Final Project Math 3329

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1 Introduction

Throughout the Spring 2024 term, we have learned about many isometries in Euclidean Geometry. Isometries are transformations of the plane $\mathbb{R}^2 \to \mathbb{R}^2$ that preserve distance. There are two types of isometries; opposite or direct. A direct isometry preserves orientation, while an opposite isometry reverses orientation. We started with learning about basic translations, reflections, rotations and glide reflections. Following that, we learned about how we write these isometries analytically. Once we were finished with that, we proceeded to learn about other types of transformations of the planes like dilations. We also learned how to write these analytically. The purpose of this paper is to thoroughly summarize these isometries and dilations in a way that is clear and concise. Included in the paper will also be some applets to explore and help build a better understanding.

1.1 Reflections, r_l

A reflection $r_l : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a transformation of the plane $\mathbb{R}^2$ along line l such that $\forall A \in \mathbb{R}^2$, $A' = r_l(A)$, where line l is the perpendicular bisector of the segment connecting A to A'. A reflection can be either a direct isometry or an opposite isometry. A product of an even number of reflections is a direct isometry since it will preserve orientation. A product of an odd number of reflections is an opposite isometry since it will reverse orientation. In notation, when there are multiple reflections happening, we reverse the order. For instance, if we wanted to do r_l, followed by r_m followed by r_n, we would write $r_n r_m r_l$. Products of reflections can be equivalent to other isometries that will be explored later in this paper. In fact, every isometry can be written as a product of at most three reflections. For instance, the identity transformation can be expressed as a product of 0 reflections, or two reflections about the same line. A reflection through the origin can be written analytically as follows

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} cos2\theta & -sin2\theta \\ sin2\theta & cos2\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \forall (x,y) \in \mathbb{R}^2$$

Where θ is the directed positive angle from the positive x-axis to the line m. This equation allows us to reflect about the x-axis and then rotate by an angle of 2 θ about the origin.

A reflection that does not pass through the original can be written analytically as follows

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} cos2\theta & -sin2\theta \\ sin2\theta & cos2\theta \end{pmatrix} \begin{pmatrix} x-h \\ -(y-k) \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} \forall (x,y) \in \mathbb{R}^2$$

This equation allows us to translate the plane to the origin, reflect about the origin, rotate by 2 θ and then translate the origin back to the original point.

Here is an **applet** that will help you visualize this isometry.

Directions: First, try moving point E around to see what happens to the triangles. Notice that only triangle $A'B'C'$ is moving. This is because it is being reflected by line f, so when the line of reflection moves, so does the reflected triangle. Also notice how all of the points are reversed. This means that this reflection is an opposite isometry.

Now select the second half of the options on the left menu to complete two reflections. Notice that the orientation of $A''B''C''$ is now the same as ABC. This is a direct isometry.

Select the option to reflect about line. Highlight the triangle $A'B'C'$ and then select line f. Notice how that new triangle is exactly the same as the initial triangle. That is the identity transformation, which was a product of two reflections made on line f.

1.2 Translations

A translation, $t_{\vec{PQ}} : \mathbb{R}^2 \to \mathbb{R}^2$ is a transformation of the plane along vector $\vec{PQ}$ such that $\forall A \in \mathbb{R}^2$, $A' = t_{\vec{PQ}}(A)$, where $\vec{PQ}$ is the vector parallel, similarly directed, and congruent to $\overline{AA'}$. A product of two reflections $r_m r_n$ is equal to a translation $t_{\vec{PQ}}$ where the $\vec{PQ}$ is perpendicular to m and n, and the magnitude of $\vec{PQ}$ is twice the distance between lines m and n. Similarly, If $t_{\vec{PQ}}$ is a translation through the vector $\vec{PQ}$, then $t_{\vec{PQ}} = r_m r_n$ where m and n are parallel lines, both perpendicular to $\vec{PQ}$ and having the property that the distance between lines m and n is $\frac{1}{2}|\vec{PQ}|$. Every direct isometry is either a translation or a rotation.
Translations are written analytically as
$t_{\vec{v}}(x', y') = (x + h, y + h)\ \forall x, y \in \mathbb{R}^2$
where (h, k) is the vector of translation.

Here is an **applet** that will help you visualize this isometry.
Directions: Begin by moving the point E around to move the vector and observe how triangle $A'B'C'$ moves depending on how the vector is directed and its length. Next, select the other half of the buttons on the left menu to show the distances between AA', BB', CC'. Notice that all of the lines are congruent and parallel to $\vec{DE}$; this is the definition of a translation. Notice that the orientation is preserved. Recall that a product of 2 reflections is a translation and a direct isometry. That means this could also be shown with a reflection between triangles ABC and $A'B'C'$.

