Math 132H Notes

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This set of notes is very informal and tries its best to simplify often hard to digest ideas. Hopefully it's useful in your learning of the material.

FINISH: put figures in right places

Contents

Lecture 1—March 29, 2021

Today felt more like an overview than anything/basic introduction, so we didn't do anything too crazy today, but it was still pretty cool.

We first define the complex numbers.

Definition 1.1. We define $\mathbb{C} \coloneqq \{x + iy : x, y \in \mathbb{R} \wedge i^2 = -1\}$.

Notice that $\mathbb{C}$ is a 2D basically $\mathbb{R}^2$ and is a field, but the i gives us some cool stuff. It's also a field since we can add, subtract, multiply, and divide.

Here are some basic operations/facts with $z \coloneqq x + iy$:

(a) $\overline{z} \coloneqq x - iy$;

(b) $z \cdot \overline{z} = |z|^2 = x^2 + y^2$;

(c) $\Re(z) \coloneqq x$ and $\Im(z) \coloneqq y$.

We also have that $\mathbb{C}$ is a metric space with respect to $|z|^2$. Thus it also satisfies the triangle inequality: $|z + w| \leq |z| + |w|$.

We then talked about Euler's Identity/Formula for a bit but it culminated with this fact:

Fact 1.2. *We may represent complex numbers as $re^{i\theta}$ for suitable r and θ.*

Now we're talking about sequences in $\mathbb{C}$.

Definition 1.3. Let (z_n) be a sequence in $\mathbb{C}$. We say that $\lim_{n\to\infty} z_n = w$, or (z_n) *converges to w* if and only if $|z_n - w| \to 0$ as $n \to \infty$. Equivalently, for all $\varepsilon > 0$ there exists an N such that $n \geq N$ implies that $|z_n - w| < \varepsilon$.

Definition 1.4. Let (z_n) be a sequence in $\mathbb{C}$. We say that (z_n) is a *Cauchy sequence* if and only if for all $\varepsilon > 0$, there exists an N such that $|z_n - z_m| < \varepsilon$ whenever $n, m \geq N$.

Now onto our first theorem!

Theorem 1.5. *The set $\mathbb{C}$ is complete (i.e., all Cauchy sequences in $\mathbb{C}$ converge in $\mathbb{C}$).*

Topology of the $\mathbb{C}$

Then we talked a bit about topology and how we're lucky we have a metric because that gives us a lot of nice properties. The first thing we really talked about regarding topology is disc in $\mathbb{C}$.

Definition 1.6. We define the following sets for $r > 0$ and $z_0 \in \mathbb{C}$:

(a) $D_r(z_0) \coloneqq \{z \in \mathbb{C} : |z - z_0| < r\}$ (*open disc of radius r in* $\mathbb{C}$);

(b) $\overline{D}_r(z_0) \coloneqq \{z \in \mathbb{C} : |z - z_0| \leq r\}$ (*closed disc of radius r in* $\mathbb{C}$);

(c) $C_r(z_0) \coloneqq \{z \in \mathbb{C} : |z - z_0| = r\}$ (*circle of radius r in* $\mathbb{C}$).

Definition 1.7. Given a set $\Omega \subseteq \mathbb{C}$ and a point $z_0 \in \Omega$, z_0 is an *interior point of* Ω if and only if there exists an open disc $D_r(z_0) \subseteq \Omega$.

The *interior of* Ω is the set of all interior points of Ω. This is denoted $\text{int}(\Omega)$.

The set Ω is open if and only if $\Omega = \text{int}(\Omega)$.

Definition 1.8. A set $\Omega \subseteq \mathbb{C}$ is closed if and only if $\mathbb{C} \setminus \Omega$ is open.

Remark. Equivalently, Ω is closed if and only if it contains all of its limit points.

This leads us into int's counterpart: closure.

Definition 1.9. The closure of Ω is the union of Ω with all of its limit points. This is denoted $\overline{\Omega}$.

We're just going to keep loading up on definitions for now.

Definition 1.10. The boundary of Ω is $\overline{\Omega} \setminus \text{int}(\Omega)$.

Because we are in a metric space, we have a sense of being bounded. This means that the distance between points is bounded. The actual definition for this that we use in this class is this:

Definition 1.11. A set $\Omega \subseteq \mathbb{C}$ is bounded if and only if it is contained in some $D_r(z_0)$ for some finite r.

Now to compactness! This guy is pretty important. The professor said Heine-Borel is good exercise. Pretty good. b)

Definition 1.12. A set $\Omega \subseteq \mathbb{C}$ is *compact* if and only if Ω is closed and bounded.

Theorem 1.13. *Let Ω be a subset of $\mathbb{C}$. Then, Ω is compact if and only if every sequence has a subsequence that converges in Ω.*

Remark. This is known as *sequential compactness*. Very important property.

Now we can get to the definition of *covering compactness*.

