Math 191 Notes

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This set of notes is very informal and tries its best to simplify often hard to digest ideas. Hopefully it's useful in your learning of the material.

FINISH: put figures in right places

Contents

Lecture 1—March 29, 2021

Categories

History: Eilenberg and MacLane needed to make sense of "naturality." Here are a bunch of examples of natural maps!

Example 1.1. (a) Let X be a set and let $\mathcal{P}(X)$ be the powerset of X. Then there is a natural function from $X \to \mathcal{P}(X)$ defined by $x \mapsto \{x\}$.

(b) For any sets X and Y, let $Y^X := \{f : X \to Y\}$. Then there exists a natural bijection from $\mathcal{P}(X)$ to $\{0,1\}^X$ by $A \mapsto \chi_A$ where χ_A is the characteristic function of A.

(c) For any sets X and Y, $X \times Y := \{(x,y) : x \in X \wedge y \in Y\}$. Then there is a natural bijection from $X \times Y$ to $Y \times X$: $(x,y) \mapsto (y,x)$.

How do we make this "naturality" precise, however? Well, there are a few problems that Eilenberg and MacLane had to face. Here they are:

- We're talking about "natural maps"—they need domains and codomains (these are called functors)
  - Functor is a "construction"
  - Functors also have inputs and outputs $\rightsquigarrow$ some kind of mapping, so they need domain and codomain, also.
  - The domain and codomain of functors are *categories*.

Now let's define what a category is. Spoiler: it's actually really long lmao.

Definition 1.2. A *category* $\mathcal{C}$ consists of

(1) a collection of objects $A, B, C, \ldots,$

(2) and a collection of morphisms (arrows) $f, g, h, \ldots$

such that

(i) each morphism has a domain and a codomain object. We write $f : A \to B$ as a shorthand for "f is a morphism with domain A and codomain B," and we write $\mathcal{C}(A,B)$ for the collection of all morphisms $f : A \to B$.

(ii) Each object A has an idntity morphism $1_A : A \to A$.

(iii) For any pair of "composable morphisms" g and f with $\mathrm{Dom}(g) = \mathrm{Cod}(f)$, there is a composite morphism $g \circ f$ with $\mathrm{Dom}(g \circ f) = \mathrm{Dom}(f)$ and $\mathrm{Cod}(g \circ f) = \mathrm{Cod}(g)$. This is exemplified in the following diagram:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

These data are subject to two axioms:

(C1) (associativity) for any $f : A \to B$, $g : B \to C$, and $h : C \to D$, $(h \circ g) \circ f = h \circ (g \circ f)$; and

(C2) (unitality) for any $f : A \to B$, $f \circ 1_A = f = 1_B \circ f$.

Finally that definition is done. It's the longest I've ever seen (so far). Now to some examples of categories.

Example 1.3. (a) The category of sets consists of all sets and all functions between sets. We write Set.

(b) A *pointed set* is a pair (X, x) where $x \in X$ is a distinguished element. A morphism $f\colon (X, x) \to (Y, y)$ is a function $f\colon X \to Y$ such that $f(x) = y$. We denote this Set_*.

(c) A *monoid* is a triple $(M, \cdot, e)$ such that

(i) M is a set,

(ii) $\cdot\colon M \times M \to M$ is a binary operation, and

(iii) $e \in M$ is a distinguished element

such that

(M1) (associativity) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in M$ and

(M2) (unitality) $e \cdot x = x = x \cdot e$ for all $x \in M$.

A *monoid homomorphism* $f\colon (M, \cdot_M, e_M) \to (N, \cdot_N, e_N)$ is a function $f\colon M \to N$ such that

(i) $f(e_M) = e_N$ and

(ii) $f(x \cdot_M y) = f(x) \cdot_N f(y)$.

These data assemble into the category Mon.

(d) A group is a quadruple $(G, \cdot, e, (-)^{-1})$ such that

(i) G is a set,

(ii) $\cdot\colon G \times G \to G$ is a binary operation,

(iii) e is a distinguished element, and

(iv) $(-)^{-1}\colon G \to G$ is a unary operation

such that

(G1) $(G, \cdot, e)$ is a monoid and

(G2) $x^{-1} \cdot x = e = x \cdot x^{-1}$ for all $x \in G$.

A *group homomorphism* $f\colon (G, \cdot_G, e_G, (-)^{-1}_G) \to (H, \cdot_H, e_H, (-)^{-1}_H)$ is a function $f\colon G \to H$ such that

(i) $f(xy) = f(x)f(y)$ for all $x, y \in G$,

(ii) $f(e_G) = e_H$, and

(iii) $f(x^{-1}) = f(x)^{-1}$ for all $x \in G$.

