remark. We can rewrite our bunch of existentials as a single number being projected many times, like $Q(\mathbf{x}) \iff (\exists a)[R(\mathbf{x},(a)_0,\ldots,(a)_{k-1})]$.

Here are some more partially decidable relations using these properties.

Example 3.10. The following relations are partially decidable for each fixed n.

(1) $R^{(n)}(e) \iff_{\text{df}} W_e^{(n)} \neq \varnothing$.

Proof. We may rewrite our relation as:

$$W_e^{(n)} \neq \varnothing \iff (\exists x_0)\cdots(\exists x_{n-1})[\varphi_e^{(n)}(\mathbf{x})\downarrow],$$

This is partially decidable since partialy decidability is closed under many existentials and $\varphi_e^{(n)}(\mathbf{x})\downarrow$ is partially decidable. Thus $R^{(n)}$ is partially decidable. ■

(2) $Q^{(n)}(e,y) \iff_{\text{df}} y \in E_e^{(n)}$.

Proof. Same many existential trick again. ■

Now here comes a way to relate decidable and partially decidable relations.

Theorem 3.11. *Let R be an n-ary relation. The relation R is decidable if and only if R and $\neg R$ are both partially decidable.*

Proof. ($\Rightarrow$) Since R is decidable, $\neg R$ is also decidable. Then by Example 3.4, R and $\neg R$ are partially decidable.

($\Leftarrow$) Suppose R and $\neg R$ are partially decidable. Let $f,g\colon \mathbb{N} \rightharpoonup \mathbb{N}$ be such that

$$R(\mathbf{x}) \iff f(\mathbf{x})\downarrow \quad\text{and}\quad \neg R(\mathbf{x}) \iff g(\mathbf{x})\downarrow.$$

Notice that for any $\mathbf{x}$, $R(\mathbf{x})$ holds (and hence $f(\mathbf{x})\downarrow$) or $\neg R(\mathbf{x})$ holds (and hence $g(\mathbf{x})\downarrow$), but not both! Then, running both $f(\mathbf{x})$ and $g(\mathbf{x})$ and seeing which halts is a decision procedure for R since if $f(\mathbf{x})$ halts, R holds, and if $g(\mathbf{x})$ halts, R does not hold. ■

Corollary 3.12. *Any relation R which is partially decidable but not decidable has $\neg R$ not partially decidable.*

3.1.1 Graphs of computable partial functions

Now let's talk about graphs! When we talked about total functions, we saw that a total function was computable if and only if its graph was decidable. It turns out that we actually have an analagous result for partial functions. Indeed, it is:

Theorem 3.13. *Let $f\colon \mathbb{N}^n \to \mathbb{N}$ be a partial function. Then, f is computable if and only if its graph relation G_f is partially decidable.*

Proof. ($\Rightarrow$) Suppose f is computable and has a code $e \in \mathbb{N}$. Then,

$$f(\mathbf{x}) \downarrow= y \iff \exists z[T_n(e, \mathbf{x}, z) \wedge U(z) = y]$$

which is partially decidable since the statement being quantified over is indeed decidable.

($\Leftarrow$) Suppose G_f is partially decidable. Pick a decidable relation R such that

$$G_f(\mathbf{x}, y) \iff \exists z[R(\mathbf{x}, y, z)].$$

We may then define f by

$$f(\mathbf{x}) = (\mu a[R(\mathbf{x}, (a)_0, (a)_1)])_0.$$

Thus f is computable since it is defined through minimalization with a decidable relation and substitution with computable functions. ■