Definition 1.14. Let Ω be a subset of $\mathbb{C}$. An *open cover of* Ω is a family of open sets $\{U_\alpha\}$ (not necessarily countable) such that

$$\Omega \subseteq \bigcup_\alpha U_\alpha.$$

Theorem 1.15. *A subset Ω of $\mathbb{C}$ is compact if and only if every open cover has a finite subcover.*

Now to our first proposition/proof (though it is said to be trivial haha). However, before this we need a quick definition.

Definition 1.16. The diameter of a set $\Omega \subseteq \mathbb{C}$ is defined to be $\sup S$ where $S := \{|z - w| : z, w \in \mathbb{C}\}$.

Proposition 1.17. *Suppose $\Omega_1 \supseteq \Omega_2 \supseteq \cdots \supseteq \Omega_n \supseteq \cdots$ is a sequence of nonempty compact sets such that $\text{diam}(\Omega_n) \to 0$ as $n \to \infty$. Then there is a unique $w \in \mathbb{C}$ such that $w \in \Omega_n$ for all n.*

Proof. Choose $z_n \in \Omega_n$ for each n. Then (z_n) is a Cauchy sequence. Let $w := \lim_{n\to\infty} z_n$ (this exists since $\mathbb{C}$ is complete). Then w is clearly in Ω_n for all n and it also trivially unique (since a convergent sequence can only have 1 limit).

My question: how did we get that there is only one sequence? Couldn't we get a bunch? ■

Now to connectedness.

Definition 1.18. A set $\Omega \subseteq \mathbb{C}$ is *connected* if and only if it is not possible to express Ω as the disjoint union of nonempty sets Ω_1 and Ω_2 such that $\overline{\Omega}_1 \cap \Omega_2 = \Omega_1 \cap \overline{\Omega}_2 = \varnothing$.

Remark. In the case of an open Ω, this definition reduces to the following: Ω is connected if and only if you cannot express Ω has the union of disjoint nonempty open sets.

Definition 1.19. A *region* is an open and connected set.

Functions on regions

Definition 1.20. Let Ω be a subset of $\mathbb{C}$ and let $f \colon \Omega \to \mathbb{C}$. The function f is *continuous at a point* $z_0 \in \Omega$ if and only if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $z \in \Omega$ and $|z - z_0| < \delta$, $|f(z) - f(z_0)| < \varepsilon$.

The function f is *continuous on* Ω if and only if it is continuous at every $z_0 \in \Omega$.

Theorem 1.21. *A continuous f on a compact Ω is bounded and attains a maximum and a minimum.*

We talk about holomorphic functions (complex differentiable functions) next!

Lecture 2—March 31, 2021

Complex Derivatives (Holomorphic functions)

Let $\Omega \subseteq \mathbb{C}$ be open. Let $f\colon \Omega \to \mathbb{C}$. Let $F(x,y) = (u(x,y), v(x,y))$ be a related vector function. Define f by $f(z) = u(x,y) + iv(x,y)$.

Definition 2.1. A function $f\colon \Omega \to \mathbb{C}$ ($\Omega \subseteq \mathbb{C}$ open) is *differentiable* at $z_0 \in \Omega$ if there is a $f'(z_0) \in \mathbb{C}$ such that

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = f'(z_0).$$

Remark. This limit must be independent of how h goes to 0. This causes it to be stronger than the real variable definition.

Definition 2.2. A function $f\colon \Omega \to \mathbb{C}$ on an open subset Ω of $\mathbb{C}$ is *holomorphic on* Ω if it differentiable at every $z_0 \in \Omega$.

If C is a closed set then f is *holomorphic on* C if it is holomorphic on an open set containing C.

If $\Omega = \mathbb{C}$ and if f is holomorphic on $\mathbb{C}$, f is said to be *entire*.

Example 2.3.

(1) $f(z) = z^n$, $n \geq 0$ is entire. In this case, $f'(z) = nz^{n-1}$.

(2) $f(z) = z^n$ where $n < 0$ is holomorphic on $\mathbb{C} \setminus \{0\}$.

(3) $f(z) = \overline{z}$ is NOT holomorphic anywhere.

Remark. Verify all of these! Good practice.

Remark. A holomorphic function is continuous.

Proposition 2.4. *Let $f, g\colon \Omega \to \mathbb{C}$ ($\Omega \subseteq \mathbb{C}$ open) be holomorphic and $c \in \mathbb{R}$. Then*

(i) $f + g$ is holomorphic on Ω and $(f+g)' = f' + g'$,

(ii) $(cf)' = cf'$,

(iii) Standard product rule holds,

(iv) Quotient rule holds,

(v) Chain rule: $f\colon \Omega \to U$, $g\colon U \to \mathbb{C}$, both holomorphic, then $(g \circ f)' = g'(f(z))f'(z)$.

Complex function as a mapping

We can identify complex functions with real functions. Like, $f(z) = u + iv$ and $F(x,y) = (u(x,y), v(x,y))$ ($F\colon \mathbb{R}^2 \to \mathbb{R}^2$).