1.3 Rotations

A rotation $R_{Q,\theta} : \mathbb{R}^2 \to \mathbb{R}^2$ is a transformation of the plane about point Q through angle θ such that $\forall A \in \mathbb{R}^2$, $A' = R_{Q,\theta}(A)$ where either $A = A' = Q$ or $\overline{QA} \cong \overline{QA'}$ and $m < AQA' = \theta$. The point Q is called the center of rotation. The product of two reflections $r_m r_n$ can be a rotation about Q through angle 2θ if the two lines m and n intersect at a point Q where the angle from m to n measures $\theta > 0$. Every direct isometry is either a translation or a rotation. A rotation about the origin is written analytically as follows

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \forall (x, y) \in \mathbb{R}^2$$

A rotation that doesn't pass through the origin is written analytically as follows

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix} \begin{pmatrix} x - h \\ -(y - k) \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} \forall (x, y) \in \mathbb{R}^2$$

This equation allows you to translate the plane to the origin, perform the rotation, and translate the plane back to its original state.

Here is an **applet** that will help you visualize this isometry.
Directions: Move the slider to change the angle of rotation and observe how the triangle moves. Then move point C around and observe the changes. Now select the rest of the list on the left side to show the reflection about lines m and n. Move the slider around again and observe. Notice that when $\theta = 56°$, triangles JAI and $F'G'E'$ are the same. Recall that a product of two reflections is a translation.

1.4 Glide Reflections

A glide reflection is the product of a translation followed by a reflection about a line that is parallel to the vector of translation. A glide reflection is a simplified transformation of a product of three reflections $r_l r_n r_m$ across any three lines in $\mathbb{R}^2$. Additionally, every opposite isometry is a glide reflection.

Here is an **applet** that will help you visualize this isometry.
Directions: Move point A around and observe how the 3 reflected triangles move. Observe the 4 triangles and their orientation. Notice that triangle $D''E''F''$ preserves triangle DEF's orientation. Recall that a translation preserves orientation. That means there must be a vector of translation between these two triangles. Select the vector (F, F') from the left menu to reveal the vector. Notice that the vector is parallel to line h. Next observe the orientation of triangle $F'''E'''D'''$. The orientation is flipped, then that transformation must be a product of one or three reflections. Select the rest if the options from the left menu to reveal the reflection of triangle $D''E''F''$. Notice that this triangle is exactly the same as $F'''E'''D'''$. Then the glide reflection is a product of a translation, followed by a reflection.

1.5 Dilations

A dilation, also called a homothety, $H_{p,k}$: $\mathbb{R}^2 \to \mathbb{R}^2$ is a transformation of the plane such that $\forall A \in \mathbb{R}^2$, $A' = H_{p,k}(A)$ such that P, A and A' are all colinear and $|PA'| = k * |PA|$ $\forall k \in \mathbb{R}^2, k \neq 0$. The notation $H_{p,k}$ expresses a dilation through point p by scale factor k. A dilation is also a similarity, s_k since all distances are changed proportionally by a scale factor k, therefore dilations preserve the properties of similarities. For instance, similar triangles preserve angles. A dilation is not always an isometry. $H_{p,1}$ is called the identity transformation. $H_{p,-1}$ is a rotation $R_{p,180}$ about point p by an angle of 180 degrees. Whenever $k \neq 1, -1$ the dilation is not an isometry. A dilation $H_{P,k}$ can be found graphically by plotting two given points, (A, B) and their images, $(A'B')$. By definition, $\frac{|A'B'|}{|AB|} = k$. We can use this equation to solve for the scale factor, k. Then we can solve for $|PA'| and |PB|$ to find the dilation mapping $|AB|$ to $|A'B'|$.

The product of two dilations $H_{p,k}$ * $H_{q,j}$ with $k * j \neq 1$ is the dilation $H_{x,k*j}$ where x is a point colinear with p and q. If $k * j = 1$ then $H_{p,k}$ * $H_{q,j}$ is a translation.

Dilations through (0,0) can be written analytically as follows:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = k * \begin{pmatrix} x \\ y \end{pmatrix}$$

Dilations not through (0,0) can be written analytically as follows:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = k * \begin{pmatrix} x - a \\ y - b \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$

This equation allows us to translate the plane to (0,0), dilate through (0,0) and translate back to its original point.