We denote this category Grp.

(e) A *preorder* is a pair $(P, \leq)$ such that P is a set and $\leq$ is a binary relation on P such that

(P1) (reflexivity) $x \leq x$ for all $x \in P$ and

(P2) (transivitiy) $x \leq y$ and $y \leq z$ implies $x \leq z$ for all $x, y, z \in P$.

A morphism of preorders $f\colon P \to Q$ is a function such that $x \leq y$ implies $f(x) \leq f(y)$ (*order-preserving functions*).

We denote this category Preord.

Definition 1.4. Let $\mathcal{C}$ be a category. A *subcategory* $\mathcal{D}$ of $\mathcal{C}$ consists of a collection of objects of $\mathcal{C}$ and a collection of morphisms of $\mathcal{C}$ such that

(1) (closed under domain/codomain) if $f: A \to B$ is in $\mathcal{D}$, then so are A and B;

(2) (closed under composition) if $f: A \to B$, $g: B \to C$ are in $\mathcal{D}$, then so is $g \circ f$; and

(3) (contains identities) if A is an object of $\mathcal{D}$, then so is 1_A.

Now to examples, again.

Example 1.5. (a) The collection of all finite sets and all the maps between them is a subcategory of Set.

We denote this category FinSet.

(b) A *commutative monoid* is a monoid $(M, \cdot, e)$ such that $x \cdot y = y \cdot x$ for all $x, y \in M$. The collection of all commutative monoids and monoid homomorphisms between tehm for a subcategory of Mon.

We denote this category CMon.

(c) An *abelian group* is a group $(G, \dots)$ such that $\cdot$ is commutative. This is a subcategory of Grp.

We denote this category Ab.

(d) A *poset* $(P, \leq)$ is a preorder $(P, \leq)$ such that $\leq$ is antisymmetric, i.e. $x \leq y$ and $y \leq x$ implies that $x = y$. The collection of all posets and order-preserving maps between them form a subcategory of Preord.

We denote this category Pos.

Definition 1.6. A subcategory $\mathcal{D}$ of $\mathcal{C}$ is *full* if and only if for any objects A and B of $\mathcal{D}$, every morphism $f: A \to B$ in $\mathcal{C}$ is also in $\mathcal{D}$.

Lecture 2—March 31, 2021

Recall

Last time introduced categories. Categories are composed of objects and morphisms. Morphisms are composed of domain and codomain, identity maps, associativity, and unital (i.e., identity acts like identity).

Example 2.1. Set, Set_*, Mon, Grp, Preord, etc. Also, subcategories: $\mathsf{CMon} \subseteq \mathsf{Mon}$, $\mathsf{Ab} \subseteq \mathsf{Grp}$, $\mathsf{Pos} \subseteq \mathsf{Preord}$.

Notice that the categories in this example are collections of "structured sets" plus structure preserving maps between them.

Question 2.2. Can we encode familiar properties of functions categorically?

Definition 2.3. Suppose that $\mathcal{C}$ is a category. A morphism $f\colon A \to B$ in $\mathcal{C}$ is an *isomorphism* if it has a two-sided inverse, i.e., if there exists a morphisms $g\colon B \to A$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$.

Two objects $A, B \in \mathcal{C}$ are *isomorphic* if there is an isomorphism between them, in which case, we write $A \cong B$.

Example 2.4. In each of the categories Set, Set_*, Mon, Grp, Preord, an isomorphism is just a bijective set map that preserves all structure (operations & distinguished elements).

Example 2.5. In each of categories Preord and Pos, an isomorphism $f\colon P \to Q$ is a bijective set map such that

$$x \leq y \iff f(x) \leq f(y).$$

Upshot: Purely categorical condition of invertibility (having a two-sided inverse) encodes bijective correspondences that identify structure (in these "concrete" cases).

Encoding injectivity and surjectivity

Definition 2.6. Let $\mathcal{C}$ be a category. A morphism $f\colon A \to B$ in $\mathcal{C}$ is a *monomorphism* if: for any $T \in \mathcal{C}$ and $h, k\colon T \rightrightarrows A$ if $fh = fk$, then $h = k$.

Similarly, a morphism $f\colon A \to B$ in $\mathcal{C}$ is an *epimorphism* if for any $T \in \mathcal{C}$ and $h, k\colon B \to T$, if $hf = kf$, tehn $h = k$.

Example 2.7. Suppose $f\colon A \to B$ is in Set.Then f is monomorphism if and only if f is injective, and f is epimorphism if and only if f is surjective.

