Math 131ABH Notes

Mealaud Mokhtarzad

This set of notes is very informal and tries its best to simplify often hard to digest ideas. Hopefully it's useful in your learning of the material.

FINISH: put figures in right places

Contents

1 Metric space topology

1.1 Introduction to metric spaces

test

2 Basics of topology

2.1 Super basics of topological spaces

We start with the titular definition.

Definition 2.1 (Topology)**.** A `topology` on a set X is a collection $\mathcal{T}$ of subsets of X such that

(1) $\varnothing, X \in \mathcal{T}$;

(2) any union of sets in $\mathcal{T}$ is in $\mathcal{T}$; and

(3) any finite intersection of sets in $\mathcal{T}$ is in $\mathcal{T}$.

The elements of $\mathcal{T}$ are called `open sets`. A topological space is a pair $(X, \mathcal{T})$.

Remark. A topology is a prescription for which sets are open.

Let's relate this back to something more familiar and intuitive: open sets in metric spaces. Recall the following properties of open sets in metric spaces: the empty set and the whole space are open; any union of open sets is open; and any finite intersection of open sets is open. I think of our definition of topology as a generalization of that.

Now let's get to some examples of topologies. (Some are gross, which seems to be a common theme in topology. Things can get pretty pathological at times.)

Example 2.2.

(1) If X is a metric space, the collection of open sets is a topology called the `metric topology`.

(2) If X is any set, the family $\mathcal{T} := \{\varnothing, X\}$ forms a topology called the `indiscrete topology`.

(3) If X is any set, the family $\mathcal{T} := \mathcal{P}(X)$ (the collection of all subsets) forms the `discrete topology`.

(4) Let $X := \{a, b\}$. Then $\mathcal{T} := \{\varnothing, \{a\}, X\}$ is a topology.

Let's start translating some of our basic metric-space-type ideas to more general topological spaces. Next we have the neighborhood (another name for an open ball in metric spaces).

Definition 2.3 (Neighborhood)**.** Let X be a topological space. An open `neighborhood` of $x \in X$ is an open set U with $x \in U$.

Recall that we would define open sets in metric spaces using interior points. We actually have a very similar idea in general topological spaces using open neighborhoods instead of the open ball.

Definition 2.4. Let X be a topological space and suppose S is a subset of X. A point $x \in X$ is an `interior point` of S if and only if there exists an open neighborhood U of x that is contained in S

The `interior` of S is the set of interior points of S and is denoted S°.

Proposition 2.5. *Let X be a topological space and let S be a subset of X. Then*

(1) S° is open and

(2) S is open if and only if $S = S^\circ$.

Proof. (1) Suppose $s \in S^\circ$. Then there exists an open neighborhood U_s of s contained in S. In fact, U_s is an open neighborhood of all elements of S in U_s. Hence, all points of U_s are interior points of S and we have that $U_s \subseteq S^\circ$. Now, notice that $\bigcup_{s \in S^\circ} U_s$ is open by properties of open sets. Moreover, notice that

$$S^\circ \subseteq \bigcup_{s \in S^\circ} U_s \subseteq S^\circ \implies S^\circ = \bigcup_{s \in S^\circ} U_s,$$

so S° is open.

(2) ($\Rightarrow$) Suppose S is open. Then S is an open neighborhood of any of its points, so $S \subseteq S^\circ \subseteq S$. Thus $S^\circ = S$. ($\Leftarrow$) If $S = S^\circ$, then S is open since S° is open. ■

Now let's get to closed sets. (They're what you think they are, if you're familiar with metric topology.)

Definition 2.6 (Closed set)**.** Let X be a topological space. A subset C of X is `closed` if and only if $X \setminus C$ is open.

It turns out that our adherent points thing from metric spaces still works but this is the more topology-core definition of closed. (We'll see this more and more in topology as we continue to study it, but things are usually defined in terms of open sets.) Let's actually get into adherent points now.

Definition 2.7 (Adherent point)**.** Let X be a topological space. A point $x \in X$ is `adherent` to a subset S of X if and only if every open neighorhood of x has nonempty intersection with S.