Recall that F is *differentiable at* $P_0 = (x_0, y_0)$ if there exists $J\colon \mathbb{R}^2 \to \mathbb{R}^2$ linear such that

$$\frac{|F(P_0 + H) - F(P_0) - J(H)|}{|H|} \to 0$$

as $|H| \to 0$. Equivalently this is:

$$|F(P_0 + H) - F(P_0) - J(H)| = |H|\psi(H)$$

where $\psi(H) \to 0$ as $|H| \to 0$.

Remark. Read this over.

If J exists it is unique and also u_x, u_y, v_x, and v_y (partial derivatives) exist. In fact,

$$J = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$

gives the linear function.

Remark.

$$x + iy \mapsto x \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + y \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} x & y \\ -y & x \end{bmatrix}$$

since

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^2 = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This actually stores all the same information as complex numbers (addition, i, multiplication, commutativity, etc.). We could actually take this further to make things like quarternions and such. Neat stuff.

The way we see truth in the comment ”It's easier to a matrix than a number” with how complex numbers are special types of matrices.

Notice that

$$J = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$

having this form would mean that $v_x = v_y$ and $u_y = -v_x$. (These are called the Cauchy-Riemann Equations.) If J satisfies this, then it's related to being complex differentiable.

Consider

$$f'(z_0) = \lim_{h_1 \to 0} \frac{f(x_0 + h_1 + iy_0) - f(x_0 + iy_0)}{h_1} = \frac{\partial f}{\partial x}(z_0) = (u_x + iv_x)(z_0).$$

where $h = h_1 + ih_2$. Also, letting $h = -ih_2$, we must have

$$f'(z_0) = \lim_{h_2 \to 0} \frac{f(x_0 + i(y_0 + h_2)) - f(x_0 + iy_0)}{ih_2} = \frac{1}{i}\frac{\partial f}{\partial y}(z_0) = -i(u_y + iv_y) = v_y - u_y.$$

Thus if f is complex differentiable at z_0, then the Cauchy-Riemann Equations $u_x = v_y$ and $u_y = -v_x$ must hold at $z_0 = x_0 + iy_0$.

So, just because a real function is differentiable, that does not mean it is complex differentiable.

Example 2.5. $F(x, y) = (x, -y)$ is real differentiable but not complex differentiable since it does not satisfy the Cauchy-Riemann equations as $u(x, y) = x$, $v(x, y) = -y$, $u_x = 1$, and $v_y = -1$.

Real sense:

$$J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

so we're real differentiable, but this clearly fails to be complex differentiable by the Cauchy-Riemann Equations.

Notation 2.6.

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

huh??? what the

Proposition 2.7. *If f is holomorphic at z_0 then*

(i) $\frac{\partial f}{\partial \bar{z}}(z_0) = 0$ *and* $f'(z_0) = \frac{\partial f}{\partial z}(z_0) = \partial \frac{\partial u}{\partial z}(z_0)$.

(ii) *Writing* $F(x,y) = (u(x,y), v(x,y)) = f(z)$. *Then* F *is differentiable in the real sense and*

$$\det(J_F(x_0,y_0)) = |f'(z_0)|^2.$$

Proof. (i)

$$\begin{aligned}
\frac{\partial f}{\partial z}\frac{2(u+iv)}{2z} &= \frac{1}{2}\left(\frac{\partial}{\partial x}(u+iv) - i\frac{\partial}{\partial y}(u+iv)\right) \\
&= \frac{1}{2}(u_x + iv_x - i(u_y + iv_y)) \\
&= \frac{1}{2}\left(2u_x - 2iu_y\right) \\
&= 2\frac{\partial u}{\partial z}.
\end{aligned}$$

Check other formula on your own.

(ii) We want to show that if $H = (h_1, h_2)$, $h = h_1 + ih_2$,

$$J_F(x_0,y_0)(H) = \left(\frac{\partial u}{\partial x} - i\frac{\partial v}{\partial y}\right)(h_1 + ih_2).$$

We first expand the LHS:

$$J_F(x_o,y_0)(H) = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} h_1u_x + h_2u_y \\ h_1v_x + h_2v_y \end{bmatrix}.$$

We now expand the RHS

$$\frac{\partial u}{\partial x}h_1 + \frac{\partial u}{\partial y}h_2 + i\left(\frac{\partial u}{\partial x}h_2 - \frac{\partial u}{\partial y}h_1\right) = \frac{\partial u}{\partial x}h_1 + \frac{\partial u}{\partial y}h_2 + i\left(\frac{\partial v}{\partial y}h_2 + \frac{\partial v}{\partial x}h_1\right).$$

Then we can identify the LHS with the RHS by treating a complex number as vector!

Now,

$$\det(J_F(x_0,y_0)) = u_xv_y - u_yv_x.$$

We then have that

$$\text{Cauchy-Riemann} \implies u_x^2 + u_y^2 = \left|2\frac{\partial u}{\partial z}\right|^2$$

since

$$\frac{\partial u}{\partial z} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right),$$

so $\det(J_F) = |f'(z_0)|^2$. ∎