Remark. These don't correspond in all categories.

We can say a bit more in Set. If $r\colon A \to B$ is and epimorphism in Set, then there is a function $s\colon B \to A$ such that $r \circ s = 1_B$.

Definition 2.8. Suppose $r\colon A \to B$ and $s\colon B \to A$ are morphisms such that $r \circ s = 1_B$. Then s is a *section* or *right inverse* to r, and r is a *retraction* or *left inverse* to s. In general, we call a morphism a *split epimorphism* if it has a section, and a *split monomorphism* if it has a retraction.

Remark. Split monomorphism $\implies$ monomorphism.

Remark. Split epimorphism $\implies$ epimorphism.

Remark. Isomorphism $\implies$ split monomorphism and split epimorphism $\implies$ monomorphism and epimorphism

WARNING: converse is false in general.

Categories as algebraic objects

We can view categories as algebraic objects.

We can regard a number of familiar mathematical objects as categories.

Example 2.9. A *preorder category* is a category $\mathcal{P}$ with at most one morphism in $\mathcal{P}(A, B) := \{f \colon A \to B\}$ for any $A, B \in \mathcal{P}$. If $\mathcal{P}$ is a preorder category with only a Set's worth of morphisms, and we define

$$A \leq B \iff \exists f \colon A \to B$$

for all $A, B \in \mathcal{P}$, then $(\mathrm{Ob}(\mathcal{P}), \leq)$ is a preorder. Conversely, if $(P, \leq)$ is a preorder, and we define a category $\mathcal{P}$ by taking $\mathrm{Ob}(\mathcal{P}) = P$ and $\mathrm{Mor}(\mathcal{P}) := \{(A, B) \in P \times P : A \leq B\}$ and set $\mathrm{Dom}(A, B) = A$ and $\mathrm{Cod}(A, B) = B$, $\mathrm{id}_A = (A, A)$, and $(B, C) \circ (A, B) = (A, C)$, then we get a preorder category.

Preorders and preorder categories are basically the same thing.

Example 2.10. A *monoid category* (nonstandard terminology) is a category with a single object:

$$\begin{tikzcd} A \arrow[loop above] \arrow[loop left] \arrow[loop right] \end{tikzcd}$$

If $\mathcal{M}$ is a monoid category with object $A \in \mathcal{M}$, then $(\mathcal{M}(A, A), \circ, 1_A)$ is a monoid. Conversely, if $(M, \cdot, e)$ is a monoid, then the category $\mathsf{B}M$, with single object $*$ and morphisms $\mathcal{M}$ (all with domain/codomain $*$) is a category with $1_* = e$ and $y \circ x = y \cdot x$.

$$\boxed{M}$$
$$\begin{tikzcd} * \arrow[loop above] \end{tikzcd}$$

Example 2.11. A *group category* (nonstandard terminology) is a monoid category $\mathcal{G}$ such that every morphism of $\mathcal{G}$ is an isomorphism. The same constructions relating monoid and monoid categories restrict to constructions relating groups and group categories.

These examples justify the following terminology:

Definition 2.12. A *groupoid* is a category in which every morphism is an isomorphism.

$$\begin{tikzcd} \bullet \arrow[r, "\cong"] \arrow[d, bend left, "\cong"] & \bullet \\ \bullet \arrow[u, bend left, "\cong"] \arrow[ur, "\cong"'] & \end{tikzcd}$$

Remark. Groupoid is $\approx$ "many object group".

Definition 2.13. The *core* of a category $\mathcal{C}$ is the subcategory with objects consisting of all objects of $\mathcal{C}$ and morphisms consisting of all isomorphisms in $\mathcal{C}$.

Size consideration

Russel's Paradox: there is not set of all sets.

The collection of all sets is "too large" to be a set. (Grothendieck universes can be used to stratify sets by their size, so we have "small sets", "large sets", "even larger sets", etc.)

Definition 2.14. Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ is *small* if $\mathrm{Mor}(\mathcal{C})$ is a set (also implies that the $\mathrm{Ob}(\mathcal{C})$ is a set). We say that $\mathcal{C}$ is *locally small* if $\mathcal{C}(A, B) := \{f : A \to B\}$ is a set for all $A, B \in \mathcal{C}$. A category $\mathcal{C}$ is *large* if it is **not** small.

Example 2.15. Set, Set_*, Mon, Grp, Preord, etc. are locally small, large categories.

Example 2.16. The preorder category associated with a preorder, monoid category associated to a monoid, and a group category associated to a group are small categories.