Now that we have that, we can get to closures.

Definition 2.8 (Closure of a set)**.** Let X be a topological space and let S be a subset of X. The `closure` of S is defined to be $X \setminus (X \setminus S)^\circ$ and is denoted $\overline{S}$. Moreover, $\overline{S}$ consists precisely of the points of X adherent to S.

Proposition 2.9. *The closure of a set is closed.*

Proof. This is clear considering that the closure is in fact the complement of an interior (which is always open). ■

Proposition 2.10. *Let X be a topological space and let S be a subset of X. Then, S is closed if and only if $S = \overline{S}$.*

Proof. ($\Leftarrow$) Suppose $S = \overline{S}$. Then S is closed (by Proposition 2.9).

($\Rightarrow$) Suppose S is closed. Then $X \setminus S$ is open (by definition), so $(X \setminus S)^\circ = X \setminus S$ by Proposition 2.5. Thus we have that

$$\overline{S} = X \setminus (X \setminus S)^\circ = X(X \setminus S) = S.$$

■

Now let's move on to sequences.

Definition 2.11. A sequence of points (x_n) in a topological space X `converges` to $x \in X$ if and only if for every open neighborhood U of x, there exists an integer N such that $x_n \in U$ whenever $n \geq N$.

Here's a neat little theorem.

Theorem 2.12. *If S is a subset of a topological space X and (x_n) is a sequence in S converging to $x \in X$, then $x \in \overline{S}$.*

Proof. Suppose towards a contradiction that $x \notin \overline{S}$. Then $x \in X \setminus \overline{S}$, which is open. Then there exists an open neighborhood of x entirely contained in $X \setminus \overline{S}$ and since (x_n) converges to x, such a neighborhood contains infinitely many terms of (x_n). But $x_n \in S$ for all n. ■

Now let's talk a bit more about adherent points. It turns out that we can use adherent points to define what the boundary of a subset of a topological space is.

Definition 2.13. Let X be a topological space and let S be a subset of X. A point $x \in X$ is a `boundary point` of S if and only if x is adherent to both S and $X \setminus S$ (i.e., $x \in \overline{S} \cap \overline{(X \setminus S)}$).

The set of boundary points, or `boundary` of S, is $\overline{S} \cap \overline{(X \setminus S)}$ and is denoted by ∂S.

Proposition 2.14. *Let X be a topological space and let S be a subset of X. Then,*

(1) $S^\circ \cap \partial S = \varnothing$ and

(2) $\overline{S} = S^\circ \cup \partial S$

Proof. (1) Suppose towards a contradiction that $S^\circ \cap \partial S$ is nonempty. Then there exists an x in the intersection. Then there exists an open neighborhood of x that is completely contained inside S. However, since $x \in \partial S$, every open neighorbood intersects $X \setminus S$ nontrivially. Thus we have reached a contradiction and $S^\circ \cap \partial S$ is empty.

(2) ($\supseteq$) Since $S^\circ \subseteq S \subseteq \overline{S}$ and $\partial S \subseteq \overline{S}$, we have that $S^\circ \cup \partial S \subseteq S$. ($\subseteq$) Suppose $x \in \overline{S}$. Then, either there exists an open neighborhood U of x with $U \subseteq S$ or not. In the former case, $x \in S^\circ$. In the latter case, any open neighorhood of x intersects $X \setminus S$, so $x \in \overline{(X \setminus S)}$, so $x \in \overline{S} \cap \overline{(X \setminus S)}$, so $x \in \partial S$. ■

Now let's move on to I think our shortest section: subspaces.

2.2 Subspaces

Definition 2.15. Let S be a subset of a topological space $(X, \mathcal{T})$. The family

$$\mathcal{S} := \{U \cap S : U \in \mathcal{T}\}$$

is a topology for S called the `subspace topology`. (This is also called the `relative topology`.) The elements of $\mathcal{S}$ are called `relatively open` subsets of S.

Here's an example of this:

Example 2.16. If X is a metric space and Y is a subspace of X, the topology on Y obtained by restricting the metric to Y coincides with the relative topology.